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Existence of Universal Locally Univalent Functions

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Dedication

To my parents , my grandfather , my brothers , my family , my colleagues and each person gave me support and assistance.

Ramzi Jafar

Declaration

I certify that this submitted for the degree of master is result of my own research, expect where otherwise acknowledge. And that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signature : ..

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Abstract

In this thesis, we proved Runge theorem and universality results for locally univalent holomorphic and meromorphic functions in compact sets and in neighborhood of the compact sets. After that , we approximate the meromorphic function in an open set containing compact set , and had new problems in approximating the continuous function .

Contents

Introduction

The Universality concept covers a wide range of phenomena in complex analysis .Generally speaking , a universal object is the object which approximates every object in some universe with some restrictions and limitations on it . For example , universality occurs when the translates of an entire function can approximate any other entire function or when the partial sums of a formal power series or a formal trigonometric series approximate all functions in some natural class. For long time , existing approximation theorems were used in constructions of universal functions and universal series .In recent years , however , constructions have required the development of new approximation theorems , thereby also enriching the area of complex analysis.

There is no single definition of a universal function , but all of them have the following in common , one considers a suitable sequence $T = T_n$ of operators acting on a space X , for example , of holomorphic functions with values in another space Y of holomorphic functions , then a function $f \in X$ is called universal with respect to T if $\{f, T_1 f, T_2 f, ...\}$ is dense in Y. One of earliest example of a universal function is due to *Birkhoff* (1929) [3] who showed that there exists an entire function f whose translates $f(z+n), n \geq 1$ can approximate any other entire function , uniformly on compact sets , in that case we have $(T_n f)(z) = f(z+n)$, and $X = Y$ is the space of entire functions with the usual compact-open topology.

Siedel and *Walsh* [32] showed that an analogue of *Birkhoff*'s university theorems holds for holomorphic functions in the unit disk , if we replace translates by non-euclidean translates , that is $T_n f = f \circ \phi_n$ is the composition of f with an automorphism ϕ_n of the unit disk $\mathbb{D}$ .At the core of studying holomorphic function in the unit disk $\mathbb{D}$ is $\mathcal{B}(\mathbb{D})$ the set of all bounded holomorphic functions on the disk.

Extending the study of functions in the unit disk ,which are universal with respect to composition with automorphism of the disk , *Mortini* talked about the universality of functions f holomorphic in a domain ω with respect to a sequence $(f \circ \phi_n)$ of compositions , where (ϕ_n) are self-maps of Ω .

In general theory of university , the talk is about a bounded operator T

defined on some separable Banach space X which is called the *Hypercyclic Operator* if there exists some vector $x \in X$ such that the orbit of x under T , namely $\{T^n x : n \geq 0\}$ is dense in X .

The main focus of the thesis is proving *Runge-theorems* and universality results for locally univalent holomorphic and meromorphic functions , Refining a result of *M.Heins* [14] .

In chapter one, we recovered some basics from complex analysis .

In chapter two, we stated two important theorem in approximation theory "Runge's and Mergelyan's ".

In chapter three, we applied Runge's theorem for locally univalent functions.

Chapter 1

Preliminaries

1.1 Introductory Geometric Considerations

The geometric theory of functions of a complex variable makes a study of analytic functions defined by some geometric property or other and a study of various geometric properties of certain classes of analytic functions. Therefore, it naturally rests on a number of general geometric concepts that are encountered in present-day mathematics. Here we propose to make some brief remarks about these concepts, in order of their occurrence, that are associated with the complex plane and that we shall need in our subsequent exposition.

Domains and curves, one of the basic geometric concepts in the theory of functions of a complex variable is the concept of a domain. A domain is defined as an open set any two points in which can be connected by a broken line consisting entirely of points of that set (the property of connectedness). The boundary points of a domain are those points in the complex plane that do not belong to the domain but are cluster points of it. If a domain is other than the entire plane, it necessarily has boundary points. The set of all boundary points of a domain is called its boundary. The boundary of a domain is a closed set . Those points in the plane that are neither interior nor boundary points of a domain are called exterior points of the domain. Every exterior point of a domain has a neighborhood containing no points of the domain.

The union of a domain and its boundary is called a closed domain. In contrast with this, a domain itself is sometimes called an open domain. A domain is said to be simply connected if its boundary consists either of a continuum or of a single point or if the domain itself is the entire complex plane. Otherwise, a domain is said to be multiply connected. Specifically, it is said to be doubly, triply, , n-connected if its boundary consists of 2, 3, , n disjoint continua (possibly degenerate). All these regions are said to finitely connected and the continua (including the degenerate ones) are called boundary continua. The role of domains in the study of closed and open sets is clear from the, following theorem:

Theorem 1.1.1 *Every open set E in the complex plane is the union of finitely or countably many domains.*

Another basic geometric concept in the theory of functions of a complex variable is that of a curve.

A continuous curve is a set of points in the rectangular coordinates x, y plane of which can be written as continuous functions

$$x = \phi(t), y = \psi(t) \tag{1.1}$$

of a real variable t in some finite interval $a \leq t \leq b$. It is easy to see that this set is a continuum.

However, the concept of a continuous curve is too general for our purposes. There are continuous curves that do not at all correspond to our intuitive idea of a curve as a one-dimensional figure. In fact, it is possible to construct a continuous curve that passes through every point of a given square. On the other hand, if we require that the curve have no multiple points, it will possess a number of clear-cut properties. Such curves are called simple curves or Jordan curves.

Thus, the continuous curve (1.1) or, more briefly, the curve

$$z = z(t) = \phi(t) + i\psi(t), \quad a \leq t \leq b \tag{1.2}$$

is called a Jordan curve if, for any two distinct values t_1 and t_2 in the interval $[a, b)$ with $t_1 \neq t_2$, we have $z(t_1) \neq z(t_2)$ and $z(t_2) \neq z(b)$. The points $z(a)$ and $z(b)$ may or may not coincide. In the first case, the curve is called a closed, in the second case a non-closed Jordan curve. We have the following important theorem (due to Jordan):

Theorem 1.1.2 *[10] A closed Jordan curve C partitions the plane (including ∞) into two simply connected domains both of which have C as their boundary. One of these domains is bounded and is called the interior of C. The other contains ∞ and is called the exterior of C. The complement of a nonclosed Jordan curve C consists of a single simply connected domain containing ∞ and having C as its boundary.*

From non-closed Jordan curves, one can construct continuous curves that are not of the Jordan type. On the other hand, even a Jordan curve is sometimes too general. Then, depending on our purpose, we introduce curves of more restricted types, for example, smooth and piecewise-smooth curves.

A curve 1.2 is said to be smooth if the function $z(t)$ has a continuous non-zero derivative $z'(t)$ everywhere in [a, b] (a one-sided derivative at the two endpoints). The requirement of smoothness is obviously equivalent to the requirement that the curve have everywhere a continuously turning tangent. A curve consisting of finitely many smooth curves is called a piecewise-smooth curve.

Finally, the simplest type of continuous curve is an analytic curve. This is a curve defined by an equation of the form $z = z(t)$ for $a \leq t \leq b$, where $z(t)$ can be expanded in a power series

$$z(t) = c_0 + c_1(t - t_0) + c_2(t - t_0)^2 + ..., \tag{1.3}$$

with $c_1 \neq 0$, about each value t_0 in [a, b]. A continuous curve consisting of a finite number of analytic curves is called a piecewise-analytic curve.

1.2 Holomorphic Functions

Throughout this discussion we identify the complex plane, $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Let Ω be an open set in the complex plane and let f be a complex valued function in the space of all continuous differentiable functions $C^1(\Omega)$. If the real coordinates are denoted by x and y, then we set $z = x + iy$ and $\bar{z} = x - iy$. Also we have :

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}$$

We define partial differential operators in the following way:

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}) \text{ and } \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})$$

Now, the differential of f can be expressed as a linear combination of dz and $d\bar{z}$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

Definition 1.2.1 *A function $f \in C^1(\Omega)$ is said to be holomorphic if $\frac{\partial f}{\partial \bar{z}} = 0$ in Ω , or equivalently if df is proportional to dz. If the function f is holomorphic we write f' rather than $\frac{\partial f}{\partial z}$ We denote the set of all holomorphic functions on Ω by $\mathcal{O}(\Omega)$.*

Lemma 1.2.1 *Let $f : U \to \mathbb{C}$ be continuous function on U with $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$ and u and v are real-valued functions. Then f is holomorphic , if we have that u and v satisfy the equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at every point of U.

The equations :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the *Cauchy-Riemann* equations.

Definition 1.2.2 *A complex function f is called analytic if around each point z_0 of its domain the function f can be computed by a convergent power series. More precisely, for each z_0 there exists $\epsilon > 0$ and a sequence of complex numbers $(a_0, a_1, ...)$ such that*

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... = \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

for $|z - z_0| < \epsilon$.

If f is analytic then f and all its derivatives are holomorphic. The derivatives can be computed as the derivatives of a convergent power series, i.e. by deriving term by term. In particular

$$f^{(n)}(z_0) = \frac{a_n}{n!}$$

which shows that the expression of f as a power series at z_0 is unique.

If the power series is convergent for all $z \in \mathbb{C}$ i.e. not just for $|z - z_0| < \epsilon$, the function f is called an entire function.

Remark: Some books use the word "analytic" instead of "holomorphic." Still others say "differentiable" or "complex differentiable" instead of "holomorphic." The use of "analytic" derives from the fact that a holomorphic function has a local power series expansion about each point of its domain. The use of "differentiable" derives from properties related to the Cauchy-Riemann equations and conformality.

Definition 1.2.3 *A meromorphic function f on $U \subset \mathbb{C}$ with singular set S is a function $f : U \backslash S \to \mathbb{C}$ such that f is holomorphic on $U \backslash S$, S is discrete and for each $p \in S$ and $r > 0$ such that $D(p,r) \subseteq U$, $S \cap D(p,r) = \{p\}$, the function $f|_{D(p,r)\backslash\{p\}}$ has a finite order pole at p .*

Definition 1.2.4 *A subset K of a metric space X is compact if for every collection $\mathcal{G}$ of open sets in X with the property $K \subset \cup\{G : G \in \mathcal{G}\}$ there is a finite sets $G_1, G_2, ..., G_n$ in $\mathcal{G}$ such that $K \subset G_1 \cup G_2 \cup ... \cup G_n$. The collection of sets $\mathcal{G}$ is called a cover of K , if each member of $\mathcal{G}$ is an open it is called an open cover of K.*

Definition 1.2.5 *A set $\Omega \subset \mathbb{C}$ is said to be disconnected if there exist nonempty sets $A, B \subset \mathbb{C}$ such that*

$$\Omega = A \cup B \text{ and } \bar{A} \cap B = A \cap \bar{B} = \emptyset$$

A set $\Omega \subset \mathbb{C}$ is said to be connected if it is not disconnected.

