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Dynamics of First Order Systems of Difference Equations

by

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Declaration

I Cerify that this thesis, submitted for the degree of Master, is the result of my own research except where otherwise acknowledged ,and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution .

signed ....

Bashir saleh Mohammed Abdellah

Date:January 29 ,2006

Dedication

To my mother ,

and my dear wife ,

and my daughter Luna ,

and my son Mohammed,

I dedicate this work.

Acknowledgments

I would like to express my gratitude to all those who helped me to prepare and complete this work ,specially and personally to my supervisor ,Dr .Tahseen Mughrabi for his help and advice through the period of study.

My thanks to my classmates of the mathematics department at Al-Quds University.

To them all, I extend my sincere love and regardes.

Abstract

The purpose of this work is to study and investigate the dynamics and qualitative behavior of the first order systems of difference equations(discrete dynamical systems) :

$$x(n+1) = f(x(n))$$

where f is some function on $\Re^k$ and $x(n) \in \Re^k$.

We study the asymptotic behavior and stability of hyperbolic and nonhyperbolic equilibrium points which leads us naturally to address the phenomenon of local bifurcation[15] which is a characteristic of systems that depends on a vector parameter and has nonhyperbolic equilibrium point.

Basically ,it is well known that the nonlinear systems are in general unsolvable ,hence carrying out geometrical and qualitative analyses is also of great concern in this thesis .To accomplish this we introduce efficient tools for two-dimensional systems : one way is using phase space analysis and an alternative way is the linearization method in which a linear system is used instead of a nonlinear system to investigate the behavior of the nonlinear system where the eigenvalues of the Jacobian matrix $D_x f\big|_{x^*}$ of $f(x(n))$ evaluated at an equilibrium point x* of the system are used to classify the type of x* with the help of a key theorem which is an analogue to Hartmann-Grobman theorem.[7] in the case of continous dynamical systems .

Furthermore,we introduce the well-known technique of Lyapunov [3] to study global behavior of the equilibrium points of a system where the idea of a special positive definite function V defined on R^k is used to classify the equilibrium point x* of a discrete dynamical system.

الخلاصة

الهدف من هذه الرسالة هو دراسة وبحث السلوك الديناميكي والنوعي لأنظمة من الدرجة الأولى

للمعادلات الفرقية(الأنظمة الديناميكية المنفصلة) التي تأخذ الشكل التالي

$$x(n+1)=f(x(n))$$

حيث أن f معرف على $\Re^k$, $x(n) \in \Re^k$

وكذلك في هذه الرسالة تطرقنا لدراسة السلوك التقاربي والثبات لنقاط الاتزان من نوع القطع

الزائد والقطع غير الزائد التي توصلنا الى ظاهرة تعرف باسم الشق والتي تعتمد على المتجه

البارامتري وتكون نقطة الاتزان ليست من نوع قطع زائد.

وبشكل عام من المعروف ان الأنظمة غير الخطية لاتحل ولهذا عملنا على التحليل النوعي

والهندسي لحلول مثل هذه الأنظمة لما فيه من اهمية عظمى في تطبيقات هذه الأنظمة. لتحقيق ذلك استخدمنا تعريفات ونظريات مهمة والتي

تفيدنا في طريقة المستوى البياني وكذلك في الطريقة الخطية. فالطريقة الخطية تمكننا من معرفة سلوك النظام غير الخطي عند نقطة الاتزان وذلك بالاعتماد على

سلوك النظام الخطي عند نقطة الاتزان

علاوة على ذلك تطرقنا الى تعريف طريقة مشهورة الا وهي طريقة ليابانوف حيث أن الفكرة الأساسية في هذه الطريقة ايجاد اقتران خاص

موجب محدد والذي يمكننا من تحديد نوع الاتزان للنظام

الديناميكي المنفصل .

Contents

Introduction

Many real-world phenomina and their essential properties can be modeled by using systems of difference equations . The resulting mathematical models and their solutions are then employed to the many applications of systems of difference equations in economics ,biology , statistics and communications just to mention some.

In this work our main goal is the study of the dynamics of solutions for systems of difference equations which means investigating the qualitative behavior of solutions rather than finding them as well as studying the asymptotic behavior and stability of equilibrium solutions .Moreover ,we study local bifurcations of hyperbolic and nonhyperbolic equilibrium points.

In chapter one we will be concerned with the basic theory underlying essential topics in dynamical systems such as existence and uniqueness of solutions where we represent key theorems and basic concepts .Moreover ,the notion of a fundamental matrix and some of its basic properties ,and the idea of a state transition matrix and its role in solving nonhomogeneous autonomous systems are presented .In the last section of chapter one we present a modern and fundamental tool used to analyze the behavior of solutions . This method is of a geometrical nature and it utilieses the invariant manifold theory results .

Chapter two is devoted to present three analytical methods used to solve first order linear dynamical systems **:**The first method called the Putzer algorithm technique which is based on the idea of calculating the matrix A^n and then using it to get the general solution of the system $x(n+1) = A(n)x(n)$ .The second method which is called Jordan method is used to solve autonomous systems ,it involves the transformation of the matrix A into a Jordan form J which in turn is employed to determine the solution of a given system .

The third method uses the Z-transform operator which is an analogue to the continous Laplace transform used to solve systems of ordinary differential equations with constant coefficient .In our case the Z-transform operator transform the dynamical system into an algebraic system ,then employing the inverse Z-transform one can get the solution.

The remaining two chapters are the core of our present work. In chapter three the stability properties of solutions for first order systems of difference equations is developed .Here,the emphasis is on qualitative and asymptotic behavior of solutions ,rather than solving the system.This investigation is so vital since in applications most of the models are nonlinear and hence they are unsolvable analytically .To complete what is initiated in section three of chapter one , concerning the geometrical analysis theory, in section two we introduce the phase space analysis for the solutions of two-dimensional systems (i.e the phase space is the xy –plane).This analysis helps us decide on the stability of equilibrium solutions depending on the eigenvalues of the transformation matrix J and their algebraic and geometric multiplicites. Furthermore ,to deal with nonlinear systems the idea of linearization is used , this means approximating the behavior of a nonlinear system by its linearized version which is confirmed by a key theorem [15] which is
a variant of Hartmann-Grobman theorem in the continous case .Also ,we introduce the well-known technique due to Lyapunov[3].This method addresses both local and global asymptotic stability behavior of equilibrium points.

An essential topic concerning the local stability of a dynamical system upon changing a parameter μ is investigated in chapter four .It is well known that such systems undergo bifurcation phenomina .To study the local bifurcation behavior we introduce many tools and ideas such as the center manifolds theorem which pave the way to studying local bifurcations .We elaporate on the notion of hyperbolic and nonhyperbolic equilibrium points regarding their bifurcations .Here ,we have three types of behavior to discuss corresponding to eigenvalues of modulus one namely : the saddle node bifurcation ,the transcritical bifurcation and the pitchfork bifurcation ,where we discuss their characteristics and show graphically their geometric structure.

Chapter One
Theory of Systems of Difference Equations

1.1 Introduction

This chapter introduces the basic theory underlying topics in difference equations related to existence and uniqueness of solutions where we represent key theorems relevant to these topics.Moreover ,the notions of a fundamental matrix and transition matrix are established. Section three represents the geometrical theory which is an essential tool in analyzing the topological and asymptotic behavior of solutions for systems of difference equations .
To illustrate these ideas ,we study two examples using the techniques and methods provided by theory.

1.2 Existence and Uniqueness Theory

This section is devoted to studying solutions of first order linear nonhomogeneous systems of difference equations.

$$
\begin{aligned}
x_1(n+1) &= a_{11}(n)\, x_1(n) + a_{12}(n)\, x_2(n) + \dots + a_{1k}(n)\, x_k(n) + g_1(n) \\
x_2(n+1) &= a_{21}(n)\, x_1(n) + a_{22}(n)\, x_2(n) + \dots + a_{2k}(n)\, x_k(n) + g_2(n) \\
&\vdots \\
x_k(n+1) &= a_{k1}(n)\, x_1(n) + a_{k2}(n)\, x_2(n) + \dots + a_{kk}(n)\, x_k(n) + g_k(n)
\end{aligned}
$$

This system can be written in the matrix form

$$x(n+1) = A(n)x(n) + g(n) \tag{1.2.1}$$

where $x(n)=[x_1(n),x_2(n),x_3(n),\dots,x_k(n)]^T$ and $A=(a_{ij}(n))$ is a k x k nonsingular matrix

$$g(n)=[g_1(n),g_2(n),\dots,g_k(n)]^T$$

Remark: if $g(n)\equiv 0$ in equation(1.2.1) then ,we get

$$x(n+1) = A(n)x(n) \qquad (1.2.2)$$

and this system (1.2.2) is said to be linear homogeneous system.

If for some $n_0 \geq 0$, $x(n_0)=x_0$ is specified then the system (1.2.1) and the initial data $x(n_0)=x_0$ is called an initial value problem .

Remark:if A(n) ≡A in equation (1.2.1) then ,we get

$$x(n+1) = A\,x(n) + g(n) \qquad (1.2.3)$$

where A is a square constant matrix .

