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Algebraic Approach To The Fractional Derivatives

By

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Declaration

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledge, and that this thesis (or any part of the same) has not been submitted for the higher degree to any other university or institution.

Signed… ………..

Nancy Jamal Zuhair Maraqa

Date: June 23 , 2012

Dedication

To my parents , to my brothers and to my daughter Natalie.

Acknowledgment

I would like to express my thanks to those helped me to prepare and complete this work.

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Abstract

Many concepts of mathematics can be generalized. In this thesis, we discuss the generalization of the concept of derivatives to include the derivatives of fractional derivatives. Two approaches to the definition of fractional derivatives are given and proved that are equal.

We introduce an approach to the fractional derivatives of the functions using the Taylor series of analytic functions. In order to calculate the fractional derivatives of *f*, it is not sufficient to know the Taylor expansion of *f*, but we should also know the constants of all consecutive integrations of *f*. The method of calculating the fractional derivatives very often requires a summation of divergent series, and thus in this note, we first introduce a method of such summation of series via analytical continuation of functions.

A derivative of a function of order a , for any real number a (called a fractional derivative) is the subject of this thesis. Here the definition will depend on the formal power series summation. We used this definition to find the fractional derivative of the constant functions and the polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved properties of the fractional derivative. And we proved that the fractional derivative of order $a \in R$ of the exponential function e^x is the exponential function e^x and this help us in finding the fractional derivatives of the trigonometric functions and hyperbolic functions.

Finally, we introduce a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives.

الخلاصة

موضوع هذا البحث هو مشتقة الاقتران من الدرجة a ، حيث أن a عدد حقيقي ليس بالضرورة أن يكون عدداً صحيحاً موجباً. لقد قمنا بهذا البحث بتقديم تعريفين للمشتقة الكسرية؛ الأول باستخدام ريمان – لوفل، والثاني بالاعتماد على متسلسلة القوة للاقتران.

وفي هذا السياق واعتماداً على التعريف الثاني قمنا بحساب المشتقة الكسرية للاقترانات الثابتة والاقترانات كثيرة الحدود، حيث كانت النتائج مطابقة للنتائج الناتجة عن تطبيق ما هو معروف إلى الآن من تعريفات المشتقة الكسرية. إضافة إلى ذلك؛ قمنا بإثبات بعض خصائص المشتقة الكسرية.

بالإضافة إلى ذلك أثبتنا أن المشتقة الكسرية للاقتران الأسي e^x هو الاقتران الأسي e^x نفسه، وهذا يساعدنا في الحصول على المشتقة الكسرية للاقترانات المثلثية والاقترانات الزائدية.

إن أهمية هذا البحث تكمن في إيجاد آلية سهلة التطبيق للإيجاد المشتقات الكسرية للاقترانات، وحيث أن كل اقتران قابل للاشتقاق يمكن كتابته على طريقة متسلسلة القوى بغض النظر عن التقارب أو عدمه. فإننا نرى أن هذه الطريقة يمكن لها إعطائنا نتائج طيبة في هذا المجال والذي بدوره سوف ينعكس على التطبيقات العملية الكثيرة.

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Introduction

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this mathematical definition can clearly become confused when considering exponents of non integer value. While almost anyone can verify that $x^3 = x.x.x$, how might one describe the mathematical meaning of $x^{3/4}$, or moreover the transcendental exponent *x¼*. One cannot conceive what it might be like to multiply a number or quantity by itself 3.4 times, or ¼ times, and yet these expressions have a definite value for any value *x*, verifiable by infinite series expansion, or more practically, by calculator.

Now, in the same way consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g. function continuity), completing *n* integrations can become as methodical as multiplication.

But the curious mind can not be restrained from asking the question what if *n* were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted, but as this report will show, fractional calculus flows quite naturally from our traditional definitions. And just as fractional exponents such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order $\frac{1}{2}$ and beyond can find practical use in many modern problems.

The fractional derivative is natural a natural extension of the familiar derivative $\frac{d^n f(x)}{dx^n}$ where n=0,1,2,…to arbitrary number α ((integral, rational, irrational or complex)). Fractional differentiation is of use in the solution of ordinary, partial, and integral equations as well as in the contexts, a few of which are indicated in the bibliography although other methods of solution are available, the fractional derivative approach to these problems often suggests methods that are not obvious in a classical formulation. The fractional calculus forms a special chapter in the non-general "Operation Calculus" which considers functions of the differential operator "D" more general that D^{α}.

Fractional Calculus is the branch of calculus that generalizes the derivative of function to non-integer order allowing calculations such as deriving a function to $1/2$ order despite generalized would be a better option, the name "Fractional" is used for denoting this kind of derivative, see [1].

The simplest approaches to the definition of fractional differentiation begin by looking at a few well-known functions, and try to find various derivatives by means of an intuitive approach. We will be making use of the usual notation for derivatives, see [2], and we get the following:

$$D^n f(x) = \lim_{h \to o} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x + (n-m)h)$$

Differentiation and integration are usually regarded as discrete operations, in the sense that we differentiate or integrate a function once, twice, or any whole number of times. However, in some circumstances it's useful to evaluate a *fractional derivative*. In a letter to L'Hospital in orders, and L'Hospital asked what would be the result of half-differentiating x. Leibniz replied The paradoxical "It leads to a paradox, from which one day useful consequences will be drawn".

The idea of generalizing the concepts of differentiation and integration to non-integer (fractional) orders has a long mathematical history. It was first discussed in the correspondence of G.W. Leibniz around 1690. Over the centuries many famous mathematicians including Euler, Riemann, Liouville and Weyl have built up a body of mathematical knowledge on fractional integrals and derivatives that is known under the name of fractional calculus.

In chapter (2) an approach to the fractional derivative of order $\alpha \in R$ of a function is given. This definition will depend on the formal power series summation. We used this new definition to find the fractional derivative of the constant functions and polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved properties of the fractional derivative. Finally, a proof of the well known fact that fractional derivative of $e^{\lambda x}$ of order $\alpha \in R$ Is equal to $\lambda^{\alpha} e^{\lambda x}$. Also, we proved that $sin^{(\alpha)}(x) = \sin(x + \alpha\pi/2)$ and $cos^{(\alpha)}(x) = \cos(x + \alpha\pi/2)$.

In chapter (3) we introduce an alternative definition of the fractional derivatives and also a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives (also integrals). Further are found the expansions of the functions $\frac{xe^x}{e^x-1}, \frac{1}{\cos(x)}, x \tanh x$, and some other functions of the form $\sum_{k=-\infty}^{\infty} a_k \frac{x^k}{k!}$, which enables us to calculate any fractional derivative of these functions at $x = 0$. These calculations lead to representations of the Bernoulli and Euler numbers B_k and E_k for any complex number k via fractional derivatives of some functions at $x = 0$.

Chapter 1

Riemann-Liouville Operator

The concept of non-integral order of integration can be traced back to the genesis of differential calculus itself: the philosopher and creator of modern calculus G.W. Leibniz made some remarks on the meaning and possibility of fractional derivative of order $\alpha \in R$ in the late 17:th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since.

Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of the time; it has been said to have lead him to construct the Gamma function for fractional powers of the factorial [2, p. 243]. An early attempt by Liouville was later purified by the Swedish mathematician Holmgren, who in 1865 made important contributions to the growing study of fractional calculus. But it was Riemann [4] who reconstructed it to fit Abel's integral equation, and thus made it vastly more useful. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types [1, p. xxxi], but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed.

1.1 The Gamma function:

As will be clear later, the gamma function is intrinsically tied to fractional calculus. The simple interpretation of the gamma function is simply the generalization of the factorial for all real numbers. The definition of the gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-u}\, u^{z-1} du\,, \qquad \text{for all } z \in R \qquad (1)$$

The beauty of the gamma function can be found in its properties. First as seen in (2), this function is unique in that the value for any quantity is , by consequence of the form of the integral, equivalent to that quantity z minus one times the gamma of the quantity minus one,

$$\Gamma(z+1) = z\,\Gamma(z), \quad \text{also, when } z \in N^+, \quad \Gamma(z) = (z-1)! \qquad (2)$$

This can be shown through a simple integration by parts. The consequence of this relation for integer values of z is the definition for factorial. Note that at negative integer values, the gamma function goes to infinity, yet is defined at non-integer values.

Now, we give an example.

Example 1.1.1

Using equation (2), then we get

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

1.2 Beta function:

Also known as the Euler Integral of the First Kind, the Beta Function is in important relationship in fractional calculus. Equation (3) demonstrates the Beta Integral and its solution in terms of the Gamma function.

$\mathrm{B}(p,q) := \int_0^1 (1-u)^{p-1}\, u^{q-1} du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(q,p)$, where $p, q \in R^+$ (3)

Now, we give an example.

Example 1.2.1

Using equation (3), then we get

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi\,.$$

1.3 Riemann-Liouville integral:

The fractional derivative of order $\alpha \in R$ of a function f is

$$\frac{d^\alpha f(x)}{dx^\alpha} = f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha+1}}\,dt$$

Where $\Gamma(n)$ is the Euler's Gamma function.

Now, we give an example.

