Deanship of Graduate Studies

Al-Quds University

Extreme Points of The Set of Univalent Functions

By

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M.Sc. Thesis

Jerusalem- Palestine

2004

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B. Sc. : College of science and Technology Abu-Dies / Palestine

A thesis Submitted in Partial fulfillment of requirements for the degree of Master of Science, Department of Mathematics / Program of Graduate Studies.

Al-Quds University

May, 2004

Graduated Studies / Mathematics

Deanship of graduate studies

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Master thesis submitted and accepted, Date: May 31, 2004.

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Dedication

To my parents . To my Family . To my husband and to my daughter Gaida .

Acknowledgment

I would like to express my thanks to those helped me to prepare and complete this work.

I owe the success of this work to Dr. Ibrahim Al –Grouz for his greate help in giving me the approparíate references and suggestions for this work.

I am also very grateful , to Dr . Mohammed Khalil and Dr. Ibrahim Al-Masri for their efforts in preparing , writing , analysis and putting the final touches

My thanks to the principle of my school , Mr. Ashraf Shakarneh , for giving me the time to go and to leave the school to complete this work .

Abstract

Let Δ be the unit disk and $\mathcal{A}$ be the set of all analytic functions in Δ. Let S be the set of all normalized univalent functions in the open unit disk

i.e. $\{ f \in \mathcal{A} : f$ is univalent , $f(0)=0$ and $f'(0)=1 \}$.

Our main goal in the thesis is to study the extreme points and the convex hulls of several subfamilies of S such as the set of all starlike functions in S $(S^{\bullet})$, the positive real functions are $(\wp)$, the family of closed- to – convex functions (CL), the family of convex function (K) and the set of all function with real coefficients in S (T), and give a description about these extreme points by the integration . Also , we prove some facts about the extreme point of the family S. An example of a function in S which is not extreme point of S was given.

In general , its proved that if $f \in S$ omits two values of equal modulus, then f is not an extreme point of S .

الخلاصة

افرض أن Δ هو قرص الوحدة وأن المجموعة $\mathcal{A}$ هي مجموعة كل الاقترانات التحليلية في Δ .

افرض أن S مجموعة الاقترانات المركبة الأحادية المعرفة داخل دائرة الوحدة.

الهدف الأساسي من هذه الرسالة هو دراسة الاقترانات القصوى والغطاء لعدة مجموعات جزئية من S مثل الاقترانات الشبه نجمية $\left(S^{\bullet}\right)$، الاقترانات المقعرة $\left(K\right)$، الاقترانات القريبة من التقعر $\left(CL\right)$ وكذلك $\left(T\right)$ و $\left(\wp\right)$. وقدمنا وصفا مفصلا حول الاقترانات القصوى باستخدام التكامل .

أيضا ، لقد أثبتنا بعض الحقائق حول الاقترانات القصوى للمجموعة S و أعطينا مثالا على اقتران من المجموعة S كونه لا ينتمي الى مجموعة الاقترانات القصوى ، و أثبتنا بعض الصفات الضرورية ليكون الاقتران من الاقترانات القصوى من S أو ليس من الاقترانات القصوى في S .

Table of Contents

Introduction

Let Δ be the open unit disk , $\Delta = \{z : |z| < 1\}$. A function f is said to be univalent on Δ

if it is one – to – one in Δ .

The theory of univalent functions is an old subject , born around the beginning of this Century with the paper by P. Koebe [30]yet it remains an active field of current research.

A number of books on the subject have been written and an extensive introduction is provided by P . L . Duren [15] ,G .M . Goluusin[16], W . K . Hyman [23] and Chr . Pomerenke[43].Some additional books about univalent functions are by J .A . Hummel 29] ,I . M . Milin [38] , P . Montele[39] , and A . C .Schaeffer And D .C . Spencer [52] .

Several books on complex analysis also contain information about univalent functions.

A number of survey articles have been written about the general theory of univalent functions and more specific developments and we mention [3] ,[10] ,[13] ,[17], [18] , [24] ,[51] ,[36] ,[46] and [54]

The book [5]edited by D . A . brannan and J . G Clunie also contains survey articles , introductory lectures and current research work on univalent functions and other fields in complex analysis .

Let S be the set of all analytic and univalent functions f in the unit disk Δ with the normalization $f(0) = 0$ and $f'(0) = 1$.

Most books of complex analysis are concerned with this family S .

Also , P . Duren .T .Macgr. is a good reference for the set S .

One of the major problems of the field is the Bieberbach conjecture , dating from the year 1916 , which asserts that the Taylor coefficients of each function of class S satisfy the inequality $|a_n| \le n$ .For many years this famous problem has stood as a challenge and has inspired the development of ingenious methods until the proof of de Brange[13] in 1985.

Most of the material in chapter two in the thesis deals with univalent functions . Also , a generalization of the subfamilies S^* and K were introduced by M .S . Roberstone in [45] . The geometric chararecterization mentioned for close – to – convex functions is due to Z . Lewandowski [32 ,33] .The class T of t typically real functions was introduced and studied by W . Rogonsinski [48] .We give a detailed proof of some theorems which we need to develop the theory of the extreme point of S and the subfamilies of S

The last two chapters give a brief introduction to the subordination which goes back to E . Lindelof [34] . Subordination was more formally introduced and studied by J . E Littlewood [35] and the later by Rogosinski [49] .

we say that f is an extreme point of the determination of convex hull and extreme point of special families of univalent functions .

The determination of the extreme points of the family S is partially proved , until now it is not known the general characterization of the extreme points of S . Some results can be found in Brickman [7],G Springer [55] .

