

Deanship of Graduate Studies


Al-Quds University

Compactness In Bitopological Spaces

Ghaleb Ali Jowhar Halabia

Ref  
QA  
611.23  
.H3  
2002

3192 A-0-0-0-1

Library		المكتبة
Acc No :		رقم المتسلسل :
College :		الكلية :

M. Sc. Thesis

مكتبة كلية العلوم والتكنولوجيا  
القدس C.I.  
College of Science &  
Technology - Library  
Jerusalem  
Acc. No. 9326 الرقم المتسلسل

2002

المكتبة الرئيسية



3192A-0-0-0-1

# Compactness In Bitopological Spaces

By

Ghaleb Ali Jowhar Halabia

B. Sc.: University of Jordan

Jordan

A thesis submitted in partial fulfillment of requirement for the degree of  
Master of Science, Department of Mathematics / Program of Graduate  
Studies.

Al-Quds University

February, 2002

The program of graduated studies / Department of Mathematics

Deanship of Graduate Studies

Compactness In Bitopological Spaces

By

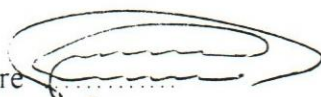
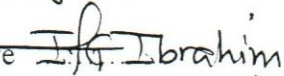
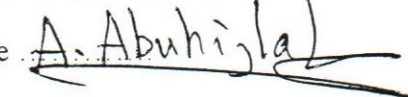
Student Name: Ghaleb Ali Jowhar Halabia

Registration No: 9720077

Supervisor: Dr. Yousef Bdeir

Master thesis submitted and accepted, Date: February 10, 2002.

The names and signatures of the examining committee members are as follows:

- |                           |                   |  |
|---------------------------|-------------------|--|
| 1. Dr. Yousef Bdeir       | Head of committee | Signature  |
| 2. Dr. Ibrahim Al- Ghrouz | Internal Examiner | Signature  |
| 3. Dr. Ayman Abu Hijlah   | External Examiner | Signature  |

Al-Quds University

Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ..... 

Ghaleb Ali Jowhar Halabia

Date: February 10, 2002.

## Acknowledgements

I owe thanks:

To my supervisor Dr. Yousef Bdeir for his helpful corrections and comments concerning the material of the thesis.

To Mr. Micheal Facusa for his translating the reference [6] from French to English.

To Miss. Khawlah Jamous for her typing the thesis .

And to Mohammed Hdidoun for the final montage of this thesis.

## Abstract

In this thesis, we introduce three concepts concerning the compactness in bitopological spaces, namely, semi compactness, pairwise compactness and Birsan compactness. Also, we introduce other concepts in bitopological spaces such as Hausdorffness, continuity, regularity and normality, and study their relations with compactness in bitopological spaces. We discuss generalizations for well known theorems and results concerning compactness in single topology, such as Alexander and Tychonoff theorems for bitopological spaces.

## Table of contents

Introduction .....	1
Chapter one: Semi compactness.....	4
1.1 Swart's definition for compactness in bitopological spaces.....	4
1.2 Continuity in bitopological spaces.....	9
1.3 Least upper bound topology.(l.u.b.) .....	12
1.4 Hausdorffness in bitopological spaces.....	17
1.5 Product topology in bitopological spac.....	21
Chapter two: Pairwise compactness.....	24
2.1 Pairwise compactness in bitopological spaces.....	24
2.2 Regularity and normality in bitopological spaces .....	35
2.3 Pairwise compactness of a subspace of bitopological spaces.....	37
2.4 $p$ -compactness of the product topology in bitopological spaces.....	41
Chapter three: Birsan compactness.....	51
3.1 Conversely compactness and B-compactness in bitopological spaces .....	51
3.2 Coversely compactness of sets in bitopological spaces.	61
3.3 Continuous (open) functions and conversely compactness in bitopological spaces .....	64

3.4 Alexander's and Tychonoff's theorems, and conversely compactness in bitopological spaces .....	67
References.....	70



## Introduction

In 1962, J.C. Kelly [8] has defined the concept of the bitopological space to be a non- empty set  $X$  on which two arbitrary topologies  $\tau_1$  and  $\tau_2$  are defined. This definition is denoted by the triple  $(X, \tau_1, \tau_2)$ . Since this initiation, several authors have considered the problem of defining the concept of compactness in bitopological spaces. And in this thesis we study the definitions for compactness in bitopological spaces, which were introduced by Swart in [1], Fletcher, Holye and Patty in [2], Kim in [5-1], Kim and Naimpaly in [5-2], Cooke and Reilly in [4], and Birsan in [6].

