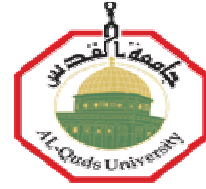


**Deanship of Graduate Studies
Al-Quds University**



**Some Norm Inequalities for Kronecker and
Hadamard Products**

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M.Sc. Thesis

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Hadamard Products**

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Al-Quds University/Palestine**

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This thesis is submitted in Partial fulfillment of requirements of the degree of Master of Science , Department of Mathematics / Program of Graduate studies.

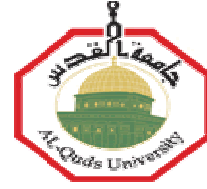
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The Program of Graduate Studies /Department of Mathematics



Thesis aproval

**Some Norm Inequalities for Kronecker and
Hadamard Products**

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Al-Quds University

1433/2012

Dedication

To my mother , Mariem

To my father , Abdelfatah

To my wife , Duaa

To my children , Yazan, Mariem

To my brothers , Mustafa, Alian, Ahmed

To my sisters, Ansaf, Eman, Hanan

To my friend , Tariq

To my colleagues "teachers"

Declaration

I certify that the thesis, submitted for the degree of master, is the result of my own research except where otherwise acknowledged, and that the thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed

Alaa saleh

Date :

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I am grateful to all the following doctors who taught me during the MA degree: Dr. Yousif Zahalqia, Dr. Ibrahem Qrouz, . Dr. Taha Abu kaf, Dr. Abedulhakeem Eidah, Dr. Mohammed Kaleel, Dr. Yousif Bedar.

Abstract

Many basic properties of the Kronecker products and Hadamard products are given , and many results for positive definite matrices are discussed. Moreover Holdert's inequality and the arithmetic, geometric mean inequalities are also applied for Kronecker and Hadamard products .

An analysis of inequalities concerning the spectral radius of Hadamard products of positive operators as space have been done in all details, including some applications for the Kronecker products in matrix equations and differential matrix equations.

Furthermore we showed that these inequalities can be extended to infinite nonnegative matrices .

A development of inequalities for Kronecker products and Hadamard products of positive definite matrices involving Kronecker powers and Hadamard powers of linear combination of matrices are given in complete details.

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Introduction

When most people multiply two matrices together, they generally use the conventional multiplication method.

We consider two types of matrix multiplication, that are very interesting, these multiplication are the Kronecker product and the Hadamard product.

In mathematics, the Kronecker product denoted by \otimes is an operation on two matrices of arbitrary sizes resulting in a block matrix. The Kronecker product should not be confused with the usual matrix multiplication which is an entirely different operation.

The Hadamard product denoted by \odot is a binary operation that takes two matrices of the same dimensions, and produces another matrix where each element is the product of the corresponding element of the original two matrices.

In chapter one, sections 1, 2 and 3, I give some basic concepts from matrix analysis.

In section 4, I give some of the basic properties of the Kronecker Product, and show

The difference between matrix multiplication and Kronecker Products matrices, by comparing some basic properties, also, we present the Kronecker sum of matrices, the vec-vector.

At the end of this chapter in section 5, we present some properties of the Hadamard products of matrices.

In chapter two, we analyze some inequalities for Kronecker products and Hadamard products of positive definite matrices in all details.

In chapter three, we analyze the Hadamard product of matrices of operators on \mathbb{C}^n , and inequalities for spectral radius of Hadamard products in all details.

Finally, in chapter four we put some applications of the Kronecker product, matrix equations, and matrix differential equations.

Index of Special Notation

	The set of all real numbers
	The set of all complex numbers
	Usually field (or)
	Square matrix of size
	Matrix of size
det()	The determinant of the matrix
	The transpose matrix of a matrix
	Conjugate of ,
	Conjugate transpose of ,
	Inverse of a nonsingular
-	Square root of matrix such that -
tr	Trace of
	Absolute value or -
	Spectrum of
	Spectral radius of
	Submatrix of ,
	Principal submatrix
	Vector of stacked columns of ,
	Kronecker product
	Hadamard product
	Kronecker sum
·	norm
·	(Euclidean) norm, frobenius

- (maximum absolute value) norm
- norm
- Singular value of ,
- eigenvalue of
- Cond Condition number
- Column vector
- Unitary matrix
- A The Hadamard inverse
- J 1 The Hadamard identity
- Summation
- Product
- The Kronecker power
- The Hadamard power
- The Hadamard sum
- The positive definite matrices

Chapter one

Preliminaries

1.1 Introduction

The contents of sections 1.1, 1.2, and 1.3 can be found in ref. [11].

Definition 1.1.1 If $A = [a_{ij}]$, then $A^T = [a_{ji}]$ is called the transpose of A and $A^H = [a_{ji}^*]$ is called the adjoint transpose of A , and the trace of A if A is defined by $\text{trace}(A) = \sum_{i=1}^n a_{ii}$.

Theorem 1.1.1 Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix then

$$\text{trace}(AB) = \text{trace}(BA).$$

Definition 1.1.2 If A is a square matrix then

- (a) A is called Hermitian if $A = A^H$.
- (b) A is called normal if $AA^H = A^H A$.
- (c) A is called unitary if $A^{-1} = A^H$, where I_n is an identity matrix of order n .
- (d) A is called orthogonal if $A^{-1} = A^T$. Therefore, $A^T A = I_n$.

Remark 1.1.3 All unitary and Hermitian matrices are normal.

Example 1.1.1 If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $A^H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, therefore $AA^H = A^H A$, thus A is normal.

Theorem 1.1.2 If A is a Hermitian matrix, then its eigenvalues are real number.

Definition 1.1.3 A matrix A is called idempotent if $A^2 = A$, and is called nilpotent if $A^n = 0$ for positive integer n .

Definition 1.1.4 Let A be a square matrix. A non-zero vector v is called an eigenvector corresponding to a scalar λ if $Av = \lambda v$. The scalar λ is called an eigenvalue of A , the set of all eigenvalues of A is called the spectrum of A and is denoted by $\sigma(A)$.

Definition 1.1.5 The spectral radius of A is the non negative real number

$$\max \{ |\lambda| : \lambda \in \sigma(A) \}.$$

Example 1.1.2 Consider the matrix $A = \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$, then we have

$$\begin{vmatrix} 7-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0, \text{ thus } (7-\lambda)(1-\lambda) - 8 = 0, \text{ which gives } \lambda = 3, 5, \text{ therefore}$$

$$\sigma(A) = \{3, 5\}, \text{ Hence } \rho(A) = 5.$$

Theorem 1.1.3 Let A be a square matrix, then $\text{trace}(A)$ equals to the sum of the eigenvalues of A and $\det(A)$ equals to the product of the eigenvalues of A .

Theorem 1.1.4 (Schurs unitary Triangularization theorem)

Given a matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$ in any prescribed order, then

there is a unitary matrix U such that $U^{-1}AU = T$ where $T = (t_{ij})$ is upper triangular matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$.

Definition 1.1.6 The matrix P is called a permutation matrix if each row and column has exactly one 1, and zeros elsewhere.

Example 1.1.3 Let
$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
, P and Q are permutation matrices.

Definition 1.1.8 (a) Let A be an $m \times n$ matrix, for index sets $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_s\}$, we denote the submatrix that lies in the rows of A indexed by I and the columns indexed by J as $A_{I,J}$.

Example 1.1.4 $A_{\{1,3\},\{1,2,3\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(b) If $I = J$ and $A_{I,I}$, then the submatrix $A_{I,I}$ is called a principal submatrix of A .

1.2 Norms of vectors and matrices

Definition 1.2.1 Let V be a vector space over a field F .

A function $\| \cdot \|$ is a vector norm if for all $x, y \in V$, we have:

- (1) $\|x\| \geq 0$
- (2) $\|x\| = 0$ if and only if $x = 0$.
- (3) $\| \alpha x \| = |\alpha| \|x\|$ for all scalars $\alpha \in F$.
- (4) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 1.2.2 Let X be a complex (or real) linear space. Then the function

$\langle \cdot, \cdot \rangle$ with the properties

- (1) $\langle x, x \rangle \geq 0$,
- (2) $\langle x, x \rangle = 0$ if and only if $x = 0$,
- (3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
- (4) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,

for all $x, y, z \in X$, $\alpha, \beta \in \mathbb{C}$ is called an inner product space on X .

Example 1.2.1 (vector norms)

(a) The Euclidean norm (or norm) on \mathbb{R}^n is

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

(b) The sum norm (or norm) on \mathbb{R}^n is $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$.

(c) The max norm (or ∞ Norm) on \mathbb{R}^n is $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$.

(d) The Norm on \mathbb{R}^n is $\|x\|_p = \sqrt[p]{|x_1|^p + \dots + |x_n|^p}$ for $p \geq 1$.

Theorem 1.2.1 (Holders Inequality)

If $p \geq 1$ and $q \geq 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|x\|_p \|y\|_q \geq |x_1 y_1 + x_2 y_2 + \dots + x_n y_n|, \text{ that is } \|x\|_p \|y\|_q \geq |x \cdot y|.$$

Theorem 1.2.2 (Cauchy – Schwarz Inequality)

If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space over field \mathbb{F} , then

$|\langle x, y \rangle| \leq \|x\| \|y\|$, $\langle x, x \rangle = \|x\|^2$. For all x, y , equality occurs if and only if x, y are linearly dependent.

Definition 1.2.3 A function $\|\cdot\|$ is said to be a matrix Norm if for all

$A \in M_n(\mathbb{F})$, it satisfies the Following :

- (a) $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = O$.
- (b) $\|cA\| = |c| \|A\|$, for all scalars $c \in \mathbb{F}$.
- (c) $\|A+B\| \leq \|A\| + \|B\|$.
- (d) $\|AB\| \leq \|A\| \|B\|$.

Some important properties of matrix norm are :

- (a) If $A \in M_n(\mathbb{F})$, $\|I\| = 1$.
- (b) $\|I\| = 1$.
- (c) If A is invertible matrix, then $\|A^{-1}\| \geq \frac{1}{\|A\|}$.
- (d) If $A \in M_n(\mathbb{F})$ such that $\|A\| = 1$.

Example 1.2.2 Let $\| \cdot \|_p$, the p-Norm is defined by

for $1 \leq p < \infty$, some special cases of the p-norm are :

(a) The ∞ -Norm defined for \mathbb{R}^n by $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. The maximum column sum matrix norm $\|A\|_\infty$ is defined on $\mathbb{R}^{m \times n}$ by $\|A\|_\infty = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$.

(b) The 1 -Norm defined for \mathbb{R}^n by $\|x\|_1 = \sum_{i=1}^n |x_i|$. The maximum row sum matrix norm $\|A\|_1$ is defined on $\mathbb{R}^{m \times n}$ by $\|A\|_1 = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$.

(c) In particular, when $p=2$ then

$\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$, is called the Frobenius norm (Euclidean norm).

(d) The spectral Norm is defined by $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$.

Definition 1.2.4 Let X and Y be normed spaces and let $A: X \rightarrow Y$ be a bounded linear operator with a bounded inverse A^{-1} . Then $\text{Cond}(A) = \frac{\|A\|}{\|A^{-1}\|}$, is called the condition number of A .

For example the invertible matrix A we have

$$\frac{\|A\|}{\|A^{-1}\|}$$

Definition 1.2.5 A matrix norm $\|\cdot\|$ is called unitarily invariant norm if

For all A and all unitary matrices U, V .

1.3 Positive definite matrices

Definition 1.3.1 A Hermitian matrix A , is said to be positive definite if

$x^*Ax > 0$ for all nonzero x , and it is called a positive semidefinite matrix if

$x^*Ax \geq 0$ for all x .