Example 1.3.1

The $(1/2)$ th derivatives of the functions $f(x) = x$ and $g(x) = \sqrt{x}$ are

$$f^{(1/2)}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} \qquad \text{and} \qquad g^{(1/2)}(x) = \frac{\sqrt{\pi}}{2}$$

Solution:

Using this definition, the $(1/2)$ th derivative of the function $f(x) = x$, is given by

$$f^{(1/2)}(\mathrm{x}) = \frac{1}{\Gamma(-(1/2))}\int_0^{\mathrm{x}} \frac{\mathrm{t}}{(\mathrm{x}-\mathrm{t})^{\frac{1}{2}+1}}\mathrm{dt}$$

$$= \frac{\sqrt{x}}{\Gamma(-(1/2)}\int_0^1 \mathrm{u}(1-\mathrm{u})^{-\frac{1}{2}-1}\mathrm{du}$$

$$= \frac{\sqrt{x}}{\Gamma(-(1/2))}\frac{\Gamma(2)\Gamma(-(1/2))}{\Gamma((3/2))}$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

Also, using the same definition, the $(1/2)$ th derivatives of the functions $g(x) = \sqrt{x}$ is given by

$$g^{(1/2)}(x) = \frac{1}{\Gamma(-(1/2))}\int_0^{x} \frac{\sqrt{t}}{(x-t)^{\frac{1}{2}+1}}dt$$

$$= \frac{1}{\Gamma(-(1/2))}\int_0^1 \sqrt{\mathrm{u}}(1-\mathrm{u})^{-\frac{1}{2}-1}\mathrm{du}$$

$$= \frac{1}{\Gamma(-(1/2))}\frac{\Gamma(3/2)\Gamma(-(1/2))}{\Gamma(1)}$$

$$= \frac{\sqrt{\pi}}{2}. \qquad \blacksquare$$

Now we will give several properties of the fractional derivatives.

Theorem 1.3.2 :[4]

If f(x) = c, where c is constant, then $f^{(\alpha)}(x) = \frac{c}{\Gamma(1-\alpha)}x^{-\alpha}$

Proof:

Using the definition, the α th derivative of the function is

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{c}{(x-t)^{\alpha+1}} dt$$

And so,

$$\begin{aligned} f^{(\alpha)}(x) &= \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{c}{(x-t)^{\alpha+1}} dt \\ &= \frac{c}{x^{\alpha}\,\Gamma(-\alpha)} \int_0^1 (1-u)^{-\alpha-1} du \\ &= \frac{c}{x^{\alpha}\,\Gamma(-\alpha)} \frac{\Gamma(1)\Gamma(-\alpha)}{\Gamma(1-\alpha)} \\ &= \frac{c}{\Gamma(1-\alpha)} x^{-\alpha}. \end{aligned}$$

Theorem 1.3.3 :[4]

If $f(x) = x^n$, $n \in Z$, then $f^{(\alpha)}(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$

Proof:

Using the definition, the α th derivative of the function is

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha+1}} dt$$

And so

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{t^n}{(x-t)^{\alpha+1}} dt$$

$$= \frac{x^{n-\alpha}}{\Gamma(-\alpha)} \int_0^1 u^n (1-u)^{-\alpha-1} du$$

$$= \frac{x^{n-\alpha}}{\Gamma(-\alpha)} \frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} \qquad \blacksquare$$

Now, we give an example.

Example 1.3.4 :[4]

The $(-1/2)$ th derivative of the function $f(x) = x^2$ is

$$f^{(-1/2)}(x) = \frac{16x^{5/2}}{15\sqrt{\pi}}$$

Solution:

Using the definition, the $(-1/2)$ th derivative of the function is

$$f^{(-1/2)}(\mathrm{x}) = \frac{1}{\Gamma(1/2)} \int_0^{\mathrm{x}} \frac{\mathrm{t}^2}{(\mathrm{x}-\mathrm{t})^{-\frac{1}{2}+1}} \mathrm{dt}$$

$$= \frac{x^{5/2}}{\Gamma(1/2)} \int_0^1 \mathrm{u}^2 (1-\mathrm{u})^{\frac{1}{2}-1} \mathrm{du}$$

$$= \frac{x^{5/2}}{\Gamma(1/2)} \frac{\Gamma(3)\Gamma(1/2)}{\Gamma(7/2)}$$

$$= \frac{16x^{5/2}}{15\sqrt{\pi}}$$

Now, assuming a function $f(x)$ that is defined for $x > 0$, from the definite integral from 0 to x, call this

$$(Jf)(x) = \int_0^x f(t)dt.$$

Repeating this process gives

$$(J^2 f)(x) = \int_0^x (Jf)(t)dt = \int_0^x \left(\int_0^t f(s)ds\right)dt,$$

and this can be extended arbitrarily.

The Cauchy formula for repeated integration, namely

$$(J^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt$$

leads to a straightforward way to a generalization for real n.

Simply, using the Gamma function to remove the discrete nature of the factorial function (recalling that $\Gamma(n+1) = n!$, or equivalently $\Gamma(n) = (n-1)!$) gives us a natural candidate for fractional applications of the integral operator.

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt$$

This is in fact a well-defined operator. It can be shown that the J operator satisfies

$$(J^\alpha)(J^\beta)f = (J^\beta)(J^\alpha)f = (J^{\alpha+\beta})f = \frac{1}{\Gamma(\alpha+\beta)}\int_0^x (x-t)^{\alpha+\beta-1} f(t)dt$$

Finally, it is clearly that if $\alpha > 0$, then

$$D^{-\alpha}(f(x)) = J^\alpha(f(x))$$

Now, we give an example.

Example 1.3.5

The $(1/2)$ th integral of the function $f(x) = x^2$ is given by

$$J^{(1/2)}(f(x)) = \frac{16x^{5/2}}{15\sqrt{\pi}}$$

Solution:

Using the definition, the $(1/2)$ th integral of the function is

$$\left(J^{\frac{1}{2}}(f(x)\right) = \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2} f(t)dt$$

and so,

$$J^{(1/2)}(f(x)) = D^{(-1/2)}(f(x))$$

using example 1.4, then the $(-1/2)$ th derivative of the function $f(x) = x^2$ is

$$f^{(-1/2)}(x) = \frac{16x^{5/2}}{15\sqrt{\pi}}$$

Therefore,

$$J^{(1/2)}(f(x)) = \frac{16x^{5/2}}{15\sqrt{\pi}}$$

Chapter 2

Series approach to the fractional derivatives

We introduce an approach to the fractional derivatives of the analytical functions using the Taylor series of the functions. In order to calculate the fractional derivatives of f, it is not sufficient to know the Taylor expansion of f, but we should also know the constants of all consecutive integrations of f. For example, any fractional derivative of e^x is e^x only if we assume that the nth consecutive integral of e^x is e^x for each positive integer n. The method of calculating the fractional derivatives very often requires a summation of divergent series, and thus, in this note, we first introduce a method of such summation of series via analytical continuation of functions.

2.1 Power series:

In this section, quite a strong method of summation is introduced, which considers a large class of series. It is used in 2.2 in order to effectively calculate the fractional derivatives of a given function. But first we recall some facts about the power series.

Definition 2.1.1

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is called a power series in x about c.

Now, we give an example.

Example 2.1.2

The series $\sum_{n=0}^{\infty} x^n$ is a power series in x about 0. In fact, we have seen that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } -1 < x < 1$$

Definition 2.1.3

If f is a function for which there exists constants $a_0, a_1, a_2, \ldots$ such that $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$,for all values of x in some open interval about c, then we say f is analytic at c. If for some $h > 0$ the equality holds for all x in the interval $I = (c-h, c+h)$, then we say f is analytic on I and we call $\sum_{n=0}^{\infty} a_n(x-c)^n$ a power series representation of f on .

Proposition 2.1.4 :[5]

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ on $(c-R, c+R)$, where $R > 0$ is the radius of convergence of the power series. Then $a_n = \frac{f^{(n)}(c)}{n!}$ for $n = 0,1,2,\dots$

Proof:

Since $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

Then,

$$f(c) = a_0 + a_1(c-c) + a_2(c-c)^2 + \cdots = a_0$$

So $a_0 = f(c)$

Next,

$$f'(c) = \sum_{n=1}^{\infty} n a_n (c-c)^{n-1} = a_1$$

So $a_1 = f'(c)$

For a_2 we have

$$f''(c) = \sum_{n=2}^{\infty} n(n-1) a_n (c-c)^{n-2} = 2a_2$$

So $a_2 = \frac{f''(c)}{2}$

In general, for $k = 0,1,2,\dots,$

$$f^{(k)}(c) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (c-c)^{n-k} = k!\, a_k,$$

from which it follows that $a_k = \frac{f^{(k)}(c)}{k!}$

Therefore, $a_n = \frac{f^{(k)}(c)}{n!}$, for $n = 0,1,2,\dots$ ■

2.2 A new approach to fractional derivatives

In this section we will consider summation of divergent series for calculating of fractional derivatives of order $\alpha \in R$ of several functions.

Discussion of the method:

Let $\sum_{i=0}^{+\infty} b_i$ be a given series. Consider the formal power series $\sum_{i=0}^{+\infty} b_i x^i$ and look for a differential equation which it satisfies, even if the radius of convergence of the power series is 0. If f is the solution of the corresponding differential equation, then we take $f(x) = \sum_{i=0}^{+\infty} b_i x^i$ for each x. Set $f(x) = \sum_{i=0}^{+\infty} a_i \frac{x^i}{i!}$ and define $\frac{x^i}{i!} = 0$ for each x and for $i = -1, -2, \ldots$, then we can write f in the form

$$f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$$

Definition 2.2.1

If $f(x) = \sum_{-\infty}^{+\infty} \frac{a_i x^i}{i!}$ for each *x*. Then, for any $\alpha \in R$, the fractional derivative of order $\alpha \in R$ of afunction f is defined to be

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha}}{(i-\alpha)!}.$$

Notes:

i) For each $x > 0, x! = \Gamma(x+1)$.

(ii)$(-\alpha)! = \frac{\Gamma(-\alpha+m)}{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+m-1)}$, $\quad m-1 < \alpha < m$, m is a positive integer.

iii) for each $x \neq 0, \frac{x^{i-\alpha}}{(i-\alpha)!} \neq 0$ if α is non-integer number.

The interpretation of the coefficients $a_{-i}, i \in Z^+$ is the following. The coefficient a_{-1} is equal to $g(0)$, where $g'(x) = f(x)$, i.e. it is the integral constant of $\int f\,dx$. Similarly, a_{-2} is equal to the integral constant of $\int(\int f\,dx)dx$, and so on.

Using this new approach we can prove the following theorem that verifies some formulas for fractional derivatives of order $\alpha \in R$ of several functions.