Remark:if A(n) ≡A in equation (1.2.2) then ,we get

$$x(n+1) = A\,x(n) \qquad (1.2.4)$$

where A is a square constant matrix .

We start by considering the existence and uniqueness of a solution .

Theorem 1.2.1 :[3]

Let $x_0 \in \Re$,$n_0 \in Z^+$ and let the matrix $A(n)$ be nonsingular ,then the difference equation

$x(n+1)=A(n)x(n)$ has a unique solution $x= x(n , n_0 , x_0)$with $x(n_0 ,n_0 ,x_0)=x_0$

Where $\Re$ is the set of the real numbers.

Proof:

From equation(1.2.2),we have

$$x(n_0+1, n_0,x_0)=A(n_0)x(n_0)=A(n_0)x_0$$

$$x(n_0+2, n_0,x_0)=A(n_0+1)\,x(n_0+1)=A(n_0+1)\,A(n_0)x_0$$

Inductively , one may conclude that

$$x(n,n_0,x_0)=\left(\prod_{i=n_0}^{n-1}A(i)\right)x_0 \tag{1.2.5}$$

where

$$\prod_{i=n_0}^{n-1}A(i)=\begin{cases}A(n-1)A(n-2)...A(n_0) & if\ \ n>n_o\\ I & if\ \ n=n_o\end{cases}$$

Formula (1.2.5)gives the unique solution with the desired properties.

We will now develop the notion of a fundamental matrix for first order systems of difference equations ,and we will show that the space of such solution is a linear space.

Definition 1.2.2:[3,13]

The solutions $x_1(n),x_2(n),...,x_k(n)$ of equation(1.2.2) are called linearly independent for all $n\geq n_0\geq 0$,whenever $c_1x_1(n)+c_2x_2(n)+...+c_kx_k(n)=0$ for all $n\geq n_0$ implies that $c_i=0$, $i=1,2,3,...,k$. .

Definition 1.2.3:[3,12]

Let $\Phi(n)$ be a kxk matrix with columns that are solutions of the system (1.2.2),and suppose that $\Phi(n)$ is invertible for all $n\geq n_0$, then $\Phi(n)$ is said to be a fundamental matrix for the system.

Theorem1.2.4:[3,10]

The system (1.2.2) , $x(n+1)=A(n)x(n)$,has k-linearly independent solutions for $n \geq n_0$,which form a k-dimensional space.

proof

Let $S=\{x_1(n), x_2(n), \ldots, x_k(n) \ : n \geq n_0\}$ denote the set of all solutions of system (1.2.2) and let $x_1(n), x_2(n) \in S$ then $\alpha_1 x_1(n)+\alpha_2 x_2(n) \in S$ since

$$(\alpha_1 x_1+\alpha_2 x_2)(n+1) = \alpha_1 x_1(n+1)+\alpha_2 x_2(n+1)$$

$$=\alpha_1 A(n) x_1(n)+\alpha_2 A(n) x_2(n)$$

$$=A(n)(\alpha_1 x_1(n)+\alpha_2 x_2(n)) \text{ , for } n \geq n_0$$

Hence S is a linear space

Next,we show that S is of dimension k this means that we must find k-linearly independent solutions $x_1(n), x_2(n), \ldots, x_k(n)$ that span S..

To this end, choose a set of k linearly independent vectors $\xi_1, \xi_2, \ldots \xi_k$ in the k- dimensional $\Re^k$ space by theorem (1.2.1) there exist k solutions $x_1, .. x_k$ of system (1.2.2) such that $x_1(n_0)= \xi_1, \ldots x_k(n_0)= \xi_k$ if on the contrary these solutions are linearly dependent , then there exist scalars $\alpha_1, \ldots, \alpha_k . \in R$,not all zero ,such that

$$\sum_{i=1}^{k} \alpha_i x_i(n) = 0$$

for all $n \geq n_0$,.this implies, in particular, for $n=n_0$,

$$\sum_{i=1}^{k} \alpha_i x_i(n_0) = \sum_{i=1}^{k} \alpha_i \xi_i = 0$$

But this contradicts the assumption that $\{\xi_1, \xi_2, \ldots \xi_k\}$ is a linearly independent set.

Therefore the solutions $x_1(n), x_2(n), \ldots, x_k(n)$ are linearly independent .Finally we must show that the solutions $x_1, \ldots, x_k$ span S .Let $x(n)$ be any solution of system (1.2.2) for $n \geq n_0$ such that $x(n_0)= \xi$ then there exist a unique scalar $\alpha_1, \ldots, \alpha_k$ .in $\Re$ such that

$$\xi = \sum_{i=1}^{k} \alpha_i \xi_i$$

since by assumption the vectors $\xi_1, \xi_2, \ldots \xi_k$ form a basis for $\Re^k$ space .Now, let $y(n)$ be another solutionof system (1.2.2) for $n \geq n_0$ such that $y(n_0)= \xi$ and

$$y(n)=\sum_{i=1}^{k}\alpha_i\, x_i(n)$$

But by theorem 1.2.1 we have

$$x(n)=y(n)=\sum_{i=1}^{k}\alpha_i\, x_i(n)$$

since x(n) was chosen arbitrary , it follows that the solutions $x_1(n),\ldots,x_k(n)$ span S

Definition1.2.5 :[3,12]

A set $S=\{x_1(n),x_2(n),\ldots,x_k(n)\}$ of k linearly independent solutions of system (1.2.2) is called a fundamental set for the system .

Some basic properties of a fundamental matrix are given by the following theorems.

Theorem 1.2.6:[3,10]

A fundamental matrix $\Phi(n)$ of system (1.2.2) satisfies the matrix difference equation

$$\Phi(n+1)=A(n)\Phi(n) \qquad (1.2.6)$$

Proof:

Let $\Phi(n)$ be a kxk matrix whose columns are solutions of the system (1.2.2) ,we write

$$\Phi(n)=[x_1(n),x_2(n),\ldots,x_k(n)].$$

Now , by equation(1.2.6),we have

$$\Phi(n+1)=[A(n)x_1(n),A(n)x_2(n),\ldots,A(n)x_k(n)].$$

$$=A(n)\,[x_1(n),x_2(n),\ldots,x_k(n)].$$

$$=A(n)\,\Phi(n)$$

Hence , $\Phi(n)$ satisfies the matrix equation (1.2.6)

Theorem 1.2.7:[3,12]

If Φ is a fundamental matrix for system (1.2.2) and if C is any nonsingular constant kxk matrix then $\Phi\ C$ is also is a fundamental matrix for the system .

Proof :

Let Φ be a fundamental matrix for the system (1.2.2) and let C be any nonsingular constant kxk matrix then ,we have

$$(\Phi\ C)(n+1)= \ (\Phi\ )(n+1)C$$

$$=A(n)\ (\Phi)(n)C$$

$$= A(n)\ (\Phi C)(n)$$

and hence ΦC is asolution of the matrix equation (1.2.6) but

$$det\ (\Phi\ C)= det\Phi\ detC\neq 0$$

Therefore ΦC is a fundamental matrix

Next ,let us study the structure of the solutions of system (1.2.1) and of system(1.2.2). In doing so ,we need to introduce the concept of the state transition matrix .

In the following definition ,we use the natural basis$\{e_1,e_2,...,e_k\}$ for the linear space S.

Definition 1.2.8 :[3]

A fundamental matrix Ψ for system (1.2.2) whose columns are determined by the linearly independent solutions $x_1(n), x_2(n),....,x_k(n)$ with $x_1(n_0)=e_1,...x_k(n_0)=e_k$ i.e $\Psi(n_0)=I$ is called the state transition matrix(resolvent matrix) for the system .

Equivalently if Φ is any fundamental matrix for system (1.2.2) then the matrix Ψ determined by $\Psi(n, n_0)= \Phi(n)\ \Phi^{-1}(n_0)$ for all $n\geq n_0$ is called the state transition matrix .

The following theorem states the most important properties of such matrices.