Properties of positive definite (semidefinite) matrices :

- (a) Any principal submatrix of a positive definite matrix is positive definite.
- (b) The sum of any two positive definite (semidefinite) matrices of the same size is positive definite (semidefinite).
- (c) Each eigenvalue of a positive definite (semidefinite) matrix is a positive (nonnegative) real number.
- (d) For a Hermitian matrices A, B , we write $A \succ B$ if $A - B$ is positive definite, similarly we write $A \succeq B$ if $A - B$ is positive semidefinite.
- (e) A Hermitian matrix with positive (nonnegative) eigenvalues is positive definite (semidefinite).

Definition 1.3.2 Let A, B , then B is a square root of A , if $B^2 = A$.

Example 1.3.1

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, be a Hermitian matrix.

Then $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{Q}$, thus $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{Q}$, which gives

10, 12. The eigenvector for $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and for $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so the matrix of the eigenvectors is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Finally, we have to convert this matrix into an

orthogonal matrix by applying the Gram-Schmidt orthonormalization process on the

column vectors to give $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, which is a unitary matrix. Thus

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Theorem 1.3.1 Let A be a positive semidefinite and let n be a given integer,

then there exists a unique positive semidefinite Hermitian matrix B such that $A = B^n$,

written as $B = A^{1/n}$.

Example 1.3.2 (1) If A (positive definite matrix) with eigenvalues $\lambda_1, \dots, \lambda_n$,

then $A = U \text{diag}(\lambda_1, \dots, \lambda_n) U^*$, where U is a unitary matrix.

(2) If $A \geq 0$, $A = U \text{diag}(\lambda_1, \dots, \lambda_n) U^*$.

(3) The function calculus for A is defined as $f(A) = U \text{diag}(f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)) U^*$.

Definition 1.3.3 A map $\phi: A \rightarrow B$ is unital if $\phi(1_A) = 1_B$

is positive if $\phi(A) \geq 0$ whenever $A \geq 0$.

$A \geq 0 \iff A^* = A$

Definition 1.3.4 A map $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is jointly concave if for any A, B, C, D

and any $\alpha, \beta, \gamma, \delta \geq 0$, $\alpha + \beta + \gamma + \delta = 1$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Definition 1.3.5 Let $A \in \mathbb{R}^{m \times n}$. Let the eigenvalues of the symmetric

matrix $A^T A$ be denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$. Where

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, then $\lambda_1, \lambda_2, \dots, \lambda_n$ are called the singular values of A .

Example 1.3.3 Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$, then $A^T A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 2 \end{pmatrix}$, thus $A^T A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 4 \\ 0 & 4 & 4 \end{pmatrix}$.

The eigenvalues of $A^T A$ are 0, 4, 9. Thus the singular values are 0, 2, 3.

Theorem 1.3.2 (Singular value Decomposition)

Let $A \in \mathbb{R}^{m \times n}$ has rank r and let $\sigma_1, \dots, \sigma_r$ be the nonzero singular value of A , then A can

be represented in the form $A = UDV$ where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are unitary and the

matrix $D \in \mathbb{R}^{m \times n}$, $D_{ii} = \sigma_i$, $D_{ij} = 0$ for all $i \neq j$, and $\sigma_i > 0$.

where $q = \min\{m, n\}$, the numbers $\{\sigma_i\}_{i=1}^q$ are the

singular values of A .

Theorem 1.3.3 (Polar Decomposition)

Let $A \in \mathbb{C}^{m \times n}$, with $\text{rank } A = m$. Then A may be written in the form $A = U|A|$, where $|A|$ is positive semidefinite, $\text{rank } |A| = \text{rank } A$, and $U \in \mathbb{C}^{m \times m}$ has orthonormal rows (that is $UU^* = I$). The matrix $|A|$ is always uniquely determined as $(|A|)^2 = A^*A$, and U is uniquely determined when A has rank m . If A is real then U and $|A|$ may be taken to be real.

1.4 The Kronecker product of matrices

Leopold Kronecker was a German mathematician was born in liegnitz, Prussia (December 7,1823-December 29,1891).

In mathematics, the Kronecker product denoted by \otimes is an operation on two matrices of arbitrary size resulting in a block matrix. The Kronecker product should not be confused with the usual matrix multiplication which is an entirely different operation.

Definition 1.4.1 Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$. Then the Kronecker product of A and B is defined as the matrix $A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$ $M^{mp \times nq}$, and has mn blocks.

Example 1.4.1 Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, then

$$A^{-1}B^{-1} = \begin{pmatrix} B & 2B & 4B \\ 3B & 0 & B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 3 & 2 & 6 & 4 & 12 & 8 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 9 & 6 & 0 & 0 & 3 & 2 \end{pmatrix}$$

And $B^{-1}A^{-1} = \begin{pmatrix} A & 0A \\ 3A & 2A \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 6 & 12 & 2 & 4 & 8 \\ 9 & 0 & 3 & 6 & 0 & 2 \end{pmatrix}$, thus $A^{-1}B^{-1} \neq B^{-1}A^{-1}$, in general.

Also if $A = I$, then $A^{-1}B^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, of size $n \times n$ where $n = 3$.

And $B^{-1}A^{-1} = \begin{pmatrix} b & \dots & 0 \\ 0 & \dots & b \\ b & \dots & 0 \\ 0 & \dots & b \end{pmatrix}$, of size $n \times n$.

We note that if $A = I$, $B = I$, then $I^{-1}I^{-1} = I$. For example

$$I^{-1}I^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $A^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B^{-1}A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, \dots ,

The following theorem states some basic properties of the Kronecker Product :

Theorem 1.4.1. 7 Let $A \in M_n(F)$, then :

(a) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, for all $A \in M_n(F)$ and $B \in M_m(F)$, $C \in M_p(F)$.

(b) $A \otimes (B + C) = A \otimes B + A \otimes C$, for $B, C \in M_m(F)$.

(c) $(A+B) \otimes C = A \otimes C + B \otimes C$ for $B \in M_m(F)$, and $C \in M_p(F)$.

(d) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for $B, C \in M_m(F)$.

(e) $(A \otimes B)^T = A^T \otimes B^T$.

(f) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(g) $O_n \otimes O_m = O_{nm}$.

Proof : a) $(A \otimes B) \otimes C = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mm} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & \dots & a_{11}b_{1m} & \dots & a_{11}b_{m1} & \dots & a_{11}b_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}b_{11} & \dots & a_{n1}b_{1m} & \dots & a_{n1}b_{m1} & \dots & a_{n1}b_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}b_{m1} & \dots & a_{n1}b_{mm} & \dots & a_{n1}b_{11} & \dots & a_{n1}b_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}b_{m1} & \dots & a_{n1}b_{mm} & \dots & a_{n1}b_{11} & \dots & a_{n1}b_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{nn}b_{11} & \dots & a_{nn}b_{1m} & \dots & a_{nn}b_{m1} & \dots & a_{nn}b_{mm} \end{pmatrix}$

$A \otimes B$

B .

e) $(A \otimes B)^T = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mm} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & \dots & a_{11}b_{1m} & \dots & a_{11}b_{m1} & \dots & a_{11}b_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}b_{11} & \dots & a_{n1}b_{1m} & \dots & a_{n1}b_{m1} & \dots & a_{n1}b_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}b_{m1} & \dots & a_{n1}b_{mm} & \dots & a_{n1}b_{11} & \dots & a_{n1}b_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}b_{m1} & \dots & a_{n1}b_{mm} & \dots & a_{n1}b_{11} & \dots & a_{n1}b_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{nn}b_{11} & \dots & a_{nn}b_{1m} & \dots & a_{nn}b_{m1} & \dots & a_{nn}b_{mm} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}^T \otimes \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mm} \end{pmatrix}^T = A^T \otimes B^T$

$$g) \begin{pmatrix} OA & OA & O & O \\ OA & OA & O & O \end{pmatrix} Q$$

In the following, we will see the difference between AB and $A B$, it is known that if

, , and Q , it is not necessary that O or Q , but the following corollary shows that if $A B = Q$, then either O or Q .

Corollary 1.4.2 Let $A \in M_n$ and $B \in M_n$. Then $A B = Q$ if and only if either

$$O \text{ or } Q.$$

Proof : if $A B = Q$, then $a_i B = a_i B = O = O$.

So O or $a_i = O$ for all $i = 1, \dots, n$, $i = 1, \dots, n$, thus either O or Q .

Conversely, let either O or Q . Then by theorem (1.4.1 (g)) then $A B = Q$.

Theorem 1.4.3 (The mixed product rule)

Let $A \in M_n$, $B \in M_m$, $C \in M_m$, and $D \in M_n$, then $A B C D = AC BD$

Proof : (see ref [13] .

Theorem 1.4.4. 1 If $A \in M_n$ and $B \in M_n$ are normal matrices then,

$$A B \text{ is normal.}$$

Proof : $A B A B = A B A B$ (by theorem 1.4.1 (f))

$$AA \quad BB \quad (\text{by theorem 1.4.3})$$

$$A \quad A \quad B \quad B \quad (\text{since } A \text{ and } B \text{ are normal})$$

$$A \quad B \quad A \quad B \quad (\text{by theorem 1.4.3})$$

$$= A \quad B \quad A \quad B .$$

From the mixed rule product, we have the following corollaries :

Corollary 1.4.5. 7 If $A \quad M$ and $B \quad M$ are nonsingular, then $A \quad B$ is also nonsingular, with $A \quad B \quad A \quad .$

Proof : $(A \quad B \quad A \quad AA \quad B \quad (\text{by theorem 1.4.3})$

$$I \quad I \quad I \quad .$$

$$A \quad A \quad B \quad A \quad I \quad I \quad I \quad .$$

Thus $A \quad = A \quad B$ under conventional matrix multiplication , so $A \quad B$ is nonsingular.

Corollary 1.4.6 If $A \quad M$ is similar to $B \quad M$ and \quad is similar to \quad then $A \quad C$ is similar to $B \quad D$.

Proof : Since A is similar to B and C is similar to D , there exist nonsingular matrices

P, Q such that $A \quad PBP$ and $C \quad QDQ$, so

$$A \quad C \quad PBP \quad QDQ$$

$$P \quad Q \quad BP \quad DQ \quad (\text{by mixed product rule})$$

$$P^T Q^T B D P Q \quad (\text{by mixed product rule})$$

$$P^T Q^T B D P Q \quad (\text{by corollary 1.4.5}).$$

The following corollaries present the orthogonal and unitary properties of Kronecker product in the usual sense :

Corollary 1.4.7 If $A \in M$ is orthogonal and $B \in M$ is orthogonal then $A \otimes B$ is orthogonal matrix.

Proof : A and B are orthogonal, so $AA^T = I$ and $BB^T = I$.

Using theorem (1.4.3), $(A \otimes B)(A \otimes B)^T = (A \otimes B)(A^T \otimes B^T) = (AA^T) \otimes (BB^T)$

$$= I \otimes I = I.$$

Therefore $A \otimes B$ is orthogonal.

Corollary 1.4.8 Let $U \in M$ and $V \in M$ be a unitary matrices, then $U \otimes V$ is a unitary matrix.

Proof : U and V are unitary implies $U^T = U^{-1}$ and $V^T = V^{-1}$. Using corollary (1.4.5)

$(U \otimes V)^T = U^T \otimes V^T = U^{-1} \otimes V^{-1} = (U \otimes V)^{-1}$. Therefore $U \otimes V$ is a unitary matrix.

Theorem 1.4.9. If $A \in M$ and $B \in M$, then $\text{tr}(A \otimes B) = \text{tr} A \text{tr} B = \text{tr} B \text{tr} A$.

Proof : $\text{tr}(A \otimes B) = \text{tr} a \otimes B = \text{tr} a \otimes B \dots \text{tr}$

$$a \otimes \text{tr} B = a \otimes \text{tr} B \dots a \otimes \text{tr} B$$

$$= a \otimes a \dots a \otimes \text{tr} B$$

$$\text{tr}(A+B) = \text{tr} A + \text{tr} B.$$

Consequently, $\text{tr}(A+B) = \text{tr} A + \text{tr} B = \text{tr} B + \text{tr} A = \text{tr}(B+A)$.