Theorem 2.2.2 :[6]

i) If $f(x) = c$, where c is constant, then $f^{(\alpha)}(x) = c\frac{x^{-\alpha}}{(-\alpha)!} = \frac{c}{\Gamma(1-\alpha)} x^{-\alpha}$.

ii) If $f(x) = x^n, n \in Z$, then $f^{(\alpha)}(x) = \frac{n! x^{n-\alpha}}{(n-\alpha)!} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$.

Proof:

(i) Write $f(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!}$, where $a_i = 0, \forall i \neq 0$ and $a_0 = c$. Applying Definition 2.2.1 to the function f, we get

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha}}{(i-\alpha)!} = \frac{a_0 x^{0-\alpha}}{(0-\alpha)!} = c\frac{x^{-\alpha}}{(-\alpha)!} = c\frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

(ii) Write $f(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!}$,where $a_i = 0, \forall i \neq n$ and $a_n = n!$. Applying Definition 2.2.1 to the function f, we get

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha}}{(i-\alpha)!} = \frac{a_n x^{n-\alpha}}{(n-\alpha)!} = \frac{n!\, x^{n-\alpha}}{(n-\alpha)!} = \frac{\Gamma(1+n)}{\Gamma(n-\alpha+1)} x^{n-\alpha}. \quad \blacksquare$$

2.3 Some Properties of fractional derivatives

We are now in a position to discuss some properties of fractional derivatives .We will show that fractional derivatives have several properties that one would expect, such as the fractional derivative operator is linear and repeated fractional differentiation is accumulative.

Theorem 2.3.1 :[6]

If $f(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!}$ and $g(x) = \sum_{i=-\infty}^{+\infty} \frac{b_i x^i}{i!}$ for each x.Then for any $\alpha, \beta, c \in R$,

(i) $\frac{d^\alpha}{dx^\alpha}\big(f(x) + g(x)\big) = f^{(\alpha)}(x) + g^{(\alpha)}(x)$.

(ii) $\frac{d^\alpha}{dx^\alpha}\left(\frac{d^\beta(f(x))}{dx^\beta}\right) = \frac{d^{\alpha+\beta}(f(x))}{dx^{\alpha+\beta}}$.

(iii) $\frac{d^\alpha}{dx^\alpha}\big(cf(x)\big) = cf^{(\alpha)}(x)$.

Proof:

(i) Since $f(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!}$ and $g(x) = \sum_{i=-\infty}^{+\infty} \frac{b_i x^i}{i!}$.Then

$$f(x) + g(x) = \sum_{i=-\infty}^{+\infty} \frac{(a_i + b_i)x^i}{i!}$$

Applying Definition 2.2.1 to the function $f(x) + g(x)$, we get

$$\left(f(x)+g(x)\right)^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} (a_i + b_i)\frac{x^{i-\alpha}}{(i-\alpha)!}$$

$$= \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha}}{(i-\alpha)!} + \sum_{i=-\infty}^{+\infty} \frac{b_i x^{i-\alpha}}{(i-\alpha)!}$$

$$= f^{(\alpha)}(x) + g^{(\alpha)}(x)$$

(ii) Since $f(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!}$, applying Definition 2.2.1 to the function $f(x)$, we get

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha}}{(i-\alpha)!}$$

Applying Definition 2.2.1 to the function $f^{(\alpha)}(x)$, we get

$$\frac{d^\beta}{dx^\beta}\left(f^{(\alpha)}(x)\right) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha-\beta}}{((i-\alpha)-\beta)!}$$

But

$$\frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}\left(f(x)\right) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-(\alpha+\beta)}}{((i-(\alpha+\beta))!} = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha-\beta}}{(i-\alpha-\beta)!}$$

Therefore,

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{d^\beta\left(f(x)\right)}{dx^\beta}\right) = \frac{d^{\alpha+\beta}(f(x))}{dx^{\alpha+\beta}}$$

(iii) Since $f(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!}$, then $cf(x) = \sum_{i=-\infty}^{+\infty} \frac{ca_i x^i}{i!}$. Applying Definition 2.2.1 to the function $cf(x)$, we get

$$(cf)^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} \frac{ca_i x^{i-\alpha}}{(i-\alpha)!} = c\sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-\alpha}}{(i-\alpha)!} = cf^{(\alpha)}(x). \quad \blacksquare$$

Note that for any positive integer n ,this definition agrees with the traditional definition of the derivative as the following examples show:

(1) $\frac{d^n}{dx^n}(c) = \frac{cx^{-n}}{\Gamma(1-n)} = \frac{cx^{-n}}{(-n)!} = 0$

(2) $\frac{d^3}{dx^3}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-3+1)}x^{n-3} = \frac{n!}{(n-2)!}x^{n-3} = n(n-1)(n-2)x^{n-3}$

Also, if $\alpha = 0$ we have $\sum_{i=-\infty}^{+\infty} \frac{a_i x^{i-0}}{(i-0)!} = \sum_{i=-\infty}^{+\infty} \frac{a_i x^i}{i!} = f(x)$.

Now the question is the case n is a negative derivative integer? Let $n = -1$, then

$$f^{(-1)}(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i+1}}{(i+1)!} = \cdots + a_{-2}\frac{x^{-1}}{(-1)!} + a_{-1}\frac{x^0}{0!} + a_1\frac{x^2}{2!} + \ldots$$

$$= a_{-1}\frac{x^0}{0!} + a_0\frac{x}{1!} + a_1\frac{x^2}{2!} + a_2\frac{x^3}{3!} + \cdots$$

$$= a_{-1} + \int_0^x f(t)dt = \int f(x)dx$$

In general, if n is a positive integer, then

$$f^{(-n)}(x) = \sum_{i=-\infty}^{+\infty} \frac{a_i x^{i+n}}{(i+n)!} = \cdots + a_{-n-1}\frac{x^{-1}}{(-1)!} + a_{-n}\frac{x^0}{0!} + a_{-n+1}\frac{x}{1!} + a_{-n+2}\frac{x^2}{2!} + \cdots$$

$$= a_{-n}\frac{x^0}{0!} + a_{-n+1}\frac{x}{1!} + a_{-n+2}\frac{x^2}{2!} + \cdots + a_0\frac{x^n}{n!} + \cdots$$

$$= a_{-n} + a_{-n+1}\frac{x}{1!} + a_{-n+2}\frac{x^2}{2!} + \cdots + a_{-1}\frac{x^{n-1}}{(n-1)!} + \int_0^x\int_0^x \ldots \int_0^x f(t)dt,$$

n-times integrations.

Hence, $f^{(-n)}(x) = \iint \ldots \int f(x)dx$, n-times integrations.

Now, if we take $f(x) = e^x$,then for any positive integer n, we have

$$f^{(n)}(x) = e^x \qquad \text{and} \qquad f^{(-n)}(x) = \iint \ldots \int e^x dx$$

Theorem 2.3.2 :[6]

If $g(x) = f(\lambda x)$, then $g^{(\alpha)}(x) = \lambda^{\alpha} f^{(\alpha)}(\lambda x)$

Proof:

It is sufficient to prove the theorem for $\alpha \notin Z$, because for $\alpha \in Z$, theorem is obvious. So, assume that $\alpha \notin Z$.

Write $f(x) = \sum_{n=-\infty}^{+\infty} a_n \frac{x^n}{n!}$,then

$$g(x) = f(\lambda x) = \sum_{n=-\infty}^{+\infty} a_n \frac{\lambda^n x^n}{n!}$$

Then

$$g^{(\alpha)}(x) = \sum_{n=-\infty}^{+\infty} a_n \frac{\lambda^n x^{n-\alpha}}{(n-\alpha)!}$$

But

$$f^{(\alpha)}(x) = \sum_{n=-\infty}^{+\infty} a_n \frac{x^{n-\alpha}}{(n-\alpha)!}$$

So

$$f^{(\alpha)}(\lambda x) = \sum_{n=-\infty}^{+\infty} a_n \frac{(\lambda x)^{n-\alpha}}{(n-\alpha)!} = \sum_{n=-\infty}^{+\infty} a_n \frac{\lambda^{n-\alpha} x^{n-\alpha}}{(n-\alpha)!}$$

$$= \lambda^{-\alpha} \sum_{n=-\infty}^{\infty} a_n \frac{\lambda^n x^{n-\alpha}}{(n-\alpha)!} = \lambda^{-\alpha} g^{(\alpha)}(x)$$

Therefore, $g^{(\alpha)}(x) = \lambda^{\alpha} f^{(\alpha)}(\lambda x)$. ■

2.4 New proofs of fractional derivatives of the exponential and trigonometric functions:

In this section we will prove that the fractional derivatives of order $\alpha \in R$, of the exponential function e^x is the exponential function e^x.

To do this, we need the following lemmas:

Lemma 2.4.1 :[6]

For any $\alpha > -1$, if

$$k(x) = -\alpha x^{1+\alpha} + \alpha(\alpha+1)x^{2+\alpha} - \alpha(\alpha+1)(\alpha+2)x^{3+\alpha} + \cdots$$

Then

$$k(1) = -e\alpha \int_0^1 e^{-1/t} t^{\alpha-1} dt.$$

Proof:

The function k satisfies the following differential equation

$$k'(x) = -\alpha(1+\alpha)x^{\alpha} + \alpha(\alpha+1)(\alpha+2)x^{1+\alpha} - \alpha(\alpha+1)(\alpha+2)(\alpha+3)x^{2+\alpha} + \cdots$$

$$= -\frac{k(x)+\alpha x^{1+\alpha}}{x^2} = -\frac{k(x)}{x^2} - \alpha x^{\alpha-1}$$

i.e. ,

$$y' + \frac{1}{x^2} y = -\alpha x^{\alpha-1}, \qquad y = k(x)$$

Also, $k(0) = 0$ because $\alpha > -1$. Hence the solution of this differential equation is the following function

$$k(x) = -\alpha e^{1/x} \int_0^x e^{-1/t} t^{\alpha-1} dt$$

So

$$k(1) = -e\alpha \int_0^1 e^{-1/t} t^{\alpha-1} dt. \qquad \blacksquare$$

Lemma 2.4.2 :[6]

For any $\alpha > -1$,

$$\int_0^1 e^{-t}\, t^{-\alpha} dt = e^{-1} \left(\frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots\right)$$

Proof:

Since $t^{-\alpha} e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n-\alpha}}{n!}$, then

(1) $$\int_0^1 e^{-t}\, t^{-\alpha} dt = \int_0^1 \left(\sum_{n=0}^{+\infty} (-1)^n \frac{t^{n-\alpha}}{n!}\right) dt$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \int_0^1 t^{n-\alpha} dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \frac{1}{n-\alpha+1}$$

i.e.