In fact, these definitions are summarized into just three definitions only, namely, semi compactness (s-compactness), pairwise compactness (p-compactness), and Birsan compactness (conversely and B-compactness).

Semi compactness is discussed in chapter one which is divided into five sections. Section one begins with Swart's definition for bitopological compactness. Also, we define quasi-open (closed) subsets of  $(X, \tau_1, \tau_2)$ , quasi-closure and some relations and results concerning them. In section two, we define four different types of continuous functions in the bitopological spaces, and deduce some useful results concerning continuity and compactness in bitopological spaces in this chapter and in the following two chapters. Least upper bound topology is introduced in section three. Also, we show that semi-compactness in bitopological space is equivalent to the compactness in least

upper bound topology. In section four we introduce two different and related definitions for Hausdorffness in bitopological spaces, and deduce many useful results and conclusions concerning Hausdorffness in bitopological spaces. In the last section we introduce the product topology and conclude a generalization of Tychonoff theorem in the single topological space.

In section one of chapter two, three different but dependent definitions of bitopological compactness, which we call each of them pairwise compactness because they are equivalent, are considered. Also we show that semi and pairwise compactness are dependent. In section two we define pairwise regularity and pairwise normality, and deduce related results between pairwise Hausdorffness and pairwise compactness. In section three, we define the notion of pairwise compactness in a subspace of a bitopological space and its relation with pairwise compactness in bitopological spaces. In section four, we discuss some definitions and theorems related to Datta [3-2], and prove a generalization of Alexander theorem in single topology, and show that a generalization of Tychonoff theorem in single topology fails for bitopological spaces, by giving a counter example.

In section one of chapter three, we introduce a new different concept for compactness in bitopological spaces. We show that this definition is independent from the two definitions, which are introduced in chapter one and chapter two. Also, we give two dependent definitions, one is called conversely compactness and the other is called B-compactness, and then shows that B-compactness implies conversely compactness. We introduce two different ways which are equivalent to conversely compactness, and deduce the relation between conversely (B-) compactness and compactness of  $\tau_1$  and  $\tau_2$ .

In fact we show that conversely (B-) compactness implies compactness of  $\tau_1$  and  $\tau_2$ . We deduce the effect of pairwise Hausdorffness in comparison of topologies. The examples at the end of this section are counter examples demonstrate the relations between conversely (B-) compactness,  $p$ -regularity and  $p$ -normality in bitopological spaces. In section two, we study the compactness in subspaces and its relations with closedness and openness. In section three, we study the effect of continuous and open functions on conversely (B-) compact bitopological spaces. Finally, in section four, we make a generalization of Alexander and Tychonoff theorems in bitopological spaces.

# Chapter One

## Semi compactness

### 1.1 Swart's definition for compactness in bitopological spaces.

#### 1.1.1 Definition [1]:

A cover  $\mathcal{U}$  of a bitopological space  $(X, \tau_1, \tau_2)$  is called  $\tau_1\tau_2$ -open if  $\mathcal{U} \subset \tau_1 \cup \tau_2$ .

#### 1.1.2 Definition [1]:

A bitopological space  $(X, \tau_1, \tau_2)$  is called semi-compact (s-compact) if every  $\tau_1\tau_2$ -open cover for  $X$  has a finite subcover.

Swart in [1] consider the above definition for compactness in bitopological spaces, and uses the term compact for s-compactness in  $(X, \tau_1, \tau_2)$ .



### 1.1.3 Definition [3-1]:

A subset  $A$  in a bitopological space  $(X, \tau_1, \tau_2)$  is said to be quasi-open if for every  $x \in A$ , there exists a  $\tau_1$ -open nhood for  $x$  (nhood stands for neighborhood)  $U_x \subset A$  or a  $\tau_2$ -open nhood for  $x$ ,  $V_x \subset A$ .

### 1.1.4 Theorem[3-1]:

Quasi-open sets in a bitopological space  $(X, \tau_1, \tau_2)$  are precisely the unions of  $\tau_1$ -open and  $\tau_2$ -open sets.

Proof:

Let  $U_1, U_2$  be two  $\tau_1$  and  $\tau_2$ -open sets respectively. Let  $x \in U_1 \cup U_2$ , then  $x \in U_1$  or  $x \in U_2$ . Either of them is contained in  $U_1 \cup U_2$ , which means that  $U_1 \cup U_2$  is quasi-open.

