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Thesis Approval

Stability Theory And Asymptotic Behavior of Difference Equations.

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Abstract

The main goal of this thesis is to study the qualitative behavior, the stability, and the asymptotic behavior and the global asymptotic behavior of solutions of difference equations without the need of hopefully solving them.

First, we present the stability theory of first order difference equations , system of linear difference equations, nonlinear system of difference equations , and higher order difference equations. Moreover, we present the stability theory of nonlinear systems of difference equations by using linearization methods.

The asymptotic behavior of difference equations is investigated through the generalization of Poincare-Perron theorems for difference equations, where we introduce a similar result for the Poincare difference systems.
The global asymptotic behavior is investigated through Liapunov stability theorem and manifold geometries as well.

We apply several results in this work for some important examples of difference equations, we study the stability and the asymptotic behavior of coupled systems of rational difference equations, Volterra difference equations of convolution type, and equations of non-convolution type.

الخلاصة

الهدف الرئيس لهذه الرسالة هو دراسة السلوك الكيفي ، نظرية الثبات والسلوك التقاربي والسلوك التقاربي العالمي لحل معادلات فرقية دون الحاجة إلى حلها .

في البداية تم عرض نظرية الثبات للمعادلات الفرقية من الدرجة الأولى ، النظام الخطي للمعادلات الفرقية ، النظام الغير خطي للنظريات الفرقية والمعادلات الفرقية من الدرجة العليا . إضافة إلى ذلك فقد تم عرض نظرية الثبات للنظام غير الخطي للمعادلات الفرقية باستخدام طرق التخطيط .

تم التحقق من السلوك التقاربي للمعادلات الفرقية من خلال تعميم نظريات (Poincare-Perron) للمعادلات الخطية ، حيث تم تقديم نتائج مشابهة لأنظمة (Poincare) الفرقية .

أما السلوك التقاربي العالمي فقد تم التحقق منه من خلال نظريات الثبات لـِ (Liapunov) والدراسات الهندسية المتنوعة أيضا .

فقد نم تطبيق العديد من نتائج هذا العمل على أمثلة مهمة للمعادلات الفرقية . كما تم دراسة نظرية الثبات والسلوك التقاربي لنظامين للمعادلات الفرقية النسبية وهي نظريات (Volterra) الفرقية الالتفافية والمعادلات الفرقية غير الالتفافية .

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Introduction

This thesis mainly consists of two topics, the first topic is the stability theory of difference equation, the second is the asymptotic behavior of difference equations. In our investigation we study the qualitative behavior of solutions of difference equations without the need of solving them (if possible).

The thesis consists of four chapters: chapter one is basically a theoretical introduction to difference equations, it contains some basic theorems and definitions, we show the relation between difference equations and dynamical systems, also we present the notion of equilibrium points, dynamics of first order differences equations, periodic points, systems of linear difference equations, and difference equations of higher order.

Chapter two investigates the stability of first order difference equations (i.e. dynamical system of one dimension), for this investigation we use formulas which were published by Faa 'di Bruno[13], in section two the stability of system of linear difference equations which was studied by Elaydi [8] is considered , furtheremore we present stability theory of nonlinear systems of difference equations by using linearization methods .In the last section of chapter two we investigate stability of higher order scalar difference equations by using Schur-Cohn Criterion[8] .

Chapter three deals with asymptotic behavior of higher order difference equations, so we investigate asymptotic behavior for difference equations of Poincare type, and Poincare's difference systems which were studied by Mate[14], Pituk[15].[16] they are considered as generalizations of Poincare- Perron theorems for Poincare's systems and pituk in[16] .

The stability and asymptotic behavior for some examples are the content of Chapter four where in section one we present the global stability of system of rational difference equations, which was studied by Clark, Kulenovic [4].

In section two we present the Volterra difference equation of convolution type this example was studied by Elaydi[10], where he used the Liapunov stability theorem , and the idea of Z – transform.

In the last section we analyse the asymptotic behavior of the Volterra difference equation of nonconvolution type which was studied by Gyor, Horvath[12].

Chapter One

Preliminaries

1.1 Introduction

Difference equation usually describe the evolution of the certain phenomena over the course of time , consider the general difference equation

$$x(n+1) = f(x(n)) \text{ , where } \quad f : R \to R \qquad (1.1.1)$$

We may look at this problem as follows:

starting from an initial point x_0, one may generate the sequence .

$$x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots$$

$f(x_0) = x_1$ is called the first iterate of x_0 under f .

$f^2(x_0) = f(f(x_0)) = x_2$ is called the second iterate of x_0 under f .

In General $f^n(x_0) = x_n$ is the nth iterate of x_0 under f.

This iterative procedure is an example of a discrete dynamical system , so dynamical systems described by a set of (n) variables $x_1, x_2, \ldots, x_n$ which is specified by the difference equation

$$x(n+1) = f(x(n))$$

Remark :

The difference equation and discrete dynamical systems are equivalent in the sense that they represent the same physical model . For instance, when we talk about difference equations we usually refer to the analytic theory of subject, and when we talk about discrete dynamical system, we generally refer to its geometrical and topological aspects.