(2) $$\int_0^1 e^{-t}\, t^{-\alpha} dt = \frac{1}{1-\alpha} - \frac{1}{1!}\frac{1}{2-\alpha} + \frac{1}{2!}\frac{1}{3-\alpha} - \frac{1}{3!}\frac{1}{4-\alpha} + \cdots$$

But in [7], p.238, is proved the following identity

$$(3)\qquad \frac{1}{\Gamma(x+1)}+\frac{1}{\Gamma(x+2)}+\cdots = \frac{e}{\Gamma(x)}\left(\frac{1}{x}-\frac{1}{1!}\frac{1}{x+1}+\frac{1}{2!}\frac{1}{x+2}-\frac{1}{3!}\frac{1}{x+3}+\cdots\right)$$

By taking $x = 1-\alpha$ in (3) we have

$$(4)\qquad \frac{1}{\Gamma(2-\alpha)}+\frac{1}{\Gamma(3-\alpha)}+\cdots = \frac{e}{\Gamma(1-\alpha)}\left(\frac{1}{1-\alpha}-\frac{1}{1!}\frac{1}{2-\alpha}+\frac{1}{2!}\frac{1}{3-\alpha}-\frac{1}{3!}\frac{1}{4-\alpha}+\cdots\right)$$

From (2) and (4), we have

$$\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}+\frac{\Gamma(1-\alpha)}{\Gamma(3-\alpha)}+\cdots = e\int_0^1 e^{-t}t^{-\alpha}dt$$

Thus

$$\int_0^1 e^{-t}t^{-\alpha}dt = e^{-1}\left(\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)}+\cdots\right). \qquad \blacksquare$$

Lemma 2.4.3 :[6]

For any $\alpha > 0$,

$$(-\alpha)! = -\alpha\int_0^1 e^{-1/t}t^{\alpha-1}dt + e^{-1}\left(1+\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)}+\cdots\right)$$

Proof:

Using the definition of Gamma function, we have

$$(-\alpha)! = \Gamma(-\alpha+1) = \int_0^\infty e^{-t}t^{-\alpha}dt = \int_0^1 e^{-t}t^{-\alpha}dt + \int_1^\infty e^{-t}t^{-\alpha}dt$$

For the first integral Lemma 2.4. 2 implies

$$\int_0^1 e^{-t}\,t^{-\alpha}dt = e^{-1}\left(\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)}+\cdots\right) \qquad (1)$$

For the second integral, using repeated integration by parts we have:

$$\int_{1}^{+\infty} e^{-t}\, t^{-\alpha} dt = e^{-1}(1 - \alpha + \alpha(\alpha+1) - \cdots + (-1)^{n}\alpha(\alpha+1)\ldots(\alpha+n-1))$$

$$+(-1)^{n+1}\alpha(\alpha+1)\ldots(\alpha+n)\int_{1}^{\infty} \frac{e^{-t}}{t^{\alpha+n+1}} dt \qquad (2)$$

but

$$\int_{0}^{1} e^{-1/t} t^{\alpha-1} dt = \int_{1}^{\infty} e^{-t} t^{\alpha-1} dt \qquad (3)$$

since $\alpha + 1 > 0$ when $\alpha > -1$, then

$$\int_{0}^{1} e^{-1/t}\, t^{\alpha-1} dt = \int_{1}^{\infty} e^{-t}\, t^{-(\alpha+1)} dt$$

$$= e^{-1}(1 - (\alpha+1) + (\alpha+1)(\alpha+2) - \cdots$$

$$+(-1)^{n-1}(\alpha+1)(\alpha+2)\ldots(\alpha+n-1))$$

$$+(-1)^{n}(\alpha+1)(\alpha+2)\ldots(\alpha+n)\int_{1}^{+\infty} \frac{e^{-t}}{t^{\alpha+n+1}} dt$$

therefore,

$$(-\alpha e\int_{0}^{1} e^{-1/t} t^{\alpha-1} dt) + 1 = e\int_{1}^{\infty} e^{-t}\, t^{-\alpha} dt$$

Now, from (1),(2) and (3) we have:

$$(-\alpha)! = e^{-1}\left(\frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \cdots\right) + (-\alpha\int_{0}^{1} e^{-1/t} t^{\alpha-1} dt) + e^{-1}$$

thus

$$((-\alpha)!\, e + \alpha\, e\int_{0}^{1} e^{-1/t} t^{\alpha-1} dt) = 1 + \frac{1}{1-\alpha} + \frac{1}{(1-\alpha)(2-\alpha)} + \cdots$$

hence,

$$(-\alpha)! = -\alpha \int_{0}^{1} e^{-1/t}\, t^{\alpha-1} dt + e^{-1}\left(1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \cdots\right) \qquad \blacksquare$$

Theorem 2.4.4 :[6]

For any $\alpha \in R, \frac{d^{(\alpha)}}{dx^{(\alpha)}}(e^x) = e^x$.

Proof:

It is sufficient to prove the theorem for $\alpha \notin Z$, because for $\alpha \in Z$, the theorem is obvious. So, assume that $\alpha \notin Z$.

Using the expansion

$$e^x = \sum_{i=-\infty}^{+\infty} \frac{x^i}{i!} = \cdots + \frac{x^{-2}}{(-2)!} + \frac{x^{-1}}{(-1)!} + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$$

Applying Definition 2.2.1, we have

$$\frac{d^{(\alpha)}}{dx^{(\alpha)}}(e^x) = \sum_{i=-\infty}^{+\infty} \frac{x^{i-\alpha}}{(i-\alpha)!}$$

$$= \cdots + \frac{x^{-2-\alpha}}{(-2-\alpha)!} + \frac{x^{-1-\alpha}}{(-1-\alpha)!} + \frac{x^{-\alpha}}{(-\alpha)!} + \frac{x^{1-\alpha}}{(1-\alpha)!} + \frac{x^{2-\alpha}}{(2-\alpha)!} + \cdots$$

and hence the theorem will be proved if we prove the identity

$$\ldots + \frac{x^{-2-\alpha}}{(-2-\alpha)!} + \frac{x^{-1-\alpha}}{(-1-\alpha)!} + \frac{x^{-\alpha}}{(-\alpha)!} + \frac{x^{1-\alpha}}{(1-\alpha)!} + \frac{x^{2-\alpha}}{(2-\alpha)!} + \cdots = e^x$$

Let

$$g(x) = \cdots + \frac{x^{-2-\alpha}}{(-2-\alpha)!} + \frac{x^{-1-\alpha}}{(-1-\alpha)!} + \frac{x^{-\alpha}}{(-\alpha)!} + \frac{x^{1-\alpha}}{(1-\alpha)!} + \frac{x^{2-\alpha}}{(2-\alpha)!} + \cdots$$

Multiplying $g(x)$ by $(-\alpha)!$ we get

$$(-\alpha)!\, g(x) = \cdots + \frac{(-\alpha)!\, x^{-2-\alpha}}{(-2-\alpha)!} + \frac{(-\alpha)!\, x^{-1-\alpha}}{(-1-\alpha)!}$$

$$+ \frac{(-\alpha)!\, x^{-\alpha}}{(-\alpha)!} + \frac{(-\alpha)!\, x^{1-\alpha}}{(1-\alpha)!} + \frac{(-\alpha)!\, x^{2-\alpha}}{(2-\alpha)!} + \cdots$$

And using the identity

$$(x+n)! = (x+n)(x+n-1)\ldots(x+1)x!$$

For any x and positive integer n, we obtain the following equivalent equality

$$(-\alpha)!\,g(x) = \cdots - x^{-3-\alpha}\alpha(\alpha+1)(\alpha+2) + x^{-2-\alpha}\alpha(\alpha+1) - x^{-1-\alpha}\alpha$$

$$+x^{-\alpha} + \frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{2-\alpha}}{(2-\alpha)(1-\alpha)} + \frac{x^{3-\alpha}}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots$$

Now, let $h(x)$ denote the right side of the previous equality. Clearly it satisfies $h'(x) = h(x)$, then $h(x) = ce^x$, where c is a constant. Hence it is sufficient to prove that $c = (-\alpha)!$. Now, let $x = 1$ then

$$(-\alpha)!\,g(1) = (\ldots - \alpha(\alpha+1)(\alpha+2) + \alpha(\alpha+1) - \alpha)$$

$$+(1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots)$$

We will consider two cases:

Case 1: let $\alpha > -1$.Then the series

$$1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots$$

Is convergent for any $\alpha \notin \{1,2,3,\ldots\}$, and by Lemma 2.4.1**,** we have

$$-\alpha + \alpha(\alpha+1) - \alpha(\alpha+1)(\alpha+2) + \cdots = k(1) = -e\alpha\int_0^1 e^{-1/t}t^{\alpha-1}dt$$

Thus,

$$(-\alpha)!\,g(1) = -e\alpha\int_0^1 e^{-1/t}t^{\alpha-1}dt$$

$$+(1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots)$$

Using Lemma 2.4.3**,** we get

$$(-\alpha)!\,g(1) = (-\alpha)!\,e$$

But, $(-\alpha)!\,g(1) = h(1) = ce = (-\alpha)!\,e$ which implies $c = (-\alpha)!$.