Remark 1.4.1 By theorem (1.4.9) $\text{tr}(A+B) = \text{tr} A + \text{tr} B$, if A and B are square matrices, but if $A \in M_n, B \in M_m$, then $\text{tr}(A+B) = \text{tr}(B+A)$ in general as will see in the following example :

Example 1.4.2 Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 2 & 5 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & 1 & 2 \\ 0 & 6 & 0 & 3 \\ 4 & 10 & 2 & 5 \\ 2 & 2 & 1 & 1 \end{pmatrix}$, then $A+B = \begin{pmatrix} 4 & 5 & 2 & 3 \\ 2 & 9 & 2 & 8 \\ 6 & 15 & 4 & 13 \\ 3 & 3 & 2 & 2 \end{pmatrix}$.

And $B+A = \begin{pmatrix} 2 & 1 & 4 & 2 \\ 0 & 0 & 6 & 3 \\ 4 & 2 & 10 & 5 \\ 2 & 1 & 2 & 1 \end{pmatrix}$. Therefore $\text{tr}(A+B) = 7$, and $\text{tr}(B+A) = 13$.

The mixed product rule can be generalized in two ways as will see in the following theorem :

Theorem 1.4.10 If $A_1, A_2, \dots, A_n \in M_m$ and $B_1, B_2, \dots, B_n \in M_n$, then

$$(a) \text{tr}(A_1 A_2 \dots A_n B_1 B_2 \dots B_n) = \text{tr}(A_1 B_1 A_2 B_2 \dots A_n B_n).$$

$$(b) \text{tr}(A_1 B_1 A_2 B_2 \dots A_n B_n) = \text{tr}(A_1 A_2 \dots A_n B_1 B_2 \dots B_n).$$

Proof : We use mathematical induction to prove (a) and (b).

(a) Let $n=2$, so by the mixed product property $\text{tr}(A_1 A_2 B_1 B_2) = \text{tr}(A_1 B_1 A_2 B_2)$.

Assume that $\text{tr}(A_1 A_2 \dots A_n B_1 B_2 \dots B_n) = \text{tr}(A_1 B_1 A_2 B_2 \dots A_n B_n)$.

Now, $(A \ A \ \dots \ A \ A \ B \ B \ \dots \ B$

$A \ A \ \dots \ A \ A \ B \ B \ \dots \ B \ B$

$A \ A \ \dots \ A \ B \ B \ \dots \ B \ A$ (by theorem 1.4.3)

$A \ B \ A \ B \ \dots \ A \ B \ A \ A \ B \ A \ B \ \dots \ A \ B \ A \ B \ .$

(b) Let \mathcal{Z} , so by the mixed product property $A \ B \ A \ B \ A \ A \ B \ B$

Assume that $(A \ B \ A \ B \ \dots \ A \ B \ A \ A \ \dots \ A \ B \ B \ \dots \ B$

Now $(A \ B \ A \ B \ \dots \ A \ B \ A \ B$

$A \ B \ A \ B \ \dots \ A \ B \ A \ B$

$A \ A \ \dots \ A \ B \ B \ \dots \ B \ A \ B$

$A \ A \ \dots \ A \ A \ B \ B \ \dots \ B \ B$ by mixed product property

$A \ A \ \dots \ A \ A \ B \ B \ \dots \ B \ B \ .$

Corollary 1.4.11 Let $A \in M$ and $B \in M$.

(a) if A and B are idempotent then $A \ B$ is an idempotent.

(b) If A and B are nilpotent then $A \ B$ is nilpotent.

Proof : (a) A and B are idempotent then $A^2 = A, B^2 = B$, so

$(A \ B)^2 = A \ B \ A \ B = A \ A \ B \ B = A \ B$

(b) A and B are nilpotent then $A^k = 0, B^l = 0$. So,

$(A \ B)^{k+l} = A \ B \ A \ B \ \dots \ A \ B \ A \ A \ \dots \ A \ B \ B \ \dots \ B \ A \ B \ 0 \ 0 \ 0$

Theorem 1.4.12. 13 Let $A \in M_n$ and $B \in M_n$, then

(a) $A^k I = A^k I$ and $I B^k = I B^k$, $k = 1, 2, \dots$

(b) For any polynomial $p(t)$, $p(A) I = p(A) I$ and $p(I) B = I p(B)$.

Proof : (a) $A^k I = \underbrace{A I A I \dots A I}_k$

$$= \underbrace{A A \dots A}_k \underbrace{I I \dots I}_k \quad (\text{by theorem 1.4.10 (b)})$$

$$= A^k I.$$

And $I B^k = \underbrace{I B I B \dots I B}_k = \underbrace{I I \dots I}_k \underbrace{B B \dots B}_k = I B^k$.

(b) Let $p(t) = a_0 + a_1 t + \dots + a_n t^n$, so

$$p(A) I = a_0 I + a_1 A I + \dots + a_n A^n I, \quad A^k I = A^k I.$$

Now, $p(A) I = a_0 I + a_1 A I + \dots + a_n A^n I$ (by part a)

$$= a_0 I + \underbrace{a_1 A I + \dots + a_n A^n I}_{A I} \quad (\text{by theorem 1.4.1 (a)})$$

$$= a_0 I + A I = A I.$$

Similarly, we can prove that $p(I) B = I p(B)$.

In the following lemma shows that the Kronecker product of two upper triangular matrices is also upper triangular.

Lemma 1.4.13. 5 If $A \in M_n$ and $B \in M_m$ be upper triangular then $A \otimes B$ is upper triangular.

Proof : A and B are upper triangular, then $A_{jj} = a_j$ where $a_j \neq 0$ for $j = 1, \dots, n$ and

$B_{jj} = b_j$ where $b_j = 0$ for $j = 1, \dots, p$. By definition,

$$A + B = \begin{pmatrix} a_1 + b_1 & & & \\ a_2 + b_2 & & & \\ \vdots & \vdots & \vdots & \vdots \\ a_n + b_n & & & \end{pmatrix}. \text{ So, } a_j + b_j = 0 \text{ for } j = 1, \dots, p$$

since $a_j \neq 0$ for $j = 1, \dots, n$. Now the block matrices $A_{jj} + B_{jj}$ are upper triangular since

A_{jj} is upper triangular, hence $A + B$ is upper triangular.

The following theorem shows the relation between A , B and $A + B$

Theorem 1.4.14. 13 Let $A \in M_n$ and $B \in M_m$, if λ is an eigenvalue of A with corresponding eigenvector x and if μ is an eigenvalue of B with corresponding eigenvector y , then $\lambda + \mu$ is an eigenvalue of $A + B$ with corresponding eigenvector

$\begin{pmatrix} x \\ y \end{pmatrix}$. If $A = \begin{pmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{pmatrix}$, \dots , and $B = \begin{pmatrix} \mu_1 & & \\ & \dots & \\ & & \mu_m \end{pmatrix}$, \dots , then

$A + B$ has eigenvalues $\lambda_i + \mu_j$, $i = 1, \dots, n, j = 1, \dots, m$ (including algebraic multiplicities).

In particular, $\lambda + \mu = \lambda + \mu$.

Proof : Suppose $Ax = \lambda x$ and $By = \mu y$, for $x, y \neq 0$. Now by the mixed product property

$$(A + B) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ By \end{pmatrix} = \begin{pmatrix} \lambda x \\ \mu y \end{pmatrix} = (\lambda + \mu) \begin{pmatrix} x \\ y \end{pmatrix}.$$

By Schur's triangularization theorem, there exist unitary matrices $U \in M_n$ and $V \in M_m$, such that $U^*AU = T$ and $V^*BV = S$ where T and S are upper triangular matrices

Then by theorems 1.4.1(f) and 1.4.10 (b) $U^*(A + B)U = U^*AU + U^*BU = T + U^*BU$ and $V^*(A + B)V = V^*AV + V^*BV = S + V^*BV$

$T^{-1} T$, is

upper triangular and is similar to $A^{-1} B$. The eigenvalues of A , B and $A^{-1} B$ are exactly the main diagonal entries of $T^{-1} T$, T and $T^{-1} T$ respectively, and the main diagonal of $T^{-1} T$ consists of pair wise products of the entries on the main diagonals of T^{-1} and T .

Corollary 1.4.15 Let $A \in M_n$ and $B \in M_n$. Then $\det(A^{-1} B) = \frac{\det B}{\det A}$.

Proof: Assume that $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n are the eigenvalues of $A \in M_n$ and $B \in M_n$, respectively. Then we have

$$\det(A^{-1} B) = \lambda_1^{-1} \mu_1 \lambda_2^{-1} \mu_2 \dots \lambda_n^{-1} \mu_n = \frac{\mu_1 \mu_2 \dots \mu_n}{\lambda_1 \lambda_2 \dots \lambda_n} = \frac{\det B}{\det A}.$$

Corollary 1.4.16. If $A \in M_n$ and $B \in M_n$, then $\det(A^{-1} B) = \frac{\det B}{\det A}$.

Proof: $\det(A^{-1} B) = \det(A^{-1}) \det B = \frac{1}{\det A} \det B = \frac{\det B}{\det A}$.

$$\dots \mu_1 \mu_2 \dots \mu_n = \det B = \det A \det(A^{-1} B).$$

Corollary 1.4.17 If $A \in M_n$ and $B \in M_n$ are positive (semi) definite Hermitian matrices

Then $A^{-1} B$ is also positive (semi) definite Hermitian.

Proof: (see ref [1]).

In the following theorem prove the relation between (S.V.D) of A , B and $A+B$:

Theorem 1.4.18. Let $A \in M_n$ and $B \in M_n$ have singular value decomposition

$$A = V_1 D_1 W_1^T \text{ and } B = V_2 D_2 W_2^T, \text{ where } D_1 \in M_n, D_2 \in M_n,$$

and let $\text{rank } A = r$ and $\text{rank } B = r$. Then $A+B = V_1 D_1 W_1^T + V_2 D_2 W_2^T$.

The nonzero singular values of $A+B$ are the r positive numbers $\{ \sigma_j(A+B) \mid 1 \leq j \leq r \}$ (including multiplicities).

Zero is a singular value of $A+B$ with multiplicity $\min\{m, n\} - r$.

In particular, the singular values of $A+B$ are the same as those of $B+A$, and $\text{rank } (A+B) = \text{rank } (B+A) = r$.

same as those of $B+A$, and $\text{rank } (A+B) = \text{rank } (B+A) = r$.

Theorem 1.4.19. If $A \in M_n$ and $B \in M_n$. Then for all p -norms $\|A+B\|_p \leq \|A\|_p + \|B\|_p$.

B .

Proof : (Case 1) For Frobenius norm, $\|A+B\|_F = \sqrt{\text{tr}((A+B)(A+B)^T)}$.

$$\|A+B\|_F = \sqrt{\text{tr}(A^T A + B^T B + A^T B + B^T A)} \quad (\text{by theorem 1.4.1 (f)})$$

$$\|A+B\|_F = \sqrt{\|A\|_F^2 + \|B\|_F^2 + 2 \text{tr}(A^T B)}$$

$$\|A+B\|_F = \sqrt{\|A\|_F^2 + \|B\|_F^2 + 2 \text{tr}(A^T B)} \quad (\text{by theorem 1.4.9})$$

$$\|A+B\|_F = \sqrt{\|A\|_F^2 + \|B\|_F^2 + 2 \text{tr}(A^T B)} \leq \sqrt{\|A\|_F^2 + \|B\|_F^2 + 2\|A\|_F \|B\|_F} = \|A\|_F + \|B\|_F.$$

Now for the 2-norm ;

$$\|A+B\|_2 = \sqrt{\lambda_{\max}(A+B)^T (A+B)} = \sqrt{\lambda_{\max}(A^T A + B^T B + A^T B + B^T A)}.$$

(Case 2) The max-norm, $\|A+B\|_{\infty} = \max_i \sum_j |a_{ij} + b_{ij}|$

$\|A\|_1 = \sum_{j=1}^n |a_{1j}|$, $\|b\|_1 = \sum_{j=1}^n |b_j|$.