Theorem 1.2.9:[3,10]

Let $\Phi(n_0)=\xi$ and let $\Psi(n, n_0)$denote the state transition matrix for the system (1.2.2)for all $n \geq n_0$,then

(i) $\Psi(n, n_0)$ is the unique solution of the matrix equation

$$\Psi(n+1, n_0)= A(n)\, \Psi(n, n_0)$$

(ii) $\Psi(n, n_0)$ is nonsingular for all $n \geq n_0$

(iii) $\Psi(n,r)\, \Psi(r, n_0)= \Psi(n, n_0)$ for all $n \geq r \geq n_0$

(iv) $\Psi^{-1}(n, n_0)= \Psi(n_0,n)$ for all $n \geq n_0 \geq 0$

(v) The unique solution $x(n, n_0,x_0)$ of the system(1.2.2) with $x(n, n_0,x_0)=x_0$ is given by

$$x(n, n_0,x_0) = \Psi(n, n_0)\, x_0 \qquad (1.2.7)$$

(vi) $$\Psi(n,n_0) = \prod_{m=n_0}^{n-1} A(m) \qquad (1.2.8)$$

proof

(i)Using the definition $\Psi(n, n_0)= \Phi(n)\, \Phi^{-1}(n_0)$,we get

$$\Psi(n+1,n_0)= \Phi(n+1)\, \Phi^{-1}(n_0)$$

$$=A(n)\, \Phi(n)\, \Phi^{-1}(n_0)$$

$$=A(n)\, \Psi(n, n_0)$$

(ii)Since for any fundamental matrix Φ of system (1.2.2) $\det\Phi(n)\neq 0$ for $n \geq n_0$,it follows that

$$\det \Psi(n, n_0)= \det(\Phi(n)\, \Phi^{-1}(n_0))= \det\Phi(n) \det \Phi^{-1}(n_0)\neq 0 \quad \text{for } n\geq n_0$$

(iii)For any fundamental matrix $\Phi$of system (1.2.2),we have

$$\Psi(n, n_0)= \Phi(n)\, \Phi^{-1}(n_0)$$

$$\Psi(n, n_0)= \Phi(n)\, I\, \Phi^{-1}(n_0)$$

$\Psi(n, n_0) = \Phi(n)\, \Phi^{-1}(r)\, \Phi(r)\, \Phi^{-1}(n_0)$

$= \Psi(n,r)\, \Psi(r, n_0)$ *for* n,,$r \geq n_0$

(iv)For any fundamental matrix Φ of system (1.2.2)

$\Psi^{-1}(n, n_0) = (\Phi\,(n)\, \Phi^{-1}(n_0))^{-1}$

$= \Phi\,(n_0)\;\; \Phi^{-1}(n)$

$= \Psi(\,n_0,n)$ for n, $n_0 \geq n_{0.}$

(v) By theorem (1.2.1) we know that the system (1.2.2) has a unique solution $x(n)$ for all $n \geq n_0$ with $x(n_0)=x_0$ .To show that (1.2.7) is indeed a solution of system (1.2.2) Note, first that $x(n_0)=x(n_0, n_0)x_0=x_0$,then we have

$x(n+1, n_0,x_0) = \Psi(n+1, n_0)\, x_0 \;=A(n)\, \Psi(n, n_0)\, x_0$

$=A(n)\;\; \Phi(n)\, \Phi^{-1}(n_0)\, \Phi(\,n_0) = A(n)\, \Phi(n)$

which shows that (1.2.7) is the desired solution

(vi) $\Phi(n)=A(n$

$-1)\, \Phi(n\text{-}1)= A(n\text{-}1)\, A(n\text{-}2)\, \Phi(n\text{-}2)$

Inductively ,we get

$$\Phi(n) = \prod_{m=n_0}^{n-1} A(m)\Phi(n_0)$$

or

$$\Psi(n, n_0) = \prod_{m=n_0}^{n-1} A(m)$$

The following lemma is the discrete analogue of Abel's formula for continous time dynamical system.

Lemma 1.2.10:[3,12] (Abel’s formula)

For any $n \ge n_0 \ge 0$

$$\det(\Phi(n)) = \left(\prod_{m=n_0}^{n-1} \det A(m)\right) \det \Phi(n_0) \qquad (1.2.9)$$

proof

Muliplying both sides of Eq(1.2.8) by $\Phi(n_0)$ and taking the determinant ,we get

$$\det \Phi(n) = \det\left(\prod_{m=n_0}^{n-1} A(m)\right) \det \Phi(n_0)$$

$$\det \Phi(n) = \prod_{m=n_0}^{n-1} \det(A(m)) \det \Phi(n_0)$$

Theorem 1.2.11:[3,10]

The solutions $x_1(n), x_2(n), \dots, x_k(n)$ are linearly independent for $n \ge n_0$ if and only if the matrix $\Phi(n)$ is nonsingular ($det(\Phi(n) \neq 0$) for all $n \ge n_0$

proof

Suppose that $\Phi(n)=[x_1(n), x_2(n), \dots, x_k(n)]$ is a fundamental matrix of system (1.2.2) then the columns of Φ form a linearly independent set . Let $x(n)$ be a nontrivial solution of system (1.2.1) by theorem (1.2.1) there exist unique scalars

$\alpha_1, \dots, \alpha_k . \in \Re$ not all zero such that

$$x(n) = \sum_{i=1}^{k} \alpha_i x_i(n)$$

or $x=\Phi a$ with $a=(\alpha_1, \dots, \alpha_k)^T$

Now, let $n= n_0$ then we have $x(n_0)=\Phi(n_0)$ which is a system of k linear equations ,by construction , this system has a unique solution for any choice of $x(n_0)$.

Hence det $\Phi(n_0) \neq 0$ .It now follows from Abel's formula that det $\Phi(n) \neq 0$ for all $n \ge n_0$.

Conversely , let $\Phi(n)$ be a solution of the matrix equation (1.2.6) and assume that

$\det\Phi(n) \neq 0$ for $n \geq n_0$ then the columns of Φ, $x_1,...x_k$, are linearly independent for all

$n \geq n_0$ .Hence $\Phi(n)$ is a fundamental matrix of system (1.2.2).

In the next results , the structure of the solution of the system(1.2.1) is established .

Theorem 1.2. 12 [3,14] (Variation of Constants Formula)

The unique solution of the initial value problem

$$x(n+1)= A(n)x(n) +g(n) \quad ,x(n_0)= y_0 \tag{1.2.10}$$

is given by

$$x(n,n_0,y_0) = \Psi(n,n_0)y_0 + \sum_{r=n_0}^{n-1} \Psi(n,r+1)g(r) \tag{1.2.11}$$

or more explicitly by

$$x(n,n_0,y_0) = \left(\prod_{m=n_0}^{n-1} A(m) \right) y_0 + \sum_{r=n_0}^{n-1} \left(\prod_{m=r+1}^{n-1} A(m) \right) g(r) \tag{1.2.12}$$

proof :

To verify that $x(n, n_0,x_0)$ given in (1.2.11)is indeed a solution of system(1.2.1) , for

$n \geq n_0$,we proceed as follows:

$$x(n+1,n_0,y_0) = \Psi(n+1,n_0)x_0 + \sum_{r=n_0}^{n} \Psi(n+1,r+1)g(r)$$

$$= A(n)\Psi(n,n_0)y_0 + \sum_{r=n_0}^{n-1} \Psi(n+1,r+1)g(r) + \Psi(n+1,n+1)g(n)$$

$$= A(n)\Psi(n,n_0)y_0 + \sum_{r=n_0}^{n-1} A(n)\Psi(n,r+1)g(r) + \Psi(n+1,n+1)g(n)$$

$$= A(n)\left(\Psi(n,n_0)y_0 + \sum_{r=n_0}^{n-1} A(n)\Psi(n,r+1)g(r)) \right) + \Psi(n+1,n+1)g(n)$$

$$= A(n)\Psi(n,n_0,y_0) + g(n)$$

From (1.2.11) we also note that $x(n_0, n_0,y_0)=y_0$. therefore $x=x(n)$ given in (1.2.10) is a solution of (1.2.1).

Remark when $y_0 = 0$,equation (1.2.11) reduces to

$$x_p(n) = \sum_{r=n_0}^{n-1} \Psi(n,r+1)g(r)$$

Remark when $y_0 \neq 0$ but $g(n)=0$,equation (1.2.11) reduces to

$$x_h(n) = \Psi(n,n_0)\mathrm{y}_0$$

Therefore the solution of equation(1.2.10) has the form

$$x(n) = x_p(n) + x_h(n) \quad ,x(n_0)=n_0$$

It is the unique solution of the system(1.2.1).

Corollary (1.2.13) : **[3]** for autonomous systems ,when A is a constant matrix ,the solution of equation (1.2.1) is given by

$$x(n,n_0,y_0) = A^{n-n_0}\mathrm{y}_0 + \sum_{r=n_0}^{n-1} A^{n-r-1}g(r)$$

Example(1.2.14) (Trade Model)

Consider a model of the trade between three countries , restricted by the following assumptions:

(i) National income =consumption outlays + net investment +exports – imports.

(ii) Domestic consumption outlays =Total consumption- imports.

(iii)Time is divided into periods of equal length,denoted by n =0,1,2….

Let for country j =1,2,3

$y_j(n)$=national income in period n

$c_j(n)$=total consumption in period n

$i_j(n)$=net investment in period n

$x_j(n)$=exports in period n

$m_j(n)$=imports in period n

$d_j(n)$=consumption of domestic products in period n.