Now, suppose that $\alpha \le 1$. Let k be any integer smaller than $\alpha + 1$. Then

$\alpha - k \ge -1$ and using the fact that

$$\frac{d^\alpha}{dx^\alpha} \circ \frac{d^\beta}{dx^\beta}\big(f(x)\big) = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}\big(f(x)\big),$$

We obtain

$$\frac{d^\alpha e^x}{dx^\alpha} = \frac{d^{\alpha-k}}{dx^{\alpha-k}} \circ \frac{d^k}{dx^k} e^x = \frac{d^{\alpha-k}}{dx^{\alpha-k}} e^x = e^x$$

Thus for any $\alpha \in R, \frac{d^{(\alpha)}}{dx^{(\alpha)}}(e^x) = e^x$. ■

Use Theorem 2.3.2 and Theorem 2.4.4, we have the following theorem:

Theorem 2.4.5 :[6]

For any $\alpha \in R, \frac{d^{(\alpha)}}{dx^{(\alpha)}}\big(e^{\lambda x}\big) = \lambda^\alpha e^{\lambda x}$

Corollary 2.4.6 :[6]

The fractional derivatives of order α of the functions $\sin x$ and $\cos x$ are given by

$$\sin^{(\alpha)}(x) = \sin\left(x + \frac{\alpha\pi}{2}\right), \cos^{(\alpha)}(x) = \cos\left(x + \frac{\alpha\pi}{2}\right)$$

Proof:

Using the identity $\cos x + i \sin x = e^{ix}$ and according to the Theorem 2.4.5 we get

$$\cos^{(\alpha)}(x) + i\sin^{(\alpha)}(x) = i^\alpha e^{ix}$$

and

$$\cos^{(\alpha)}(x) - i\sin^{(\alpha)}(x) = (-i)^{(\alpha)} e^{-ix}$$

Hence we obtain

$$\cos^{(\alpha)}(x) = \frac{i^\alpha e^{ix} + (-i)^\alpha e^{-ix}}{2}$$

$$= \frac{e^{\alpha(i\pi/2)} e^{ix} + e^{\alpha(-i\pi/2)} e^{-ix}}{2}$$

Using the fact that

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

We have

$$\cos^\alpha(x) = \frac{(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2))e^{ix} + (\cos(\pi\alpha/2) - i\sin(\pi\alpha/2))e^{-ix}}{2}$$

$$= \cos(\pi\alpha/2)\frac{\left(e^{ix}+e^{-ix}\right)}{2} - \sin(\pi\alpha/2)\frac{\left(e^{ix}-e^{-ix}\right)}{2i}$$

$$= \cos(\pi\alpha/2)\cos(x) - \sin(\pi\alpha/2)\sin(x) = \cos(x + \frac{\alpha\pi}{2})$$

Similar argument gives

$$\sin^{(\alpha)}(x) = \sin\left(x + \frac{\alpha\pi}{2}\right)$$

For $f(x) = \sin(x)$, using the identity $\cos x + i\sin x = e^{ix}$ and according to the Theorem 2.4.5 we get

$$\cos^{(\alpha)}(x) + i\sin^{(\alpha)}(x) = i^{\alpha}e^{ix}$$

And

$$\cos^{(\alpha)}(x) - i\sin^{(\alpha)}(x) = (-i)^{(\alpha)}e^{-ix}$$

Hence we obtain

$$\sin^{(\alpha)}(x) = \frac{i^{\alpha}e^{ix} - (-i)^{\alpha}e^{-ix}}{2i}$$

$$= \frac{e^{\alpha(i\pi/2)}e^{ix} - e^{\alpha(-i\pi/2)}e^{-ix}}{2i}$$

Using the fact that

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

We have

$$\sin^{(\alpha)}(x) = \frac{(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2))e^{ix} - (\cos(\pi\alpha/2) - i\sin(\pi\alpha/2))e^{-ix}}{2i}$$

$$= \cos(\pi\alpha/2)\frac{\left(e^{ix}-e^{-ix}\right)}{2i} + \sin(\pi\alpha/2)\frac{\left(e^{ix}+e^{-ix}\right)}{2}$$

$$= \cos(\pi\alpha/2)\sin(x) + \sin(\pi\alpha/2)\cos(x) = \sin\left(x + \frac{\alpha\pi}{2}\right)$$ ■

Chapter III

Algebraic Approach to the Fractional Derivatives

In this chapter we introduce a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives (also integrals).
Further are found the expansions of the functions $\frac{xe^x}{e^x-1}, \frac{1}{\cos x}$, $x\tanh x$ and some other functions of the form $\sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!}$, which enables us to calculate any fractional derivative of these functions at x = 0. These calculations lead to representations of the Bernoulli B_k and Euler numbers E_k for any complex number k, via fractional derivatives of some functions at x = 0.

3.1 Theoretical results for fractional derivatives

In this section we represent an improved version of this idea, by distinguishing a class of analytical functions which have "natural" representations for $i \epsilon Z$.

Now let us assume that an analytical function f can be written in the form

$$f(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad (x \neq 0 \ \ if \ \alpha \notin Z),$$

which means that the sum of the right side (including the summation of divergent series) converges to f(x), for $x \neq 0$. The formal calculation of the $(\alpha+i)$ −th derivative at x = 0 yields that $f^{(\alpha+i)}(0) = a_i$. Hence

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(0) \frac{x^{\alpha+i}}{(\alpha+i)!}$$

where α is an arbitrary real or complex number. More generally

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0) \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, (x \neq x_0 \ \ if \ \alpha \notin Z)$$

which generalizes the ordinary Taylor's series.

On the other hand, if f admits fractional derivatives (integrations are also included) of arbitrary order, let $g(x) = f^{(\alpha)}(x)$. If we write g as a Laurent's series

$$g(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^i}{(i)!},$$

Then

$$f(x) = g^{(-\alpha)}(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!},$$

and f can be written in the required form. Namely, we proved the following proposition.

Proposition 3.1.1 :[8]

If f admits fractional derivatives of arbitrary order, then f satisfies the equalities

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(0) \frac{x^{\alpha+i}}{(\alpha+i)!}$$

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0) \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, (x \neq x_0 \ if \ \alpha \notin Z)$$

The previous discussion naturally yields to the following definition of fractional derivatives.

Definition 3.1.2

Assume that an analytical mapping f can be written in the following form

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \quad (x \neq x_0 \ if \ \alpha \notin Z),$$

for each $\alpha \in R$ (or $\alpha \in C$). Then f together with the above representations is called ideal function and we define $f^{(\alpha+i)}(x_0) := C_i, (i \in Z)$.

Now, we give examples.

Example 3.1.3

$f(x) = e^x$ is an ideal function.

Solution:

We know that

$$e^x = \sum_{i=-\infty}^{\infty} \frac{x^i}{i!}$$

Now,

$$f^{(-\alpha)}(x) = \sum_{i=-\infty}^{\infty} \frac{x^{i+\alpha}}{(i+\alpha)!} = e^x$$

Therefore, e^x is an ideal function.

Example 3.1.4

The function $f(x) = \sin(x)$ is an ideal, such that

$$\sin(x) = \sum_{j=-\infty}^{\infty} sin\frac{(\alpha+j)\pi}{2} \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

Solution:

We start with the right hand side,

$$\sum_{j=-\infty}^{\infty} sin\frac{(\alpha+j)\pi}{2} \cdot \frac{x^{\alpha+j}}{(\alpha+j)!} = \sum_{j=-\infty}^{\infty} [\frac{e^{i\frac{(\alpha+j)\pi}{2}} - e^{-i\frac{(\alpha+j)\pi}{2}}}{2i}] \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{i=-\infty}^{\infty} \left[\frac{\left(e^{i\pi/2}\right)^{\alpha+j} - \left(e^{-i\pi/2}\right)^{\alpha+j}}{2i}\right] \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \frac{1}{2i} \sum_{j=-\infty}^{\infty} \left[(i)^{\alpha+j} - (-i)^{\alpha+j}\right] \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \frac{1}{2i} [\sum_{j=-\infty}^{\infty} \frac{(ix)^{\alpha+j}}{(\alpha+j)!} - \sum_{j=-\infty}^{\infty} \frac{(-ix)^{\alpha+j}}{(\alpha+j)!}]$$

$$= \frac{1}{2i} \left[e^{ix} - e^{-ix}\right] = \sin(x)$$

Example 3.1.5

The function $f(x) = \cos(x)$ is an ideal, such that

$$\cos(x) = \sum_{j=-\infty}^{\infty} cos\frac{(\alpha+j)\pi}{2} \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

Solution:

We start with the right hand side,

$$\sum_{j=-\infty}^{\infty} cos\frac{(\alpha+j)\pi}{2} \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{j=-\infty}^{\infty} [\frac{e^{i\frac{(\alpha+j)\pi}{2}} + e^{-i\frac{(\alpha+j)\pi}{2}}}{2}] \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{i=-\infty}^{\infty} \left[\frac{\left(e^{i\pi/2}\right)^{\alpha+j} + \left(e^{-i\pi/2}\right)^{\alpha+j}}{2}\right] \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \frac{1}{2}\sum_{j=-\infty}^{\infty} \left[(i)^{\alpha+j} + (-i)^{\alpha+j}\right] \cdot \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \frac{1}{2}[\sum_{j=-\infty}^{\infty} \frac{(ix)^{\alpha+j}}{(\alpha+j)!} + \sum_{j=-\infty}^{\infty} \frac{(-ix)^{\alpha+j}}{(\alpha+j)!}]$$

$$= \frac{1}{2}\left[e^{ix} + e^{-ix}\right] = \cos(x)$$

3.2 Properties of ideal functions (I)

In this section, we derive some of the properties of the class of ideal functions. The set of ideal functions I can be separated in a quotient set I/f, where the equivalence relation f is defined by $f \sim g$ iff there exists $\alpha \in C$, such that $f^{(\alpha)} = g$. Each such class determines unique sequence a_i, (i$\in Z$), such that $\sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!} \in I$. Namely, then $\sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!} \sim \sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!}$ for arbitrary $\alpha \in C$. Also, I is a nonempty set because $e^x, \sin(x), \cos(x) \in I$. The zero function is also an ideal function.