$\|a\|_1 = \sum_{j=1}^n |a_{1j}|$.

(Case 3) The ∞ -norm is similar to the max-norm except the largest absolute row sum is used rather than the largest absolute column sum, by taking the transpose.

(Case 4) The spectral-Norm $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$, $\|sA + tB\|_2 = \sqrt{\lambda_{\max}(sA + tB)^T (sA + tB)}$.

$\|sA\|_2 = |s| \|A\|_2$, $\|sB\|_2 = |s| \|B\|_2$.

Corollary 1.4.20. If $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times m}$ are nonsingular, then $\text{cond}(A^{-1}B) = \text{cond}(A) \text{cond}(B)$.

Proof : $\text{cond}(A^{-1}B) = \frac{\|A^{-1}B\|_2}{\|A^{-1}B\|_1} = \frac{\|A^{-1}\|_2 \|B\|_2}{\|A^{-1}\|_1 \|B\|_1} = \frac{\|A^{-1}\|_2}{\|A^{-1}\|_1} \frac{\|B\|_2}{\|B\|_1} = \text{cond}(A) \text{cond}(B)$.

$\|A^{-1}B\|_2 = \|A^{-1}\|_2 \|B\|_2$ (by corollary 1.4.4)

$\|A^{-1}B\|_1 = \|A^{-1}\|_1 \|B\|_1$.

The following will concern the Kronecker sum of matrices :

Definition 1.4.2 Let $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{n \times n}$. Then the Kronecker sum of A and B is the $(m+n)$ -by- $(m+n)$ matrix denoted by $A \oplus B$ and defined as $A \oplus B = A \otimes I_n + I_m \otimes B$.

the following example shows that $(A \oplus B)^{-1} = B^{-1} \oplus A^{-1}$ in general.

Example 1.4.3 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Then

$$\begin{array}{cccccc}
 & & & & 3 & 2 & 3 & 1 & 0 & 0 \\
 & & & & 3 & 4 & 1 & 0 & 1 & 0 \\
 A & B & I & A & B & I & & & & \\
 & & & & 1 & 1 & 6 & 0 & 0 & 1 \\
 & & & & 2 & 0 & 0 & 4 & 2 & 3 \\
 & & & & 0 & 2 & 0 & 3 & 5 & 1 \\
 & & & & 0 & 0 & 2 & 1 & 1 & 7
 \end{array}$$

$$\begin{array}{cccccc}
 & & & & 3 & 1 & 2 & 0 & 3 & 0 \\
 & & & & 2 & 4 & 0 & 2 & 0 & 3 \\
 B & A & I & B & A & I & & & & \\
 & & & & 3 & 0 & 4 & 1 & 1 & 0 \\
 & & & & 0 & 3 & 2 & 5 & 0 & 1 \\
 & & & & 1 & 0 & 1 & 0 & 6 & 1 \\
 & & & & 0 & 1 & 0 & 1 & 2 & 7
 \end{array}$$

We saw the Kronecker product of two matrices A and B has as its eigenvalues all possible pairwise products of the eigenvalues of A and B . The following theorem shows that the Kronecker sum of A and B has as its eigenvalues all possible pairwise sums of the eigenvalues of A and B .

Theorem 1.4.21 Let $A \in M_n$ and $B \in M_m$. If λ is an eigenvalue of A and x is a corresponding eigenvector of A , and if μ is an eigenvalue of B and y is a corresponding eigenvector of B , then $\lambda + \mu$ is an eigenvalue of the Kronecker sum $I_m \otimes A + B \otimes I_n$ and $x \otimes y$ is a corresponding eigenvector of the Kronecker sum. In fact $(I_m \otimes A + B \otimes I_n)(x \otimes y) = (\lambda + \mu)(x \otimes y)$.

Proof : (see ref .)

Remark 1.4.2. Let $A \in M_n$ and $B \in M_m$, then $I_m \otimes A$ commutes with $B \otimes I_n$.

Proof : $(I_m \otimes A)(B \otimes I_n) = (I_m \otimes A)B \otimes I_n = B \otimes I_n = (B \otimes I_n)(I_m \otimes A)$

$$B I I A .$$

Theorem 1.4.22 Let $A \in M_n$ and $B \in M_n$ be matrices then $\text{tr}(A+B) = \text{tr} A + \text{tr} B$.

Proof : $\text{tr}(A+B) = \text{tr}(I(A+B)I)$

$$= \text{tr}(IA + BI)$$

$$= \text{tr} IA + \text{tr} BI \quad (\text{by theorem 1.4.9})$$

$$= \text{tr} A + \text{tr} B .$$

Theorem 1.4.23 Let $A \in M_n$ and $B \in M_n$. Then for $\lambda \in \mathbb{C}$,

$$\text{tr}(\lambda A + B) = \lambda \text{tr} A + \text{tr} B .$$

Proof : $\text{tr}(\lambda A + B) = \text{tr}(I(\lambda A + B)I) = \text{tr}(I\lambda A I + IBI)$

$$= \text{tr} \lambda IA + \text{tr} BI \quad (\text{by theorem 1.4.19})$$

$$= \lambda \text{tr} A + \text{tr} B .$$

We consider members of M_n as vectors by ordering their entries in a conventional way

from left to right, which is given in the following definition :

Definition 1.4.3 Let $A = (a_{ij}) \in M_n$, we associate the vector $\text{vec} A$ defined by

$$\text{vec} A = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}) .$$

Remark 1.4.3. Let $A, B \in M$ and $\vec{a}, \vec{b} \in \text{Vec } A$. Then $\text{Vec } (A + B) = \text{Vec } A + \text{Vec } B$.

Proof : $\text{Vec } (A + B) = \text{Vec } \begin{pmatrix} a_1 & b_1 & \dots & b_1 \\ \dots & \dots & \dots & b_2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_1 \end{pmatrix}$

$$= \begin{pmatrix} a_1 & b_1, \dots, & \dots, & b_1, \dots, a_1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} a_1, \dots, & \dots, & \dots, a_1 & b_1, \dots, & \dots, b_1, \dots, \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} a_1, \dots, & \dots, & \dots, a_1 & b_1, \dots, & \dots, b_1, \dots, \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$\text{Vec } A + \text{Vec } B$.

The next theorem indicates to the close relationship between the Vec-vector and the

Kronecker Product :

Theorem 1.4.24. Let $A \in M$, $B \in M$ and $X \in M$, then $\text{Vec } (AXB) = (B^T \otimes A) \text{Vec } X$.

$\text{Vec } X$.

Proof : Denote the k -th column of AXB by $(AXB)_k$. Then $(AXB)_k = A (XB)_k$

$(AXB)_k = \begin{pmatrix} x_{1k} b_{11} & x_{1k} b_{12} & \dots & x_{1k} b_{1q} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$. This implies that $(AXB)_k = \begin{pmatrix} x_{1k} b_{11} & x_{1k} b_{12} & \dots & x_{1k} b_{1q} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$

$$= \begin{pmatrix} x_{1k} & \dots & x_{1k} & \dots & x_{1k} & \dots & x_{1k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} b_{11} \\ \dots \\ b_{1q} \end{pmatrix} = \begin{pmatrix} b_{11} A & \dots & b_{1q} A \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} x_{1k} \\ \dots \\ x_{1k} \end{pmatrix}$$

$\begin{pmatrix} b_{11} A & \dots & b_{1q} A \end{pmatrix} \text{Vec } X$ for $k = 1, 2, \dots, q$. So,

$$\text{Vec } AXB = \begin{bmatrix} B & A \\ B & A \\ B & A \end{bmatrix} \text{Vec } X.$$

Corollary 1.4.25. Let $A \in M_{m,n}$, $B \in M_{n,p}$ and $X \in M_{n,1}$. Then

(a) $\text{Vec } AX = I_m A \text{Vec } X.$

(b) $\text{Vec } XB = B^{-1} \text{Vec } X.$

(c) $\text{Vec } AXB = A B \text{Vec } X.$

Proof : (c) $\text{Vec } AXB = \text{Vec } AX \text{Vec } XB$ (by remark 1.4.3)

$$\text{Vec } AXI = \text{Vec } I XB$$

$$I_m A \text{Vec } X = B^{-1} \text{Vec } X \text{ (by theorem 1.4.24)}$$

$$I_m A B^{-1} \text{Vec } X$$

$$A B \text{Vec } X \text{ (by definition 1.4.2).}$$

Corollary 1.4.26. Let $A \in M_{m,n}$ and $B \in M_{n,p}$. Then $\text{Vec } AB = I_m A \text{Vec } B$

$$B^{-1} A \text{Vec } I_n = B^{-1} \text{Vec } A.$$

Proof : $\text{Vec } AB = \text{Vec } ABI = I_m A \text{Vec } B$ (by theorem 1.4.24)

$$I_m A \text{Vec } B.$$

Next, I_m is equivalent to $\text{Vec } AB = B^{-1} A \text{Vec } I_n$. Finally I_m

is equivalent to $\text{Vec } AB = B^{-1} \text{Vec } A.$

The following lemma describes the relation between $\|A\|_1$ and $\|A\|_\infty$.

Lemma 1.4.27. Let $A \in \mathbb{C}^{m \times n}$. Then $\|A\|_1 = \|A^T\|_\infty$, where M is a permutation matrix this matrix P is given by $P = [e_{j_i}]$ where each e_{j_i} has entry 1 in position i, j and all other entries are zero.

The previous lemma leads us to the following theorem :

Theorem 1.4.28. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $\|AB\|_1 \leq \|A\|_1 \|B\|_1$ where P, Q are permutation matrices such that $A = PAQ$, $B = BQ$.

Proof : Let $Y = AXB$, where $M \in \mathbb{C}^{m \times m}$. Then $Y = M^{-1}AXB$. So $\text{Vec}Y = (M^{-1} \otimes I) \text{Vec}X$.

And $\|Y\|_1 = \|M^{-1}AXB\|_1 \leq \|M^{-1}\|_1 \|A\|_1 \|B\|_1 \|X\|_1$ (by theorem 1.4.24).

But $\|Y\|_1 = \|M^{-1}AXB\|_1 = \|M^{-1}\|_1 \|A\|_1 \|B\|_1 \|X\|_1$, where M is a permutation matrix, and $\|M^{-1}\|_1 = 1$.

Where $M \in \mathbb{C}^{m \times m}$, is a permutation matrix. So,

$$\|Y\|_1 = \|M^{-1}AXB\|_1 = \|M^{-1}\|_1 \|A\|_1 \|B\|_1 \|X\|_1, \text{ i.e.}$$

$$\|M^{-1}AXB\|_1 = \|A\|_1 \|B\|_1 \|X\|_1. \text{ But } \|M^{-1}AXB\|_1 = \|M^{-1}\|_1 \|A\|_1 \|B\|_1 \|X\|_1, \text{ so}$$

$$\|M^{-1}\|_1 = 1, \text{ for all } M \text{ and this implies } \|M\|_1 = 1.$$

Corollary 1.4.29 Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $\|AB\|_1 \leq \|A\|_1 \|B\|_1$ for any

unitarily invariant norm $\|\cdot\|$ on M_n .

Proof :

Since U, V are unitary matrices.

1.5 The Hadamard product of matrices

The Hadamard product is a binary operation that takes two matrices of the same size, and produces another matrix where each element ij is the product of element ij of the original two matrices.

Definition 1.5.1 The Hadamard product of $A = [a_{ij}]_m \times n$ and $B = [b_{ij}]_m \times n$ is

defined by $A \circ B = [a_{ij} b_{ij}]_m \times n$.