For country 1 we then have

$y_1(n)=c_1(n) +i_1(n) +x_1(n) -m_1(n)$,

$d_1(n)= c_1(n) - m_1(n)$

which , combining those two equations, gives

$$y_1(n)= d_1(n)+x_1(n) + i_1(n). \quad (1.2.13)$$

Likewise, for country 2 , we have

$$y_2(n)=d_2(n)+x_2(n) + i_2(n) \quad (1.2.14)$$

Like wise ,. for country 3 , we have

$$y_3(n) =d_3(n) +x_3(n) + i_3(n) \quad (1.2.15)$$

We now make the following reasonable assumption : the domestic consumption $d_j(n)$ and the imports $m_j(n)$ of each country at period $(n+1)$ are proportional to the country's national income $y_i(n)$ one time period earlier.Thus,

$$d_1(n+1) = \alpha_1 y_1(n), \qquad m_1(n+1) = \beta_1 y_1(n). \tag{1.2.16}$$

$$d_2(n+1) = \alpha_2 y_2(n). \qquad m_2(n+1) = \beta_2 y_2(n) \tag{1.2.17}$$

$$d_3(n+1) = \alpha_3 y_3(n) \qquad m_3(n+1) = \beta_3 y_3(n). \tag{1.2.18}$$

The constants $\alpha_{i,,}$ β_i are called marginal propensities. Furthermore,α_i , $\beta_i > 0$, for i=1,2,3.Since we are considering a world with three countries,the exports of one must be equal to the imports of the others, i.e.,

$$x_1(n) = m_2(n) + m_3(n) \tag{1.2.19}$$

,

$$x_2(n) = m_1(n) + m_3(n) \tag{1.2.20}$$

$$x_3(n) = m_1(n) + m_2(n) \quad . \tag{1.2.21}$$

Substituting from Eqs.(1.2.16},(1.2.17), (1.2.18) .(1.2.19) ,(1.2.20) ,(1.2.21) into Eqs.(1.2.13)and (1.2.14) ,(1.2.15) leads to

$$\begin{bmatrix} y_1(n+1) \\ y_2(n+1) \\ y_3(n+1) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_2 & \beta_3 \\ \beta_1 & \alpha_2 & \beta_3 \\ \beta_1 & \beta_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} y_1(n) \\ y_2(n) \\ y_3(n) \end{bmatrix} + \begin{bmatrix} i_1(n) \\ i_2(n) \\ i_3(n) \end{bmatrix} \tag{1.2.22}$$

Let us further assume that the net investiments $i_1(n)=i_1$, $i_2(n)=i_2$ *and* $i_3(n)=i_3$ are constants Then Eq(1.2.22) becomes

$$\begin{bmatrix} y_1(n+1) \\ y_2(n+1) \\ y_3(n+1) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_2 & \beta_3 \\ \beta_1 & \alpha_2 & \beta_3 \\ \beta_1 & \beta_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} y_1(n) \\ y_2(n) \\ y_3(n) \end{bmatrix} + \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

by the variation of constants Formula (1.2.11) ,we obtain

$$y(n) = A^n y(0) + \sum_{r=0}^{n-1} i A^{n-r-1}$$

$$= A^n y(0) + i\sum_{r=0}^{n-1} A^r \qquad (1.2.23)$$

Remark: The matrix A^n can be computed by the methods of the next chapter (Putzer algorithm , Jordan–canonical form and Z-transform methods).

Remark:see comments on page 60

Example(1.2.15)

Consider the k[th] order equation

$$y(n+k)+p_1(n)y(n+k\text{-}1)+\ldots+p_k(n)y(n)=g(n) \qquad (1.2.24)$$

This equation may be written as a system of first order equations of dimention k ,let

$z_1(n)=y(n)$

$z_2(n)=y(n+1)=z_1(n+1)$

$z_3(n)=y(n+2)=z_2(n+1),\ldots,z_k(n)=y(n+k\text{-}1)=z_{k\text{-}1}(n+1)$

Let $z(n)=(z_1(n),z_2(n),\ldots,z_k(n))$

Hence

$z_1(n+1)=z_2(n)$

$z_2(n+1)=z_3(n)$

$z_{k\text{-}1}(n+1)=z_k(n)$

$z_k(n+1)=-p_k(n)z_1(n)-p_{k\text{-}1}(n)z_2(n)-\ldots-p_1(n)z_k(n)+g(n)$

In vector notation , the system takes the form

$z(n+1)=A(n)z(n)+h(n)$

where

$$A(n)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ -p_k(n) & -p_{k-1}(n) & -p_{k-2}(n) & \cdots & -p_1(n) \end{bmatrix}$$

is a kxk matrix , and

$$h(n)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g(n) \end{bmatrix}$$

is a column vector of order k

Note if $g(n)=0$,we arrive at the homogeneous system

$$z(n+1)=A(n)z(n) \qquad (1.2.25)$$

As a special case ,we consider the 2nd order equation:

$$y(n+2) - a(n)y(n+k-1) - a(n)y(n)=g(n) \qquad (1.2.26)$$

we may write this equation as a system of first order equations

$z(n+1)=A(n)z(n)+h(n)$,where

$$A(n)=\begin{bmatrix} 0 & 1 \\ a(n) & a(n) \end{bmatrix} \quad ,h(n)=\begin{bmatrix} 0 \\ g(n) \end{bmatrix}$$

$$\Psi(n,n_0)=\prod_{i=n_0}^{n-1} A(i)=\begin{bmatrix} D(n-3)a(n-1) & D(n-2) \\ C(n-2)a(n-1) & C(n-1) \end{bmatrix}$$

where

$$C(n)=C(n-1)a(n)+C(n-2)a(n-1)$$

$$D(n)=D(n-1)a(n+1)+D(n-2)a(n) \qquad \text{for } n \geq n_0+2$$

$$C(n_0-1)=D(n_0-1)=1 \ ,$$

$$C(n_0) = a(n_0)\,,\; D(n_0) = a(n_0+1)$$

$$C(n_0+1) = a(n_0) + a(n_0)a(n_0+1)$$

$$D(n_0+1) = a(n_0+1) + a(n_0+1)a(n_0+2)$$

Hence,the general solution is

$$x(n, n_0, y_0) = \Psi(n, n_0)y_0 + \sum_{r=n_0}^{n-1} \Psi(n, r+1)g(r)$$

1.3 Geometrical Theory

In this section we present the basic theorems underlying the geometric analysis of solutions.

Now consider the following system

$x(n+1)= f(n, x(n))$, $x(n_0)=x_0$ (1.3.1)

where $x(n) \in \Re^k$, $f: Z^+ \times \Re^k \to \Re^k$.,and suppose that $f(n, x)$ is continuous in x .

Definitions(1.3.1):[3,7, 15]

(a) A point $x*$ in $\Re^k$ is called an equilibrium point of system(1.3.1) if $f(n, x*)= x*$ for all $n \geq n_0$. Usually $x*$ is assumed to be the origin 0 and is called the zero solution .

(b) The equilibrium point $x*$of equation(1.3.1) is said to be

(1) Stable if for any $\varepsilon>0$ and $n_0 \geq 0$ there exists $\delta=\delta(\varepsilon,n_0)>0$ such that $||x_0- x*||<\delta$ implies $||x(n,n_0,x_0)- x*||<\varepsilon$ for all $n \geq n_0$.

(2) uniformly stable if δ in (1) may be chosen independent of n_0 ,otherwise $x*$ is said to be unstable.

(3) Attractive if there exist $\mu = \mu(n_0)$ such that $||x_0- x*||<\mu$ implies $\lim_{n\to\infty} x(n, n_0,x_0)= x*$.

(4)Uniformly Attractive if the choice of μ in(3) is independent of n_0 the condition for uniformly attractivity may be paraphrased by saying there exists μ>0 such that for every ε and no there exists $N=N(\varepsilon)$ independent of n_0 , such that

$||x(n,n_0,x_0)- x*||<\varepsilon$ for all $n\geq n_0 + N$ whenever $||x_0- x*||< \mu$.

(5) Asymtotically stable if it is stable and attractive and uniformly Asymtotically stable if it is uniformly stable and uniformly attractive .

(6) Exponentially stable if there exists δ>0,c>0 such that

$||x(n,n_0,x_0)- x*||< c||x_0- x*|| \mu^{n-no}$ whenever $||x_0- x*||<\delta$.

(c) A solution $x(n,n_0,x_0)$ is bounded if for some positive constant M,we have

$||x(n, n_0,x_0)||<M$ for all $n\geq n_0$,where M may depend on each solution .

Definition (1.3.2):[15]

Let $S \subset \Re^n$ be a set ,then S is said to be invariant under the system (1.3.1) if for any $x_0 \in S$ we have $x(n,n_0,x_0) \in S$ for all n.

Definition (1.3.3):[11,15]

An invariant $S \subset \Re^n$ is said to be a C^r (r≥1) invariant manifold if S has the structure of a C^r differentiable manifold.