Theorem 3.2.1 :[8]

I is a vector space.

Proof:

For any $\alpha, \beta \in C$. Let

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!} ,$$

$$g(x) = \sum_{i=-\infty}^{\infty} g^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!},$$

$$h(x) = \sum_{i=-\infty}^{\infty} h^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!},$$

Then

(a) $f(x) + g(x) = \sum_{i=-\infty}^{\infty}[f^{(\alpha+i)}(x_0) + g^{(\alpha+i)}(x_0)]\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$$= \sum_{i=-\infty}^{\infty}(f+g)^{(\alpha+i)}(x_0)\ \frac{(x-x_0)^{(\alpha+i)}}{(\alpha+i)!}$$

Therefore, $f + g \in I$

(b) $f(x) + g(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0) + \sum_{i=-\infty}^{\infty} g^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$$= \sum_{i=-\infty}^{\infty} g^{(\alpha+i)}(x_0) + \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$$

$$= g(x) + f(x).$$

Therefore, $f + g = g + f$.

(c)$(f+g)(x) + h(x) = \sum_{i=-\infty}^{\infty}[(f+g)^{(\alpha+i)}(x_0) + h^{(\alpha+i)}(x_0)]\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$$= \sum_{i=-\infty}^{\infty}[f^{(\alpha+i)}(x_0) + (g+h)^{(\alpha+i)}(x_0)]\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$$

$$= \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!} + \sum_{i=-\infty}^{\infty}(g+h)^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$$

$$= f(x) + (g+h)(x)$$

Therefore, $(f+g) + h = f + (g+h)$.

(d) There exists a zero function in I such that $\left(f+0\right)(x) = f(x)$.

(e) For every f in I , $(-f)(x) = -f(x) = -\sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$$= \sum_{i=-\infty}^{\infty} (-f)^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$$

Therefore, $(-f) \in I$ and $f + (-f) = 0$.

(f) $(\alpha f)(x) = \alpha f(x) = \alpha \sum_{i=-\infty}^{\infty}(f)^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$$= \sum_{i=-\infty}^{\infty} (\alpha f)^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$$

Therefore, $(\alpha f) \in I$.

(g) $\alpha(\beta f)(x) = \alpha \sum_{i=-\infty}^{\infty}(\beta f)^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!} \quad = \alpha\beta \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!} =$ $(\alpha\beta)f(x)$

Therefore, $\alpha(\beta f) = (\alpha\beta)f$.

(h) $\alpha(f+g)(x) = \alpha \sum_{i=-\infty}^{\infty}(f+g)^{(\alpha+i)}(x_0)\ \frac{(x-x_0)^{(\alpha+i)}}{(\alpha+i)!}$

$$= \alpha \sum_{i=-\infty}^{\infty} [f^{(\alpha+i)}(x_0) + g^{(\alpha+i)}(x_0)]\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$$

$$= \alpha(\sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!} + \sum_{i=-\infty}^{\infty} g^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!})$$

$$= \alpha f(x) + \alpha g(x)$$

Therefore, $\alpha(f+g) = \alpha f + \beta g$

(i) $(\alpha+\beta)f(x) = (\alpha+\beta)\ \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$= \alpha f(x) + \beta f(x)$

Therefore, $(\alpha+\beta)f = \alpha f + \beta f$.

(j) $1f(x) = 1\sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$

$= f(x)$.

Therefore, $1f = f$. ■

Theorem 3.2.2 :[8]

If f is an ideal function, then $f(\lambda x) \in I$ if $\lambda \neq 0$.

Proof:

Let $f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0)\frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}$,$\lambda \neq 0$, t hen

$$f(\lambda x) = \sum_{i=-\infty}^{\infty} f^{\alpha+i}(x_0)\frac{(\lambda x - x_0)^{\alpha+i}}{(\alpha+i)!}$$

$$f(\lambda x) = \sum_{i=-\infty}^{\infty} \lambda^{\alpha+i} f^{(\alpha+i)}(x_0)\frac{(x - \frac{x_0}{\lambda})^{\alpha+i}}{(\alpha+i)!}$$

Therefore, $f(\lambda x) \in I$ if $\lambda \neq 0$. ■

Also, if P is a polynomial, $P(x) = \sum_{k=-\infty}^{n} a_k \frac{x^k}{k!}$, with known coefficients $a_0, a_1, \dots, a_n$, then $a_{-1}, a_{-2}, a_{-3}, \dots$ are not uniquely determined such that P is an ideal function. In the remark of section 3.5 is constructed a wide class of polynomials P which are ideal functions.

3.3 Expanding of the Bernoulli numbers.

Now we shall expand the Bernoulli numbers B_k for integer k and also for complex number. Although the Bernoulli numbers are defined by the development

$$\frac{x}{e^x - 1} = B_0 + B_1\frac{x^1}{1!} + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!} + \cdots$$

It appears more convenient to consider modified Bernoulli number B_i^* via the development of the function $\frac{x}{e^x-1} + x = \frac{xe^x}{e^x-1}$ as

$$\frac{xe^x}{e^x - 1} = B_0^* + B_1^*\frac{x^1}{1!} + B_2^*\frac{x^2}{2!} + B_3^*\frac{x^3}{3!} + \cdots$$

Now we get:

$$B_0^* = 1, B_1^* = \frac{1}{2}, B_2^* = \frac{1}{6}, B_4^* = -\frac{1}{30}, B_6^* = \frac{1}{42}, B_8^* = -\frac{1}{30}, B_{10}^* = \frac{5}{66}, \dots,$$

while

$$B_3^* = B_5^* = B_7^* = B_9^* = \cdots = 0.$$

Note that $B_i = B_i^*$ for $i \neq 1$ and $B_1 = -B_1^*$. According to the formula (2.4.3) in [9, p.19] we have

$$B^*_{2m} = -2m\zeta(1-2m), \qquad m = 1,2,3\ldots \qquad (3.1)$$

Where ζ is the Riemann Zeta function. Since $\zeta(-2m) = 0$ for $m = 1,2,3,\ldots$, from (3.1) we can write

$$B^*_p = -p\,.\,\zeta(1-p), \qquad p = 1,2,3\ldots \qquad (3.2)$$

Moreover, we accept by definition that for each complex number α,

$$B^*_\alpha = -\,\alpha\,.\,\zeta\big(1-\alpha\big), \qquad \text{for } |\zeta(1-\alpha)| \neq \infty \qquad (3.3)$$

3.4 Natural Representation Of $(1/cosx)$ and Euler numbers

In this section we shall consider the function

$$\frac{1}{\cos x} = \sum_{n=0}^{\infty}\left(E_{2n}\frac{x^{2n}}{(2n)!}\right), \qquad |x| < \frac{\pi}{2}$$

Since $\frac{1}{\cos(0)} = 1 = E_0$, $\frac{d}{dx}\left(\frac{1}{\cos x}\right)\Big|_{x=0} = o$, $\frac{d^2}{dx^2}\left(\frac{1}{\cos x}\right)\Big|_{x=0} = 1 = E_2$ continuing in this way,

The numbers E_{2k} satisfy the recurrent relation

$$E_0 = 1, \quad E_0 - \binom{2n}{2}E_2 + \binom{2n}{4}E_4 - \ldots. + (-1)^n\binom{2n}{2n}E_{2n} = 0$$

And the numbers $(-1)^k E_k$ are known as Euler numbers.

We start from the development of the function $\frac{1}{\cos x}$ in Fourier series. Since $\frac{1}{\cos x} \in L^2(-\pi,\pi)$ and $e_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx)$ forms an orthonormal system. Then

$$\langle\frac{1}{\cos(x)}, e_1\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}\frac{1}{\cos(x)}\cos(x)\,dx = 2\sqrt{\pi}$$

and $\langle\frac{1}{\cos(x)}, e_2\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}\frac{1}{\cos(x)}\cos(2x)\,dx = 0$. Continuing in this way, the Fourier series of $\frac{1}{\cos x}$ with respect to this orthonormal system is

$$\sum_{n=1}^{\infty}\langle\frac{1}{\cos(x)}, e_n(x)\rangle\, e_n(x)$$

Therefore,

$$\frac{1}{\cos(x)} = 2\sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}.\cos nx.$$

This equality implies the following consequence. Replacing $\cos nx = \sum_{k=-\infty}^{\infty}(-1)^k n^{2k}\frac{x^{2k}}{(2k)!}$, we obtain

$$\frac{1}{\cos x} = 2\sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{k=-\infty}^{\infty}(-1)^k n^{2k}\,\frac{x^{2k}}{(2k)!}$$

$$= \sum_{k=-\infty}^{\infty}(-1)^k 2\left(\sum_{n=1}^{\infty} n^{2k}.\cos(n-1)\frac{\pi}{2}\right)\frac{x^{2k}}{(2k)!} = \sum_{k=-\infty}^{\infty} E_{2k}\frac{x^{2k}}{(2k)!},$$

Where

$$E_{2k} = (-1)^k 2\left(\sum_{n=1}^{\infty} n^{2k}.\cos(n-1)\frac{\pi}{2}\right) .$$

Hence we obtain the required natural representation of $\frac{1}{\cos x}$. Note that the numbers E_{-2k} can easily be calculated numerically, because

$$E_{-2k} = (-1)^k 2\left(1-\frac{1}{3^{2K}}+\frac{1}{5^{2K}}+\frac{1}{7^{2K}}+\cdots\right). \qquad K \in N.$$

For example $E_{-2} = -2\left(1-\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)$.