Example 1.5.1 If $A = \begin{bmatrix} 2 & 3 & i \\ 1 & 7 & 9 \\ 3i & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 & 6 \\ 2 & 5 & 0 \\ i & 1 & 2 \end{bmatrix}$. Then

$$A \circ B = \begin{bmatrix} 2 & 27 & 6i \\ 2 & 35 & 0 \\ 3 & 0 & 10 \end{bmatrix}$$

The following theorem Shows the set of $n \times n$ matrices with nonzero entries form an abelian group under the Hadamard product :

Theorem 1.5.1 Let $A, B \in M_n(\mathbb{C})$. Then $A \circ B = B \circ A$.

Proof : Let A and B be $n \times n$ matrices with entries in \mathbb{C} . Then $A \circ B$

$= B A$ and therefore $A B = B A$.

Definition 1.5.2 The Hadamard identity is the matrix J defined by $J_{ij} = 1$

for all $1 \leq i, j \leq n$.

Theorem 1.5.2 Let A be an $n \times n$ matrix. Then $A J = J A$.

Proof: $A J = J A$ (by theorem 1.5.1)

$J_{ij} = 1$ (by definition H.P)

$J_{ij} = 1$ (by definition HID)

. Therefore $A J = J A$.

Definition 1.5.3 Let A be an $n \times n$ matrix and suppose $a_{ij} \neq 0$ for all $1 \leq i, j \leq n$.

Then the Hadamard inverse denoted by A^{-1} is $A^{-1} = \frac{1}{|A|} \text{adj}(A)$,

for $1 \leq i, j \leq n$.

Theorem 1.5.3 Let A be an $n \times n$ matrix such that $a_{ij} \neq 0$ for all $1 \leq i, j \leq n$.

Then $A^{-1} A = A A^{-1} = J$.

Proof: $A^{-1} A = A A^{-1}$ (by theorem 1.5.1)

$A^{-1} A = J$ (by definition H.P)

$A A^{-1} = J$.

Therefore $A^{-1} A = A A^{-1} = J$.

The following theorem states some basic properties of the Hadamard Product :

Theorem 1.5.4 Suppose $A, B, C \in M_n$, then

(a) $A \circ B = A \circ B = A \circ B$, for all $A, B \in M_n$.

(b) $C \circ (A \circ B) = (C \circ A) \circ B$.

(c) $A \circ B = (A \circ B) \circ I$.

Proof : (a) $A \circ B = A \circ B = A \circ B$

So, $A \circ B = A \circ B$. And

$$A \circ B = A \circ B$$

B . Therefore $A \circ B = A \circ B$.

(b) $C \circ (A \circ B) = (C \circ A) \circ B$

$$C \circ (A \circ B) = (C \circ A) \circ B$$

$$C \circ (A \circ B) = (C \circ A) \circ B$$

Therefore $C \circ (A \circ B) = (C \circ A) \circ B$.

(c) $A \circ B = A \circ B = A \circ B$.

From the previous results, we conclude the following corollary :

Corollary 1.5.5 If $A, B \in M_n$, then $A \circ B = A \circ B$ such that O and

O .

Proof : $A B = A B = A B$

$A B$

$J J$ (by theorem 1.5.3)

$$= 1 \cdot 1 = 1 = J$$

Therefore, $A B = A B$

Remark 1.5.1 Let $A, B \in M_n$, if A and B are diagonal matrices then $A B = AB$.

Proof : $A B = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ a & b & 0 & 0 \\ 0 & a & b & 0 \end{pmatrix} AB$

The following theorem gives the relation between diagonal matrices and the matrix products on the Hadamard multiplication :

Theorem 1.5.6 If $A, B \in M_n$ and if $D, E \in M_n$ are diagonal then

$$D \circ E = (DA) \circ (BE)$$

Proof : $D \circ E = D \circ E$

$$D \circ A \circ B \circ E$$

$$D A B E \quad (\text{by definition HP})$$

$$D A B E \quad (E O \text{ for all })$$

$$D A B E \quad (D O \text{ for all })$$

$$D A E B$$

$$D A E B \quad (E O \text{ for all })$$

$$D AE B \quad (\text{by theorem entries matrix products})$$

$$D AE B \quad (D O \text{ for all })$$

$$B DAE B . \text{ Therefore } D E B.$$

Also,

$$D B B B$$

$$B (E O \text{ for all })$$

$$B$$

$$B , E O \text{ for all } .$$

Therefore, $D E DA BE .$

Definition 1.5.4 Define the diagonal matrix M with entries from a vector

$$\text{by } M_{ij} = \begin{cases} x_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Theorem 1.5.7 Let A be an $n \times n$ matrix and let x be a vector. Then the i th diagonal entry of the matrix MA coincides with the i th entry of the vector x , $i = 1, \dots, n$.

Proof : If $A = (a_{ij})$, then

$$(MA)_{ij} = \sum_{k=1}^n M_{ik} a_{kj}, \text{ for } i, j = 1, \dots, n.$$

The following lemma relate the Hadamard product to the Kronecker product by identifying $A \circ B$ as a submatrix of $A \otimes B$.

Lemma 1.5.8 If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $m \times n$ matrix, then $A \circ B$ is a principal submatrix of $A \otimes B$, in which $A \circ B$ is a principal submatrix of $A \otimes B$. In particular if $A \circ B$ is a principal submatrix of $A \otimes B$.

Theorem 1.5.9 If A is an $m \times n$ matrix, then $\text{rank}(A \circ B) \leq \text{rank}(A)$.

Proof : By lemma 1.5.8 the Hadamard product is a submatrix of the Kronecker product, but the rank of the submatrix is not greater than the rank of the matrix, thus

$$\text{rank}(A \circ B) \leq \text{rank}(A \otimes B) \quad (\text{by theorem 1.4.18})$$

Therefore $\text{rank}(A \circ B) \leq \text{rank}(A)$.

Theorem 1.5.10 Let A be an $m \times n$ matrix, B be an $m \times n$ matrix, O be an $m \times n$ zero matrix, and C be an $m \times n$ matrix, then $(A \circ B) \circ C = A \circ (B \circ C)$.

Proof : We have $A \succeq B$, by corollary (1.4.16). But $A \succeq O$ and

A_{11} is a principal submatrix of $A \succeq B$ by Lemma (1.5.8),

$A_{11} \succeq B_{11}$. Therefore,

.

Based on lemma (1.5.8) we will give the proof of the schur's product theorem in a new style as follows :

Theorem 1.5.11 (schur's product theorem)

If A, B are positive semidefinite, then $A \circ B$ is also positive semidefinite.

Proof : $A \succeq O$ given, it follows that $A_{11} \succeq O$ (by corollary 1.4.17), but

A_{11} is a principal submatrix of $A \succeq B$ (by lemma 1.5.8). So, $A_{11} \succeq B_{11}$.

The following theorem compares the determinant of the matrices A, B and $A \circ B$:

Theorem 1.5.12 (Oppenheimer inequality)

If A, B are positive semidefinite, then

1) $\det(A \circ B) \geq \det(A) \det(B)$.

2) $\det(A \circ B) \leq \det(A) \det(B)$.

Theorem (1.5.12) implies the Hadamard inequality in the usual way as follows :

Theorem 1.5.13 (Hadamard inequality)

If A is positive semidefinite, then $\det(A) \leq \prod_{i=1}^n a_{ii}$.

Proof : Let A be any positive semidefinite matrix of size n . Note that I is positive semidefinite matrix of size n . Now we have the following

$$\det(A + I) = \det \begin{pmatrix} a_{11} + 1 & & \\ & \dots & \\ & & a_{nn} + 1 \end{pmatrix} = \det(A) \det(I) \quad (\text{by theorem 1.5.12})$$

Corollary 1.5.15 Let A, B are positive semidefinite. Then

$$\det(A + B) \geq \det(A) + \det(B).$$

Proof : $\det(A + B) \geq \det(A) + \det(B)$ (by theorem 1.5.12)

$$\det(A + B) \geq \det(A) + \det(B) \quad (\text{by theorem 1.5.14}).$$

Chapter two

Inequalities for Kronecker products and Hadamard products of positive definite matrices

In this chapter, we will see some inequalities for Kronecker products and Hadamard products of positive definite matrices. The contents of this chapter can be found in [10].

2.1 Introduction

The following property involving Kronecker products of matrices can be derived from

The mixed-product property (1.4.3).

Theorem 2.1.1 Let A and B , then $A \otimes B$ for any natural number k .

Proof : \dots (k - times)
 \dots \dots (by theorem 1.4.3)
.

Corollary 2.1.2 For any A , B and C , we have $A \otimes B \otimes C$.

Proof : A , B , so

($A \otimes B$, for any positive integer n , so it follows that

$A \otimes B$. Now $A \otimes B \otimes C$ for any positive integer

m, n . Therefore $A \otimes B \otimes C$ for any m, n .

The following lemma generalizing theorem (2.1.1) :

Lemma 2.1.3 Let A and B , are positive definite matrices. Then for any non-zero real number r

Proof : A, B are positive definite matrices, assures that there exists unitary matrix U and V , such that

$$A = U \Lambda U^* \quad , \text{ where } U \text{ is a unitary matrix and } \Lambda = \text{diag} \{ \mu_1, \mu_2, \dots, \mu_n \} .$$

$$B = V \Sigma V^* \quad , \text{ where } V \text{ is a unitary matrix and } \Sigma = \text{diag} \{ \mu_1, \mu_2, \dots, \mu_n \} .$$

Thus,

$$A^{-1} = U \Lambda^{-1} U^* \quad (\text{ by theorem 1.4.3 })$$

$$B^{-1} = V \Sigma^{-1} V^* \quad (\text{ by (2) in example 1.3.2 })$$

$$A^{-1} B^{-1} = U \Lambda^{-1} U^* V \Sigma^{-1} V^* \quad (\text{ by theorem 1.4.3 })$$

Remark 2.1.4 Let $A \in M_n$ and $B \in M_n$ are matrices with polar decomposition (i.e)

$$A = U |A| \text{ and } B = V |B| . \text{ Then } A^{-1} B^{-1} = U^{-1} |A|^{-1} U^* V^{-1} |B|^{-1} V^*$$

$$U^{-1} U^* |A|^{-1} |B|^{-1} \quad (\text{ by theorem 1.4.3 })$$

$$U^{-1} U^* A A^{-1} B B^{-1} \quad (\text{ where } |A| = A A^{-1}, |B| = B B^{-1})$$

$U^* U^* A A^* B B^*^{-1}$, $A B^* A^* B$ for any positive real number

$U^* U^* A B A B^*^{-1} U^* U^* A B A B^*^{-1} U^* U^* |A B|$.

Lemma 2.1.5. \exists A map defined by $A, B \mapsto A^* B$ for $A, B \in \mathcal{H}$ is jointly concave.

Theorem 2.1.6. \exists The following identity holds for any $A, B \in \mathcal{H}$ and $s > 0$

$$A^* I + I B^* = (A + B)^* I + I (A + B) + s(I - I B^* I - I A^* I).$$

Proof: $A, B \in \mathcal{H}$ and s is positive, take $X = A + B$, $Y = A$ and $Z = B$.

It follows from the mixed-product property of the Kronecker product that

$$I \otimes I + I \otimes I = P \otimes s \otimes Z$$

$$I \otimes I + I \otimes I + I \otimes I = X \otimes Y + s \otimes Z$$

$$I \otimes I + X \otimes Y + s \otimes Z$$

$$I \otimes I + X \otimes Y = X \otimes Y$$

$$I \otimes I + X \otimes Y = I \otimes I.$$

That is

$$I \otimes I + P \otimes s \otimes Z = I \otimes I.$$

Again, the mixed-product property yields

$$I I \quad X Y \quad s Z$$

$$I I \quad .$$

Thus, $I I \quad .$ Which is

$$A I \quad I B \quad A B \quad A B \quad sI I \quad I B .$$

2.2 Inequalities for Kronecker products

In this section we derive inequalities for the Kronecker product of positive definite matrices in the form

and $C \quad B \quad D$ where $\quad , \quad , \quad ,$

are positive definite matrices and $\quad , \quad , \quad ,$ are positive real numbers such that

1.