Definition (1.3.4):[6]

Let x* be an equilibrium point of the nonlinear system of difference equation $x(n+1) = f(x(n))$, and let u be a neighborhood of x* , then we define

(1)The local stable manifold , $W^S_{loc}(x^*)$,of an equilibrium point x* is

$$W^S_{loc}(x^*) := \{x \in u : \lim_{n\to\infty} f^n(x) = x^* and\ f^{(n)}(x) \in u \quad \forall n \in N\} .$$

(2)The local unstable manifold $W_{loc}^{u}(x^*)$ of an equilibrium point x* is

$$W_{loc}^{u}(x^*) := \{x \in u : \lim_{n\to\infty} f^{-n}(x) = x^* \, and \;\; f^{(n)}(x) \in u \quad \forall n \in N\}.$$

(3)The global stable manifold $W^s(x^*)$ of an equilibrium point x* by

$$W^{S}(x^*) := \bigcup_{n\in N} \left\{ f^{(-n)}(W_{loc}^{S}(x^*)) \right\}.$$

(4)The global unstable manifold $W^s(x^*)$ of an equilibrium point x* by

$$W^{u}(x^*) := \bigcup_{n\in N} \left\{ f^{(n)}(W_{loc}^{S}(x^*)) \right\}.$$

Where u ≡B(x*,ε) for some ε >0 ,and $f^n(x)$ is the n^{th} iterate of x under f.

Example (1.3.5): Consider the following system

$$x(n+1) = x^2(n)$$

Let $f(x) = x^2$,and let u∈[-1/2,2] ,then we have

$$f^n(x) = x^{2n}$$

we use definition (1.3.4) to get

$$W_{loc}^{S}(x^*) = [-1/2, 1] \text{ and } W_{loc}^{S}(x^*) =]1, 2].$$

Definition (1.3.6) :[6,15]

Let $f : \Re^k \to \Re^k$ and let $D_x f\big|_{x^*}$ be the Jacobian matrix of $f(x)$ evaluated at an equilibrium point x*, then we define

1) The stable eigenspace , $E^S(x^*)$, of the equilibrium point ,x* by

$E^S(x^*)$ = span {eigenvectors of $D_x f\big|_{x^*}$ whose eigenvalues have modulus less than 1}.

2) The unstable eigenspace , $E^u(x^*)$, of the equilibrium point ,x* by

$E^u(x^*)$ = span {eigenvectors of $D_x f\big|_{x^*}$ whose eigenvalues have modulus greater than 1}.

Theorem(1.3.7):[11,9]

Let $f:\Re^k \to \Re^k$ be a C^1diffeomorphism with a hyperbolic equilibrium point x* .Then there exist locally stable and unstable manifolds $W^s_{loc}(x^*)$ and $W^u_{loc}(x^*)$,that are tangent ,at x* to the eigenspaces $E^S(x^*)$ and $E^u(x^*)$ of the Jacobian matrix $D_x f\big|_{x^*}$,and are of corresponding dimension.

Definition (1.3.8) :[6,11]

Let $p \geq 1$ be an integer . A periodic orbit of period *p* of the map *f* is a set of points

$\{x_1^*, x_2^*, \ldots, x_p^*\}$ such that

$f(x_1^*)=x_1^*, \ldots f(x_{p-1}^*)=x_{p-1}^*, f(x_p^*)=x_p^*$.and $f(x_{p+1}^*) \neq x_{p+1}^*$.

Chapter Two

Indirect Methods For Solving First Order Linear Systems

This chapter deals with linear first order system of difference equations and gives analytical methods to determine their solution by using Putzer algorithm , Jordan–canonical form and Z-transform methods .

2.1 The Putzer Algorithm Method

The Putzer Algorithm method is used to calculate A^n and then uses it for solving the system (1.2.3),

$x(n+1) = Ax(n) + g(n)$,where A is a constant matrix .

The algorithm proceeds as follows:

Let A be a kxk real matrix ,we look for a representation of A^n in the form

$$A^n = \sum_{j=1}^{k} u_j(n) M(j-1) \qquad (2.1.1)$$

where $u_j(n)$ is scalar function ,and

$$M(j) = (A-\lambda_j I)M(j-1)\ , M(0)=I \quad \text{for } j=1,2,...,k . \qquad (2.1.2)$$

$$M(j+1) = (A-\lambda_{j+1} I)M(j)\ , M(0)=I$$

The $u_j(n)$'s will be determined recursively.

By iteration ,we have

$$M(1) = (A-\lambda_1 I)M(0)$$

$$M(2) = (A-\lambda_2 I)(A-\lambda_1 I)M(0)$$

$$M(3) = (A-\lambda_3 I)(A-\lambda_2 I)(A-\lambda_1 I)M(0)$$

Hence, $M(n) = (A-\lambda_n I)(A-\lambda_{n-1} I)....(A-\lambda 1I)$

or

$$M(n)=\prod_{j=1}^{n}(A-\lambda_j I) \tag{2.1.3}$$

Now,using the Cayley –Hamilton theorem ,we obtain

$$M(k)=\prod_{j=1}^{k}(A-\lambda_j I)=0$$

and consequently ,$M(n)=0$ for all $n \geq k$

Thus ,we can rewrite formula (2.1.1) as follows

$$A^n=\sum_{j=1}^{k}u_j(n)M(j-1) \tag{2.1.4}$$

Leting $n=0$ in formula(2.1.4),we get

$$A^0=I=u_1(0)\,I+u_2(0)M(1)+\ldots+u_k(0)M(k-1) \tag{2.1.5}$$

Now, equation(2.1.5) is satisfied if

$$u_1(0)=1 \textit{ and } u_2(0)=u_3(0)=\ldots.=u_k(0)=0 \tag{2.1.6}$$

From formula (2.1.4) ,we have

$$\sum_{j=1}^{k}u_j(n+1)M(j-1)=A^{n+1}=AA^n$$

$$A\sum_{j=1}^{k}u_j(n)M(j-1)=\sum_{j=1}^{k}u_j(n)AM(j-1)$$

Substitution for $AM(j\text{-}1)$ from equation (2.1.2) in the a bove equation gives

$$\sum_{j=1}^{k}u_j(n+1)M(j-1)=\sum_{j=1}^{k}u_j(n)[M(j)+\lambda_j M(j-1)] \tag{2.1.7}$$

Comparing the coefficient of $M(j)$ $1\leq j\leq k$ in equation(2.1.7) and applying condition (2.1.5) we obtain

$$u_1(n+1)=\lambda_1 u_1(n) \qquad u_1(0)=1$$

$$u_j(n+1)=\lambda_j u_j(n)+u_{j-1}(n) \qquad u_j(0)=1 \qquad j=1,2,3,..k \tag{2.1.8}$$

The solutions of equation (2.1.8) are given by

$$u_1(n)=\lambda_1^{\,n} \qquad u_j(n)=\sum_{i=0}^{n-1}\lambda_j^{\,n-1-i}u_{j-1}(i) \qquad j=2,3,\ldots,k \tag{2.1 .9}$$

The two equations (2.1.3) and (2.1.9) together define the algorithm for computing A^n

which is called the Putzer algorithm .

Therefore , the solution of the system (1. 2.3) is given by

$$x(n,n_0,y_0)=A^{n-n_0}y_0+\sum_{r=n_0}^{n-1}A^{n-r-1}g(r)\,.$$

Example (2.1.1)

Consider the following nonhomogeneous system

$x(n+1)=Ax(n)+g(n)$, where

$$A=\begin{bmatrix}2 & 2 & -2\\ 0 & 3 & 1\\ 0 & 1 & 3\end{bmatrix} \quad ,g(n)=\begin{bmatrix}n\\ 1\\ 0\end{bmatrix} \quad ,x_0=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

The eigenvalues of the matrix A are

$$\lambda_1=4,\lambda_2=\lambda_3=2$$

Now , $M(0)=I$

$$M(1)=(A-4I)=\begin{bmatrix}-2 & 2 & -2\\ 0 & -1 & 1\\ 0 & 1 & -1\end{bmatrix}$$

$$M(2)=(A-2I)M(1)=\begin{bmatrix}0 & -4 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Next,we find:

$$u_1(n)=4^{\,n}$$

$$u_2(n) = \sum_{i=0}^{n-1`}(2^{n-1-i})4^i$$

$$= 2^{n-1}\left(\sum_{i=0}^{n-1} 2^i\right)$$

$$= 2^{2n-1} - 2^{n-1}$$

$$u_3(n) = \sum_{i=0}^{n-1`}(2^{n-1-i})(2^{2i-1} - 2^{i-1})$$

$$= 2^{n-2}\left(\sum_{i=0}^{n-1`}(2^i - 1)\right)$$

$$= 2^{2n-2} - 2^{n-2} - n2^{n-2}$$

Using equation(2.1.4),we obtain

$$A^n = 4^n\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + (2^{2n-1} - 2^{n-1})\begin{bmatrix}-2 & 2 & -2\\ 0 & -1 & 1\\ 0 & 1 & -1\end{bmatrix} + (2^{2n-2} - 2^{n-2} - n2^{n-2})\begin{bmatrix}0 & -4 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$A^n = \begin{bmatrix}4^n & 0 & 0\\ 0 & 4^n & 0\\ 0 & 0 & 4^n\end{bmatrix} + \begin{bmatrix}-2^{2n} + 2^n & n2^n & -n2^n\\ 0 & -2^{2n-1} + 2^{n-1} & 2^{2n-1} - 2^{n-1}\\ 0 & 2^{2n-1} - 2^{n-1} & -2^{2n-1} + 2^{n-1}\end{bmatrix}$$