If $\frac{1}{\cos x}$ is an ideal function, then $\frac{1}{\cos(x)} = \sum_{i=-\infty}^{\infty}\frac{d^{\alpha+i}}{dx^{\alpha+i}}\left(\frac{1}{\cos x}\right)\Big|_{x=0}\frac{x^{\alpha+i}}{(\alpha+i)!}$

then the α-th derivative derivative of $\frac{1}{\cos x}$ at $x = 0$ is

$$E_\alpha = \frac{d^\alpha}{dx^\alpha}\left(\frac{1}{\cos x}\right)\Big|_{x=0},$$

And using the Fourier series, we can obtain

$$E_\alpha = 2\cos\frac{\alpha\pi}{2}\sum_{n=1}^{\infty} n^\alpha \;.\cos(n-1)\,\frac{\pi}{2}$$

Specially, $E_{2k+1} = 0$ for $k \in Z$, by the following

Since $\frac{1}{\cos(x)} = 2\cos(x) - 2\cos(3x) + 2\cos(5x) - 2\cos(7x) + \cdots$

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{1}{\cos(x)}\right) = 2\cos\left(x+\frac{\alpha\pi}{2}\right) - 23^\alpha \cos\left(3x+\frac{\alpha\pi}{2}\right) + 25^\alpha\left(5x+\frac{\alpha\pi}{2}\right) - 27^\alpha\cos\left(7x+\frac{\alpha\pi}{2}\right) + \cdots$$

$$\left.\frac{d^\alpha}{dx^\alpha}\left(\frac{1}{\cos x}\right)\right|_{x=0} = 2\cos\left(\frac{\alpha\pi}{2}\right) - 23^\alpha\cos\left(\frac{\alpha\pi}{2}\right) + 25^\alpha cos\left(\frac{\alpha\pi}{2}\right) - 27^\alpha\cos\left(\frac{\alpha\pi}{2}\right) + \cdots$$

Therefore,

$$E_\alpha = 2\cos\frac{\alpha\pi}{2}\sum_{n=1}^{\infty} n^\alpha \ .\cos(n-1)\ \frac{\pi}{2}$$

Proposition 4.1.1 :[8]

The function $\frac{1}{\cos x}$ is an ideal, such that

$$\frac{1}{\cos x} = \sum_{j=-\infty}^{\infty} E_{\alpha+j}\ \frac{x^{\alpha+j}}{(\alpha+j)!}$$

Proof:

We start with the right side

$$\sum_{j=-\infty}^{\infty} E_{\alpha+j}\ \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{j=-\infty}^{\infty} 2\cos\frac{(\alpha+j)\pi}{2}\left(\sum_{n=1}^{\infty} n^{\alpha+j}.\cos(n-1)\frac{\pi}{2}\right)\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty} 2\cos\frac{(\alpha+j)\pi}{2}\ n^{\alpha+j}.\ \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty}\left[e^{i}\ \frac{(\alpha+j)\pi}{2} + e^{-i}\ \frac{(\alpha+j)\pi}{2}\right] n^{\alpha+j}.\ \frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty}\left[\frac{\left(e^{i\pi/2}.nx\right)^{\alpha+j}}{(\alpha+j)!} + \frac{\left(e^{-i\pi/2}.nx\right)^{\alpha+j}}{(\alpha+j)!}\right]$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\left(e^{inx} - e^{-inx}\right) = 2\sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\cos nx = \frac{1}{\cos x}$$

Corollary 4.1.2 :[8]

The function $\frac{1}{\cosh x}$ is an ideal function, such that

$$\frac{1}{\cosh x} = \sum_{j=-\infty}^{\infty} \frac{(i)^{\alpha+j}\, E_{\alpha+j}}{(\alpha+j)!}\, x^{\alpha+j}$$

Proof:

Let $f(x) = \frac{1}{\cos x}$, then $f(ix) = \frac{1}{\cos(ix)} = \frac{2}{e^{-x}+e^{x}} = \frac{1}{\cosh x}$. Therefore, using theorem 3.2.2, the function $\frac{1}{\cosh x}$ is an ideal function.

3.5 Representations and natural representations of other ideal functions.

In this section we shall consider some functions whose coefficients of the Taylor series contain Bernoulli numbers.

Proposition 3.5.1 :[8]

The function $\frac{x}{e^x-1}$ is an ideal function, such that

$$\frac{x}{e^x-1} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha}\, B^{*}_{\alpha+i}\, \frac{x^{\alpha+i}}{(\alpha+i)!} . \qquad \alpha \in C. \qquad (4.1)$$

Proof:

Using the indefinite (3.3) we obtain

$$\sum_{i=-\infty}^{\infty} (-1)^{i+\alpha}\, B^{*}_{\alpha+i}\, \frac{x^{\alpha+i}}{(\alpha+i)!} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} \left[\sum_{i=-\infty}^{\infty} -(\alpha+i)n^{\alpha+i-1}\right] \frac{x^{\alpha+i}}{(\alpha+i)!}$$

$$= x\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty} \frac{(-nx)^{\alpha+i-1}}{(\alpha+i)!} = x\sum_{n=1}^{\infty} e^{-nx} = x\,\frac{e^{-x}}{1-e^{-x}} = \frac{x}{e^x-1}$$

Proposition 3.5.2 :[8]

The function $\frac{xe^x}{e^x-1}$ is an ideal function, such that

$$\frac{xe^x}{e^x-1} = \sum_{i=-\infty}^{\infty} B^{*}_{\alpha+i}\, \frac{x^{\alpha+i}}{(\alpha+i)!} . \qquad \alpha \in C. \qquad (4.2)$$

Proof:

Let $f(x) = \frac{x}{e^x - 1}$, then $f(-x) = \frac{-x}{e^{-x} - 1} = \frac{xe^x}{e^x - 1}$. Therefore, using theorem 3.2.2, the function $\frac{xe^x}{e^x - 1}$ is an ideal function.

Corollary 3.5.3 :[8]

For each $\alpha \in C$,

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{xe^x}{e^x - 1}\right)\bigg|_{x=0} = B_\alpha^*. \qquad (4.3)$$

Proof:

Using the fact that $\frac{d^\alpha}{dx^\alpha}\left(\frac{xe^x}{e^x-1}\right) = \sum_{i=-\infty}^{\infty} B_{\alpha+i}^* \frac{x^i}{(i)!}$, therefore, $\frac{d^\alpha}{dx^\alpha}\left(\frac{xe^x}{e^x-1}\right)\Big|_{x=0} = B_\alpha^*$.

Remark

In Proposition 3.5.1 and 3.5. 2 were obtained the natural representations for $\frac{x}{e^x-1}$ and $\frac{xe^x}{e^x-1}$:

$$\frac{x}{e^x - 1} = \sum_{i=-\infty}^{\infty} (-1)^i B_i^* \frac{x^i}{(i)!}. \qquad \frac{xe^x}{e^x - 1} = \sum_{i=-\infty}^{\infty} B_i^* \frac{x^i}{(i)!}$$

Both function $\frac{x}{e^x-1}$ and $\frac{xe^x}{e^x-1}$ with the corresponding representation are ideal and hence their difference

$$h(x) = \frac{xe^x}{e^x - 1} - \frac{x}{e^x - 1} = x + 2B_{-1}^* \frac{x^{-1}}{(-1)!} + 2B_{-3}^* \frac{x^{-3}}{(-3)!} + 2B_{-5}^* \frac{x^{-5}}{(-5)!} + \ \dots \qquad (4.4)$$

By theorem 3.2.1

This function h generates a family of ideal polynomials P, using the following generator transformation:

i. If $p \in P$, then $p^{(\alpha)} \in P$ for each $\alpha \in Z$.
ii. If $p \in P$, then $x^n p \in P$ for each nonnegative integer n.
iii. If $p, q \in P$, then $\lambda p + \mu q \in P$ for each scalars λ, μ.

Proposition 3.5.4 :[8]

The function $x.\cot x$ is an ideal function, such that

$$x.\cot x = \sum_{j=-\infty}^{\infty} 2^{\alpha+j} \cos\frac{(\alpha+j)\pi}{2}.\ B^*_{\alpha+j}\ \frac{x^{\alpha+j}}{(\alpha+j)!}\ . \qquad \alpha \in C. \qquad (4.5)$$

Proof :

Let $f(x) = \frac{xe^x}{e^x-1} = \sum_{k=-\infty}^{\infty} B^*_k\ \frac{x^k}{(k)!}$, then

$$\frac{f(2ix)+f(-2ix)}{2} = \frac{ix\, e^{2ix}}{e^{2ix}-1} - \frac{ix\, e^{-2ix}}{e^{-2ix}-1} = x\left(\frac{ie^{ix}}{e^{ix}-e^{-ix}} - \frac{ie^{-ix}}{e^{-ix}-e^{ix}}\right)$$

$$= x\left(\frac{ie^{ix}}{e^{ix}-e^{-ix}} + \frac{ie^{-ix}}{e^{ix}-e^{-ix}}\right) = xcot(x).$$

Hence the function $x.\cot x$ is an ideal function and its natural representation is given by

$$x.\cot x = \sum_{k=-\infty}^{\infty} (-1)^k 2^{2k} B^*_{2k}\ \frac{x^{2k}}{(2k)!} \qquad (4.6)$$

According to Proposition 3.5.2, the general representation for arbitrary α is given by

$$x.\cot x = \frac{1}{2}\sum_{j=-\infty}^{\infty}\left(B^*_{\alpha+j}\ \frac{(2ix)^{\alpha+j}}{(\alpha+j)!} + B^*_{\alpha+j}\ \frac{(-2ix)^{\alpha+j}}{(\alpha+j)!}\right) =$$

$$= \sum_{j=-\infty}^{\infty}\left(2^{\alpha+j}\cos\frac{(\alpha+j)\pi}{2}.\ B^*_{\alpha+j}\ \frac{(x)^{\alpha+j}}{(\alpha+j)!}\right)$$

As a consequence we obtain the following representation for the Bernoulli numbers.