Conversely, assume  $A$  is quasi-open in  $(X, \tau_1, \tau_2)$ , and let  $x \in A$ . Then there is a  $\tau_1$ -open nhood  $U_1$  s.t.  $U_1 \subset A$  or  $\tau_2$ -open nhood  $U_2$  s.t.  $U_2 \subset A$ . Let

$A_1 = \{ x \in A \mid \exists U_1 \in \tau_1 \text{ such that } x \in U_1 \subset A \}$  and

$A_2 = \{ x \in A \mid \exists U_2 \in \tau_2 \text{ s.t. } x \in U_2 \subset A \}$ . Then  $A = A_1 \cup A_2$ . Now  $A_1$  is

$\tau_1$ -open. For let  $x \in A_1$ , then there exist  $U_1 \in \tau_1$  such that  $x \in U_1 \subset A$ . Let  $y \in U_1$ , then  $y \in U_1 \subset A$ , and  $U_1 \in \tau_1$ , so  $y \in A_1$ . By the Definition of  $A_1$ , we have  $U_1 \subset A_1$ . Now  $x \in U_1 \subset A_1$ ,  $U_1 \in \tau_1$ , and  $x$  was arbitrary in  $A_1$ , so  $A_1 \in \tau_1$ , similarly, we can show that  $A_2 \in \tau_2$ .

From this theorem and the definition of quasi-open sets we conclude the following results.

1.1.5 Corollary:

In a bitopological space  $(X, \tau_1, \tau_2)$ :

- a) Every  $\tau_1$ -open ( $\tau_2$ -open) set is quasi-open.
- b) Arbitrary unions of quasi-open sets is quasi-open.

The converse of (a) does not necessarily hold. For let  $\tau_1 = \{\{a\}, \{a,b\}, X, \emptyset\}$ ,  $\tau_2 = \{\{c\}, \{c,b\}, X, \emptyset\}$ ,  $X = \{a,b,c\}$ . Then  $\{a,c\}$  is quasi-open but not in  $\tau_1$  nor in  $\tau_2$ .

Proof of (a): follows from definition (1.1.3).

Proof of (b):

Let  $\{A_\alpha : \alpha \in \Delta\}$  be any collection of quasi-open sets. Then  $\forall \alpha \in \Delta$ ,

$A_\alpha = A'_\alpha \cup A''_\alpha$ ; where  $A'_\alpha \in \tau_1$  and  $A''_\alpha \in \tau_2$ . So,

$$\bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{\alpha \in \Delta} (A'_\alpha \cup A''_\alpha) = \left( \bigcup_{\alpha \in \Delta} A'_\alpha \right) \cup \left( \bigcup_{\alpha \in \Delta} A''_\alpha \right). \text{ But } \bigcup_{\alpha \in \Delta} A'_\alpha \in \tau_1$$

and  $\bigcup_{\alpha \in \Delta} A''_\alpha \in \tau_2$ , therefore  $\bigcup_{\alpha \in \Delta} A_\alpha$  is quasi-open.

Finite intersection of quasi-open sets need not be quasi-open as the following example shows.

1.1.6 Example:

Let  $X = \mathbb{R}$ ; the real line, let  $\tau_1 =$  the topology with the base

$\{[a,b]: a,b \in \mathfrak{R}\}$  and  $\tau_2 =$  the topology with the base  $\{(c,d]: c,d \in \mathfrak{R}\}$ .

Let  $a < b < c$ . Then  $(a,b)$  and  $[b,c)$  are quasi-open sets, but

$(a,b) \cap [b,c) = \{b\}$  is not a quasi-open set.

#### 1.1.7 Definition [3-1]:

A quasi-closed set is the complement of a quasi-open set.

From this definition and Thm. (1.1.4) and its results we conclude the following results.

#### 1.1.8 Corollary:

In a bitopological space  $(X, \tau_1, \tau_2)$ :

- a) Every  $\tau_1$ -closed ( $\tau_2$ -closed) set is quasi-closed.
- b) Arbitrary intersection of quasi-closed sets is quasi-closed.
- c) Finite union of quasi-closed sets need not be quasi-closed.
- d) Every quasi-closed set is the intersection of a  $\tau_1$ -closed set and a  $\tau_2$ -closed set.

Proof of (a):

Holds since every  $\tau_1$ -closed ( $\tau_2$ -closed) set is the complement of a  $\tau_1$ -open ( $\tau_2$ -open) set which is quasi-open.

Proof of (b):

Let  $F = \{F_\alpha : \alpha \in \Delta\}$  be any collection of quasi-closed sets, then  $\forall \alpha \in \Delta$ ,