So

$$A^n = \begin{bmatrix}2^n & n2^n & -n2^n\\ 0 & 2^{2n-1} + 2^{n-1} & 2^{2n-1} - 2^{n-1}\\ 0 & 2^{2n-1} - 2^{n-1} & 2^{2n-1} + 2^{n-1}\end{bmatrix}$$

$$x_h = A^{n-n_0}x_0 = \begin{bmatrix}2^n + n2^n\\ 2^{2n-1} + 2^{n-1}\\ 2^{2n-1} - 2^{n-1}\end{bmatrix}$$

$$A^{n-r-1}g(r) = \begin{bmatrix}r2^{n-r-1} - (n-r-1)2^{n-r-1}\\ 2^{2n-2r-3} + 2^{n-r-2}\\ 2^{2n-2r-3} - 2^{n-r-2}\end{bmatrix}$$

$$x_c = \sum_{r=0}^{n-1} A^{n-r-1} g(r) = \begin{bmatrix} 2^n \sum_{r=0}^{n-1} r2^{-r} - 2^{n-1}(n-1)\sum_{r=0}^{n-1} 2^{-r} \\ 2^{2n-3}\sum_{r=0}^{n-1} 2^{-2r} + 2^{n-2}\sum_{r=0}^{n-1} 2^{-r} \\ 2^{2n-3}\sum_{r=0}^{n-1} 2^{-2r} - 2^{n-2}\sum_{r=0}^{n-1} 2^{-r} \end{bmatrix}$$

$$x_c = \begin{bmatrix} 2^n(-n2^{-n+1} - 2^{-n+1} + 2) - 2^{n-1}(n-1)(2 - 2^{-n+1}) \\ 2^{2n-3}\left((2^{-2n+2} - 4)/3\right) + 2^{n-2}(2 - 2^{-n+1}) \\ 2^{2n-3}\left((2^{-2n+2} - 4)/3\right) - 2^{n-2}(2 - 2^{-n+1}) \end{bmatrix}$$

The general solution is

$$x = x_c + x_h = \begin{bmatrix} 2^n + n2^n + 2^n(-n2^{-n+1} - 2^{-n+1} + 2) - 2^{n-1}(n-1)(2 - 2^{-n+1}) \\ 2^{2n-1} + 2^{n-1} + 2^{2n-3}\left((2^{-2n+2} - 4)/3\right) + 2^{n-2}(2 - 2^{-n+1}) \\ 2^{2n-1} - 2^{n-1} + 2^{2n-3}\left((2^{-2n+2} - 4)/3\right) - 2^{n-2}(2 - 2^{-n+1}) \end{bmatrix}$$

2.2 The Jordan form : Autonomous systems

This section presents a second method for solving system (1.2.3),

$x(n+1) = Ax(n) + g(n)$,where A is a constant matrix.

This method involves the transformation of A into a Jordan canonical form ,denoted by J which can then be used to determine the solution .

Now ,consider the non-homogeneous system of first order linear difference equations (1.2.3), $x(n+1) = A\,x(n) + g(n)$.

This system can be transformed into a homogeneous system of first order linear difference equation by letting $z(n) = x(n) - x^*$,and $x^* = [I - A]^{-1} g(n)$ provided that all the eigenvalues λ of A less than one , |λ|<1 , the system (1.2.3) becomes

$$z(n+1) = A(z(n) + x^*) + g(n) - x^* = Az(n) - [I - A]x^* + g(n).$$

since $x^* = [I - A]^{-1} g(n)$,we have

$$z(n+1) = A z(n).$$

Thus ,the non-homogeneous system is transformed into a homogeneous one by shifting the origin of the non-homogeneous system to the equilibrium point x*.

The following proposition establishes the solution of the nonhomogeneous system (1.2.3).

Proposition(2.2.1):[7]

The solution of the non-homogeneous first order linear system:

$$x(n+1) = A x(n) + g(n)$$

is given by

$$x(n) = PJ^n P^{-1} (x_0 - x^*) + x^*$$

where J is the Jordan matrix form corresponding to A

Proof

Let . $z(n) \equiv x(n) - x^*$,then we have

$$z(n+1) = A z(n)$$

where $A = PJP^{-1}$ and J is the Jordan matrix.Thus

$$z(n+1) = PJP^{-1} z(n)$$

Pre-multypling both sides by P^{-1} and letting $y(n) = P^{-1} z(n)$,it follows

$$y(n+1) = J\, y(n).$$

Thus

$$y(n) = J^n\, y_0 = J^n P^{-1} z_0 = J^n P^{-1}(x_0 - x^*).$$

Furthermore,since $P^{-1} z(n) = y(n)$,it follows that $z(n) = P\, y(n)$

Hence

$$x(n) = PJ^{n}P^{-1}(x_0 - x^*) + x^*.$$

Example(2.2.2)

Consider the system of $x(n+1)=Ax(n)+g(n)$, where

$$A = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -3 & 1/2 & 0 & 1 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \quad , \quad g(n) = \begin{bmatrix} 1 \\ 1 \\ n \\ 1 \end{bmatrix} \quad , x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The eigenvalues of A may be obtained by the solving the characteristic equation

$$\det(A - \lambda I) = \det\begin{bmatrix} 1/2-\lambda & 0 & 0 & 0 \\ -3 & 1/2-\lambda & 0 & 1 \\ 0 & 0 & 1/2-\lambda & 1 \\ 0 & 0 & 0 & 1/2-\lambda \end{bmatrix} = 0$$

or

$$det(A\text{-}\lambda I)=(\lambda\text{-}\ 1/2)^4=0$$

Thus $\lambda=\lambda_1=\lambda_2=\lambda_3=\lambda_4=1/2$

The eigenvectors for $\lambda = 1/2$ are

$$\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \xi_3 = \begin{bmatrix} 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_4 = \begin{bmatrix} -1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 0 & 0 & 1/3 & -1/3 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We return the characteristic polynomial is $c(\lambda)=(\lambda\text{-}1/2)^4$

so, the algebraic multiplicity = 4

Now we want to find dim(Ker(A-1/2I)) to know the geometric multiplicity

Ker(A-1/2I) = {$(0,0,1,0)^T$,$(0,1,0,0)^T$ } so,dim(Ker(A-1/2I))=2

so, the geometric multiplicity = 2 means that there are two blocks associated with this eigenvalue (λ=1/2), to determine the order of the two blocks (2 and 2 or 3 and 1), we calculate that $(A-\lambda I)^2= 0$ therefore the number s_2 =2 , hence there are two blocks of order 2

$$J = diag\left[\begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}\right]$$

or

$$J = \begin{bmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

and

$$J^n = \begin{bmatrix} 2^{-n} & n2^{-n+1} & 0 & 0 \\ 0 & 2^{-n} & 0 & 0 \\ 0 & 0 & 2^{-n} & n2^{-n+1} \\ 0 & 0 & 0 & 2^{-n} \end{bmatrix}$$

The equilibrium point is

$$x^* = [I - A]^{-1} g(n) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -12 & 2 & 0 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ n \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 2n+4 \\ 3 \end{bmatrix}$$

$$x_0 - x^* = \begin{bmatrix} 2 \\ 3 \\ n \\ 2 \end{bmatrix}$$

Hence the general solution is

$$x(n) = PJ^n(x_0 - x^*) + x^* = \begin{bmatrix} 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2^{-n} & n2^{-n+1} & 0 & 0 \\ 0 & 2^{-n} & 0 & 0 \\ 0 & 0 & 2^{-n} & n2^{-n+1} \\ 0 & 0 & 0 & 2^{-n} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ n \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \\ 2n+4 \\ 3 \end{bmatrix}$$

2.3 The Z-transform Method

This section present a third method to solve the linear first order systems of difference equations.The method uses the idea of a discrete transform which produce an algebraic equation in the transform of the unknown function which can be obtained by an inversion process.

First,we introduce the notion of the Z-transform operator which is defined on a specified vector space.