Chapter One

Univalent Functions

In this chapter we deal with analytic functions on (a simply connected) domain that maps conformaly this domain to the unit disk Δ .

With some normalization. Also, gives some elementary properties of these functions.

To do this, we want to define what we mean by the univalent functions.

1.1 The Set of Analytic Functions on Δ

Definition 1.1.1

Let Δ be the open unit disk , in notation $\{ z : |z| < 1 \}$. Its boundary is the unit circle , is denoted by $\partial\Delta$.

Definition 1.1.2

A complex function f is continuously differentiable if f' is continuous on Δ .

Definition 1.1.3

The function f is analytic on Δ if f is continuously differentiable on Δ . In notation $\mathcal{A}$ denotes the set of all analytic functions.

In particular f is analytic at a point z_0 if it is analytic in a neighborhood of z_0 .

It is easy to show that if f and g are two analytic functions on Δ then

$f \pm g$ *and* $f \cdot g$ are also analytic on Δ.

Also if $g(z) \neq 0$, for all $z \in \Delta$ then $\frac{f}{g}$ is analytic on Δ, as we see in the following example.

Example 1.1.1.

$f(z)=\frac{z}{1-z}$ is analytic function on Δ since $f'(z)=\frac{(1-z)+z}{(1-z)^2}=\frac{1}{(1-z)^2}$ .Which is continuous on Δ since $1-z \neq 0$ .Hence f is analytic.

Example 1.1.2

Let $f(z)=\frac{z}{(1-z)^2}$ be a function on Δ . To show that f is analytic, want to find f'

Then $f'(z)=\frac{1+z}{(1-z)^3}$ since $1-z \neq 0$,then f' is continuous on Δ .Hence f is analytic on Δ.

1.2 The Univalent Function

Definition 1.2.1

Let f be a complex function defined on Δ. We say f is univalent in domain Δ if whenever $f(z_1)=f(z_2)$ implies $z_1=z_2$, or when $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \Delta$.

Example 1.2.1

Let $f(z)=\frac{z}{1-z}$ be a function on Δ.

To show that f is univalent, suppose $f(z_1)=f(z_2)$ for some $z_1, z_2 \in \Delta$. Then

$$f(z_1)=\frac{z_1}{1-z_1}=\frac{z_2}{1-z_2}=f(z_2)$$

since $1-z_1 \neq 0$, $1-z_2 \neq 0$ then multiplying both sides by $(1-z_1)(1-z_2)$, we get

$z_1(1-z_2)=z_2(1-z_1)$, there fore $\rightarrow z_1-z_1z_2=z_2-z_1z_2$ and hence $z_1=z_2$.

So f is univalent in Δ.

Example 1.2.2

Let $f(z)=\dfrac{z}{(1-z)^2}=z+2z^2+\cdots+nz^n+\cdots$

(Koebe function) be a function on Δ.

To show that f is univalent, suppose $f(z_1)=f(z_2)$ for some $z_1, z_2 \in \Delta$ then

$\dfrac{z_1}{(1-z_1)^2}=\dfrac{z_2}{(1-z_2)^2}$, since $1-z_1 \neq 0$, $1-z_2 \neq 0$, then multiplying both sides by

$(1-z_1)^2 \cdot (1-z_2)^2$, we get

$$z_1(1-z_2)^2=z_2(1-z_1)^2$$

Therefore

$$z_1(1-2z_2+z_2^2)=z_2(1-2z_1+z_1^2)$$

$$z_1-2z_1z_2+z_1z_2^2=z_2-2z_1z_2+z_2z_1^2$$

$$z_1+z_1z_2^2=z_2+z_2z_1^2$$

$$z_1+z_1z_2^2-z_2-z_2z_1^2=0$$

$$z_1-z_2+z_1z_2^2-z_2z_1^2=0$$

$$(z_1-z_2)+z_1z_2(z_2-z_1)=0$$

$$(z_1-z_2)(1-z_1z_2)=0$$

Since $1-z_1z_2 \neq 0 \quad (z_1, z_2 \in \Delta)$, so $z_1 = z_2$. Hence f is a univalent in Δ.

Example 1.2.3

Let $f(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ be a function on Δ .

To show that f is univalent, suppose $f(z_1) = f(z_2)$ for some $z_1, z_2 \in \Delta$ then

$\frac{1}{2}\log\frac{1+z_1}{1-z_1} = \frac{1}{2}\log\frac{1+z_2}{1-z_2}$ it is clearly that $\log\frac{1+z_1}{1-z_1} = \log\frac{1+z_2}{1-z_2}$, we get

$\frac{1+z_1}{1-z_1} = \frac{1+z_2}{1-z_2}$,since $1-z_1 \neq 0, 1-z_2 \neq 0$ then multiplying both sides by

$(1-z_1)(1-z_2)$,therefore

$$(1+z_1)(1-z_2) = (1+z_2)(1-z_1)$$

$$1-z_2+z_1-z_1z_2 = 1-z_1+z_2-z_1z_2$$

It is clearly that $-z_2-z_2+z_1+z_1 = 0$,and so $2z_1 - 2z_2 = 0$ .So clearly $z_1 = z_2$

Hence f is univalent on Δ.

To know how the univalent function behave in a neighborhood of some point z_0, let f be analytic at z_0 and $f'(z_0) \neq 0$, then there is a neighborhood $N(z_0)$ of z_0 such that f is univalent in $N(z_0)$.

Theorem 1.2.1

Let $f'(z_0) = 0$ then $f(z)$ is not univalent in any neighborhood of z_0.