Now , we introduce the notion of a connected component.

Definition 1.2.6 *Given $\Omega \subset \mathbb{C}$ we say that a connected set $A \subset \Omega$ is a connected component of Ω if any connected set $B \subset \Omega$ containing A is equal to A.*

We note that if a set $\Omega \subset \mathbb{C}$ is connected, then it is its own unique connected component.

Now, In order to define the integral of a complex function, we first introduce the notion of a path:

Definition 1.2.7 *A continuous function $\gamma : [a, b] \to \Omega$ is called a path in Ω, and its image $\gamma([a, b])$ is called a curve in Ω.*

Now, we define two operations, the first is the inverse of a path.

Definition 1.2.8 *Given a path $\gamma : [a, b] \to \Omega$ we define the path $-\gamma : [a, b] \to \Omega$ by $-\gamma(t) = \gamma(a + b - t)$ for each $t \in [a, b]$.*

The second operation is the sum of paths.

Definition 1.2.9 *Given a path $\gamma_1 : [a_1, b_1] \to \Omega$ and $\gamma_2 : [a_2, b_2] \to \Omega$ such that $\gamma_1(b_1) = \gamma_2(a_2)$ we define the path $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \Omega$ by*

$$\gamma_1 + \gamma_2 = \begin{cases} \gamma_1(t) & \text{if } t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

We also consider the notions of a regular path and a piecewise regular path

Definition 1.2.10 *A path $\gamma : [a, b] \to \Omega$ is said to be regular if it is of class C^1 and $\gamma'(t) \neq 0$ for every $t \in [a, b]$, taking the right-sided derivative at a and the left-sided derivative at b.*

More precisely, the path $\gamma : [a,b] \to \Omega$ is regular if there exists a path $\alpha : (c,d) \to \Omega$ of class C^1 in some open interval (c,d) containing $[a,b]$ such that $\alpha(t) = \gamma(t)$ and $\alpha'(t) \neq 0$ for every $t \in [a,b]$.

Definition 1.2.11 *A path $\gamma : [a,b] \to \Omega$ is said to be piecewise regular if there exists a partition of $[a,b]$ into a finite number of subintervals $[a_j, b_j]$ (intersecting at most at their endpoints) such that each path $\gamma_j : [a_j, b_j] \to \Omega$ defined by $\gamma_j = \gamma$ for $t \in [a_j, b_j]$ is regular, taking the right-sided derivative at a_j and the left-sided derivative at b_j .*

Definition 1.2.12 *Let $\gamma(t), a \leq t \leq b$, be a closed path .We define the trace of γ to be the image $\Gamma = \gamma([a,b])$*

Definition 1.2.13 *For $z_0 \notin \Gamma$, we define the winding number $W(\gamma, z_0)$ of γ around z_0 to be the increase in the argument of $z - z_0$ around γ, normalized by dividing by 2π. If γ is piecewise smooth, the winding number is the integer*

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$$

Definition 1.2.14 *Let U be a connected open set. Let $\gamma : [0,1] \to U$ be a continuous closed curve. We say that γ is homologous to 0 if $Ind_\gamma(P) = 0$ for all points $P \in \mathbb{C} \backslash U$.*

Definition 1.2.15 *We say γ_1 is homologous to γ_2 if $\gamma_1 \circ \gamma_2^{-1}$ is homologous to 0.*

Now we introduce the notion of the integral along a path.

Definition 1.2.16 *Let $f : \Omega \to \mathbb{C}$ be a continuous function and let $\gamma : [a,b] \to \Omega$ be a piecewise regular path. We define the integral of f along γ by*

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt$$

The integral has the following properties :

Theorem 1.2.1 *If $f, g : \Omega \to \mathbb{C}$ are continuous functions and $\gamma : [a,b] \to \Omega$ is a piecewise regular path, then :*

1. *For any $c, d \in \mathbb{C}$, we have $\int_\gamma (cf + dg) = c \int_\gamma f + d \int_\gamma g$*
2. $\int_{-\gamma} f = - \int_\gamma f$
3. *For any piecewise regular path $\alpha : [p,q] \to \Omega$ with $\alpha(p) = \gamma(b)$, we have $\int_{\gamma+\alpha} f = \int_\gamma f + \int_\alpha f$*

[10] We also describe two additional properties. For the first one we need the notion of equivalent paths.

Definition 1.2.17 *Two paths $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ are said to be equivalent if there exists a differentiable function $\phi : [a_2, b_2] \to [a_1, b_1]$ with $\phi' > 0$, $\phi(a_2) = a_1$, and $\phi(b_2) = b_1$, such that $\gamma_2 = \gamma_1 \circ \phi$.*

We can now formulate the following results from [6]

Theorem 1.2.2 *If $f : \Omega \to \mathbb{C}$ is continuous function and let γ_1 and γ_2 are equivalent piecewise regular paths in Ω , then*

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Now , we will introduce the concept of primitive, the concept of primitive is useful for the computation of integrals. Let us consider a function $f : \Omega \to \mathbb{C}$ in an open set $\Omega \subset \mathbb{C}$

Definition 1.2.18 *A function $F : \Omega \to \mathbb{C}$ is said to be a primitive of f in the set Ω if F is holomorphic in Ω and $F' = f$ in Ω.*

We first show that in connected open sets all primitives differ by a constant.

Proposition 1.2.1 *If F and G are primitives of f in some connected open set $\Omega \subset \mathbb{C}$ then $F - G$ is constant in Ω.*

Primitives can be used to compute integrals as follows:

Proposition 1.2.2 *If F is a primitive of a continuous function $f : \Omega \to \mathbb{C}$ in an open set $\Omega \subset \mathbb{C}$ and $\gamma : [a, b] \to \Omega$ is a piecewise regular path, then*

$$\int_{\gamma} f = F(\gamma(b)) - F(\gamma(a))$$

We also consider paths with the same initial and final points.

Definition 1.2.19 *A path $\gamma : [a, b] \to \mathbb{C}$ is said to be closed if $\gamma(a) = \gamma(b)$*

The following property is an immediate consequence of Proposition 1.2.2

Proposition 1.2.3 *If $f : \Omega \to \mathbb{C}$ is continuous function having a primitive in an open set $\Omega \subset \mathbb{C}$ and $\gamma : [a, b] \to \Omega$ is a closed piecewise regular path, then*

$$\int_{\gamma} f = 0$$

Now we show that any holomorphic function has primitives. We recall that a set $\Omega \subset \mathbb{C}$ is said to be convex if

$$tz + (1-t)w \in \Omega$$

for each $z, w \in \Omega$ and $t \in [0,1]$.

Theorem 1.2.3 *If $f : \Omega \to \mathbb{C}$ is holomorphic function in a convex open set $\Omega \subset \mathbb{C}$, then f has a primitive in Ω.*

More generally, we have the following result :

Theorem 1.2.4 *If $f : \Omega \to \mathbb{C}$ is continuous function in a convex open set $\Omega \subset \mathbb{C}$ and there exists $p \in \Omega$ such that f is holomorphic in $\Omega \backslash \{p\}$, then f has a primitive in Ω.*

With all these results , now we can introduce Cauchy's theorem.

1.3 Cauchy's theorem

Cauchy's theorem is a big theorem which is used almost everywhere in analysis .Right away it will reveal a number of interesting and useful properties of analytic functions.

We start with a statement of the theorem for functions :

Theorem 1.3.1 *(Cauchy's integral formula for functions)[26] Suppose γ is a simple closed curve and the function $f(z)$ is analytic on a region containing γ and its interior. We assume γ is oriented counterclockwise. Then for any z_0 inside γ:*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$

With a slight change of notation (z becomes w and z_0 becomes z) we often write the formula as

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw$$

Cauchy's integral formula is worth repeating several times. So, now we give it for all derivatives $f^{(n)}(z)$ of f. This will include the formula for functions as a special case:

Theorem 1.3.2 *(Cauchy's integral formula for derivatives)[26] If $f(z)$ and γ satisfy the same hypotheses as for Cauchy's integral formula then, for all z inside γ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw, n = 0, 1, 2, ...$$

where, γ is a simple closed curve, oriented counterclockwise, z is inside γ and $f(w)$ is analytic on and inside γ.

Let C_R be the circle $|z - z_0| = R$. Assume that $f(z)$ is analytic on C_R and its interior, i.e. on the disk $|z - z_0| \leq R$. Finally let $M_R = max|f(z)|$ over z on C_R. Using Cauchy's integral formula for derivatives , we have :

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi i}\int_\gamma \frac{|f(w)|}{|w-z|^{n+1}}|dw| \leq \frac{n!}{2\pi i}\frac{M_R}{R^{n+1}}\int_\gamma |dw| = \frac{n!}{2\pi i}\frac{M_R}{R^{n+1}}.2\pi R$$

Therefore , we have

Theorem 1.3.3 *(Cauchy's inequality)*

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$$

Now , we will state the Maximum modulus principle ,Briefly, the maximum modulus principle states that if f is analytic and not constant in a domain A then $|f(z)|$ has no relative maximum in A and the absolute maximum of $|f|$ occurs on the boundary of A. In order to get this technically correct we need to state it carefully with the proper assumptions.

Theorem 1.3.4 *(maximum modulus principle)[6] Suppose $f(z)$ is analytic in a connected region A and z_0 is a point in A :*

1. *If $|f|$ has a relative maximum at z_0 then $f(z)$ is constant in a neighborhood of z_0.*
2. *If A is bounded and connected, and f is continuous on A and its boundary, then either f is constant or the absolute maximum of $|f|$ occurs only on the boundary of A.*

1.4 Polynomial Approx. Theory

Approximation theory plays a fundamental role in complex analysis , sometimes it is inestimably useful to be able to approximate functions by polynomials. A local study of differentiable functions is unthinkable without Taylor polynomials, e.g., the Taylor polynomials of degree two is the one in use in calculus when finding local extrema. In a somehow more global and advanced setting the Weierstrass approximation theorem is fundamental and gives a very strong result.

Theorem 1.4.1 *Let K be a compact subset of $\mathbb{R}^n$ and f be a continuous function in K. For every $\epsilon > 0$ there exists a polynomial P such that $\sup_{x\in K} |P(x) - f(x)| < \epsilon$.*

When we look at functions defined on $\mathbb{C}$, the results become more complicated. As our primary concern in this thesis is the holomorphic functions, and the question naturally becomes this: Given a holomorphic function in a domain Ω and a compact set $K \subset \Omega$, when can f be approximated by polynomials uniformly on K? That is, when can one find a polynomial $P(z)$ and for any $\epsilon > 0$ with

$$\sup_{z\in k} |f(z) - P(z)| < \epsilon?$$

The first comment is that a positive answer , but it is depends on the domain , so the domains of functions are crucial to rational approximation, and they have been characterized in terms of geometric features, topological properties, continuous analytic capacity, peak points, etc.

