FOURIER SERIES AND ANALYSIS AND APPLICATIONS

by

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Fourier Series and Analysis and Applications

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اعادة كتابة وصياغة وحدات ٣ و ٤ ، ترتيب المراجع ، تصحيح الأخطاء اللغوية ، إتباع نظام الترقيم المناسب ، اضافة وحذف للاشكال المبينة ، توضيح بعض المعادلات

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Dedication

This thesis is respectfully dedicated to my mother, my father, my brothers and my sisters.

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Summary

Because of the important role that the *Fourier* series and *Fourier* transforms play in physics and engineering, we have therefore focussed our attention in this thesis on the theory of *Fourier* series and its applications.
Convolution theory and its relation to the transform methods has been investigated. Orthogonal and trigonometric system in two variables together with double *Fourier* series for a function with different periods have also been widely discussed.
Solutions to some boundary value problems in the field of heat flow and wave propagation have been obtained using the separation of variables method and the Eigenfunction_ expansion technique.
The *Fourier* expansion with respect to the Bessel's functions and Bessel's inequality have been used in the solutions of the boundary value problems.

CONTENTS

Chapter one

Fourier series on T

In this chapter we discuss the *Fourier* coefficients in section (1), the summability in norm and homogenous Banach space on T in section (2), the order of magnitude of *Fourier* series of square summable functions and absolutely convergent *Fourier* series in section (3), *Fourier* series of square summable functions and orthogonal and trigonometric system in two variables in section (4), double *Fourier* series for a function with different periods in x and y in section (5), absolutely convergent *Fourier* series in section (6) and *Fourier* coefficients of linear series in section (7).

1.1 Fourier coefficients

Let T be defined as the quotient $R/2\pi Z$ group, where $2\pi Z$ is the group of the integral multiples of 2π and we denote by $L^1(T)$ the space of all complex-valued Lebegue integrable functions on T.

For $f \in L^1(T)$ we define the norm of f by $\| f \|_{L^1(T)} = \frac{1}{2\pi} \int_T | f(t) | \, dt$.

Note: If $f \in L^1(T)$ then $\int f(t) \, dt$ is defined on T.

Definition (1.1.1):[20] A trigonometric polynomial on T is an expression of the form

$$P(t) = \sum_{n=-N}^{N} a_n e^{\text{int}} \quad . \qquad (1.1.1)$$

The numbers (n) appearing in (1.1.1) are called the frequencies of P.
The largest integer (n) such that $(a_n)+(a_{-n}) \neq 0$ is called the degree of p.
The values assumed by the numbers (n) are integers so that each of the summands in (1.1.1) is a function on T.

We can compute the coefficients (a_n) by the formula

$$a_n = \frac{1}{2\pi} \int_T p(t) \, e^{-\text{int}} dt \qquad (1.1.2)$$

which follows immediately from the fact that for each integer J we have

$$\frac{1}{2\pi}\int e^{iJt}\,dt = \begin{cases} 1 & , \text{ if } \quad J = 0 \\ 0 & , \text{ if } \quad J \neq 0 \end{cases} \qquad (1.1.3)$$

Definition(1.1.2):[8], [9], [10] A trigonometric series on T is an expression of the form

$$S = \sum_{n=-\infty}^{\infty} a_n e^{int} \qquad (1.1.4)$$

where n assumes integral values; however, the number of terms in (1.1.4) may be finite and there is no assumption whatsoever about the size of the coefficients or about convergence .

The conjugate $\overline{S}$ of (1.1.4) is the series

$$\overline{S} = \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n) a_n e^{int} \qquad (1.1.5)$$

where $\mathrm{sgn}\,(n) = \begin{cases} 0 & , \text{ if } \quad n = 0 \\ \dfrac{n}{|n|} & , \text{ if } \quad n \neq 0 \end{cases}$.

Let $f \in L^1(T)$ be motivated by (1.1.3) we define the nth *Fourier* coefficient of f by

$$\hat{f}(n) = \frac{1}{2\pi}\int_T f(t)\, e^{-int}\, dt \ . \qquad (1.1.6)$$

Definition (1.1.3): [9], [20] The *Fourier* series $S\,[f]$ of a function $f \in L^1(T)$ is the trigonometric series

$$S[f] = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} = f(t) \qquad (1.1.7)$$

where $\hat{f}(n)$ is the *Fourier* coefficients of f.

The series conjugate to $S\,[f]$ will be denoted by $\overline{S}\,[f]$ and is given in the form

$$\overline{S}[f] = \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\hat{f}(n)\, e^{int} \qquad (1.1.8)$$

This is also referred to as the conjugate *Fourier* series of f.

