Deanship of Graduate Studies
Al-Quds University

Existence of Universal Locally Univalent Functions

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Jerusalem-Palestine

1440/2019

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Award of Master of Science in Mathematics at Al-Quds University

1440/2019

Al-Quds University

Deanship of Graduate Studies

Graduate Studies / Mathematics

Thesis Approval

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Master Thesis submitted and accepted, Date: 15/5/2019

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Jerusalem-Palestine

May 2019

Dedication

To my parents , my grandfather , my brothers , my family , my colleagues and each person gave me support and assistance.

Ramzi Jafar

Declaration

I certify that this submitted for the degree of master is result of my own research, expect where otherwise acknowledge. And that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signature : ..

Ramzi Tarek Hashim Jafar

Date : 15/5/2019

Acknowledgment

Thanks is given first to God, the Cherisher and Sustainer of the words. Peace and blessing be upon my first teacher and educator, Prophet Mohammad, who has taught the world.

I am thankful to my supervisor Dr. Ibrahim Alghrouz for his encouragement, guidance and support from the initial to the final level enabled me to develop and understand the subject.

My gratitude especially presents to my department "Mathematics" and to all my Doctors.

Abstract

In this thesis, we proved Runge theorem and universality results for locally univalent holomorphic and meromorphic functions in compact sets and in neighborhood of the compact sets. After that , we approximate the meromorphic function in an open set containing compact set , and had new problems in approximating the continuous function .

Contents

Introduction

The Universality concept covers a wide range of phenomena in complex analysis .Generally speaking , a universal object is the object which approximates every object in some universe with some restrictions and limitations on it . For example , universality occurs when the translates of an entire function can approximate any other entire function or when the partial sums of a formal power series or a formal trigonometric series approximate all functions in some natural class. For long time , existing approximation theorems were used in constructions of universal functions and universal series .In recent years , however , constructions have required the development of new approximation theorems , thereby also enriching the area of complex analysis.

There is no single definition of a universal function , but all of them have the following in common , one considers a suitable sequence $T = T_n$ of operators acting on a space X , for example , of holomorphic functions with values in another space Y of holomorphic functions , then a function $f \in X$ is called universal with respect to T if $\{f, T_1 f, T_2 f, ...\}$ is dense in Y. One of earliest example of a universal function is due to *Birkhoff* (1929) [3] who showed that there exists an entire function f whose translates $f(z+n), n \geq 1$ can approximate any other entire function , uniformly on compact sets , in that case we have $(T_n f)(z) = f(z+n)$, and $X = Y$ is the space of entire functions with the usual compact-open topology.

Siedel and *Walsh* [32] showed that an analogue of *Birkhoff*'s university theorems holds for holomorphic functions in the unit disk , if we replace translates by non-euclidean translates , that is $T_n f = f \circ \phi_n$ is the composition of f with an automorphism ϕ_n of the unit disk $\mathbb{D}$ .At the core of studying holomorphic function in the unit disk $\mathbb{D}$ is $\mathcal{B}(\mathbb{D})$ the set of all bounded holomorphic functions on the disk.

Extending the study of functions in the unit disk ,which are universal with respect to composition with automorphism of the disk , *Mortini* talked about the universality of functions f holomorphic in a domain ω with respect to a sequence $(f \circ \phi_n)$ of compositions , where (ϕ_n) are self-maps of Ω .

In general theory of university , the talk is about a bounded operator T

defined on some separable Banach space X which is called the *Hypercyclic Operator* if there exists some vector $x \in X$ such that the orbit of x under T , namely $\{T^n x : n \geq 0\}$ is dense in X .

The main focus of the thesis is proving *Runge-theorems* and universality results for locally univalent holomorphic and meromorphic functions , Refining a result of *M.Heins* [14] .

In chapter one, we recovered some basics from complex analysis .

In chapter two, we stated two important theorem in approximation theory "Runge's and Mergelyan's ".

In chapter three, we applied Runge's theorem for locally univalent functions.

Chapter 1

Preliminaries

1.1 Introductory Geometric Considerations

The geometric theory of functions of a complex variable makes a study of analytic functions defined by some geometric property or other and a study of various geometric properties of certain classes of analytic functions. Therefore, it naturally rests on a number of general geometric concepts that are encountered in present-day mathematics. Here we propose to make some brief remarks about these concepts, in order of their occurrence, that are associated with the complex plane and that we shall need in our subsequent exposition.

Domains and curves, one of the basic geometric concepts in the theory of functions of a complex variable is the concept of a domain. A domain is defined as an open set any two points in which can be connected by a broken line consisting entirely of points of that set (the property of connectedness). The boundary points of a domain are those points in the complex plane that do not belong to the domain but are cluster points of it. If a domain is other than the entire plane, it necessarily has boundary points. The set of all boundary points of a domain is called its boundary. The boundary of a domain is a closed set . Those points in the plane that are neither interior nor boundary points of a domain are called exterior points of the domain. Every exterior point of a domain has a neighborhood containing no points of the domain.