Now , we have to find some necessary and/or sufficient conditions such that a function f on a compact set K can be approximated by polynomials. Let us start with this theorem [16] .

Theorem 1.4.2 *If $\Omega \subset \mathbb{C}$ is an open set,$f_n \in \mathcal{O}(\Omega)$ for all $n \in \mathbb{N}$ and $f_n \to f$uniformly on compact subsets of Ω then $f \in \mathcal{O}(\Omega)$. where $\mathcal{O}(\Omega)$ is the set of functions holomorphic on Ω.*

The theorem states that a function f is to be approximated uniformly by polynomials on an open set must be holomorphic since all polynomials in the variable z are holomorphic.

From theorem 1.4.2 it follows that the following conditions are necessary on f

1. f is continuous on K.

2. f is holomorphic on the interior of K (K°).

We denote the set of functions which satisfy these conditions by $A(K)$,so here we suppose that f is holomorphic on an open set containing K, whereas in Weierstrass one assumes only that f is continuous on K. But can we find some sufficient or necessary conditions on the set K so that every function in $A(K)$ can be approximated in K? Suppose its compliment, K^c , has a bounded component U and assume we can approximate every function $g \in A(K)$ uniformly in K by polynomials. We choose some $\zeta \in U$ and consider the function on K given by $g(z) = (z - \zeta)^{-1}$.By assumption we can find a sequence $(g_n)_{n\in\mathbb{N}}$ of polynomials such that $g_n \to g$ uniformly on K ,by the maximum principle we have

$$\sup_{z\in\bar{U}} |g_n(z) - g_m(z)| = \sup_{z\in\partial U} |g_n(z) - g_m(z)| \leq \sup_{z\in K} |g_n(z) - g_m(z)|$$

So g_n converges uniformly in $\bar{U}$ to some limit $G \in A(\bar{U})$.We notice that $G(z)(z - \zeta) = 1$ on ∂U , so $G(z)(z - \zeta) = 1$ on all of U.This gives a contradiction when $z = \zeta$.

We will see that if every function $f \in A(K)$ can be approximated by polynomials on K, then K^c is connected, so we need to define a new set of functions.

Definition 1.4.1 *Let $E \subset \mathbb{C}$ be an arbitrary set and f be a function on E. We say that $f \in \mathcal{O}(E)$ if there exists some open set $\Omega \subset \mathbb{C}$ and a holomorphic function, $g \in \mathcal{O}(\Omega)$ such that $E \subset \Omega$ and $g|_E = f$.*

In 1885, Runge proved the following considerably general result stated above [27]:

Theorem 1.4.3 *(Runge) Let K be a compact set. Every function in $\mathcal{O}(K)$ can be uniformly approximated by polynomials on K if and only if the set K^c is connected.*

The famous Runge's theorem states that any function which is holomorphic on a neighborhood of a compact set admits rational approximation, as discussed in the later chapters.

Quite obviously we have $\mathcal{O}(K) \subset A(K)$. Therefore it is rather natural to wonder if Runge's theorem can be extended to the set $A(K)$. The answer to that question is quite satisfyingly yes, and was proved by Mergelyan in 1952. It is startling that a period of sixty-seven years elapsed between the appearance of Runge's theorem and that of Mergelyan's theorem. Especially if one examines the large number of papers written on this subject during those years. The explanation is perhaps that people thought that Mergelyan's theorem was too good to be true[28].

Theorem 1.4.4 *(Mergelyan) Let K be a compact set. Every function in $A(K)$ can be uniformly approximated on K by polynomials if and only if the set K^c is connected.*

So, Mergelyan's theorem is the ultimate development and generalization of the Weierstrass approximation theorem and Runge's theorem. It gives the complete solution of the classical problem of approximation by polynomials.

Chapter 2

Rational Approximation Theory

2.1 Runge's Theorem

Let Ω be an open set in $\mathbb{C}$ and let K be a compact set in Ω . For any continuous function ϕ on K, we set

$$|\phi|_K = \sup_{z \in K} |\phi(z)|$$

We define a topology from [27] on $\mathcal{O}(\Omega)$ by taking as a fundamental system of neighborhoods of $f \in \mathcal{O}(\Omega)$ the sets

$$\{g \in \mathcal{O}(\Omega) | |f - g|_K < \epsilon\}$$

where K runs over the compact subsets of Ω and ϵ over the positive real numbers.

This topology is metrizable ; in fact, the topology is defined by the following metric:

Let $\{K_n\}_{n \geq 1}$ be a sequence of compact sets in Ω such that $K_n \subset K_{n+1}^\circ$, $\cup_n K_n = \Omega$ .For $f, g \in \mathcal{O}(\Omega)$, we set

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{|f - g|_{K_n}}{1 + |f - g|_{K_n}}$$

This defines a metric which induces the topology defined above. It also makes $\mathcal{O}(\Omega)$ a complete metric space.

The above topology is also called the topology of uniform convergence on compact sets (or the topology of compact convergence) for the following reason. Let $\{f_n\}_{n \geq 1}$ be a sequence of functions in $\mathcal{O}(\Omega)$,then f_n converges in $\mathcal{O}(\Omega)$ if and only if f_n converges uniformly on any compact set in Ω. This topology is also sometimes called the compact open topology.

Let K be a compact set in $\mathbb{C}$. We denote by $\mathcal{A}(K)$ the space of continuous functions ϕ on K such that there exists an open set $U \supset K$ (depending on ϕ) and $f \in \mathcal{O}(U)$ such that $f|_K = \phi$. We shall denote by $\mathcal{C}(K)$ the space of all continuous functions ϕ on K with the norm $||\phi|| = |\phi|_K$. $\mathcal{C}(K)$ is a Banach space.

We consider $\mathcal{A}(K)$ as a subspace of $\mathcal{C}(K)$; it is not, in general, a closed subspace.

Definition 2.1.1 *Let X be a topological space and f a complex valued function defined on X. We define the support of f to be the closure in X of the set$\{x \in X : f(x) \neq 0\}$ and denote this set by supp(f). When it is necessary to emphasize the space X, we shall speak of the support of f in X and write supp$_X(f)$. If Ω is an open set in $\mathbb{R}^n$ or in $\mathbb{C}^n$ and $1 \leq k \leq \infty$, we denote by $C_0^\infty(\Omega)$ the set of $f \in C^k(\Omega)$ such that supp$_X(f)$ is compact.*

Definition 2.1.2 *A subset of a topological space is said to be precompact if its closure is compact. [18]*

"Relatively compact" is then used to mean "precompact" as it is defined here [7, 20]

Definition 2.1.3 *Let K ba a compact subset of Ω,then the convex hull of K in Ω with respect to $\mathcal{F}$ is defined to be*

$$\hat{K}_{\mathcal{F}} \equiv \{z \in \Omega : |f(z)| \leq \sup_{t\in K} f(t) \textit{ for all } f \in \mathcal{F}\}$$

where $\mathcal{F}$ is the family of holomorphic function on Ω. We say that Ω is holomorphically-convex or "$\mathcal{O}$-convex with respect to $\mathcal{F}$ provided $\hat{K}_{\mathcal{F}}$ is compact in Ω whenever K is. [19]

Finally, if Ω is open in $\mathbb{C}$ and K is a compact subset of Ω, we denote by $\rho = \rho_{\Omega,K}$ the restriction map

$$\rho : \mathcal{O}(\Omega) \to \mathcal{A}(K),\ \rho(f) = f|_K$$

Theorem 2.1.1 (Runge's Theorem: First Form) *[27]Let Ω be open set in $\mathbb{C}$ and let K be a compact subset of Ω . Then, the following statements are pairwise equivalent*

1. *$\rho(\mathcal{O}(\Omega))$ is dense in $\mathcal{A}(K)$ (with the topology induced from $\mathcal{C}(K)$; ρ is the restriction map defined above).*

2. No connected component of $\Omega - K$ is relatively compact in Ω.

3. For any $a \in \Omega$, $a \notin K$, there exists $f \in \mathcal{O}(\Omega)$ with

$$|f(a)| > |f|_K$$

Proof: (i) $\Rightarrow$ (ii). Suppose that $\Omega - K$ has a connected component U which is relatively compact in Ω. We claim that $\partial U \subset K$. To see this, let $a \in \partial U$, $a \notin K$, and let D be a disc centered at a with $D \subset \Omega - K$. Since $a \in \bar{U}$, we have $D \cap U \neq \emptyset$. Consequently, $D \cup U$ is connected (since D and U are connected), and $D \cup U \subset \Omega - K$. Since U is a connected component of $\Omega - K$, it follows that $D \subset U$, and a cannot be on ∂U, i.e contraction.

Thus $\partial U \subset K$.

Let $z_0 \in U$ and let $f(z) = (z - z_0)^{-1}$. Then $f|_K \in \mathcal{A}(K)$. Suppose that there is a sequence $\{f_n\}_{n \geq 1}$, $f_n \in \mathcal{O}(\Omega)$, which converges to $f|_K$, uniformly on K. Then, by the maximum principle,

$$\sup_{z \in \bar{U}} |f_n(z) - f_m(z)| = \sup_{z \in \partial U} |f_n(z) - f_m(z)| \leq |f_n(z) - f_m(z)|_k$$

Hence $\{f_n|_U\}$ converges to a function $g \in \mathcal{O}(U)$, uniformly on U.

Now, as $n \to \infty$, $(z - z_0)f_n(z) \to 1$, uniformly for $z \in \partial U \subset K$. Hence, again by the maximum principle, $(z - z_0)f_n(z) \to 1$ for $z \in U$; hence $(z - z_0)g(z) = 1$ for $z \in U$, which is absurd, since $z - z_0 = 0$ for $z = z_0$. This contradiction proves that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let $\rho(\mathcal{O}(\Omega)) = E$; then $E \subset \mathcal{A}(K) \subset \mathcal{C}(K)$. By corollary of Hahn-Banach theorem [5] (If X is a locally convex space and Y is a linear subspace of X, then Y is dense in X iff the only continuous linear functional on X that vanishes on Y is identical to the zero functional) applying to the space $\mathcal{A}(K)$, E is dense in $\mathcal{C}(K)$ if and only if the following holds:

Let λ be a continuous linear form (= functional) on $\mathcal{C}(K)$. If $\lambda|_E = 0$, we have

$$\lambda|_{\mathcal{O}}(K) = 0. \tag{2.1}$$

For any continuous linear form λ on $\mathcal{C}(K)$, we define a function $\Phi = \Phi_\lambda$ on $\mathbb{C} - K$ as follows:

$$\Phi(w) = \lambda(\phi_w) \ , \ w \notin K$$

where ϕ_w is the function $z \mapsto 1/(z - w)$, $z \in K$.