Difference equation (1.1.1) is called autonomous or time-invariant .

If the function f in (1.1.1) is replaced by a function g of two variables $g: Z^+ \times R \to R, Z^+$ is the set of nonnegative integers then we have

$$x(n+1) = g(n, x(n)) \qquad (1.1.2)$$

This equation is called nonautonomous or time-variant

Remark :

The study of (1.1.2) is much more complicated and does not lend itself to the discrete dynamical system theory of first-order equation .

1.2 Equilibrium Points :

The notion of equilibrium point is central in the study of the dynamics of any physical system .

Definition 1.2.1 .[8,18]

A point $\bar{x}$ in the domain of f is said to be an equilibrium point of (1.1.1) if it is a fixed point of f, i.e. $f(\bar{x}) = \bar{x}$.

In other word $\bar{x}$ is a constant solution of (1.1.1) , because if $x(0) = \bar{x}$ is an initial point , then $x(1) = f(x(0)) = f(\bar{x}) = \bar{x}$ and $x(2) = f(x(1)) = f(\bar{x}) = \bar{x}$ and so on.

Example 1.2.1

Take the difference equation.

$$x(n+1) = x^3(n)$$

Where $f(x) = x^3(n)$, there are three equilibrium points

If we let $f(\bar{x}) = \bar{x}.$ $i.e. \bar{x}^3 = \bar{x}$ then $x = -1, 0, 1$

Example 1.2.2

The equation

$$x(n+1) = x^2(n) - x(n) + 1$$

has only one equilibrium point , $\bar{x} = 1$.

Graphically , an equilibrium point is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$.

We can see from Figure (1.2.1) , Figure (1.2.2) the equilibrium points in examples 1.2.1 , 1.2.2 respectively .

1.3 Dynamics Of First Order Difference Equation

In this section we will study the possible behaviors of one dimensional discrete dynamical systems.

We consider equation (1.1.1) with. $f(x) = ax(n) + b, x(0) = x_0$

Where a and b are constants .

Suppose $b = 0$ i.e $x(n+1) = ax(n)$

Then we have by iteration $x(n) = a^n x_0$

If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$ and so $x(n) \to 0$

If $|a| > 1$, a^n explodes as $n \to \infty$. thus unless x_0=0 , we have $|x(n)| \to \infty$

If $|a| = 1$, we have two cases

1) If $a = 1$ then $x(0) = x(1) = ..., i.e. x(n) = x_0$

2) If $a = -1$ then $x(0) = -x(1) = x(2) = -x(3) = ...$ that is $x(n)$ alternates between x_0 and - x_0 forever .