Theorem 2.2.1. 10 For $\quad , \quad , \quad ,$ and $\quad , \quad , \quad ,$ 0 such that $\quad 1,$

$$C \quad B \quad D .$$

Proof : Let \quad be a real-valued function defined by \quad for $\quad 0$ and $\quad 0 \quad 1.$

Clearly, ϕ is continuous, and ϕ is representation for

$\phi(A, B) = \int \phi(A, B) d\mu$. write $\phi(A, B) = \int \phi(A, B) d\mu$. Hence, the functional calculus

for A, B is $\phi(A, B) = \int \phi(A, B) d\mu$ can be written as

$\phi(A, B) = \int \phi(A, B) d\mu$. It follows from lemma 2.1.3 that

$$\phi(A, B) = \int \phi(A, B) d\mu$$

Hence, by lemma 2.1.6 we obtain

$$\phi(A, B) = \int \phi(A, B) d\mu$$

$$\phi(A, B) = \int \phi(A, B) d\mu$$

$$\phi(A, B) = \int \phi(A, B) d\mu$$

$$\phi(A, B) = \int \phi(A, B) d\mu \quad . \text{ (by lemma 2.1.6)}$$

Since $\phi(A, B)$ and $\phi(A, B)$ are positive definite, by lemma 2.1.5 we have that the map

defined by

$\phi(A, B)$ is jointly concave. It is well-known

that the positive linear combination of the jointly concave maps is jointly concave.

Hence, from the viewpoint of the Riemann integral, the integrand is also jointly concave

and so is $\phi(A, B)$. This means that for any λ, μ, ν and scalar $0 \leq \lambda, \mu, \nu \leq 1$,

$$\phi(A, B) = \int \phi(A, B) d\mu$$

and $\phi(A, B) = \int \phi(A, B) d\mu$. Let $\lambda / \mu = \nu$, thus $0 \leq \nu \leq 1$.

$$\begin{aligned}
 \text{So, } & \frac{1}{\|C\|_p} \frac{1}{\|B\|_q} \frac{1}{\|D\|_r} \\
 & \frac{1}{\|C\|_p} \frac{1}{\|B\|_q} \frac{1}{\|D\|_r} \\
 & \frac{1}{\|C\|_p} \frac{1}{\|B\|_q} \frac{1}{\|D\|_r} \\
 & \frac{1}{\|C\|_p} \frac{1}{\|B\|_q} \frac{1}{\|D\|_r} \quad (\text{since } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1) \\
 & \frac{1}{\|C\|_p} \frac{1}{\|B\|_q} \frac{1}{\|D\|_r} \\
 & \frac{1}{\|C\|_p} \frac{1}{\|B\|_q} \frac{1}{\|D\|_r}
 \end{aligned}$$

Therefore $\|C\|_p \|B\|_q \|D\|_r$.

From theorem (2.2.1), we obtain the Hölder inequality for positive definite matrices as a special case.

Recall that the real numbers p, q are conjugate exponents if p, q are positive and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Corollary 2.2.2 For A, B, C and conjugate exponents p, q , we have

$$\|A\|_p \|B\|_q \leq \|A+B\|_r.$$

Proof : take $A=1, B=1$ and $C=1$ in theorem 2.2.1. Then

$$\|A\|_p \|C\|_q \leq \|B\|_r$$

Replacing $\| \cdot \|_q$ with $\| \cdot \|_r$, Hence

$$\|A\|_p \|B\|_q \leq \|B\|_r$$

Finally we replace $\| \cdot \|_q$, $\| \cdot \|_r$, with $\| \cdot \|_p$, $\| \cdot \|_q$, $\| \cdot \|_r$ respectively we have

$$\|A\|_p \|B\|_q \leq \|B\|_r$$

Therefore $\|A\|_p \|B\|_q \leq \|B\|_r$.

Remarks 2.2.1 The Cauchy-Schwarz inequality is obtained from corollary (2.2.2) by

taking \mathbb{R} , since $\| \cdot \|_2$.

Corollary 2.2.3 For \mathbb{R} and conjugate exponents p, q , we have

$$\|f\|_p \|g\|_q \leq \|fg\|_1.$$

Proof : Let $f, g \in L^1(\mathbb{R})$. By corollary (2.2.2) we get

$$\|f\|_p \|g\|_q \leq \|fg\|_1. \text{ Now let } f, g \text{ then}$$

$$\|f\|_p \|g\|_q \leq \|fg\|_1$$

Hence $\|f\|_p \|g\|_q \leq \|fg\|_1$ (since $\|f\|_p \|g\|_q \geq \|fg\|_1$).

For \mathbb{R} , \mathbb{C} , and \mathbb{O} such that 1. Patrawut Chansangiam,

Patcharin Hemchote, Praiboon Pantaragphong in 10, developed the following results :

(1) $A \leq B \leq C$.

Proof : Take $\lambda = 1$, in theorem 2.2.1, we get the inequality

$$A \leq B \leq C.$$

(2) $B \leq C \leq A$.

Proof : Let $\lambda = 1$, in theorem 2.2.1 then we get the inequality

$$B \leq C \leq A.$$

(3) $A \leq C \leq B \leq D$.

Proof : Let $\lambda = 1$ in theorem (2.2.1) we get

$$A \leq C \leq B \leq D.$$

Then by corollary (2.1.2), we get the inequality

$$A \leq C \leq B \leq D.$$

(4) $A \leq B \leq C \leq D$.

Proof : Take $\lambda = 1$ in theorem 2.2.1 we get to

$$C \leq B \leq D, \text{ then}$$

$$A \leq B \leq C \leq D \leq C \leq B \leq D, \text{ then}$$

$$A \leq B \leq C \leq D \leq C \leq B \leq D, \text{ (by theorem 1.4.1 (a))}$$

Then $A \leq B \leq C \leq D \leq C \leq B \leq D$, (since $\lambda = 1$

Hence $A \leq B \leq C \leq D$.

(5) $\overline{A \cap C} = \overline{B \cap D}$.

Proof : Let $x \in \overline{A \cap C}$ in result (4) then

$x \in \overline{A \cap C} = \overline{A} \cup \overline{C}$, but $x \notin \overline{B \cap D}$, hence

$$\overline{A \cap C} = \overline{B \cap D}.$$

(6) $A \cap B = A \cap B$.

Proof : Let $x \in A \cap B$ in result (1) we get to

$$x \in A \cap B = A \cap B, \text{ then}$$

$$x \in A \cap B = A \cap B, \text{ then}$$

$$x \in A \cap B = A \cap B, \text{ (since } 1 \text{)}$$

Hence $A \cap B = A \cap B$.

(7) $A \cap B = A \cap B$.

Proof : Let $x \in A \cap B$ in result (2), then we get the inequality

$$A \cap B = A \cap B.$$

(8) $\overline{A \cap B} = \overline{B \cap A}$.

Proof : Let $x \in \overline{A \cap B}$ in result (2) and by $\overline{A \cap B} = \overline{B \cap A}$ then we get the inequality

$$\overline{A \cap B} = \overline{B \cap A}.$$

Definition 2.2.1 Let $A \in M_n(\mathbb{R})$. The Kronecker power $A^{\otimes k}$ is defined inductively

for all positive integer k by $A^{\otimes 1} = A$ and $A^{\otimes k+1} = A^{\otimes k} \otimes A$ for $2, 3, \dots$ i.e

$A^{\otimes k} = \underbrace{A \otimes A \otimes \dots \otimes A}_{k\text{-times}}$. This definition implies that $A^{\otimes k} \in M_{n^k}(\mathbb{R})$, the

matrix $A^{\otimes k}$.

Theorem 2.2.4 For any $A \in M_n(\mathbb{R})$, positive integer k , and real number λ , then

.

Proof : Let P_k be the statement $A^{\otimes k} = (A^{\otimes k})$. If $k=1$, then

$A^{\otimes 1} = A$, which is true. Therefore P_1 is satisfied.

Assume that P_k is satisfied, $A^{\otimes k} = (A^{\otimes k})$. Now

.

Thus, P_1 is true, thus P_k is true for all k .

Corollary 2.2.5. Let A_1, A_2, \dots, A_k be a set of arbitrary square matrices with the same size.

Then the Kronecker product has the following

$(A_1 \otimes A_2 \otimes \dots \otimes A_k) = (A_1 \otimes A_2 \otimes \dots \otimes A_k)$, For any positive integer k .

Proof : $(A_1 \otimes A_2 \otimes \dots \otimes A_k) = (A_1 \otimes A_2 \otimes \dots \otimes A_k)$ (k-times)

$(A_1 \otimes A_2 \otimes \dots \otimes A_k) = (A_1 \otimes A_2 \otimes \dots \otimes A_k)$ (by corollary 1.4.9)

$(A_1 \otimes A_2 \otimes \dots \otimes A_k) = (A_1 \otimes A_2 \otimes \dots \otimes A_k)$.

Corollaries 2.2.6 If A, B, C , and O , then

(1) $(A+B)^{-1} = A^{-1} - A^{-1}B^{-1}A^{-1}$.

(2) $(A+B)^{-1} = B^{-1} - B^{-1}A^{-1}B^{-1}$.

(3) $(A+B)^{-1} = A^{-1} - A^{-1}B^{-1}A^{-1}$.

Proof : (1) Take $C = B^{-1}A^{-1}$ in result (1) we get to

$$(A+B)^{-1} = A^{-1} - A^{-1}B^{-1}A^{-1}, \text{ then}$$

$$(A+B)^{-1} = B^{-1} - B^{-1}A^{-1}B^{-1} \quad (\text{by definition 2.2.1}).$$

(2) From 1 in corollary 2.2.6 with $A = B$ 1.

(3) Take $C = A^{-1}B^{-1}$ in result (7) we get to

$$(A+B)^{-1} = B^{-1} - B^{-1}A^{-1}B^{-1}, \text{ then}$$

$$(A+B)^{-1} = A^{-1} - A^{-1}B^{-1}A^{-1} \quad (\text{by definition 2.2.1 and lemma 2.1.3}).$$

The next result is the AM-GM inequality for the Kronecker product of matrices :

Corollary 2.2.7 If A, B commute under the Kronecker product, then

$$\frac{A+B}{2} \geq \sqrt{AB}, \text{ with equality iff } A=B.$$

Proof : $(A+B)^{-1} = A^{-1} - A^{-1}B^{-1}A^{-1}$ (by corollary 2.2.6 (3))

$$(A+B)^{-1} = B^{-1} - B^{-1}A^{-1}B^{-1} \quad (\text{by definition 2.2.1 and lemma 2.1.3}), \text{ given}$$

2.3 Inequalities for Hadamard products

In this section we derive inequalities for Hadamard products of positive definite matrices

Theorem 2.3.1. For A, B, C, D and $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta + \gamma + \delta = 1$,

$$C \geq \alpha A + \beta B + \gamma D.$$

Proof : Define ϕ by $\phi(A, B) = \alpha A + \beta B$. The Hadamard product of

matrices is a principal submatrix of the Kronecker product of matrices. Consequently,

there exists a unital positive linear map ψ such that $\psi(C) = \alpha A + \beta B + \gamma D$.