Definition (2.3.1) [3,10,12]

If $x(n)=[x_1(n),x_2(n),...,x_k(n)]$ where x_i:Z→$\Re$,i=1,...,k and if each x_i(n) is Z-transformable then we define the Z-transform of the vector x as

$$\tilde{x}(z) = [\tilde{x}_1(z), \tilde{x}_2(z), ...\tilde{x}_k(z)]$$

where

$$\tilde{x}(z) = Z(x(n)) = \sum_{i=0}^{\infty} x(i) z^{-i}$$

Definition(2.3.2):[3,10,12]

If $A(n)=(a_{ij}(n))$ where a_{ij}: Z→R and if each a_{ij} is Z-transformable

then we define the Z-transform of the matix $A(n)$ as

$$\tilde{A}(z) = Z(a_{ij}(n)) = (Z(a_{ij}(n))) = (\tilde{a}_{ij}(z))$$

Now consider the system

$$x(n+1)=Ax(n) \quad ,x(0)=x_0 \tag{2.3.1}$$

Taking the Z transform of both sides of Equation (2.3.1),we get

$$z\tilde{x}(z) - zx(0) = A\tilde{x}(z)$$

which gives

$$(z\mathrm{I} - A)\tilde{x}(z) = zx(0)$$

or

$$\tilde{x}(z) = z(z\mathrm{I} - A)^{-1}x(0) \tag{2.3.2}$$

Taking the inverse Z-transform of (2.3.2) then ,we obtain the solution of (2.3.1)

$$x(n) = Z^{-1}\left[z(zI - A)^{-1}\right]x(0) = \Psi(n,0)x(0) = A^n x(0)$$

It follows from (2.3.2) and (2.3.1) that

$$\tilde{\Phi}(z) = z(zI - A)^{-1}$$

$$\Psi(n,0) = \Phi(n) = Z^{-1}\left[z(zI - A)^{-1}\right] = A^n$$

Next ,consider the non homogeneous system of the form

$$x(n+1)=Ax(n)+g(n) \qquad x(0)=x_0 \tag{2.3.3}$$

Taking the Z transform of both sides of Equation (2.3.3),we get

$$z\tilde{x}(z) - zx(0) = A\tilde{x}(z) + \tilde{g}(z)$$

which gives

$$(z\mathrm{I} - A)\tilde{x}(z) = zx(0) + \tilde{g}(z)$$

or

$$\tilde{x}(z) = z(z\mathrm{I} - A)^{-1}x(0) + (z\mathrm{I} - A)^{-1}\tilde{g}(z) \tag{2.3.4}$$

Taking the inverse Z-transform of (2.3.4) then, we obtain the solution of (2.3.3)

$$x(n) = x_h(n) + x_p(n) = Z^{-1}\left[z(zI - A)^{-1}\right]x(0) + Z^{-1}\left[(zI - A)^{-1}\tilde{g}(z)\right]$$

$$= \Psi(n,0)x(0) + \sum_{r=0}^{n-1}\Psi(n,r+1)g(r)$$

Therefore

$$x_h(n) = \sum_{r=0}^{n-1}\Psi(n,r+1)g(r) \quad \text{and}$$

$$x_h(n) = \Psi(n,0)x(0)$$

Next , consider the system of the form (volterra system of Convolution type)

$$x(n+1) = Ax(n) + \sum_{j=0}^{n} B(n-j)x(j) \tag{2.3.5}$$

where $\sum_{j=0}^{\infty} B(j) < \infty$

Taking the Z transform of both sides of Equation (2.3.5),we get

$$z\tilde{x}(z) - zx(0) = A\tilde{x}(z) + \tilde{x}(z)\tilde{B}(z) \qquad |z| > R \quad ,$$

which gives

$$(z\mathrm{I} - A - \tilde{B}(z))\tilde{x}(z) = zx(0) \qquad |z| > R$$

or

$$\tilde{x}(z) = z(z\mathrm{I} - A - \tilde{B}(z))^{-1}x(0) \tag{2.3.6}$$

where $\det(zI - A - \tilde{B}(z_0)) \neq 0$ for all , |z |≥R

Taking the inverse Z-transform of (2.3.6) then we obtain the solution of (2.3.5)

$$x(n) = Z^{-1}\left[z(zI - A - \tilde{B}(z))^{-1}\right]x(0) = \Psi(n,0)x(0) = A^n x(0)\,.$$

Finally ,consider the non homogeneous system of the form (Convolution type)

$$x(n+1) = Ax(n) + \sum_{j=0}^{n} B(n-j)x(j) + g(n) \tag{2.3.7}$$

Taking the Z transform of both sides of Equation (2.3.7),we get

$$z\tilde{x}(z) - zx(0) = A\tilde{x}(z) + \tilde{g}(z) + \tilde{x}(z)\tilde{B}(z)$$

which gives

$$(z\mathrm{I} - A - \tilde{B}(z))\tilde{x}(z) = zx(0) + \tilde{g}(z)$$

or

$$\tilde{x}(z) = z(z\mathrm{I} - A - \tilde{B}(z))^{-1}x(0) + (z\mathrm{I} - A - \tilde{B}(z))^{-1}\tilde{g}(z) \qquad (2.3.8)$$

Taking the inverse Z-transform of (2.3.8) then we obtain the solution of (2.3.7)

$$x(n) = x_h(n) + x_p(n) = Z^{-1}\left[z(zI - A - \tilde{B}(z))^{-1}\right]x(0) + Z^{-1}\left[(zI - A - \tilde{B}(z))^{-1}\tilde{g}(z)\right] \quad \text{or}$$

or

$$x(n) = \Psi(n,0)x(0) + \sum_{r=0}^{n-1}\Psi(n, r+1)g(r)$$

Therefore

$$x_p(n) = \sum_{r=0}^{n-1}\Psi(n, r+1)g(r)$$

and

$$x_h(n) = \Psi(n,0)x(0) = A^n x(0)$$

Example(2.3.3)

Consider the following system

$$x(n+1) = Ax(n) + \sum_{j=0}^{n-1}B(n-j-1)x(j) + g(n)$$

where

$$A = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{6} \end{bmatrix} \quad , B(n) = \begin{bmatrix} 2^{n/2} & 0 \\ 0 & 6^{n/2} \end{bmatrix} \quad ,, \quad g(n) = \begin{bmatrix} n \\ 0 \end{bmatrix} \quad , and\ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From (3.3.8),we have

$$\widetilde{B}(z)=\begin{bmatrix}\dfrac{z}{z-\sqrt{2}} & 0\\ 0 & \dfrac{z}{z-\sqrt{6}}\end{bmatrix}$$

$$\widetilde{x}(z)=z\left(\begin{bmatrix}z & 0\\ 0 & z\end{bmatrix}-\begin{bmatrix}-\sqrt{2} & 0\\ 0 & -\sqrt{6}\end{bmatrix}-\begin{bmatrix}\dfrac{z}{z-\sqrt{2}} & 0\\ 0 & \dfrac{z}{z-\sqrt{6}}\end{bmatrix}\right)^{-1}$$

$$\widetilde{x}(z)=z\begin{bmatrix}\dfrac{z^2-z-2}{z-\sqrt{2}} & 0\\ 0 & \dfrac{z^2-z-6}{z-\sqrt{6}}\end{bmatrix}^{-1}$$

$$\widetilde{x}(z)=z\begin{bmatrix}\dfrac{z-\sqrt{2}}{z^2-z-2} & 0\\ 0 & \dfrac{z-\sqrt{6}}{z^2-z-6}\end{bmatrix}$$

Now ,we take the inverse Z-transform and use table (2) to get the homogeneous solution of the system

$$x_h(n)=\begin{bmatrix}\dfrac{2^n(2-\sqrt{2})+(-1)^n(1+\sqrt{2})}{3} & 0\\ 0 & \dfrac{2^n(3-\sqrt{6})+(-1)^n(2+\sqrt{6})}{5}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

or

$$x_h(n)=\begin{bmatrix}\left(2^n(2-\sqrt{2})+(-1)^n(1+\sqrt{2})\right)/3\\ 0\end{bmatrix}$$

$$(zI-A-\widetilde{B}(z))^{-1}\widetilde{g}(z)=\begin{bmatrix}\dfrac{z-\sqrt{2}}{z^2-z-2} & 0\\ 0 & \dfrac{z-\sqrt{6}}{z^2-z-6}\end{bmatrix}\begin{bmatrix}(z/(z-1)^2\\ 0\end{bmatrix}$$

$$=\begin{bmatrix}z(z-\sqrt{2})/(z-2)(z+1)(z-1)^2\\ 0\end{bmatrix}$$

Similarly as before taking the inverse and use table (2) we find the particular solution

$$x_p(n) = Z^{-1}\left[(zI - A - \tilde{B}(z))^{-1}\tilde{g}(z)\right] = \begin{bmatrix} \left(2^{n+2}(2-\sqrt{2}) + (-1)^n(1+\sqrt{2}) + 3 + 3\sqrt{2}\right)/12 \\ 0 \end{bmatrix}$$

Hence ,the general solution is given by

$$x(n) = x_h(n) + x_p(n) = \begin{bmatrix} \left(2^{n+4}(2-\sqrt{2}) + 5(-1)^n(1+\sqrt{2}) + 3 + 3\sqrt{2}\right)/12 \\ 0 \end{bmatrix}$$

Chapter Three

Stability Theory and Asymptotic Behavior

In this chapter the stability properties for system of first order difference equations is developed .Here, the focus is on the qualitative behavior of solutions rather than finding them .This investigation is of great importance since in practice most of the problems are nonlinear and mostly unsolvable .Various notions of stability are presented together with some examples to illustrate these notions.

3.1 Stability Of Linear Systems

In this section , we study the stability of the linear nonautonomous (time-variant) system (1.2.2),$x(n+1)=A(n)x(n)$, $n \geq n_0 \geq 0$

where ,$A(n)$ is nonsingular for all $n \geq n_0$

The results obtained for this system (1.2.2) are general ,so they include the case of an autonomous system (1.2.4),: $x(n+1)=Ax(n)$, $n \geq n_0 \geq 0$

where A is independent of n.

The following theorem state necessary and sufficient conditions for stability using the fundamental matrix $\Phi(n)$ for the system.

