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Compactness In Bitopological Spaces

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Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ....

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Abstract

In this thesis, we introduce three concepts concerning the compactness in bitopological spaces, namely, semi compactness, pairwise compactness and Birsan compactness. Also, we introduce other concepts in bitopological spaces such as Hausdorffness, continuity, regularity and normality, and study their relations with compactness in bitopological spaces. We discuss generalizations for well known theorems and results concerning compactness in single topology, such as Alexander and Tychonoff theorems for bitopological spaces.

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Introduction

In 1962, J.C. Kelly [8] has defined the concept of the bitopological space to be a non- empty set X on which two arbitrary topologies τ_1 and τ_2 are defined. This definition is denoted by the triple (X,τ_1,τ_2). Since this initiation, several authors have considered the problem of defining the concept of compactness in bitopological spaces. And in this thesis we study the definitions for compactness in bitopological spaces, which were introduced by Swart in [1], Flectcher, Holye and Patty in [2], Kim in [5-1], Kim and Naimpaly in [5-2], Cooke and Reilly in [4], and Birsan in [6].

In fact, these definitions are summarized into just three definitions only, namely, semi compactness (s-compactness), pairwise compactness (p-compactness), and Birsan compactness (conversely and B-compactness).

Semi compactness is discussed in chapter one which is divided into five sections. Section one begins with Swart's definition for bitopological compactness. Also, we define quasi-open (closed) subsets of (X,τ_1,τ_2), quasi-closure and some relations and results concerning them. In section two, we define four different types of continuous functions in the bitopological spaces, and deduce some useful results concerning continuity and compactness in bitopological spaces in this chapter and in the following two chapters. Least upper bound topology is introduced in section three. Also, we show that semi-compactness in bitopological space is equivalent to the compactness in least upper bound topology. In section four we introduce two different and related definitions for Hausdorffness in bitopological spaces, and deduce many useful results and conclusions concerning Hausdorffness in bitopological spaces. In the last section we introduce the product topology and conclude a generalization of Tychonoff theorem in the single topological space.

In section one of chapter two, three different but dependent definitions of bitopological compactness, which we call each of them pairwise compactness because they are equivalent, are considered. Also we show that semi and pairwise compactness are dependent. In section two we define pairwise regularity and pairwise normality, and deduce related results between pairwise Hausdorffness and pairwise compactness. In section three, we define the notion of pairwise compactness in a subspace of a bitopological space and its relation with pairwise compactness in bitopological spaces. In section four, we discuss some definitions and theorems related to Datta [3-2], and prove a generalization of Alexander theorem in single topology, and show that a generalization of Tychonoff theorem in single topology fails for bitopological spaces, by giving a counter example.

In section one of chapter three, we introduce a new different concept for compactness in bitopological spaces. We show that this definition is independent from the two definitions, which are introduced in chapter one and chapter two. Also, we give two dependent definitions, one is called conversely compactness and the other is called B-compactness, and then shows that B-compactness implies conversely compactness. We introduce two different ways which are equivalent to conversely compactness, and deduce the relation between conversely (B-) compactness and compactness of τ_1 and τ_2.

In fact we show that conversely (B-) compactness implies compactness of τ_1 and τ_2. We deduce the effect of pairwise Hausdorffness in comparison of topologies. The examples at the end of this section are counter examples demonstrate the relations between conversely (B-) compactness, p-regularity and p-normality in bitopological spaces. In section two, we study the compactness in subspaces and its relations with closedness and openness. In section three, we study the effect of continuous and open functions on conversely (B-) compact bitopological spaces. Finally, in section four, we make a generalization of Alexander and Tychonoff theorems in bitopological spaces.

Chapter One

Semi compactness

1.1 Swart's definition for compactness in bitopological spaces.

1.1.1 Definition [1]:

A cover υ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2-$ open if $\upsilon \subset \tau_1 \cup \tau_2$.

1.1.2 Definition [1]:

A bitopological space (X, τ_1, τ_2) is called semi-compact (s- compact) if every $\tau_1\tau_2$–open cover for X has a finite subcover .

Swart in [1] consider the above definition for compactness in bitopological spaces, and uses the term compact for s-compactness in (X, τ_1, τ_2).

1.1.3 Definition [3-1]:

A subset A in a bitopological space (X,τ_1,τ_2) is said to be quasi-open if for every $x \in A$, there exists a τ_1-open nhood for x (nhood stands for neighborhood) $U_x \subset A$ or a τ_2-open nhood for x, $V_x \subset A$.