The union of a domain and its boundary is called a closed domain. In contrast with this, a domain itself is sometimes called an open domain. A domain is said to be simply connected if its boundary consists either of a continuum or of a single point or if the domain itself is the entire complex plane. Otherwise, a domain is said to be multiply connected. Specifically, it is said to be doubly, triply, , n-connected if its boundary consists of 2, 3, , n disjoint continua (possibly degenerate). All these regions are said to finitely connected and the continua (including the degenerate ones) are called boundary continua. The role of domains in the study of closed and open sets is clear from the, following theorem:

Theorem 1.1.1 *Every open set E in the complex plane is the union of finitely or countably many domains.*

Another basic geometric concept in the theory of functions of a complex variable is that of a curve.

A continuous curve is a set of points in the rectangular coordinates x, y plane of which can be written as continuous functions

$$x = \phi(t), y = \psi(t) \tag{1.1}$$

of a real variable t in some finite interval $a \leq t \leq b$. It is easy to see that this set is a continuum.

However, the concept of a continuous curve is too general for our purposes. There are continuous curves that do not at all correspond to our intuitive idea of a curve as a one-dimensional figure. In fact, it is possible to construct a continuous curve that passes through every point of a given square. On the other hand, if we require that the curve have no multiple points, it will possess a number of clear-cut properties. Such curves are called simple curves or Jordan curves.

Thus, the continuous curve (1.1) or, more briefly, the curve

$$z = z(t) = \phi(t) + i\psi(t), \quad a \leq t \leq b \tag{1.2}$$

is called a Jordan curve if, for any two distinct values t_1 and t_2 in the interval $[a, b)$ with $t_1 \neq t_2$, we have $z(t_1) \neq z(t_2)$ and $z(t_2) \neq z(b)$. The points $z(a)$ and $z(b)$ may or may not coincide. In the first case, the curve is called a closed, in the second case a non-closed Jordan curve. We have the following important theorem (due to Jordan):

Theorem 1.1.2 *[10] A closed Jordan curve C partitions the plane (including ∞) into two simply connected domains both of which have C as their boundary. One of these domains is bounded and is called the interior of C. The other contains ∞ and is called the exterior of C. The complement of a nonclosed Jordan curve C consists of a single simply connected domain containing ∞ and having C as its boundary.*

From non-closed Jordan curves, one can construct continuous curves that are not of the Jordan type. On the other hand, even a Jordan curve is sometimes too general. Then, depending on our purpose, we introduce curves of more restricted types, for example, smooth and piecewise-smooth curves.

A curve 1.2 is said to be smooth if the function $z(t)$ has a continuous non-zero derivative $z'(t)$ everywhere in [a, b] (a one-sided derivative at the two endpoints). The requirement of smoothness is obviously equivalent to the requirement that the curve have everywhere a continuously turning tangent. A curve consisting of finitely many smooth curves is called a piecewise-smooth curve.

Finally, the simplest type of continuous curve is an analytic curve. This is a curve defined by an equation of the form $z = z(t)$ for $a \leq t \leq b$, where $z(t)$ can be expanded in a power series

$$z(t) = c_0 + c_1(t - t_0) + c_2(t - t_0)^2 + ..., \tag{1.3}$$

with $c_1 \neq 0$, about each value t_0 in [a, b]. A continuous curve consisting of a finite number of analytic curves is called a piecewise-analytic curve.

1.2 Holomorphic Functions

Throughout this discussion we identify the complex plane, $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Let Ω be an open set in the complex plane and let f be a complex valued function in the space of all continuous differentiable functions $C^1(\Omega)$. If the real coordinates are denoted by x and y, then we set $z = x + iy$ and $\bar{z} = x - iy$. Also we have :

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}$$

We define partial differential operators in the following way:

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}) \text{ and } \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})$$

Now, the differential of f can be expressed as a linear combination of dz and $d\bar{z}$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

Definition 1.2.1 *A function $f \in C^1(\Omega)$ is said to be holomorphic if $\frac{\partial f}{\partial \bar{z}} = 0$ in Ω , or equivalently if df is proportional to dz. If the function f is holomorphic we write f' rather than $\frac{\partial f}{\partial z}$ We denote the set of all holomorphic functions on Ω by $\mathcal{O}(\Omega)$.*

Lemma 1.2.1 *Let $f : U \to \mathbb{C}$ be continuous function on U with $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$ and u and v are real-valued functions. Then f is holomorphic , if we have that u and v satisfy the equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \textit{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at every point of U.

The equations :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the *Cauchy-Riemann* equations.