Hence, $\phi(A, B) = \alpha A + \beta B = \alpha A + \beta B + \gamma D - \gamma D$. Since ϕ is jointly concave

by theorem 2.2.1 and ψ is positive and linear, the composition $\psi \circ \phi$ is also jointly con-

cave. This means that for any A, B, C, D and any scalar $\theta \in [0, 1]$,

$$\psi(\theta C + (1-\theta)D) \geq \theta \psi(C) + (1-\theta) \psi(D). \text{ Since}$$

$\psi(D) = \gamma D + \delta(A+B)$, by replacing θ by θ/γ , we get

$$\frac{\psi(C) - \delta(A+B)}{\gamma} \geq \frac{\psi(C) - \delta(A+B)}{\gamma} + \frac{\delta(A+B)}{\gamma} - \frac{\delta(A+B)}{\gamma}.$$

$$\frac{\psi(C) - \delta(A+B)}{\gamma} \geq \frac{\psi(C) - \delta(A+B)}{\gamma} + \frac{\delta(A+B)}{\gamma} - \frac{\delta(A+B)}{\gamma}.$$

$$\frac{\psi(C) - \delta(A+B)}{\gamma} \geq \frac{\psi(C) - \delta(A+B)}{\gamma} + \frac{\delta(A+B)}{\gamma} - \frac{\delta(A+B)}{\gamma}.$$

$$\frac{\psi(C) - \delta(A+B)}{\gamma} \geq \frac{\psi(C) - \delta(A+B)}{\gamma} + \frac{\delta(A+B)}{\gamma} - \frac{\delta(A+B)}{\gamma}.$$

$$\frac{\psi(C) - \delta(A+B)}{\gamma} \geq \frac{\psi(C) - \delta(A+B)}{\gamma} + \frac{\delta(A+B)}{\gamma} - \frac{\delta(A+B)}{\gamma} \quad (\text{since } \gamma \leq 1)$$

$$\|C\|_B = \|D\|_C$$

Therefore $\|C\|_B = \|D\|_C$.

From this theorem(2.3.1), we obtain the Holder inequality for positive definite matrices as a special case.

Corollary 2.3.2 For A, B, C and conjugate exponents p, q , we have

$$\|A \circ B\|_C \leq \|A\|_B \|B\|_C$$

Proof : Take $p=1, q=\infty$ and $r=\infty$ in theorem 2.3.1. Then

$$\|A \circ C\|_B \leq \|A\|_B \|C\|_B$$

Replacing C with B , hence

$$\|A \circ B\|_B \leq \|A\|_B \|B\|_B$$

Finally we replace A, B, C with A, B, C respectively we have

$$\|A \circ B\|_C \leq \|A\|_B \|B\|_C$$

Therefore, $\|A \circ B\|_C \leq \|A\|_B \|B\|_C$.

Remarks 2.3.1 The Cauchy-Schwarz inequality is obtained from corollary (2.3.2) by

taking $A=B=I$, since $\|I\|_B = \|I\|_C = 1$.

Definition 2.3.1 The Hadamard sum of A, B is denoted by $A \circ B$ where

$$(A \circ B)_{ij} = A_{ij} B_{ij}$$

By corollary (2.3.2), and taking $\theta = 1$ we obtain the following :

Corollary 2.3.3 For A, B, C and conjugate exponents p, q , we have

$$\|A + B\|_p \leq \|A\|_p + \|B\|_p.$$

Proof : $\|A + B\|_p \leq \|A\|_p + \|B\|_p$ (by corollary 2.3.2)

$$\|A + B\|_q \leq \|A\|_q + \|B\|_q.$$

For A, B, C and $\theta, \lambda \in \mathbb{R}$ such that $\theta + \lambda = 1$. Patrawut Chansangiam,

Patcharin Hemchote, Praiboon Pantaragphong in [10], developed the following results :

(1) $\|A + B\|_p \leq \|A\|_p + \|B\|_p.$

Proof : Take $\theta = 1, \lambda = 0$ in theorem 2.3.1, we get

$$\|A + B\|_p \leq \|A\|_p + \|B\|_p.$$

(2) $\|B + A\|_p \leq \|B\|_p + \|A\|_p.$

Proof : Let $\theta = 0, \lambda = 1$ in theorem (2.3.1) then we get the inequality

$$\|B + A\|_p \leq \|B\|_p + \|A\|_p.$$

(3) $\|A + B\|_p \leq \|C\|_p + \|B + D\|_p.$

Proof : Let $\theta = 1, \lambda = 0$ in theorem (2.3.1) then we get the inequality

$$\|A + B\|_p \leq \|C\|_p + \|B + D\|_p.$$

(4) $A \leq B \leq C \leq D$.

Proof : Take α in theorem 2.3.1 we get to

$$C - \alpha B \leq D - \alpha B, \text{ then}$$

$$A - \alpha B \leq C - \alpha B \leq D - \alpha B, \text{ then}$$

$$A - \alpha B \leq C - \alpha B \leq D - \alpha B, \text{ (by theorem 1.5.3 (a))}$$

Then $A - \alpha B \leq C - \alpha B \leq D - \alpha B$, (since $\alpha > 0$)

Hence $A \leq B \leq C \leq D$.

(5) $A \leq B \leq A \leq B \leq A \leq B$.

Proof : Let $\alpha = 1$ in result (2), we have

$$A - \alpha B \leq A - \alpha B \leq A - \alpha B .$$

(6) $A \leq B \leq A \leq B \leq A \leq B \leq A \leq B$.

Proof : Let $\alpha = 1$ in result (2) then we get the inequality

$$A - \alpha B \leq A - \alpha B \leq A - \alpha B \leq A - \alpha B .$$

Thus, $A \leq B \leq A \leq B \leq A \leq B \leq A \leq B$.

Definition 2.3.2 Let A then the Hadamard power of A is

$$A^{(r)} = (a_{ij}^r), \quad r = 2, 3, \dots$$

Hence, if $A \leq B$, then $A^{(r)} \leq B^{(r)}$.

Corollaries 2.3.4 If $A, B \in \mathbb{R}^{n \times n}$, and $A, B \geq O$, then

$$(1) \quad \frac{1}{2} \text{tr}(A+B) \geq \frac{1}{2} \text{tr}(A) + \frac{1}{2} \text{tr}(B).$$

$$(2) \quad \frac{1}{2} \text{tr}(A+B) \geq \frac{1}{2} \text{tr}(A) + \frac{1}{2} \text{tr}(B).$$

$$(3) \quad \frac{1}{2} \text{tr}(A+B) \geq 2 \sqrt{\frac{1}{2} \text{tr}(A) \frac{1}{2} \text{tr}(B)}.$$

Proof : (1) Take $\alpha = \frac{1}{2}$ in result (1) then we get to

$$\frac{1}{2} \text{tr}(A+B) \geq \frac{1}{2} \text{tr}(A) + \frac{1}{2} \text{tr}(B). \quad \text{Then we have}$$

$$\frac{1}{2} \text{tr}(A+B) \geq \frac{1}{2} \text{tr}(A) + \frac{1}{2} \text{tr}(B) \quad (\text{ by definition 2.3.2 }).$$

(2) Let $\alpha = 1$ in (1), we get

$$\frac{1}{2} \text{tr}(A+B) \geq \frac{1}{2} \text{tr}(A) + \frac{1}{2} \text{tr}(B).$$

(3) Let $\alpha = 1$ in result (6), we get

$$\frac{1}{2} \text{tr}(A+B) \geq 2 \sqrt{\frac{1}{2} \text{tr}(A) \frac{1}{2} \text{tr}(B)}.$$

The next result is the AM-GM inequality for matrices involving the Hadamard product :

Corollary 2.2.5 For $A, B \in \mathbb{R}^{n \times n}$, we have the following inequality

$$\frac{1}{2} \text{tr}(A \circ B) \geq \frac{1}{2} \text{tr}(A) \frac{1}{2} \text{tr}(B).$$

Proof : From (3) in Corollary (2.3.4), dividing both sides on 2, we get the inequality

$$\frac{1}{2} \text{tr}(A \circ B) \geq \frac{1}{2} \text{tr}(A) \frac{1}{2} \text{tr}(B).$$

Chapter three

Bounds on the Spectral Radius of Hadamard Products

of Positive Operators on ℓ_p -Spaces

3.1 Hadamard product of matrices of operators on

Definition 3.1.1 The space ℓ_p is the space of all sequences $(x_n)_{n=1}^\infty$, $x_n \in \mathbb{C}$, ...

of numbers such that $\sum_{n=1}^\infty |x_n|^p < \infty$ converges, thus

$$\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p \right)^{1/p}.$$

Definition 3.1.2 A linear operator T from a normed space X into a normed space Y is called bounded if there exists a positive number C such that

$$\|Tx\| \leq C\|x\|, \text{ for all } x \in X.$$

We write $T \in B(X, Y)$ for $T \in \mathcal{L}(X, Y)$, whenever $T \in \mathcal{L}(X, Y)$ for all X, Y , and we denote by $B(X)$ the set of all $T \in \mathcal{L}(X, X)$. A bounded linear operator T is called positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in X$. As we assume $X = \ell_p$, every bounded operator on ℓ_p has a matrix representation with respect to the standard basis, and we will identify the operator with its matrix.

In case $T \in B(\ell_p)$, we have $T = (t_{ij})$, where each $t_{ij} \in \mathbb{C}$. We will use frequently that if

$T \geq 0$ on ℓ_p , then $t_{ij} = \overline{t_{ji}}$.

Theorem 3.1.1. Let $\|\cdot\|$ be a matrix norm on M_n . Then

$$\lim_{n \rightarrow \infty} \|\cdot\|^{-1/n} = 1, \text{ for all } A \in M_n.$$

In the following theorems we will see upper bounds for some Hadamards products of positive operator on H :

Theorem 3.1.2 [4] Let A, B, C and D be positive operators on H . Then we have

$$\sum_{l=1}^{\infty} A_l B_l \leq \sum_{l=1}^{\infty} C_l D_l.$$

Proof : Let A, B, C , and D denote the matrices of the operators A, B, C , and D respectively. Then the matrix of the operator product $\sum_{l=1}^{\infty} A_l B_l$ is given

by $\sum_{l=1}^{\infty} A_l B_l$. From Cauchy-Schwarz inequality we get

$$\sum_{l=1}^{\infty} A_l B_l \leq \sum_{l=1}^{\infty} C_l D_l,$$

Corollary 3.1.3 Let A and B be positive linear operator on H . Then we have

$$\sum_{l=1}^{\infty} A_l B_l \leq \sum_{l=1}^{\infty} C_l D_l.$$

Proof : Take A, B and C, D in theorem (3.1.2) so,

$$\sum_{l=1}^{\infty} A_l B_l \leq \sum_{l=1}^{\infty} C_l D_l.$$

Thus, $\sum_{l=1}^{\infty} A_l B_l \leq \sum_{l=1}^{\infty} C_l D_l$ (since A, B, C, D).

Corollary 3.1.4 Let A and B be positive linear operators on \mathcal{H} . Then we have

$$A \# B \leq A \#_s B \leq A \#_t B \leq A \#_r B \leq A \#_p B \leq A \#_q B \leq A \#_s B \leq A \# B.$$

Proof : We substitute A^{-1} for both A and B , and A^{-1} for A and B in theorem (3.1.2) then,

$$A \# B \leq A \#_s B \leq A \#_t B \leq A \#_r B \leq A \#_p B \leq A \#_q B \leq A \#_s B \leq A \# B.$$

So,

Thus, $A \# B \leq A \#_s B \leq A \#_t B \leq A \#_r B \leq A \#_p B \leq A \#_q B \leq A \#_s B \leq A \# B$ (by definition (2.3.2)).

Therefore,

Theorem 3.1.5 [4] Let A and B be positive linear operators on \mathcal{H} . Then we have

$$\cdot$$

Proof : $Q, 1, \dots, n$.

Let P be the statement, $\dots, 1, 2, \dots$.

For $2,$ \dots .

$$\cdot$$

But, Q . Therefore, \dots .

Assume that P is true, so Q is true.

.

.

But,

Q. Therefore,

P is true.

Hence, P is true.

Take x .

Let A and B denote the matrices of A and B , respectively.

Then the (i, j) entry of A is a_{ij} , and

The following lemma shows that the Hadamard product of two positive linear operators on \mathcal{H} is bounded :

Lemma 3.1.6 [4] Let A and B be a positive linear operators on \mathcal{H} . Then $A \circ B$ is a positive linear operator on \mathcal{H} and $\|A \circ B\| \leq \|A\| \|B\|$.