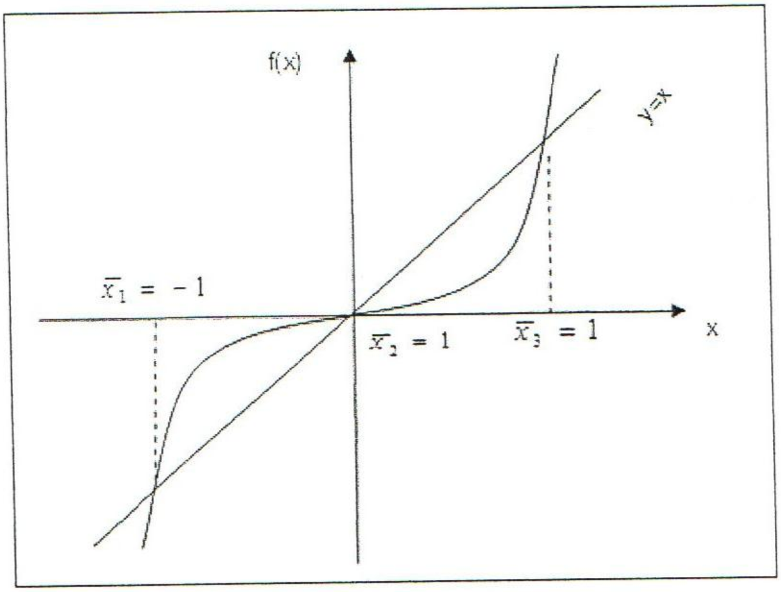

Fig (1.2.1) . Fixed points of $f(x) = x^3$

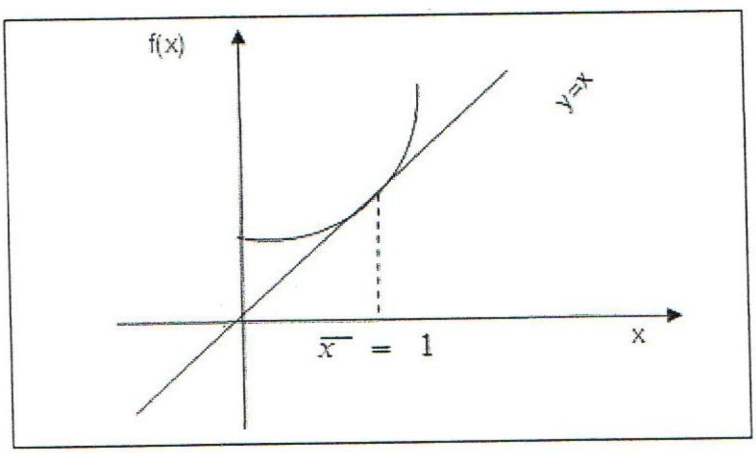

Fig (1.2.2) . Fixed points of $f(x) = x^2 - x + 1$

$x(n+1) = ax(n) + b$

$x(0) = x_0$

$x(1) = ax(0) + b = ax_0 + b$

$x(2) = ax(1) + b = a^2 x_0 + ab + b$

$x(3) = ax(2) + b = a^3 x_0 + a^2 b + ab + b$

So we have

$x(n) = a^n x_0 + (a^{n-1} + a^{n-2} + ... + a + 1)b$.

By noticing that $a^{n-1} + a^{n-2} + ... + a + 1$ is a geometric series which is equal to

$\frac{a^n - 1}{a - 1}$, provided , $a \neq 1$

$$x(n) = \begin{cases} a^n x_0 + \left(\frac{a^n - 1}{a - 1}\right)b & , a \neq 1 \\ x_0 + nb & , a = 1 \end{cases}$$

We will analyzes this solution in three cases .

$|a| < 1$, $|a| > 1$, $|a| = 1$

Case (1) : If $|a| < 1$,then $a^n \to 0$ as $n \to \infty$ and so

$x(n) \to \frac{b}{1-a}$, this special number $\bar{x} = \frac{b}{1-a}$ is a fixed point , i.e. $f(\bar{x}) = \bar{x}$

we call $\bar{x}$ an attractive or stable fixed point of the dynamical system because the system is attracted to this point .

Case (2) : If $|a| > 1$

$a^n \to \infty$, as $n \to \infty$, we can show how this affects $x(n)$;

$$x(n) = a^n x_0 + \left(\frac{a^n - 1}{a - 1}\right)b = a^n \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

if $x_0 \neq \frac{b}{1-a}$, then $|x(n)| \to \infty$, as $n \to \infty$, and

if $x_0 = \bar{x}$, then $x(n) = \bar{x}$,for all time

case (3) : If $|a| = 1$

1) if $a=1$, then $x(n)=x_0+nb$, so if $b \neq 0$ then

$|x(n)| \to \infty$ otherwise $(b=0)$ $x(n)$ is stuck at x_0 regardless of the value of x_0

2) if $a=-1$ then

$$x(0)=x_0$$
$$x(1)=-x_0+b$$
$$x(2)=-(-x_0+b)+b=x_0$$
$$x(3)=-x_0+b$$
$$x(4)=x_0$$
$$\vdots$$

Thus $x(n)$ oscillates between two values x_0 and $b-x_0$

But if $x_0=b-x_0$ i.e $x_0=b/2=\dfrac{b}{1--1}=\bar{x}$ then $x(n)$ is stuck at the fixed point $\bar{x}$.

We can summarize the behavior of the system

$x(n+1)=ax(n)+b \quad ,x(0)=x_0$

Table 1.1 : The possible behavior of one dimensional linear discrete dynamical system

Discrete : $x(n+1) = ax(n) + b$ with $x(0) = x_0$			
Conditions on			Behavior of
a	b	x_0	$x(n)$ as $n \to \infty$
$\lvert a\rvert < 1$	-	-	$\to \dfrac{b}{1-a}$
$\lvert a\rvert > 1$	-	$\neq \dfrac{b}{1-a}$	blows up
	-	$= \dfrac{b}{1-a}$	fixed at $\dfrac{b}{1-a}$
$a = 1$	$b \neq 0$	-	blows up
	$b = 0$	-	fixed at x_0
$a = -1$	-	$\neq b/2$	oscillates : x_0, $b - x_0$, ...
	-	$= b/2$	Fixed at $b/2$

We can understand the behavior by plotting graphs , this is known as graphical analysis.

Figures 1.3.1 through 1.3.6 explain most of the cases in the table.

In each figure we have drawn the function $y = f(x)$, varying the value of a, and we have drawn the line $y = x$, then we choose a starting point $x_0 = x(0)$ on the x – axis .

Draw a line straight up to the line $y = f(x)$, the y-coordinate of this point is the next number to which we wish to apply f , then we draw a line horizontally from $(x(0), f(x(0)))$ to line $y = x$, we have found the point $(f(x(0)), f(x(0))) = (x(1), x(1))$ if we drop a line down to x-axis , this point is $x(1)$ By continuing this procedure we can find $x(n)$ for all $n \geq 0$ and understand the behavior of solutions.

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