Theorem (1.1.4): **[12]** Let $f, g \in L^1(T)$ then

(a) $(f+g\hat{)}(n) = \hat{f}(n) + \hat{g}(n)$

(b) For any complex number k, $(kf\hat{)}(n) = k\hat{f}(n)$

(c) If $\overline{f}$ is the complex conjegate of f then $(\overline{f}\hat{)} = \overline{\hat{f}(-n)}$

(d) Denote $f_a(t)=f(t-a), a \in T$ then $\hat{f}_a(n) = \hat{f}(n)\ e^{-ina}$

(e) $\left|\hat{f}(n)\right| \leq \frac{1}{2\pi}\int |f(t)|\, dt = \| f \|_{L^1(T)}$

Proof:

(a) By definition $(f+g\hat{)}(n) = \frac{1}{2\pi}\int (f+g)(t) e^{-\text{int}}\, dt$

$$= \frac{1}{2\pi}\int (f(t)+g(t))e^{-\text{int}}\, dt = \frac{1}{2\pi}\int f(t)e^{-\text{int}}\, dt + \frac{1}{2\pi}\int g(t)e^{-\text{int}}\, dt$$

$$= \hat{f}(n) + \hat{g}(n)$$

hence $(f+g\hat{)}(n) = \hat{f}(n) + \hat{g}(n)$.

(b) Let k be any complex number then

$$(kf\hat{)}(n) = \frac{1}{2\pi}\int kf(t)e^{-\text{int}}\, dt = k\frac{1}{2\pi}\int f(t)e^{-\text{int}}\, dt = k\hat{f}(n)$$

hence $(kf\hat{)}(n) = k\ \hat{f}(n)$.

(c) $$(\overline{f}\hat{)}(n) = \frac{1}{2\pi}\int \overline{f(n)}\ e^{-\text{int}}\, dt$$

$$\overline{\hat{f}}(-n) = \frac{1}{2\pi}\int \overline{f(t)e^{--\text{int}}}\, dt = \frac{1}{2\pi}\int \overline{f}(t)e^{-\text{int}}\, dt = (\overline{f}\hat{)}(n)$$

hence $(\overline{f}\hat{)}(n) = \overline{\hat{f}}(-n)$.

(d) $$\hat{f}_a(n) = \frac{1}{2\pi}\int f_a(t)e^{-\text{int}}\, dt = \frac{1}{2\pi}\int f(t-a)e^{-\text{int}}\, dt$$

let $u = t - a$ then $du = dt$ and $t = u + a$ thus

$$\hat{f}_a(n) = \frac{1}{2\pi}\int f(u)e^{-in(u+a)}du = \frac{1}{2\pi}e^{-ina}\int f(u)e^{-inu}du = e^{-ina}\hat{f}(n)$$

hence $\hat{f}_a(n) = e^{-ina}\hat{f}(n)$.

(e) $|\hat{f}(n)| = \left|\frac{1}{2\pi}\int f(t)e^{-\text{int}}dt\right| = \frac{1}{2\pi}\left|\int f(t)e^{-\text{int}}dt\right| \le \frac{1}{2\pi}\int |f(t)e^{-\text{int}}|\,dt$

$= \frac{1}{2\pi}\int |f(t)|\,|e^{-\text{int}}|\,dt = \frac{1}{2\pi}\int |f(t)|\,dt = \|f\|_{L^1(T)}$,

since $|e^{-\text{int}}| = 1$, hence $|\hat{f}(n)| \le \|f\|_{L^1(T)}$.

Lemma(1.1.5):[12] Assume $f_J \in L^1(T)$, $J = 0,1,2,\ldots$ and $\|f_J - f_0\|_{L^1(T)} \to 0$ then $\hat{f}_J(n)$ converges uniformly to $\hat{f}_0(n)$

Proof :

$$\hat{f}_J(n) = \frac{1}{2\pi}\int f_J(t)e^{-\text{int}}dt,\ \hat{f}_0(n) = \frac{1}{2\pi}\int f_0(t)e^{-\text{int}}dt,$$

and

$$\|f_J - f_0\|_{L^1(T)} = \frac{1}{2\pi}\int |f_J(t) - f_0(t)|\,dt \to 0 \quad .$$

$$\|\hat{f}_J(n) - \hat{f}_0(n)\| = \frac{1}{2\pi}\int_T |\hat{f}_J(t) - \hat{f}_0(t)|\,dt$$

$$= \frac{1}{2\pi}\int_T \left|\frac{1}{2\pi}\int (f_J(t) - f_0(t))e^{-\text{int}}dt\right| \le \frac{1}{2\pi}\int_T \frac{1}{2\pi}\int_T |f_J(t) - f_0(t)|\,dt\,dt \le \int 0\,dt = 0$$

then $\|\hat{f}_J(n) - \hat{f}_0(n)\| \to 0$.

hence $\hat{f}_J(n) \to \hat{f}_0(n)$ uniformly .