1.1.4 Theorem[3-1]:

Quasi-open sets in a bitopological space (X,τ_1,τ_2) are precisely the unions of τ_1-open and τ_2-open sets.

Proof:

Let U_1, U_2 be two τ_1 and τ_2-open sets respectively. Let $x \in U_1 \cup U_2$, then $x \in U_1$ or $x \in U_2$. Either of them is contained in $U_1 \cup U_2$, which means that $U_1 \cup U_2$ is quasi-open.

Conversely, assume A is quasi-open in (X,τ_1,τ_2), and let $x \in A$. Then there is a τ_1-open nhood U_1 s.t. $U_1 \subset A$ or τ_2-open nhood U_2 s.t. $U_2 \subset A$. Let

$A_1 = \{ x \in A \mid \exists\, U_1 \in \tau_1$ such that $x \in U_1 \subset A\}$ and

$A_2 = \{ x \in A \mid \exists\, U_2 \in \tau_2$ s.t. $x \in U_2 \subset A\}$. Then $A = A_1 \cup A_2$. Now A_1 is τ_1- open. For let $x \in A_1$, then there exist $U_1 \in \tau_1$ such that $x \in U_1 \subset A$. Let $y \in U_1$, then $y \in U_1 \subset A$, and $U_1 \in \tau_1$, so $y \in A_1$. By the Definition of A_1, we have $U_1 \subset A_1$. Now $x \in U_1 \subset A_1$, $U_1 \in \tau_1$, and x was arbitrary in A_1, so $A_1 \in \tau_1$, similarly, we can show that $A_2 \in \tau_2$.

From this theorem and the definition of quasi-open sets we conclude the following results.

1.1.5 Corollary:

In a bitopological space (X,τ_1,τ_2):

a) Every τ_1-open (τ_2-open) set is quasi-open.

b) Arbitrary unions of quasi-open sets is quasi-open.

The converse of (a) does not necessarily hold. For let $\tau_1 = \{\{a\},\{a,b\},X, \emptyset\}$, $\tau_2 = \{\{c\},\{c,b\},X, \emptyset\}$, $X=\{a,b,c\}$. Then $\{a,c\}$ is quasi-open but not in τ_1 nor in τ_2 .

Proof of (a): follows from definition (1.1.3).

Proof of (b):

Let $\{ A_\alpha : \alpha \in \Delta \}$ be any collection of quasi-open sets . Then $\forall\ \alpha \in \Delta$,

$A_\alpha = A`_\alpha \cup A``_\alpha$ ; where $A`_\alpha \in \tau_1$ and $A``_\alpha \in \tau_2$. So,

$\bigcup_{\alpha\in A} A_\alpha = \bigcup_{\alpha\in A} (A`_\alpha \cup A``_\alpha ) = ( \bigcup_{\alpha\in A} A`_\alpha ) \cup ( \bigcup_{\alpha\in A} A``_\alpha)$. But $\bigcup_{\alpha\in A} A`_\alpha \in \tau_1$

and $\bigcup_{\alpha\in A} A``_\alpha \in \tau_2$, therefore $\bigcup_{\alpha\in A} A_\alpha$ is quasi-open.

Finite intersection of quasi-open sets need not be quasi-open as the following example shows.

1.1.6 Example:

Let $X = \Re$; the real line, let τ_1 = the topology with the base

$\{[a,b): a,b \in \Re\}$ and τ_2 = the topology with the base $\{(c,d]: c,d \in \Re\}$.

Let $a < b < c$. Then $(a,b]$ and $[b,c)$ are quasi- open sets , but

$(a,b] \cap [b,c) = \{b\}$ is not a quasi-open set .

1.1.7 Definition [3-1]:

A quasi-closed set is the complement of a quasi-open set.

From this definition and Thm. (1.1.4) and its results we conclude the following results.

1.1.8 Corollary:

In a bitopological space (X,τ_1,τ_2):

a) Every τ_1- closed (τ_2-closed) set is quasi-closed.

b) Arbitrary intersection of quasi-closed sets is quasi-closed.

c) Finite union of quasi-closed sets need not be quasi-closed.

d) Every quasi-closed set is the intersection of a τ_1- closed set and a τ_2-closed set.

Proof of (a):

Holds since every τ_1- closed (τ_2-closed) set is the complement of a τ_1-open (τ_2-open) set which is quasi-open.

Proof of (b):

Let $\boldsymbol{F} = \{F_\alpha : \alpha \in \Delta\}$ be any collection of quasi-closed sets , then $\forall\ \alpha \in \Delta$,