We claim that $\Phi \in \mathcal{O}(\mathbb{C} - K)$, and that $\Phi^{(n)}(w) = n!\lambda(\phi_{w,n})$ for $n \geq 0$, where $\phi_{w,n}(z) = (z - w)^{-n-1}$,$z \in K$. In fact , if $a \notin K$ and $r > 0$ is so chosen that $\bar{D}(a, r) \cap K = \emptyset$, we have

$$\phi_w(z) = \frac{1}{z-w} = \frac{1}{(z-a)(1-\frac{w-a}{z-a})} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

where the series converges uniformly for $|w-a| < r$, $z \in K$. Hence, since λ is continuous on $\mathcal{C}(K)$, we have

$$\Phi(w) = \lambda(\phi_w) = \sum_{n=0}^{\infty} (w-a)^n \lambda(\phi_{a,n})$$

Thus $\Phi \in \mathcal{O}(D(a,r))$ and $\Phi^{(n)}(a) = n!\lambda(\Phi_{w,n})$

Suppose now that $\lambda|_E = 0$. This implies that $\Phi \equiv 0$ on $\mathbb{C} - K$ if $\Omega - K$ has no connected component relatively compact in Ω. To prove this, let U be a connected component of $\mathbb{C} - K$.

Case (a). U is unbounded .

Choose $R > 0$ such that $K \subset D(0,R)$ and let $w \in U$,$|w| > R$. Now $\phi_w(z) = -\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$, the series converging uniformly for $z \in K$. Hence we have $\Phi(z) = -\sum_{n=0}^{\infty} w^{-n-1}\lambda(z^n|_K) = 0$ since $z^n|_K \in E$. Thus $\Phi = 0$ on the open set $U \cap \{w \in \mathbb{C} : |w| > R\}$ which is nonempty since U is unbounded .Thus, U being connected , we have $\Phi|_U = 0$.

Case (b). U is bounded

We claim that $U \not\subset \Omega$. In fact, we have $\partial U \subset K$ (see the proof that (i) $\Rightarrow$ (ii) above). Hence, if $U \subset \Omega$, we have $\bar{U} \subset \Omega$; moreover, if $U \subset \Omega$, U would have to be a connected component of $\Omega - K$ (since there is no larger connected set in $\mathbb{C} - K$, hence in $\Omega - K$, containing U). Thus U is compact in $\mathbb{C}$ (U being bounded) and $\bar{U} \subset \Omega$. Consequently, the connected component U of $\Omega - K$ would be relatively compact in Ω, contrary to hypothesis. Thus $U \not\subset \Omega$.

Let $a \in U$, $a \notin \Omega$. We have, for $n \geq 0$,

$$\Phi^{(n)}(a) = n!\lambda(\Phi_{w,n}) = 0$$

since $z \mapsto (z-a)^{-n-l}$ defines a function in $\mathcal{O}(\Omega)$ for $a \notin \Omega$. Since U is connected and $\Phi^{(n)}(a) = 0$ for all $n \geq 0$, we have $\Phi|_K = 0$.

We have therefore proved that $\Phi \equiv 0$ on $\mathbb{C} - K$.

Let $f \in \mathcal{A}(K)$, and let W be an open set containing K and $F \in \mathcal{O}(W)$ be such that $F|_K = f$. Choose $\alpha \in \mathcal{C}_0^{\infty}(W)$ such that $\alpha = 1$ on a neighborhood W_0 of K. For $z \in K$, we have

$$\begin{aligned} f(z) = F(z) &= -\frac{1}{\pi} \iint_W \frac{\partial \alpha}{\partial \bar{\zeta}} F(\zeta) \frac{1}{\zeta - z} d\zeta d\eta \\ &= -\frac{1}{\pi} \iint_{W-W_0} \frac{\partial \alpha}{\partial \bar{\zeta}} F(\zeta) \frac{1}{\zeta - z} d\zeta d\eta \end{aligned}$$

If $\{R_v\}$ is a finite set of rectangles whose interiors are disjoint and which cover $supp(\alpha) - W_0$, and if ζ_v is a point in $R_v \cap supp(a)$, then as the maximum of the diameters of R_v tend to 0, the sum

$$-\frac{1}{\pi}\sum_v \frac{\partial\alpha}{\partial\bar{\zeta}}(\zeta_v)F(\zeta)\frac{1}{\zeta_v - z}m_2(R_v)$$

(m_2 is the two-dimensional Lebesgue measure of R_v) converges to

$$= -\frac{1}{\pi}\iint_{W-W_0}\frac{\partial\alpha}{\partial\bar{\zeta}}F(\zeta)\frac{1}{\zeta - z}d\zeta d\eta$$

uniformly for $z \in K$ (since $K \subset W_0$). Hence

$$\begin{aligned}\lambda(f) &= -\frac{1}{\pi}\iint_{W-W_0}\frac{\partial\alpha}{\partial\bar{\zeta}}F(\zeta)\lambda(z \mapsto \frac{1}{\zeta - z})d\zeta d\eta \\ &= \frac{1}{\pi}\iint_{W-W_0}\frac{\partial\alpha}{\partial\bar{\zeta}}F(\zeta)\Phi(\zeta)d\zeta d\eta = 0\end{aligned}$$

since $\Phi = 0$ on $\mathbb{C} - K \supset W - W_0$ Thus $\lambda|E = 0 \Rightarrow \lambda|\mathcal{A}(K) = 0$, which proves 2.1 and therefore the implication (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii).Suppose that U is a connected component of $\Omega - K$ with U is relatively compact in Ω. Then $\partial U \subset K$. By the maximum principle, if $a \in U$ (so that $a \notin K$), we have

$$|f(a)| \leq \sup_{z\in\partial U}|f(z)| \leq |f|_k \text{ for all } f \in \mathcal{O}(\Omega)$$

contradicting the assumption made in (iii).

(ii) $\Rightarrow$ (iii) Let $\Omega - K = \cup_{v\in\Gamma}U_v$ be the decomposition of $\Omega - K$ into connected components. By assumption, none of the sets $\bar{U}_v$ can be a compact set contained in Ω. Let $a \in \Omega$, $a \notin K$, then $a \in U_\mu$ for some $\mu \in \Gamma$. Let $L = KU\{a\}$. Then

$$\Omega - L = \cup_{v\neq\mu}U_v \cup (U_\mu - \{a\})$$

is the decomposition of $\Omega - L$ into connected components. Since the closure of $U_\mu - \{a\}$ is the same as that of U_μ, no component of $\Omega - L$ is relatively compact in Ω. We have proved above (in the implication (ii) $\Rightarrow$ (i)) that we therefore have the following:

$$\{f|_L : f \in \mathcal{O}(\Omega)\} \text{ is dense in } \mathcal{A}(L)$$

Now, the function Φ defined by $\Phi = 0$ on K, $\Phi(a) = 1$ clearly belongs to $\mathcal{A}(L)$ (since $a \notin K$). If now $f \in \mathcal{O}(\Omega)$ is such that $|f - \Phi|_L < 1/2$, then

$$|f(a)| > 1/2 > |f|_K$$

This proves that (i) $\Rightarrow$ (iii) and thus the theorem is proved.

Now , we define new concepts and then prove the second form and the classical Runge theorem :

Definition 2.1.4 *Let Ω be an open set in $\mathbb{C}$ and let A be a subset of Ω. We define*

$$\hat{A} = \hat{A}_\Omega$$

to be the union of A with those connected components of $\Omega - A$ which are relatively compact in Ω.

We can now proceed to the second form of Runge's theorem without proof.For more details see [27].

Definition 2.1.5 *Let Ω be an open set in $\mathbb{C}$ and Ω' an open set with $\Omega' \subset \Omega$. We call (Ω, Ω') a Runge pair if the restriction map*

$$r^\Omega_{\Omega'} = r : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega'),\ f \mapsto f|\Omega'$$

has dense image , i.e.,$r(\mathcal{O}(\Omega))$ is dense in $\mathcal{O}(\Omega')$. We also say that Ω' is Runge in Ω if this is the case .

Theorem 2.1.2 (Runge's Theorem: Second Form) *[27] Let Ω, Ω' be open sets in $\mathbb{C}$ with $\Omega' \subset \Omega$. Then (Ω, Ω') is a Runge pair if and only if no connected component of $\Omega - \Omega'$ is compact.*

Now ,we state the classical theorem but we will state and prove the other version of the theorem.

Theorem 2.1.3 (The Classical Runge Theorem) *[27] Let Ω be open in $\mathbb{C}$ and let $\mathbb{C} - \Omega = \cup_{\alpha \in A} C_\alpha$ be the decomposition of $\mathbb{C} - \Omega$ into connected components C_α. Let $A' \subset A$ be the set $A' = \{\alpha \in A : C_\alpha \text{ is compact }\}$.For each $\alpha \in A'$,choose $a_\alpha \in C_\alpha$.* Then *,any $f \in \mathcal{O}(\Omega)$ can be approximated ,uniformly on compact subsets of Ω,by rational functions , all of whose poles are contained in the set $\{a_\alpha\}_{\alpha \in A'}$.*

Theorem 2.1.4 *Let K be a compact subset of the complex plane. If $f(z)$ is analytic on an open set containing K, then $f(z)$ can be approximated uniformly on K by rational functions with poles off K.*

Proof: let D be an open set with piecewise smooth boundary such that $K \subset D$ and such that $f(z)$ is analytic on $D \cup \partial D$. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta \ , \ z \in D$$

We chop ∂D up into a union of short curves γ_j such that each γ_j is contained in a disk $\{|z - c_j| < r_j\}$ that is at a positive distance from K. Then $f(z) = \sum f_j(z)$,where

$$f_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} d\zeta \ , \ z \notin \gamma_j$$

The function $f_j(z)$ is analytic off γ_j (Morera's theorem) and vanishes at ∞. Each $f_j(z)$ has a Laurent expansion in descending powers of $z - c_j$ that converges uniformly for $|z - c_j| > r_j$, hence uniformly on K. Thus $f_j(z)$ is uniformly approximable on K by polynomials in $1/(z - c_j)$. Adding these approximants, we see that J(z) is uniformly approximable on K by rational functions with poles off K.

Note:There is some flexibility with respect to the location of the poles of the approximating rational functions.