Theorem (1.1.6): [12] Let $f \in L^1(T)$, assume $\hat{f}(0) = 0$ and define

$$F(t) = \int_0^t f(u)\,du$$

then F is continuous, 2π periodic function and $\hat{F}(n) = \frac{1}{in}\hat{f}(n),\ n \ne 0$.

Proof :

To prove the continuity, let $t_0, t_1 \in T$, then

$$|F(t_0) - F(t_1)| = \left|\int_0^{t_0} f(u)du - \int_0^{t_1} f(u)\ du\right| = \left|\int_0^{t_0} f(u)\ du + \int_{t_1}^{0} f(u)\ du\right|$$

$$= \left|\int_{t_1}^{t_0} f(u)du\right| \le \int_{t_1}^{t_0} |f(u)|\, du \to 0 \text{ , as } t_1 \to t_0.$$

Hence $F(t)$ is continuous and the periodicity follows from the fact that

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(u)du = 2\pi \hat{f}(0) = 0 \ ,$$

therefore $F(t+2\pi) = F(t)$.

and $\hat{F}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F(t)\ e^{-\text{int}}\, dt$

if we let $u = F(t)$, $dv = e^{-int}$ and using integration by parts formula, we obtain

$$\hat{F}(n) = \frac{-1}{2\pi}\int_0^{2\pi} F'(t)\frac{1}{-in}e^{-\text{int}}\, dt = \frac{1}{in}\hat{f}(n) \ .$$

Definition (1.1.7):[7] The *Fourier* transform of a function $f \in L^1(R)$ is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\ e^{-inx} dx$$

for $x \in R$.

Some of the basic properties of $\hat{f}(n)$ for every $f \in L^1(R)$ are summarized in the following theorem :

Theorem(1.1.8):**[7]** Let $f \in L^1(R)$, then the *Fourier* transform $\hat{f}(n)$ satisfies:

(1) $\hat{f} \in L^\infty(R)$, with $\|\hat{f}\|_\infty \le \|f\|_{L^1(R)}$ where $\|f\|_\infty = \sup\limits_{-\infty < x < \infty} |f(x)|$

(2) $\hat{f}(n)$ is uniformly continuous on R

(3) $\hat{f}(n) \to 0$ as $n \to \pm\infty$.

Proof:

(1) To prove the first property, we have by definition

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-inx}\,dx$$

by taking the norm

$$\|\hat{f}\|_{\infty}= \sup_{-\infty<n<\infty} \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-inx}\,dx\right| \le \sup_{-\infty<n<\infty} \int_{-\infty}^{\infty} |f(x)\, e^{-inx}\,dx|$$

$$= \sup_{-\infty<n<\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|\,dx = \|f\|_{L^1(R)}$$

$$\Rightarrow \|\hat{f}\|_{\infty} \le \|f\|_{L^1(R)}.$$

(2) To prove the second property, let δ be chosen arbitrary and consider

$$\sup_n |\hat{f}(n+\delta)-\hat{f}(n)| = \sup_n \left|\int_{-\infty}^{\infty} e^{-inx}(e^{-i\delta x}-1)f(x)\,dx\right|$$

$$\le \int_{-\infty}^{\infty} |e^{-i\delta x}-1|\,|f(x)|\,dx .$$

Now, since

$$|e^{-i\delta x}-1|\,|f(x)| \le 2|f(x)| \in L^1(R)$$

and $|e^{-i\delta x}-1| \to 0$ as $\delta \to 0$,

then when $\delta \to 0$, the last integral $\to 0$. Therefore $\hat{f}$ is uniformly continuous on R.

(3) Let $n\to\pm\infty$, then for any $\varepsilon>0$ we can find g such that $g, g' \in L^1(R)$ and

$$\|f-g\|_{L^1(R)} < \varepsilon$$

therefore from (1) we have

$$|\hat{f}(n)| \le |\hat{f}(n)-\hat{g}(n)| + |\hat{g}(n)| \le \|f-g\|_{L^1(R)} + |\hat{g}(n)|$$

$$< \varepsilon + |\hat{g}(n)| < \varepsilon + \frac{1}{in}|g'(n)| \to 0 \quad \text{as} \quad n\to\pm\infty .$$

$$\Rightarrow \lim_{n\to\pm\infty} \hat{f}(n)=0 \quad \text{and hence} \quad f\in L^1(R) .$$