Proof : It is sufficient to prove that $\langle (A \circ B)x, x \rangle \geq 0$, whenever $\|x\| = 1$.

Assume $\|x\| = 1$. From $\langle Ax, x \rangle \geq 0$ it follows that $\langle Ax, x \rangle \geq -\|A\|$ for all x , so that

$0 \leq \langle Ax, x \rangle + \|A\|$. This implies immediately that $\langle Ax, x \rangle \geq -\|A\|$ is a positive operator from

to \mathcal{H} and $\|Ax + \|A\|x\| \geq 0$. Thus we take

$$\langle (Ax + \|A\|x), (Ax + \|A\|x) \rangle \geq 0. \text{ Then } \langle Ax, x \rangle \geq -\|A\| \text{ and } \langle Bx, x \rangle \geq -\|B\|,$$

thus $\langle (A \circ B)x, x \rangle \geq -\|A\| \|B\|$. Therefore

$$\langle (A \circ B)x, x \rangle \geq -\|A\| \|B\|, \text{ then } \langle (A \circ B)x, x \rangle \geq 0.$$

Therefore, $A \circ B$ is a positive operator.

Lemma 3.1.7 [4] Let A and B be positive linear operators on \mathcal{H} . Then $A \circ B$ is a positive operator on \mathcal{H} and $\|A \circ B\| \leq \|A\| \|B\|$.

Proof : By the identity $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$; $a, b \geq 0$, where a, b are positive numbers, which refers to Krivine calculus in Banach lattices, we get

$$\langle (A \circ B)x, x \rangle \geq \frac{1}{2}(\langle Ax, x \rangle + \langle Bx, x \rangle - |\langle Ax, x \rangle - \langle Bx, x \rangle|), \text{ for all } \|x\| = 1.$$

This implies that $A - \lambda I$ is a positive operator on V , and

$$\langle (A - \lambda I)x, x \rangle \geq 0, \text{ for all } x \in V.$$

By taking the minimum over x , we get

$$\lambda \leq \langle Ax, x \rangle, \text{ for all } x \in V. \text{ Then,}$$

$$\lambda \leq \langle Ax, x \rangle.$$

$$\lambda \leq \lambda.$$

3.2 Inequalities for spectral radius of Hadamard products

In this section, we will see some inequalities for spectral radius of Hadamard products of positive operators on \mathcal{H} .

Lemma 3.2.1 [4] Let A and B be positive linear operator on \mathcal{H} . Then we have

$$r(A \circ B) \leq r(A) r(B).$$

Proof : From corollary (3.1.4), it follows that

$$A \circ B \leq r(A) B.$$

Taking norms on both sides we get,

$$r(A \circ B) \leq r(A) r(B), \text{ then}$$

$$r(A \circ B) \leq r(A) r(B) \quad (\text{by lemma (3.1.7)}).$$

Taking (2) roots on both sides we get,

$$\sqrt{r(A \circ B)} \leq \sqrt{r(A)} \sqrt{r(B)}.$$

And taking limit for $n \rightarrow \infty$ on both sides we have,

$$\lim_{n \rightarrow \infty} \sqrt[n]{r(A \circ B)} \leq \lim_{n \rightarrow \infty} \sqrt[n]{r(A)} \lim_{n \rightarrow \infty} \sqrt[n]{r(B)}.$$

So, $r(A \circ B) \leq r(A) r(B)$ (by theorem (3.1.1)).

Lemma 3.2.2 [4] Let A and B be positive linear operators on \mathcal{H} . Then we have

.

Proof : Take λ in theorem (3.1.5) we get,

.

Then $\lambda^{-1} (A + \lambda B)^{-1} \lambda$, taking norms in both sides we get,

(by lemma(3.1.6)).

Taking λ^{-1} root and limit as $\lambda \rightarrow \infty$, on both sides we have

...

Similarly, $\lambda^{-1} (A + \lambda B)^{-1} \lambda$.

In theorem (3.1.2), take λ^{-1} , we get

- - .

Thus, - - .

So, .

Then, .

Taking norms in both sides we get,

.

Taking λ^{-1} root and limit as $\lambda \rightarrow \infty$ on both sides we have

From (a) and (b) we get,

Therefore,

Theorem 3.2.3 [4] Let A and B be positive linear operator on \mathcal{H} . Then,

Proof : From corollary (3.1.3), it follows that

Taking norms in both sides we get,

(by lemma (3.1.7)).

Taking 2 root and limit as $n \rightarrow \infty$ on both sides we have

(since $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} A^k B^k = \frac{A+B}{2}$).

From $\frac{A+B}{2} \leq \sqrt{AB}$, we get

(by lemma (3.2.2)).

Therefore,

Chapter four

Applications on Kronecker product.

In this section we present application of the Kronecker product to matrix equations, matrix differential equations :

4.1 Matrix equations

Knowledge of the Kronecker product and its application facilitates our analysis of matrix equations, since the Kronecker product can be used to give a convenient representation for linear matrix equations.

We start by studying the simplest matrix equation as the following theorem :

Theorem 4.1.1 7 Let A , B , C , and D , such that

A , then the system $AX + BY = C$. Has a unique solution

if and only if A is invertible if and only if A and B both are invertible.

If either A or B are not invertible, then there exist a solution if and only if

$\text{rank}(A) = \text{rank}(A, C)$. Where (A, C) is the augmented

matrix of A and C ; otherwise the system has no solution.

This equation can be generalized as follows :

$A_1 X_1 + A_2 X_2 + \dots + A_n X_n = C$, where A_i , X_i , C , $1, \dots, n$,

and X_i is $n_i \times 1$.

With the same technique we can rewrite this equation as :

So, $(A \otimes I + I \otimes B)x = c$ (by theorem 1.4.26).
 $(A \otimes I + I \otimes B)^{-1}c = x$

The unique solution is obtained if and only if $(A \otimes I + I \otimes B)$ is invertible.

The following theorem examine if the $(A \otimes I + I \otimes B)$ has a unique solution. By using eigenvalue of the Kronecker sum.

Theorem 4.1.2 13 Let A and B be $n \times n$ matrices. The equation $(A \otimes I + I \otimes B)x = c$ has a unique solution x for each c , if and only if $(A \otimes I + I \otimes B)$ is invertible.

Proof : The eigenvalue of $(A \otimes I + I \otimes B)$ are the same as those of $A + B$. Now, if we take the $\text{Vec}(\cdot)$ of both sides in equation $(A \otimes I + I \otimes B)x = c$ we get $(A + B)\text{vec}(x) = \text{vec}(c)$ (by corollary (1.4.27)). And this system of equations has a unique solution if and only if

$(A + B)$ is invertible, that is if and only if non of the eigenvalues of $(A + B)$ is zero. But

$$\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n, \text{ where } \lambda_1, \dots, \lambda_n \text{ are eigenvalues of } A \text{ and } \mu_1, \dots, \mu_n \text{ are eigenvalues of } B.$$

and $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$. So The equation $(A \otimes I + I \otimes B)x = c$ has a unique solution

if and only if $\lambda_i + \mu_j \neq 0$ for all $i, j = 1, \dots, n$, if and only if $(A + B)$ is invertible if and only if

$(A + B)$ and $(A \otimes I + I \otimes B)$ have no common eigenvalue if and only if $(A + B)$ is invertible.

If on other hand A and $-A$ have an eigenvalue in common, the existence of the solution depends on the rank of the augmented matrix $[A \ b]$: $\text{rank}[A \ b] = \text{rank} A$. If the rank of this matrix is equal to the rank of A , then the solution exist otherwise they do not.

Theorem 4.1.3 7 If A and b are $n \times 1$ column matrices. The equation $Ax = b$, which has a nontrivial solution if and only if λ is an eigenvalue of A . But the eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$. Hence $Ax = b$ has a nontrivial solution if and only if b is orthogonal to v_i for some i , $i = 1, 2, \dots, n$.

Lemma 4.1.1.4 8 Let A, B, C and D $n \times n$ matrices such that A is invertible. Then $A + B$ is

Invertible if and only if $A^{-1}(A + B)$ is invertible and $\det(A + B) = \det(A) \det(A^{-1}(A + B))$.

Theorem 4.1.1.5 8 Let $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z$ be given matrices such that $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z$ are invertible. Then the system

has a unique solution if and only if $A^{-1}(A + B)$ is invertible.

Corollary 4.1.6 Let A, B, C, D, E, F $n \times n$ be given matrices. Then the system

has a unique solution if and only if $A^{-1}(A + B)$ is invertible.

Corollary 4.1.7 Let $A, B, C, D, E,$ and $F \in M$ be given matrices. Then the system

has a unique solution if and only if $A - BCB^{-1}D$ is invertible.

If we assume that B is invertible, then the system

Corollary 4.1.8 Let $A, B, C, D, E,$ and $F \in M$ be given matrices. Then the system

has a unique solution if and only if $A - BCB^{-1}D$ is invertible.

Corollary 4.1.9 Let $A, B, C, D, E,$ and $F \in M$ be given matrices. Then the system

has a unique solution if and only if $A - BCB^{-1}D$ is invertible.

The important application of the theorem (1.4.12.(b)) are for \sin , \cos , \sin , \cos ,

\cos , lead to the following result :

Corollary 4.1.10 Let λ be a scalar matrix. Then

(1) $\sin(\lambda A) = \lambda \sin A$.

(2) $\sin A \cos \lambda A = \lambda \sin A \cos A$.

Proof (1) : We can write e^x as a power series such as :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Proof (2) : We can write $\sin A$ as a power series such as :

$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \dots$$

4.2 Matrix differential equations

In this section we present another application of the Kronecker product that deals with matrix differential equations of the form

Definition 4.2.1 Given the matrix $A(t)$, where each $a_{ij}(t)$ is a differentiable function, then the derivative of the matrix $A(t)$ with respect to the scalar t is defined as :—

Similarly, the integral of the matrix is defined as :

Theorem 4.2.1 7 Let $A(t)$ and $B(t)$, be differentiable matrices (each matrix is assumed to be a function of t . Then

Proof : On differentiating the (i, j) th block of $A(t)B(t)$, we obtain

Corollary 4.2.2 Let $A(t)$ and $B(t)$ be differentiable matrices (each matrix is assumed to be a function of t . Then

Proof : —I Q, and using definition 1.4.2, then we have

$$\begin{aligned} & - \quad - I \quad - \quad I \\ & I \quad - \quad - \quad I \\ & - \quad - \quad . \end{aligned}$$

The simplest form of matrix differential equations as the following theorem :

Theorem 4.2.3 $\dot{X} = AX + B$; $X(0) = X_0$, where $A, B \in \mathbb{R}^{n \times 1}$

This equation has the following solution :

Using this fact we can solve the matrix differential equation :

$$\dot{X} = AX + B \quad \dots 2, \text{ where } A, B \in \mathbb{R}^{n \times 1}, \text{ and } X(0) = X_0$$

Proof: use the e^{At} -notation, then we get $\frac{d}{dt}(e^{-At} X) = e^{-At} B$, and

$$e^{-At} X(0) = X_0 . \text{ Let } Y = e^{-At} X, \text{ and } \dot{Y} = e^{-At} B .$$

Then (1) becomes $\dot{Y} = e^{-At} B$; $Y(0) = X_0$. By the solution (2) we have

$$\begin{aligned} Y(t) &= \int_0^t e^{-A(t-s)} B ds + e^{-At} X_0 \\ &= \int_0^t e^{-A(t-s)} B ds + e^{-At} X_0 \\ &= \int_0^t e^{-A(t-s)} B ds + e^{-At} X_0 \end{aligned}$$

so, $X(t) = e^{At} \left[\int_0^t e^{-A(t-s)} B ds + e^{-At} X_0 \right]$; . . .

Thus, $X(t) = e^{At} X_0 + \int_0^t e^{A(t-s)} B ds$.

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