Lemma 2.1.1 *Let K be a compact subset of the complex plane, let U be a connected open subset of the extended complex plane C^* disjoint from K, and let $z_0 \in U$. Every rational function with poles in U can be approximated uniformly on K by rational functions with poles at z_0.*

Proof: Here we consider a polynomial in z to be a rational function with pole at ∞. The lemma is established by a "translation of poles" argument. We define the set V to consist of those points $\zeta \in U$ such that $1/(z - \zeta)$ is approximable uniformly on K by rational functions with pole at z_0 Thus if $\zeta \in V$, then each power $1/(z - \zeta)^k$ is also uniformly approximable on K by rational functions with pole at z_0 . Since $z_0 \in V$, the set V is nonempty. But U is connected, to show that $V = U$, it suffices to show that V is open and closed in U. The "closed" assertion is easy. If $\zeta \in U$ is a limit of a sequence $\zeta_j \in V$, then $1/(z - \zeta_j)$ converges uniformly on K to $1/(z - \zeta)$, so by the definition of V , also $\zeta \in V$. The crux of the proof then is to show that V is open, and this reduces to showing that if $\zeta_0 \in V$, and if ζ is near ζ_0, then $1/(z - \zeta)$ is uniformly approximable on K by polynomials in $1/(z - \zeta_0)$ We begin with the special case $\zeta_0 = \infty$. We must show that if ζ is near ∞ (that is, $|\zeta|$ is large), then $1/(z - \zeta)$ is uniformly approximable on K by polynomials in z. For this, we expand $1/(z - \zeta)$ in a geometric series,

$$\frac{1}{z - \zeta} = -\frac{1}{\zeta} \frac{1}{1 - (z/\zeta)}$$

If $|z| \leq \mathbb{C}$ for all $z \in K$, and if $|\zeta| > 2C$, then the kth term of the series is dominated by $1/2^k$. By the Weierstrass M-test, the series converges uniformly on K, and consequently, $\zeta \in V$ whenever $|\zeta| > 2C$. If ζ_0 is finite, the proof is essentially the same, with a change of variable $z \mapsto 1/(z - \zeta_0)$ to place ζ_0 at ∞. We choose $\epsilon > 0$ less that the distance from ζ_0 to K, so that $|z - \zeta_0| \geq \epsilon$ for $z \in K$. If $|\zeta - \zeta_0| < \epsilon/2$, then $|(\zeta - \zeta_0)^k/(z - \zeta_0)^k| < 1/2^k$ for $z \in K$, so the geometric series

$$\frac{1}{z-\zeta} = \frac{1}{z-\zeta_0}\frac{1}{1-(\zeta-\zeta_0)/(z-\zeta_0)} = \frac{1}{z-\zeta_0}\sum_{k=0}^{\infty}\frac{(\zeta-\zeta_0)^k}{(z-\zeta_0)^k}$$

is uniformly convergent for $z \in K$, by the Weierstrass M-test. Thus all ζ satisfying $|\zeta - \zeta_0| < \epsilon/2$ belong to V, and so V is an open set. This proves the lemma.

By approximating with rational functions and then using the lemma to translate the poles, we obtain immediately the following sharper version of Runge's theorem.

Theorem 2.1.5 *Let K be a compact subset of the complex plane, and suppose that $f(z)$ is analytic on an open set containing K. Let S be a subset of $\mathbb{C}^*\backslash K$ such that each connected component of $\mathbb{C}^*\backslash K$ contains a point of S. Then $f(z)$ can be approximated uniformly on K by rational functions with poles in S.*

In particular, if K is a compact subset of the complex plane, and if the complement of K is connected, then each function analytic in a neighborhood of K can be approximated uniformly on K by polynomials in z. There is a considerably more difficult theorem on polynomial approximation, *Mergelyan's theorem*, which asserts that any function that is continuous on K and analytic on the interior of K can be approximated uniformly on K by polynomials in z. The analogous statement for rational approximation is false.

2.2 Mergelyan's Theorem

Let K be a compact set in the plane and suppose that f is a complex function on K that can be uniformly approximated by analytic polynomials on K. It then follows that f is continuous on K and analytic in the interior. If any such f can be approximated uniformly by polynomials, then the complement of K must be connected.

We will state the theorem without proof, for proof see [2].

Theorem 2.2.1 (Mergelyan's Theorem) *Let K be a compact set in the plane such that the complement is connected, and suppose that f is continuous on K and analytic in the interior of K. To each $\epsilon > 0$ there is a polynomial p such that $|f - p| < \epsilon$ on K.*

The difference between Runge's and Mergelyan's that Runge's theorem applies only if f is analytic in a neighborhood of K ,but Mergelyan's theorem applies if f is analytic in the interior of K, and therefore Mergelyan's theorem is considerably stronger. In particular, if the interior of K is empty, any continuous function can be approximated uniformly by analytic polynomials. If K is an interval, this is the classical Weierstrass' theorem.

Chapter 3

Universal Functions

In this chapter, we prove Runge-type theorems and universality results for locally univalent holomorphic and meromorphic functions. Refining a result of M. Heins, we also show that there is a universal bounded locally univalent function on the unit disk.

We will start with some basic definitions

Definition 3.0.1 *A single-valued function f is said to be univalent in a domain $D \subset \mathbb{C}$ if it never takes the same value twice , that is ,if $f(z_1) \neq f(z_2)$ for all points z_1, z_2 in D with $z_1 \neq z_2$*

Definition 3.0.2 *The function f is said to be locally univalent at point $z_0 \in D$ if it is univalent in some neighborhood of z_0*

Definition 3.0.3 *An analytic univalent function is called a conformal mapping.*

Definition 3.0.4 *Let Ω be a domain in the complex plane $\mathbb{C}$, a function $f \in \mathcal{O}(\Omega)$ is called universal if the set $\{f \circ \phi : \phi \in Aut(\Omega)\}$ is dense in $\mathcal{O}(\Omega)$, where $Aut(\Omega)$ the group of all conformal automorphisms .*

For a set $M \subseteq \mathbb{C}$ we denote by $\mathcal{O}_{l.u.}(M)$ the family of all functions which are holomorphic and locally univalent on some open neighborhood of M in $\mathbb{C}$

We denote by $B(\Omega)$ the set of all $f \in \mathcal{O}(\Omega)$ such that $|f(z)| \leq 1$ on Ω , and write $B_{l.u.}(\Omega) := B(\Omega) \cap \mathcal{O}_{l.u.}(\Omega)$ for the set of bounded locally univalent functions .

We denote $\mathcal{M}(\Omega)$ the set of all meromorphic functions on Ω.

We think of $M(\Omega)$ as a metric space equipped with (metrizable) topology of locally uniform convergence w.r.t. the chordal metric χ on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ where χ is defined as usual by

1. $\chi(z_1, z_2) := \frac{|z_2 - z_1|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$, if $z_1, z_2 \in \mathbb{C}$

2. $\chi(z_1, \infty) = \chi(\infty, z_1) := \frac{1}{\sqrt{1+|z_1|^2}}$

Now, we are stating two important and famous theorems :

Theorem 3.0.1 *[31] Every open set Ω in the plane is the union of a sequence $\{K_n\}$, $n = 1, 2, 3, ...,$ of a compact sets such that*

- *K_n lies in the interior of K_{n+1} for $n = 1, 2, 3, ...$*
- *Every compact subset of Ω lies in some K_n*

Theorem 3.0.2 (Hurwitz's Theorem) *[30] Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$, and suppose that $f(z)$ has a zero of order N at z_0 Then there exists $p > 0$ such that for k large, $f_k(z)$ has exactly N zeros in the disk $|z - z_0| < p$, counting multiplicity, and these zeros converge to z_0 as $k \to \infty$.*

We denote by tr(γ) the trace of a curve in $\mathbb{C}$ and by $ind(z)$ the winding number of around z. Let U be an open set in $\mathbb{C}$ and let $\mathcal{O}_{\neq 0}(U)$ be the set of all functions in $\mathcal{O}(U)$ with no zeros in U. For a set M in $\mathbb{C}$ we write $f \in \mathcal{O}_{\neq 0}(M)$ if there is an open neighborhood U of M such that $f \in \mathcal{O}_{\neq 0}(U)$. For a compact set K in $\mathbb{C}$ we set

$$||f - g||_k := \max_{z \in K} |f(z) - g(z)|$$

where f and g are holomorphic functions in a neighborhood of K, and

$$\chi_K(f, g) := \max_{z \in K} \chi(f(z), g(z))$$

where f and g are meromorphic functions in a neighborhood of K.

Proposition 3.0.1 *Let Ω be a domain in $\mathbb{C}$, K be a compact $\mathcal{O}$-convex set in Ω and $\epsilon > 0$,*

(a) Suppose $f \in \mathcal{O}_{\neq 0}(K)$. Then there exists a connected compact $\mathcal{O}$-convex set M in Ω with piecewise differentiable boundary ∂M such that $K \subseteq M$ and a function $g \in \mathcal{O}_{\neq 0}(M)$ with $||f - g||_K < \epsilon$.

(b) Suppose $f \in \mathcal{M}_{l.u.}(K)$. Then there exists a connected compact $\mathcal{O}$-convex set M in Ω with connected interior M° such that $K \subseteq M^\circ$ and a function $g \in \mathcal{M}_{l.u.}(M)$ with $\chi_K(f, g) < \epsilon$. If $f \in \mathcal{O}_{l.u.}(K)$, then $g \in \mathcal{O}_{l.u.}(M)$ with $||f - g||_K < \epsilon$.

Proof: We only prove part (a) , the proof of part (b) is similar. By the classical theorem of Runge and by Hurwitz' theorem, there exists a rational function $g \in \mathcal{O}(\Omega) \cap \mathcal{O}_{\neq 0}(K)$ such that $||f - g||_K < \epsilon$. Let $z_1, ..., z_N$ be the zeros of g in Ω. Since K is $\mathcal{O}$-convex, there exist paths $\gamma_j : [0, 1) \to \Omega \backslash K$ with $\gamma_j(0) = z_j$, $\gamma_j(t) \to \partial\Omega$ for $t \to 1$ and such that $W := \Omega\backslash(tr(\gamma_1)\cup...\cup tr(\gamma_N))$ is connected. Note that W is open and $K \subseteq W$.By 3.0.1 there exists compact sets (M_n) such that (M_n) is the compact exhaustion of W with connected compact $\mathcal{O}-$convex sets in W such that ∂M_n is piecewise differentiable for each $n \in \mathbb{N}$. Since a compact set in W is $\mathcal{O}-$convex in W if and only if it is $\mathcal{O}-$convex in Ω , we can take $M = M_n$ with $n \in \mathbb{N}$ sufficiently large so that $M = M_n \supseteq K$.Clearly , $g \in \mathcal{O}_{\neq 0}(M)$.