Remark: It is important to note that in the last part of theorem (1.7) $\hat{f}(n) \to 0$ as $n\to\pm\infty$ is not necessary that $\hat{f}(n)\in L^1(\mathrm{R})$. This can be proved by the following example

$$u_a(x)=\begin{cases}1 & , \quad x\geq a \\ 0 & , \quad x<a\end{cases}$$

where $a\in R$.
Let

$$f(x)=e^{-x}u_0(x) \in L^1(R) \quad \text{then} \quad \hat{f}(n)=\frac{1}{1+in}\notin L^1(R) .$$

Definition (1.1.9):[7] Let $\hat{f}(n) \in L^1(R)$ be the *Fourier* transform of some function $f\in L^1(R)$. Then the inverse *Fourier* transform of $\hat{f}$ is defined by

$$(F^{-1}\hat{f})(x)=\int_{-\infty}^{\infty} e^{inx}\hat{f}(n)\, dn .$$

1.2 Summmability in Norm and Homogenous Banach spaces on T

In this section we want to establish some of the main facts of the *Fourier* transforms. We shall see that $\hat{f}$ determines f uniquely and we show how we can find f if we know $\hat{f}$.

Two very important properties of the Banach Space $L^1(T)$ are the following:

(a) If $f\in L^1(T)$ and $\mathrm{a}\in T$ then $f_{\mathrm{a}}(t)=f(t\text{-a}) \in L^1(T)$ and $\|f_{\mathrm{a}}\|_{L^1(T)}=\|f\|_{L^1(T)}$.

(b) The $L^1(T)$ valued function $\mathrm{a}\to f_{\mathrm{a}}$ is continuous on T, that is for $f\in L^1(T)$ and $\mathrm{a}_0\in T$, we have

$$\lim_{a\to a_0} \|f_a-f_{a_0}\|_{L^1(T)} =0 . \tag{1.2.1}$$

We shall refer to (a) as the translation invariance of $L^1(T)$; it's an immediate consequence of the translation invariance of the measure dt , (where the translation invariance is $\forall t_0 \in T$ and f defined on T.

$\int f(t-t_0)dt=\int f(t)dt$, the integrals are taken over T) such that

$$\| f_a \| = \frac{1}{2\pi} \int | f(t-a) | \, dt$$

let $u = t - a$, $dt = du$ then

$$\| f_a \| = \frac{1}{2\pi} \int | f(u) | \, du = \| f \|_{L^1(T)} \quad .$$

In order to establish (b), we note that (1.2.1) is valid if f is continuous (the inverse is not true), and the continuous function is dense in $L^1(T)$.

(Where dense here means for an arbitrary $\varepsilon > 0$ and for every continuous function f there exist $g \in L^1(T)$ with $\|f - g\| \leq \varepsilon$).

Let f be arbitrary function such that $f \in L^1(T)$ and $\varepsilon > 0$ be given, furthermore let g be continuous function on T such that $\| g - f \| < \varepsilon/2$ then

$$\| f_a - f_{a_0} \|_{L^1(T)} \leq \| f_a - g_a \|_{L^1(T)} + \| g_a - g_{a_0} \|_{L^1(T)} + \| g_{a_0} - f_{a_0} \|_{L^1(T)}$$

$$= \| (f - g)_a \|_{L^1(a)} + \| g_a - g_{a_0} \| + \| (g - f)_{a_0} \|$$

$$< \varepsilon + \| g_a - g_{a_0} \|_{L^1(T)} \, .$$

Hence $\overline{Lim} \| f_a - f_{a_0} \| < \varepsilon$ and ε being an arbitrary positive number. This proves (b).

Definition(1.2.1): [12] A summability kernel is a sequence $\{k_n\}$ of 2π -periodic continuous functions satisfying :

(1) $\frac{1}{2\pi} \int k_n(t) dt = 1, \; n = 1,2,3,...$

(2) $\frac{1}{2\pi} \int | k_n(t) | \, dt \leq \text{constant}$

(3) For all $0 < \delta < \pi, \underset{n \to \infty}{Lim} \int_{\delta}^{2\pi - \delta} | k_n(t) | \, dt = 0$.

A positive summability kernel is a kernel in which $k_n(t) \geq 0$ for all t and n. For positive kernels the assumption (2) is redundant .

Lemma (1.2.2): [12] Let B be a Banach space, Q is a continuous B-valued function on T and $\{k_n\}$ a summability Kernel then

$$\underset{n \to \infty}{Lim} \frac{1}{2\pi} \int k_n(T) Q(T) dT = Q(0)$$

Proof: (see **[12]** page(10)).