Corollary 3.5.5 :[8]

For each α ∈ C,

$$\frac{d^\alpha}{dx^\alpha}(x.\cot x)\bigg|_{x=0} = 2^\alpha \cos\frac{\alpha\pi}{2}.B^*_\alpha. \qquad (4.7)$$

Proof:

$$\frac{d^\alpha}{dx^\alpha}(x.\cot x) = \sum_{j=-\infty}^{\infty}\left(2^{\alpha+j}\cos\frac{(\alpha+j)\pi}{2}.\ B^*_{\alpha+j}\ \frac{(x)^j}{(j)!}\right)$$

Therefore, $\frac{d^\alpha}{dx^\alpha}(x.\cot x)\Big|_{x=0} = 2^\alpha \cos\frac{\alpha\pi}{2}.B^*_\alpha.$

Proposition 3.5.6 :[8]

The function $\frac{x}{\sin x}$ is an ideal function, such that

$$\frac{x}{\sin x} = \sum_{i=-\infty}^{\infty} B^*_{\alpha+i}\left(2-2^{\alpha+i}\right)\cos\frac{(\alpha+i)\pi}{2}.\frac{x^{\alpha+i}}{(\alpha+i)!}. \quad \alpha \in C. \qquad (4.8)$$

Proof:

We start from the identity $\frac{1}{\sin x} = \cot\frac{x}{2} - \cot x$ and thus $\frac{x}{\sin x}$ is an ideal function. Further, from

$$\frac{x}{\sin x} = 2\frac{x}{2}.\cot\frac{x}{2} - x.\cot x \qquad (4.9)$$

and from (4.6) we obtain the natural representation of $\frac{x}{\sin x}$:

$$\frac{x}{\sin x} = \sum_{k=-\infty}^{\infty}(-1)^k\left(2-2^{2k}\right)B^*_{2k}\frac{x^{2k}}{(2k)!}$$

Using the formulas (4.9) and (4.5), the general representation for arbitrary α is given by (4.8).

Corollary 3.5.7 :[8]

For each, $\alpha \in C$,

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{x}{\sin x}\right)\Big|_{x=0} = (2-2^\alpha)\cos\frac{\alpha\pi}{2}.B^*_\alpha. \qquad (4.10)$$

Proof:

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{x}{\sin x}\right) = \sum_{i=-\infty}^{\infty} B^*_{\alpha+i}\left(2-2^{\alpha+i}\right)\cos\frac{(\alpha+i)\pi}{2}.\frac{x^i}{(i)!}.$$

Therefore, $\frac{d^\alpha}{dx^\alpha}\left(\frac{x}{\sin x}\right)\Big|_{x=0} = (2-2^\alpha)\cos\frac{\alpha\pi}{2}.B^*_\alpha.$

Proposition 3.5.8 :[8]

The function $\frac{x}{e^x+1}$ is an ideal function such that

$$\frac{x}{e^x + 1} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha}\left(1 - 2^{i+\alpha}\right) B^*_{i+\alpha} \frac{x^{i+\alpha}}{(i + \alpha)!}$$

Proof:

Using the identity (3.3), we obtain

$$\sum_{i=-\infty}^{\infty} (-1)^{\alpha+i}\left(1 - 2^{\alpha+i}\right) B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha + i)!} =$$

$$\sum_{i=-\infty}^{\infty} (-1)^{\alpha+i}\left(1 - 2^{\alpha+i}\right) \left[-\sum_{n=1}^{\infty} (\alpha + i) n^{\alpha+i-1}\right] \cdot \frac{x^{\alpha+i}}{(\alpha + i)!} =$$

$$= x \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \frac{(-nx)^{\alpha+i-1}\left(-2^{\alpha+i} + 1\right)}{(\alpha + i - 1)!} =$$

$$= x \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \frac{(-nx)^{\alpha+i-1}}{(\alpha + i - 1)!} - 2x \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \frac{(-2nx)^{\alpha+i-1}}{(\alpha + i - 1)!} =$$

$$= x \sum_{n=1}^{\infty} e^{-nx} - 2x \sum_{n=1}^{\infty} e^{-2nx} = \frac{x}{e^x - 1} - \frac{2x}{e^{2x} - 1} = \frac{x}{e^x + 1}$$

Corollary 3.5.9 :[8]

The function $\frac{xe^x}{e^x+1}$ is an ideal function, such that

$$\frac{xe^x}{e^x + 1} = \sum_{i=-\infty}^{\infty} \left(2^{\alpha+i} - 1\right) B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha + i)!} \qquad (4.11)$$

whence

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{xe^x}{e^x + 1}\right)\Bigg|_{x=0} = (2^\alpha - 1)\, B^*_\alpha. \qquad (4.12)$$

Proof:

$f(x) = \frac{x}{e^x+1}$ is an ideal function, and hence $-f(-x) = -\left(\frac{-x}{e^{-x}+1}\right) = \frac{xe^x}{e^x+1}$ is also an ideal function, i.e. the equalities (4.11) and (4.12) hold.

Proposition 3.5.10 :[8]

The function x tanh x is an ideal function such that

$$x \tanh x = \sum_{i=-\infty}^{\infty} \left(2^{\alpha+i} - 1\right) B_{\alpha+i}^{*} \,.\, 2^{\alpha+i} \frac{x^{\alpha+i}}{(\alpha + i)!}$$

Proof:

$$x \tanh x = x \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = x \frac{e^{2x} - 1}{e^{2x} + 1} = -\frac{f(2x) + f(-2x)}{2}, \qquad (4.13)$$

$$-\frac{f(2x) + f(-2x)}{2} = -\left(\frac{x}{e^{2x} + 1} + \frac{-x}{e^{-2x} + 1}\right) = -\left(\frac{xe^{-x}}{e^{x} + e^{-x}} - \frac{xe^{x}}{e^{-x} + e^{x}}\right) = x\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)$$

$$= x \tanh x.$$

Where $f(x) = \frac{x}{e^{x}+1}$. According to Proposition 3.5.8, by (4.13) we obtain that x tanh x is also an ideal function.

Since

$$f(x) = \sum_{i=-\infty}^{\infty} (-1)^{i} \left(1 - 2^{i}\right) B_{\alpha+i}^{*} \frac{x^{i}}{(i)!},$$

We obtain

$$f(2x) + f(-2x) = \sum_{i=-\infty}^{\infty} \left(1 - 2^{i}\right) B_{i}^{*} \frac{2^{i} x^{i}}{(i)!} \left[(-1)^{i} + 1\right] = 2 \sum_{i=-\infty}^{\infty} \left(1 - 2^{2i}\right) B_{2i}^{*} 2^{2i} \frac{x^{2i}}{(2i)!},$$

i.e.

$$x \tanh x = \sum_{i=-\infty}^{\infty} \left(2^{2i} - 1\right) B_{2i}^{*} .\, 2^{2i} \frac{x^{2i}}{(2i)!},$$

According to Proposition 3.5.8, the general representation for arbitrary α is given by

$$x \tanh x = \sum_{i=-\infty}^{\infty} \left(2^{\alpha+i} - 1\right) B_{\alpha+i}^{*} .\, 2^{\alpha+i} \frac{x^{\alpha+i}}{(\alpha + i)!}.$$

Corollary 3.5.11 :[8]

For each $\alpha \in C$**,**

$$\frac{d^{\alpha}}{dx^{\alpha}}(\mathrm{xtanhx}) = (2^{\alpha}-1)B_{\alpha}^{*}2^{\alpha}$$

Proof:

$$\frac{d^{\alpha}}{dx^{\alpha}}(\,\mathrm{xtanhx}) = \sum_{i=-\infty}^{\infty}(2^{\alpha+i}-1)B_{\alpha+i}^{*}.\,2^{\alpha+i}\,\frac{x^{i}}{(i)!}$$

Therefore, $\frac{d^{\alpha}}{dx^{\alpha}}(\mathrm{xtanhx})\Big|_{x=0} = (2^{\alpha}-1)B_{\alpha}^{*}2^{\alpha}.$

Proposition 3.5.12 :[8]

The function $x\tan x$ is an ideal function such that

$$x\tan x = \sum_{j=-\infty}^{\infty}\left(1-2^{\alpha+j}\right)B_{\alpha+j}^{*}.\,(2i)^{\alpha+j}\frac{x^{\alpha+j}}{(\alpha+j)!}$$

Proof:

Using the formula $x\tan(x) = -f(ix)$, where $f(x) = \mathrm{xtanhx}$, and according to proposition 3.5.10, the general representation for arbitrary α is given by

$$x\tan x = \sum_{j=-\infty}^{\infty}\left(1-2^{\alpha+j}\right)B_{\alpha+j}^{*}.\,(2i)^{\alpha+j}\frac{x^{\alpha+j}}{(\alpha+j)!}.$$

Example 3.5.13

The function $f(x) = \frac{1}{1-x}$ is not ideal function.

Solution:

As we know $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^{n}$ in Taylor series expansion. Write

$$\sum_{n=0}^{\infty}x^{n} = \sum_{n=0}^{\infty}n!\frac{x^{n}}{n!},\qquad 0<|x|<1$$

Now if we try to expand $\sum_{n=0}^{\infty}n!\frac{x^{n}}{n!}$ to $\sum_{n=-\infty}^{\infty}n!\frac{x^{n}}{n!}$ we got for the negative integers defined value, since $\frac{-\infty}{-\infty}$ is undefined. Therefore, we can't write the function $\frac{1}{1-x}$ of the form of an ideal function.

Conclusion

In this thesis, a study of the background of the concept of fractional derivatives, fractional differentiation and fractional integration was done, by providing two different
approaches of the concept of fractional derivatives. At the beginning, we have introduced the concept of fractional derivatives as defined by the Riemann - Liouville which has been generalized to include also the real numbers, not integers only. Then, this thesis talks discusses many of the characteristics of this definition. The second concept
depends on the formal power series summation, which is used to find the fractional derivatives of the constant functions and polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved several properties of the fractional derivatives.
Finally, an alternative definition of the fractional derivatives and also a characteristic class of so called ideal functions was introduced, which admit arbitrary
fractional derivatives (also integrals). Further, we are found the expansions of the functions
$\frac{xe^x}{e^x-1}, \frac{1}{\cos(x)}, x\tanh x$, and some other functions of the form $\sum_{k=-\infty}^{\infty} a_k \frac{x^k}{k!}$, which enables us to calculate any fractional derivative of these functions at $x = 0$.

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