3.1 Runge-type theorems for locally univalent functions

We will start with a basic theorem which is needed to prove the following theorems

Theorem 3.1.1 *Let Ω be a domain in $\mathbb{C}$, K be a $\mathcal{O}-$convex compact set in Ω and $g \in \mathcal{O}_{\neq 0}(K)$. Then there exists a sequence $(f_m) \in \mathcal{O}_{\neq 0}(\Omega)$ such that $\lim\limits_{m\to\infty} f_m = g$ uniformly on K and*

$$\int_\gamma f_m(z)dz = \int_\gamma g(z)dz \tag{3.1}$$

for every closed curve $\gamma \subseteq K$ and every $m \in \mathbb{N}$.

Proof: By Proposition 3.0.1 (a) we may assume that K is connected and ∂K is piecewise differentiable. Let $D_1, ..., D_n$ be the bounded connected components of $\mathbb{C}\backslash K$. For $j = 1, ..., n$ choose $z_j \in D_j\backslash\Omega$ and let γ_j be a parametrization of the positively oriented boundary ∂D_j . Then $ind_{\gamma_k}(z_j) = \delta_{kj}$,where $\delta_{kj} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$. The connectedness of K implies that $\Gamma := \cup_{k=1}^n tr(\gamma_k)$ is a compact $\mathcal{O}-$convex set in Ω . Since every closed curve in K is homologous to a linear combination of the curves $\gamma_1, ..., \gamma_n$ with integer coefficients, it suffices to find a sequence $(f_m) \in \mathcal{O}_{\neq 0}(\Omega)$ such that $\lim_{m\to\infty} f_m = g$ uniformly on K and equation(3.1) holds for $\gamma = \gamma_k$ for every $k = 1, ..., n$.

Now for any $j = 1, ..., n$, Runge's Theorem implies that there is a sequence $(w_{j,m})_m$ in $\mathcal{O}(\Omega)$ with

$$\lim_{m\to\infty} w_{j,m}(z) = \frac{1}{g(z)(z-z_j)}$$

uniformly in Γ. In particular,

$$(\lim_{m\to\infty}(\int_{\gamma_k} w_{j,m}(z)g(z)))_{k,j=1,...,n} = E_n$$

where $E_n \in \mathbb{C}^{n\times n}$ is the identity matrix. Hence we can find a $\mu \in \mathbb{N}$ such that the matrix

$$A := \int_{\gamma_k} w_{j,\mu} g(z))_{k,j=1,...,n}$$

is non-singular.

By a well-known extension of Runge's Theorem there exists a sequence $(g_m) \in \mathcal{O}_{\neq 0}(\Omega)$ such that $\lim_{m\to\infty} g_m = g$ uniformly on K. Consider the functions

$$\psi_k : \mathbb{C}^n \to \mathbb{C}, \quad (s_1, ..., s_n) \mapsto \int_{\gamma_k} \exp(\sum_{j=1}^n s_j w_{j,\mu}) g(z) dz,$$

$$\psi_{k,m} : \mathbb{C}^n \to \mathbb{C}, \quad (s_1, ..., s_n) \mapsto \int_{\gamma_k} \exp(\sum_{j=1}^n s_j w_{j,\mu}) g_m(z) dz$$

,

and the entire functions $\Psi, \Psi_m : \mathbb{C}^n \to \mathbb{C}^n$ defined by $Ppsi(s) := (\psi_1(s), ..., \psi_2(s))$ and $\Psi_m(s) := (\psi_{1,m}(s), ..., \psi_{n,m}(s))$.Then $\lim_{m\to\infty} \Psi_m = \Psi$ locally uniformly on $\mathbb{C}^n$ and $D\Psi(0) = A$ is non-singular.Hence there exists a sequence $(s_m) = (s_{1,m}, ..., s_{n,m})$ in $\mathbb{C}^n$ with $\lim_{m\to\infty} s_m = 0$ and $\Psi_m(s_m) = \Psi(0)$. This concludes the proof with

$$f_m(z) = \exp(\sum_{j=1}^n s_{j,m} w_{j,\mu}(z)) g_m(z)$$

Now, we will use the previous theorem along with proposition 3.0.1 to prove Runge-type theorem for locally univalent holomorphic functions ,

Theorem 3.1.2 *Let Ω be a domain in $\mathbb{C}$ and let K be a compact set in Ω such that $\Omega\backslash K$ has no relatively compact components in Ω. Then*

(a) every function $f \in \mathcal{O}_{l.u.}(K)$ can be approximated uniformly on K by functions in $\mathcal{O}_{l.u.}(\Omega)$.

(b) every function $f \in \mathcal{M}_{l.u.}(K)$ can be approximated χ-uniformly (uniformly with respect to the chordal metric) on K by functions in $\mathcal{M}_{l.u.}(\mathbb{C})$,provided that $\mathbb{C}\backslash K$ is connected.

Proof of (a): By Proposition 3.0.1(b) we may assume that $f \in \mathcal{O}_{l.u.}(M)$ for some connected $\mathcal{O}-$convex compact set M of with smooth boundary and such that $K \subseteq M^{\circ}$. Hence we can apply Theorem 3.1.1 to $f' \in \mathcal{O}_{\neq 0}(M)$, so there exists a sequence $(g_n) \subseteq \mathcal{O}_{\neq 0}(\Omega)$ with $\lim_{n\to\infty} g_n = f'$ uniformly on M and

$$\int_\gamma g_n(z)dz = \int_\gamma f'(z)dz = 0$$

for every closed curve γ in M.

Now we choose a compact exhaustion $(K_k)_k$ of Ω by connected $\mathcal{O}-$convex sets in Ω with smooth boundaries and such that $K_1 = M$. Suppose we have fixed arbitrary numbers $\epsilon > 0$, $k \in \mathbb{N}$ and a function $h \in \mathcal{O}_{\neq 0}\Omega$ with $\int_\gamma h(z)dz = 0$ for every closed curve γ in K_k. Then by [[13], Lemma 4] there exists a function $v \in \mathcal{O}(\Omega)$ with $||v||_{K_k} < \epsilon$ and $\int_\gamma e^{v(z)}h(z)dz = 0$ for every closed curve γ in K_{k+1}.From this fact and an obvious induction argument, we can deduce that there exists a sequence $(v_{n,k})_k$ in $\mathcal{O}(\Omega)$ with $||v_{n,k}||_{K_k} < \frac{1}{2^k n}$ and such that for every closed curveγ in K_k we have

$$\int_\gamma \exp(\sum_{j=1}^{n} v_{n,j}(z))g_n(z)dz = 0$$

We define a holomorphic function $w_n \in \mathcal{O}(\Omega)$ by

$$w_n(z) := \textstyle\sum_{j=1}^{\infty} v_{n,j}(z)$$

Clearly we have $||w_n||_K < \frac{1}{n}$ and

$$\int_\gamma e^{w_n(z)} g_n dz = 0$$

for every closed curveγ in Ω . This means that for fixed $z_0 \in K$ and for each n there is an anti-derivative $G_n \in \mathcal{O}(\Omega)$ of $e^{w_n} g_n$ with $G_n(z_0) = f(z_0)$.By construction, $G_n \in \mathcal{O}_{l.u.}(\Omega)$. Since M is connected and $\lim_{n\to\infty} G_n' = f'$ uniformly on M we conclude $\lim_{n\to\infty} G_n = f$ uniformly on M and hence on K.

Proof of (b): By Proposition 3.0.1 (b) we may assume that $f \in \mathcal{M}_{l.u.}(M)$ for some compact $\mathcal{O}-$convex set M in $\mathbb{C}$ whose interior $G := M^\circ$ is connected and contains K. Since f is locally univalent in a neighborhood of M, its Schwarzian derivative

$$S_f := (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$$

is holomorphic there, so $S_f \in \mathcal{O}(M)$. According to some basic facts about complex differential equations, see e.g. ([22], Theorem 6.1), we can recover f from S_f by writing f as the quotient,

$$f = \frac{u_1}{u_2}$$

of two linearly independent solutions $u_1, u_2 \in \mathcal{O}(\Omega)$ of the homogeneous linear differential equation

$$w'' + \frac{1}{2} S_f . w = 0 \tag{3.2}$$

Since $S_f \in \mathcal{O}(M)$ and $\mathbb{C}\backslash M$ has no bounded components, the classical Runge theorem shows that there exist polynomials $p_n : \mathbb{C} \to \mathbb{C}$ such that

$$p_n \to S_f \quad \text{uniformly on } M.$$

We now consider the homogeneous linear differential equations corresponding to the polynomials p_n. Fix $z_0 \in G$ with $u_2(z_0) \neq 0$, and let $v_n \in \mathcal{O}(\mathbb{C})$ be the unique solution of the initial value problem

$$v_n'' + \tfrac{1}{2} p_n . v_n = 0, \quad v_n(z_0) = u_1(z_0), \quad v_n' = u_1'(z_0).$$

Then we clearly have

$$v_n(z) = u_1(z_0) + u_1'(z_0)(z - z_0) - \frac{1}{2} \int_{z_0}^{z} (z - \zeta) p_n(\zeta) v_n(\zeta) d\zeta, \quad z \in \mathbb{C}.$$

Hence a standard application of Gronwall's lemma(Let u, v be nonnegative integrable functions in $[1, t]$ and $c > 0$ be any constant.if $u(t) \leq c + \int_1^t u(s)v(s)ds$ holds for all t ,then $u(t) \leq c\exp(\int_1^t v(s)ds)$ forall t) ([22]) shows that the sequence (v_n) is locally bounded in G. We are therefore in a position to apply Montel's theorem(simpler version of the theorem states that a uniformly bounded family of holomorphic functions defined on an open subset of the complex numbers is normal.) and conclude that $\{v_n : n \in \mathbb{N}\}$ is a normal family. Clearly, every subsequential limit function $v \in \mathcal{O}(G)$ of (v_n) is a solution of 3.2 with $v(z_0) = u_1(z_0)$ and $v'(z_0) = u_1'(z_0)$ in G. By uniqueness of this solution, we conclude $v = u_1$. Consequently, we have

$$v_n \to u_1 \quad \text{locally uniformly in } G.$$

For the unique solution $w_n \in \mathcal{O}(\mathbb{C})$ of the initial value problem

$$w_n'' + \tfrac{1}{2}p_n.w_n = 0, \quad w_n(z_0) = u_2(z_0), \quad w_n'(z_0) = u_2'(z_0),$$

we arrive a similar way at

$$w_b \to u_2 \quad \text{locally uniformly in } G.$$

We claim that v_n and w_n are linearly independent for large n. For this purpose we consider the Wronskian

$$W(h, g) = hg' - h'g \in \mathcal{O}(G) \text{ for } g, h \in \mathcal{O}(G).$$

Since u_1 and u_2 are solutions of the linear differential equation 3.2, there is a constant $\lambda \in \mathbb{C}$ such that $W(u_1, u_2)(z) = \lambda$ for all $z \in G$, see ([22], Proposition 1.4.8). In a similar way, we see that for each $n \in \mathbb{N}$ there is $\lambda_n \in \mathbb{C}$ such that $W(v_n, w_n)(z) = \lambda_n$ for all $z \in \mathbb{C}$. By what we have already shown, $\lambda_n \to \lambda$ as $n \to \infty$. Since u_1 and u_2 are linearly independent, we have $\lambda \neq 0$, (, Proposition 1.4.2). Hence $\lambda_n \neq 0$, so v_n and w_n are linearly independent for all but finitely many $n \in \mathbb{N}$.
We can therefore apply Theorem 6.1 in [22] which implies that

$$g_n := \frac{v_n}{w_n} \in \mathcal{M}_{l.u.}(\mathbb{C})$$

Since $v_n \to u_1$ and $w_n \to u_2$ locally uniformly in G, we see that $g_n \to u_1/u_2 = f$ locally uniformly in G w.r.t. the chordal metric, so in particular $\chi-$uniformly on K.