Definition 1.2.2 *A complex function f is called analytic if around each point z_0 of its domain the function f can be computed by a convergent power series. More precisely, for each z_0 there exists $\epsilon > 0$ and a sequence of complex numbers $(a_0, a_1, ...)$ such that*

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... = \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

for $|z - z_0| < \epsilon$.

If f is analytic then f and all its derivatives are holomorphic. The derivatives can be computed as the derivatives of a convergent power series, i.e. by deriving term by term. In particular

$$f^{(n)}(z_0) = \frac{a_n}{n!}$$

which shows that the expression of f as a power series at z_0 is unique.

If the power series is convergent for all $z \in \mathbb{C}$ i.e. not just for $|z - z_0| < \epsilon$, the function f is called an entire function.

Remark: Some books use the word "analytic" instead of "holomorphic." Still others say "differentiable" or "complex differentiable" instead of "holomorphic." The use of "analytic" derives from the fact that a holomorphic function has a local power series expansion about each point of its domain. The use of "differentiable" derives from properties related to the Cauchy-Riemann equations and conformality.

Definition 1.2.3 *A meromorphic function f on $U \subset \mathbb{C}$ with singular set S is a function $f : U \backslash S \to \mathbb{C}$ such that f is holomorphic on $U \backslash S$, S is discrete and for each $p \in S$ and $r > 0$ such that $D(p,r) \subseteq U$, $S \cap D(p,r) = \{p\}$, the function $f|_{D(p,r)\backslash\{p\}}$ has a finite order pole at p .*

Definition 1.2.4 *A subset K of a metric space X is compact if for every collection $\mathcal{G}$ of open sets in X with the property $K \subset \cup\{G : G \in \mathcal{G}\}$ there is a finite sets $G_1, G_2, ..., G_n$ in $\mathcal{G}$ such that $K \subset G_1 \cup G_2 \cup ... \cup G_n$. The collection of sets $\mathcal{G}$ is called a cover of K , if each member of $\mathcal{G}$ is an open it is called an open cover of K.*

Definition 1.2.5 *A set $\Omega \subset \mathbb{C}$ is said to be disconnected if there exist nonempty sets $A, B \subset \mathbb{C}$ such that*

$$\Omega = A \cup B \text{ and } \bar{A} \cap B = A \cap \bar{B} = \emptyset$$

A set $\Omega \subset \mathbb{C}$ is said to be connected if it is not disconnected.

Now , we introduce the notion of a connected component.

Definition 1.2.6 *Given $\Omega \subset \mathbb{C}$ we say that a connected set $A \subset \Omega$ is a connected component of Ω if any connected set $B \subset \Omega$ containing A is equal to A.*

We note that if a set $\Omega \subset \mathbb{C}$ is connected, then it is its own unique connected component.

Now, In order to define the integral of a complex function, we first introduce the notion of a path:

Definition 1.2.7 *A continuous function $\gamma : [a, b] \to \Omega$ is called a path in Ω, and its image $\gamma([a, b])$ is called a curve in Ω.*

Now, we define two operations, the first is the inverse of a path.

Definition 1.2.8 *Given a path $\gamma : [a, b] \to \Omega$ we define the path $-\gamma : [a, b] \to \Omega$ by $-\gamma(t) = \gamma(a + b - t)$ for each $t \in [a, b]$.*

The second operation is the sum of paths.

Definition 1.2.9 *Given a path $\gamma_1 : [a_1, b_1] \to \Omega$ and $\gamma_2 : [a_2, b_2] \to \Omega$ such that $\gamma_1(b_1) = \gamma_2(a_2)$ we define the path $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \Omega$ by*

$$\gamma_1 + \gamma_2 = \begin{cases} \gamma_1(t) & \text{if } t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

We also consider the notions of a regular path and a piecewise regular path

Definition 1.2.10 *A path $\gamma : [a, b] \to \Omega$ is said to be regular if it is of class C^1 and $\gamma'(t) \neq 0$ for every $t \in [a, b]$, taking the right-sided derivative at a and the left-sided derivative at b.*

وجود اقترانات عالمية واحد لواحد محلية

الاعداد: رمزي طارق هاشم جعفر

الأشراف: د. ابراهيم الغروز

ملخص:

في هذه الرسالة , أثبتنا نظرية رانج والنتائج العالمية للاقترانات التحليلية و الاقترانات القريبة من التحليلية والتي تكون واحد لواحد محليا في مجموعات مرتصة وايضا بجوار مجموعات مرتصة . بعد ذلك , قمنا بتقريب الاقترانات القريبة من التحليلية في مجموعات مفتوحة تحتوي على مجموعات مرتصة . كذلك قمنا بحل مسائل في هذا المجال كانت غير محلولة من قبل حول الاقترانات المتصلة وتقريبها .