This brings up the question "Can every $f \in \mathcal{M}_{l.u.}(K)$ be approximated χ-uniformly on K by functions in $\mathcal{M}_{l.u.}(\Omega)$?" ,the answer is yes, and now we will prove it :

From theorem 3.1.2, we can approximate every function in $\mathcal{M}_{l.u.}(K)$ by functions in $\mathcal{M}_{l.u.}(\mathbb{C})$, so, we have $\{f_n\} \to f$, where $\{f_n\} \subset \mathcal{M}_{l.u.}(\mathbb{C})$ and $f \in \mathcal{M}_{l.u}(K)$. Let $\{K_n\}$ be the compact exhaustion of Ω i.e, $\Omega = \cup_{m\in\mathbb{N}} K_m$, so, we have $K \subset K_l$ for some $l \in \mathbb{N}$.

We know from basic concepts if $\{x_n\} \to x$ in a set A, then $\{x_n|_B\} \to x|_B$ for some subset B of A. And the proof of this is easy , for any $\epsilon > 0$, we have

$$0 \le |x_n - x|_B \le |x_n - x|_A < \epsilon$$

Now, f_n is locally univalent for all n, and so, $f_n|_{K_m}$, since $f_n' \neq 0, \forall z \in \mathbb{C}$. Moreover $f_n|_{K_m}$ is meromorphic.

Thus, $\{f_n|_{K_m}\} \to f|_{K_m}$, for $m \ge l$, but $f|_{K_m} = f|_K = f$, so, $\{f_n|_{K_m}\} \to f$. As $m \to \infty$, $\{f_n|_\Omega\} \to f$. Hence, we can approximate every function in $\mathcal{M}_{l.u.}(K)$ by functions in $\mathcal{M}_{l.u.}(\Omega)$.

After proving theorems for holomorphic functions, we will try to prove that if a function $f : K \to \mathbb{C}$ is continuous on a compact set K with connected complement and locally univalent in K° , then we can approximate f by entire locally univalent functions:

Before we start the proof, we will state "Tietze extension theorem"

Theorem 3.1.3 *[1] Let S be a closed subset of a metric space X and suppose that f is continuous on S, then f can be extended to a continuous function g defined on all X.Furthermore if $|f(x)| \le M$ on S , then $|g(x)| \le M$ on X.*

From Tietze theorem, we extend f to be continuous on all $\mathbb{C}$, as f is continuous on closed (compact) set K and $\mathbb{C}$ is a metric space.
Now, to transform the function from continuous to holomorphic, we use Cauchy's transform :-

Theorem 3.1.4 *[29] Let Γ be rectifiable curve and $f(\zeta)$ be a continuous function defined on Γ, then the integral of Cauchy type*

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

is holomorphic on $\mathbb{C} - \Gamma$.

Since K is compact, we can find a closed contour containing K, taking the closed contour to be a circle with radius R, "C_R".

Now, apply the Cauchy transform for f on C_R , we get $\hat{f} = \int_{C_R} \frac{f(w)}{w - z} dw$ is holomorphic on $\mathbb{C} - C_R$. As $R \to \infty$, $\hat{f}$ becomes entire , i.e $\hat{f} \in \mathcal{O}(\mathbb{C})$, so we can find $\{f_n\} \subset \mathcal{O}(\mathbb{C})$ such that $\{f_n\} \to \hat{f}$ uniformly.

By Cauchy integral formula, we have $f_n = \int_{C_R} \frac{f_n(w)}{w-z} dw$. Now, $f_n = \int_{C_R} \frac{f_n(w)}{w-z} dw \to \hat{f} = \int_{C_R} \frac{f(w)}{w-z} dw$ as $R \to \infty$. But the question here , does $f_n \to f$ uniformly ?

Conclusion: In proving the approximation of continuous functions, we got several problem :

- Does the sequence of entire function as defined above converge to the continuous function on compact set ,
- if Yes , Does the restriction of the sequence on the compact set stills entire ,
- Or we need to take the set K as a closed unbounded set .

3.2 Universal locally univalent functions

The aim of this section is to provide necessary and also sufficient conditions for the existence of Φ-universal functions for families of locally univalent holomorphic or meromorphic functions on a domain in $\mathbb{C}$. For this purpose, the following concepts, which have been introduced in [21] and [12], will play a crucial role.

Definition 3.2.1 *Let Ω be a domain in $\mathbb{C}$, $\mathcal{G} \subseteq \mathcal{M}(\Omega)$ and Φ a family of holomorphic self-maps of Ω . A function $g \in \mathcal{G}$ is called Φ-universal in $\mathcal{G}$ if $\{g \circ \phi : \phi \in \Phi\}$ is dense in $\mathcal{G}$. If $g \in \mathcal{G}$ is $Aut(\Omega)$-universal in $\mathcal{G}$, we simply call g universal in $\mathcal{G}$.*

Note that a Φ-universal function in $\mathcal{G}$ is always supposed to belong to $\mathcal{G}$.

Definition 3.2.2 *Let Ω be a domain in $\mathbb{C}$ and let (ϕ_n) be a sequence of holomorphic self-maps of Ω .*

1. *We say that (ϕ_n) is run-away , if for every compact set $K \subseteq \Omega$ there exists $n \in \mathbb{N}$ with $\phi_n(K) \cap K = \emptyset$.*
2. *We say that (ϕ_n) is eventually injective , if for every compact set $K \subseteq \Omega$ there exists $N \in \mathbb{N}$ such that the restriction $\phi_n|_K$ is injective for all $n \geq N$.*

These conditions turn out to be necessary for the existence of Φ-universal functions in $\mathcal{O}_{l.u.}(\Omega)$:

Proposition 3.2.1 *Let Ω be a domain in $\mathbb{C}$ and let Φ be a family of locally univalent self-maps of Ω . Suppose that there is a Φ-universal function in $\mathcal{O}_{l.u.}(\Omega)$, Then Φ contains a run-away and eventually injective sequence.*

Proof: Suppose that u is Φ-universal in $\mathcal{O}_{l.u.}(\Omega)$, and let K be a compact set in Ω . Choose a compact set L in which contains K in its interior and which is the closure of its interior. Let

$$\delta := \frac{1}{2}dist(K, \partial L) > 0, \quad M := \sup_{z \in L} |z|$$

Then $f(z) := z + 2M + 2\delta$ belongs to $\mathcal{O}_{l.u.}(\Omega)$. Since u is Φ-universal in $\mathcal{O}_{l.u.}(\Omega)$ there exists $\phi \in \Phi$ such that

$$||u \circ \phi - f||_L < \delta \tag{3.3}$$

This in particular implies $|u(\phi(z))| \geq |f(z)| - \delta \geq M + \delta$ for all $z \in K$, so

$$\min_{z \in K} |u(\phi(z))| \geq M + \delta > M \geq \max_{\phi(z) \in K} |\phi(z)|,$$

and thus $\phi(K) \cap K = \emptyset =$. Next , we fix $z_0 \in K$. Then the estimate 3.3 shows that for every $z \in \partial L$ we have

$$|[u(\phi(z_0)) - u(\phi(z))] - [z_0 - z]| < 2\delta \leq |z_0 - z|.$$

Hence, by Rouche's theorem (states that for any two complex-valued functions f and g holomorphic inside some region K with closed contour ∂K, if $|g(z)| < |f(z)|$ on ∂K , then f and $f + g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity.), $u(\phi(z_0)) - u(\phi(z))$ and $z_0 - z$ have the same numbers of zeros in L°. This implies that ϕ is injective on K.
Finally let (K_n) be compact exhaustion of Ω. For each j, we can find ϕ_j such that $\phi_j(K_j) \cap K_j = \emptyset$ and ϕ_j injective on K_j. Thus, $\{\phi_j\}$ is a run-away and eventually injective sequence in Φ.

We next turn to sufficient conditions, but restricting the discussion to the cases when Ω is either simply connected or of infinite connectivity. The reason for this is the fact that for domains of finite connectivity $N > 1$ there are Φ-universal functions f for $\mathcal{O}(\Omega)$ such that the family Φ of locally univalent self-maps of Ω is mainly responsible for the denseness of $\{f \circ \phi : \phi \in \Phi\}$ in $\mathcal{O}(\Omega)$ and not $f \in \mathcal{O}(\Omega)$, [12]. For simply connected domains, we have a complete picture:

We are going to apply a fairly standard universality criterion. Let $\mathcal{T}$ be a collection of continuous self-maps of a topological space X. We say that $\mathcal{T}$ acts transitively on X if for every pair of open sets U and V in X there is an $\tau \in \mathcal{T}$ such that $\tau(U) \cap V \neq \emptyset$. An element $u \in X$ is called universal for $\mathcal{T}$ if the orbit $\{\tau(u) : \tau \in \mathcal{T}\}$ is dense in X.

Theorem 3.2.1 (Birkhoff transitivity criterion) *Let X be a second countable Baire-space and $\mathcal{T}$ a family of continuous self-maps of X. Suppose that $\mathcal{T}$ acts transitively on X. Then there exists an universal element for $\mathcal{T}$ and the set of all universal elements for $\mathcal{T}$ is a dense G_δ-subset of X.*

For a proof see for instance [[11], Theorem 1]. For later use, we note that if X is a separable metric space, then a collection $\mathcal{T}$ of continuous self-maps of X acts transitively on X if and only if for every pair of points v and w in X there exist a sequence (v_n) in X and a sequence (τ_n) in T such that $v_n \to v$ and $\tau_n(v_n) \to w$.
We now apply these concepts to investigate universality for holomorphic and meromorphic functions. Let Ω be a domain in $\mathbb{C}$. We associate to any holomorphic self-map ϕ of Ω the composition operator

$$C_\phi : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega), \quad f \mapsto f \circ \phi$$

If ϕ is locally univalent, then C_ϕ maps $\mathcal{O}_{l.u.}(\Omega)$ into $\mathcal{O}_{l.u.}(\Omega)$. Since the union of $\mathcal{O}_{l.u.}(\Omega)$ with all constant functions is a complete metric space and $\mathcal{O}_{l.u.}(\Omega)$ is an open subset of this space, $\mathcal{O}_{l.u.}(\Omega)$ is a Baire-space.

Theorem 3.2.2 *Let Ω be a simply connected domain in $\mathbb{C}$ .Suppose that Φ is a family of locally univalent self-maps of Ω which contains a run-away and eventually injective sequence . Then there is a Φ-universal function in $\mathcal{O}_{l.u.}(\Omega)$ and the set of all Φ-universal functions in $\mathcal{O}_{l.u.}(\Omega)$ is a dense G_δ-subset of $\mathcal{O}_{l.u.}(\Omega)$*

Proof :In view of Theorem 3.2.1 it suffices to show that the family $\{C_\phi : \phi \in \Phi\}$ acts transitively on $\mathcal{O}_{l.u.}(\Omega)$. Let $f, g \in \mathcal{O}_{l.u.}(\Omega)$. Since Ω is simply connected there is an exhaustion (K_n) of Ω with compact sets K_n in Ω such that each K_n has connected complement. By assumption there is a sequence (ϕ_n) in Φ such that ϕ_n is injective on K_n and $\phi_n(K_n) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Define $L_n := K_n \cup \phi_n(K_n)$ and $h_n \in \mathcal{O}_{l.u.}(L_n)$ by

$$h_n := \begin{cases} f(z), & z \in K_n \\ g(\phi^{-1}(z)), & z \in \phi_n(K_n) \end{cases} \tag{3.4}$$

Note that each L_n has connected complement, hence by Theorem 3.1.2 (a) there exists a function $f_n \in \mathcal{O}_{l.u.}(\Omega)$ with $||f_n - h_n||_{K_n} \leq 1/n$. This implies $f_n \to f$ and $f_n \circ \phi_n \to g$ locally uniformly in Ω.

The next result of this section is concerned with universal locally univalent meromorphic functions. Chan [4] has shown that there exists a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ such that the set $T_f := \{f(. + n) : n \in \mathbb{N}\}$ is dense in $\mathcal{M}(\Omega)$ for every domain $\Omega \in \mathbb{C}$. In the locally univalent situation we need to restrict the discussion to simply connected domains since the same reasoning as in next example shows that if T_f is dense in $\mathcal{M}_{l.u.}(\Omega)$ for some $f \in \mathcal{M}_{l.u.}(\mathbb{C})$, then has to be simply connected.

Example: Let $K := \{z \in \mathbb{C} : 1/2 \leq |z| \leq 2\}$ and $f(z) := -1/z^2 \in \mathcal{M}_{l.u.}(K)$.Suppose that there is a sequence (g_n) in $\mathcal{M}_{l.u.}(\mathbb{C})$ which converges to f χ-uniformly on K. Then we have $S_{g_n} \to S_f$ uniformly on $\partial\mathbb{D}$. But since $S_{g_n} \in \mathcal{O}(\mathbb{C})$ for all $n \in \mathbb{N}$ the maximum principle implies $S_{g_n} \to h$ uniformly in $\mathbb{D}$ for a function $h \in \mathcal{O}(\mathbb{D})$.We have $S_f \equiv h$ on $\mathbb{D} \cap K$ and hence on $\mathbb{D}\backslash\{0\}$. This, however, contradicts the fact that 0 is a critical point of f.

Theorem 3.2.3 *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let Φ be a family of locally univalent self-maps of Ω which contains a run-away and eventually injective sequence (ϕ_n) .Then there is a Φ-universal function in $\mathcal{M}_{l.u.}(\Omega)$ and the set of all such functions is a dense G_δ-subset of $\mathcal{M}_{l.u.}(\Omega)$*

Proof: is identical to the prove of Theorem 3.2.2 except for that we need to apply part (b) of Theorem 3.1.2 instead of part (a).

Corollary 3.2.1 *There exists a function $f \in \mathcal{M}_{l.u.}(\mathbb{C})$ such that $T_f = \{f(.+ n) : n \in \mathbb{N}\}$ is dense in $\mathcal{M}_{l.u.}(\Omega)$ for every simply connected domain Ω .*

Proof: By Theorem 3.2.3 there exists a universal function $f \in \mathcal{M}_{l.u.}(\mathbb{C})$ such that T_f is dense in $\mathcal{M}_{l.u.}(\mathbb{C})$. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. Then, as a consequence of Theorem 3.1.2 (b), $\mathcal{M}_{l.u.}(\mathbb{C})$ is dense in $\mathcal{M}_{l.u.}(\Omega)$. This fact together with the universality of f implies that T_f is dense in $\mathcal{M}_{l.u.}(\Omega)$.

Now ,we recall that a compact subset K of a domain in $\mathbb{C}$ is called $\mathcal{O}$-convex if $\Omega\backslash K$ has no relatively compact components in Ω .

Theorem 3.2.4 *Let Ω be a domain of $\mathbb{C}$ of infinite connectivity and let Φ be a family of locally univalent self-maps of Ω .Suppose that there exists a sequence (ϕ_n) in Φ such that*

1. *(ϕ_n) is eventually injective , and*

2. *for every $\mathcal{O}$-convex compact set K in Ω and every $N \in \mathbb{N}$ there exists $n \geq \mathbb{N}$ such that $\phi_n(K) \cap K = \emptyset$ and $\phi_n(K) \cup K$ is $\mathcal{O}$-convex.*

Then there is a Φ*-universal function in* $\mathcal{O}_{l.u.}(\Omega)$ *and the set of all such functions is dense* G_δ*-subset of* $\mathcal{O}_{l.u.}(\Omega)$*.*

Proof: We start with an exhaustion (K_n) of Ω with compact sets K_n in Ω . We can assume that each K_n is $\mathcal{O}$-convex in Ω . By hypothesis, there is a sequence (ϕ_n) in Φ that has all the properties we need in order to proceed as in the proof of Theorem 3.2.2.

We now take a closer look at the case $\Phi \subseteq Aut(\Omega)$. It has been shown by Bernal- Gonzáles and Montes-Rodríguez [21] that if is not conformally equivalent to $\mathbb{C}\backslash\{0\}$ then there is a Φ-universal function in $\mathcal{O}(\Omega)$ if and only if Φ contains a run-away sequence. This result also holds in the setting of locally univalent functions:

Theorem 3.2.5 *Let* Ω *be a domain in* $\mathbb{C}$ *which is not conformally to* $\mathbb{C}\backslash\{0\}$ *and let* $(\phi_n) \subseteq Aut(\Omega)$ *. Then there is a* (ϕ_n)*-universal function in* $\mathcal{O}_{l.u.}(\Omega)$ *iff* (ϕ_n) *is run-away.*

Proof: By Proposition 3.2.1 we only need to show the "if"-part. Since by hypothesis, Ω is not conformally equivalent to $\mathbb{C}\backslash\{0\}$, the existence of a run-away sequence in $Aut(\Omega)$ implies that Ω is either simply connected or of infinite connectivity, see the discussion following Lemma 2.9 in ([21], p. 51-52). In the first case, when Ω is simply connected, we can simply apply Theorem 3.2.2 In the second case, when Ω is of infinite connectivity, we start with an exhaustion (K_n) of Ω with $\mathcal{O}$-convex compact sets K_n in Ω . Since ϕ_n is run-away we may assume that $\phi_n(K_n) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Then, by a key observation ([21], Lemma 2.12), it follows that $\phi_n(K) \cup K$ is $\mathcal{O}$-convex in Ω .

If Ω is conformally equivalent to $\mathbb{C}\backslash\{0\}$ then there are no universal functions in $\mathcal{O}_{l.u.}(\Omega)$. In fact, it was observed in [21] that there are no universal functions for $\mathcal{O}(\Omega)$ in this case. The argument is based on the maximum principle and stays valid for locally univalent functions.

Index

$\mathbb{C}$: Complex plane
K : Compact subset
Ω : Subset domain of $\mathbb{C}$
$\mathcal{O}(\Omega)$: The space of holomorphic functions on Ω
K° : Interior of the set K
K^c : Complement of K
$A(K)$: The space of continuous functions on K and holomorphic on K°
$C(K)$: The space of continuous functions on K
$\mathcal{A}(K)$: The space of continuous functions g on K s.t. $\exists f \in \mathcal{O}(U)$, $K \subset U$, U is open s.t. $f|_K = g$
ρ : the restriction map from $\mathcal{O}(\Omega)$ to $\mathcal{A}(K)$, $\rho(f) = f|_K$
$\hat{A}_\Omega$: The union of A with connected components of $\Omega - A$ which are relatively compact in Ω
$\mathcal{O}_{l.u.}(\Omega)$: The space of all holomorphic functions in Ω which are locally univalent
$\mathcal{O}_{l.u.}(M)$: The space of all holomorphic functions in Ω which are locally univalent in some open neighborhood of M
$Aut(\Omega)$: Automorphism map on Ω
$B(\Omega)$: the set of all $f \in \mathcal{O}(\Omega)$ s.t. $|f(z)| \leq 1$
$B_{l.u.}(\Omega)$: $B(\Omega) \cap \mathcal{O}_{l.u.}(\Omega)$
Φ : The family of holomorphic self-maps of Ω
$\mathcal{M}(\Omega)$: The set of all meromorphic functions on Ω
χ : The chordal metric
$\mathbb{D}$: The unit dist
Ω_f : The set of all non-critical points of f
$tr(\gamma)$: the trace of a curve γ
$\mathcal{O}_{\neq 0}(U)$: the set of all functions in $\mathcal{O}(U)$ with no zeros in U

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وجود اقترانات عالمية واحد لواحد محلية

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ملخص:

في هذه الرسالة , أثبتنا نظرية رانج والنتائج العالمية للاقترانات التحليلية و الاقترانات القريبة من التحليلية والتي تكون واحد لواحد محليا في مجموعات مرتصة وايضا بجوار مجموعات مرتصة . بعد ذلك , قمنا بتقريب الاقترانات القريبة من التحليلية في مجموعات مفتوحة تحتوي على مجموعات مرتصة . كذلك قمنا بحل مسائل في هذا المجال كانت غير محلولة من قبل حول الاقترانات المتصلة وتقريبها .