Theorem(3.1.1):[3]

The equilibrium solution of the system (1.2.2) is

(1) Stable if and only if there exists a positve constant M such that

$$\|\Phi(n)\| \leq M \quad \text{for } n \geq n_0 \geq 0 \qquad (3.1.1).$$

(2)Uniformly stable if and only if there exists a positve constant M such that

$$\|\Phi(n,m)\| \leq M \quad \text{for } n_0 \leq m \leq n < \infty \qquad (3.1.2)$$

(3)Asymptotically stable if and only if

$$\lim_{n \to \infty} \|\Phi(n)\| = 0 \qquad (3.1.3)$$

(4)Uniformly asymptotically stable if and only if there exists a positve constant M and $\eta\in(0,1)$ such that

$$\|\Phi(n,m)\|\leq M\,\eta^{n-m} \quad \text{for } n_0\leq m\leq n<\infty \tag{3.1.4}$$

.

proof: (1) ,(2) and (3) see [3]

Proof (4)

Suppose (3.1.4) holds . The zero solution of system (1.2.2) would then be uniformly stable by (2). furthermore ,for $\varepsilon>0$,$0<\varepsilon<M$,take $\mu=1$ in definition (1.3.1) and choose N such $\eta^N<\varepsilon/M$ .Hence ,if $\|x_0\|<1$,then $\|x(n, n_0,x_0)\|=\|\Phi(n, n_0)x_0\|\leq M\eta^{n-n_0}<\varepsilon$ for $n\geq n_0+N$ .The zero solution would be uniformly asymptotically stable, conversely ,suppose that the zero solution is uniformly asymptotically stable .It is also then uniformly stable and thus by (2)., $\|\Phi(n,m)\|\leq M$ for

$0\leq n_0\leq m\leq n<\infty$, from uniform attractivity , there exist $\eta>0$ such that for ε with $0<\varepsilon<1$ there exists N such that $\|\Phi(n,m)\|<\varepsilon$ for $n\geq n_0+N$ whenever $\|x_0\|<\eta$, then for

$n\in[n_0+mN, n_0+(m+1)N]$,$m>0$ we have

$$\|\Phi(n, n_0)\|\leq\|\Phi(n, n_0+mN)\|\,\|\Phi(n_0+mN, n_0+(m-1)N)\|\ldots\|\Phi(n_0+N, n_0)\|$$

$$\leq\|M\varepsilon^m\leq(M/\varepsilon)(\varepsilon^{-N})^{(m+1)N}=T\eta^{(m+1)N} \qquad \text{where } T=M/\varepsilon\ ,\eta=\varepsilon^{-N}$$

$$\leq T\eta^{(n-n_0)} \quad \text{for } mN\leq n-n_0\leq(m+1)N.$$

Remarks

(1)the system (1.2.2)is stable if and only if all solutions are bounded.

(2) the system (1.2.2) is exponentially stable if and only if it is uniformly asymptotically stable.

Theorem(3.1.2):[3]

(i) if $\sum_{i=1}^{k} |a_{ij}(n)| \le 1,\ 1 \le j \le k,\quad n \ge n_0$, then the zero solution of system (1.2.2) is uniformly stable .

(i) if $\sum_{i=1}^{k} |a_{ij}(n)| \le 1-\nu,\quad \nu > 0,\ 1 \le j \le k,\quad n \ge n_0$, then the zero solution of system (1.2.2) is uniformly asymptotically stable .

Proof :see[3]

Theorem(3.1.3):[3]

The equilibrium solution x=0 of the system (1.2.4), $x(n+1)=Ax(n)$ is stable if and only if $\rho(A)\le 1$ and the eigenvalues λ_i with $\|\lambda_i\|=1$ are semi simple. In case $\rho(A)<1$ then x=0 is asymptotically stable.

Proof :

Let $J=\mathrm{diag}(J_1,J_2,\ldots,J_r)$ be the Jordan form of the matrix A ,then $A=PJP^{-1}$ for some nonsingular matrix P ,where J_i is a Jordan block.

By theorem (3.1.1) the zero solution of equation(1.2.4) is stable if and only if

$\|A^n\|=\|PJ^nP^{-1}\|\le M$ or $\|J^n\|\le K$ where $K=M/(\|P\|\ \|P^{-1}\|)$

Now , $J^n=\mathrm{diag}(J_1^n,J_2^n,\ldots,J_r^n)$,where

$$J_i^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \cdots & \binom{n}{s_i-1}\lambda_i^{n-s_i+1} \\ 0 & \lambda_i^n & \ddots & \vdots \\ & & \ddots & \binom{n}{1}\lambda_i^{n-1} \\ 0 & & \cdots & \lambda_i^n \end{bmatrix}$$

If $|\lambda_i|>1$ or $|\lambda_i|=1$ and J_i is not a scalar ,then J^n becomes unbounded,on the other hand

If $|\lambda_i|<1$,then $|\lambda_i|^n n^m \to 0$ for any positive integer m ,Hence $J^n \to 0$ as $n\to\infty$.

Example (3.1.4)

We consider the zero solution of the system $x(n+1)=Ax(n)$,where

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1=\lambda_2=1$

Hence ,$\rho(A)=1$ since $\dim(A-\lambda I)=1$,then $\lambda=1$ is not semi simple Thus the solution $x=0$is unstable. so the equilibrium solution is not stable

.

Example(3.1.5)

Consider the stability of the system $x(n+1)=Ax(n)$

$$A = \begin{bmatrix} \frac{5}{12} & 0 & \frac{1}{2} \\ -1 & \frac{-1}{2} & \frac{5}{4} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

$\det(A-\lambda I)=(-1/2-\lambda)(12\lambda^2-5\lambda-2)=0$

$\lambda_1=2/3$, $\lambda_2==-1/4.$,$\lambda_3=-1/2$

$\rho(A)= 2/3<1$

so the equilibrium solution is asymptotically stable

Example(3.1.6)

Consider the stability of the system $x(n+1)=Ax(n)$,where

$$A=\begin{bmatrix} n/(n+1) & 0 \\ -1 & 1 \end{bmatrix}$$

We use theorem (3.1.2) to get

$$\sum_{i=1}^{2}|a_{i1}|=\frac{n}{n+1}+1>1$$

so, the equilibrium solution is not stable .

Example(3.1.7)

Consider the stability of the system $x(n+1)=Ax(n)$

$$A=\begin{bmatrix} \frac{1}{n+1} & 0 & \frac{\sin(n)}{2} \\ \frac{1}{4} & \frac{\sin(n)}{2} & \frac{\cos(n)}{2} \\ \frac{1}{5} & 0 & 0 \end{bmatrix}$$

We use theorem (3.1.2) to get

$$\sum_{i=1}^{3}|a_{i1}|=\frac{1}{n+1}+\frac{1}{4}+\frac{1}{5}<1$$

$$\sum_{i=1}^{3}|a_{i2}|=\frac{|\sin(n)|}{2}<1$$

$$\sum_{i=1}^{3}|a_{i3}|=\tfrac{1}{2}|\sin(n)|+\tfrac{1}{2}|\cos(n)|<1$$

so, the equilibrium solution is uniformly asymptotically stable

3.2 Phase Space Analysis

This section introduces the phase space analysis for the solutions which is a geometrical method used to study the stability properties of autonomous system(1.2.4) :

$x(n+1)=Ax(n)$, $n \geq n_0 \geq 0$

where A is a square constant matrix .

Here we consider the case of two dimensional systems.Hence the phase space is the xy-plane.The equilibrium solution of system (1.2.4) $x^*=0$ is the only equilibrium point of system(1.2.4) if (A-I) is nonsingular .But if (A-I) is singular ,then there is a family of equilibrium points .In the later case we let $y(n)=x(n)- x^*$ in equation(1.2.4) to obtain the system $y(n+1)=Ay(n)$.

Hence,the stability properties of any equilibrium point $x^*\neq 0$ are the same as those of the equilibrium point $x^* =0$ .Therefore $x^* =0$is the only equilibrium point of this system.

To establish the phase-space analysis for the system (1.2.4),$x(n+1)=Ax(n)$.We let $J=P^{-1} A P$ be the Jordan form of A ,then J may take one of the following forms

$$R(\lambda,\mu)=\begin{bmatrix}\lambda & 0\\ & \mu\end{bmatrix} \qquad R_a(\lambda)=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} \qquad K(\alpha,\beta)=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}$$

where in $R(\lambda,\mu)$ the eigenvalues λ , μ are distinct real numbers,and in $R_a(\lambda$) the eigenvalues λ are repeated real numbers,while in $K(\alpha,\beta)$ the eigenvalues λ , μ are complex numbers with $\lambda=\alpha + i\beta$ and $\mu=\alpha - i\beta$, $\beta > 0$.

To complete the analysis , Introduce the the transformation $y(n)= P^{-1}x(n)$ which is equivalent to $x(n)=Py(n)$ .Hence the system (1.2.4) takes the form

$$y(n+1)=Jy(n). \qquad (3.2.1)$$

Further , If $x(0)=x_0$ is an initial data for the system (1.2.4),then $y(0)=y_0=p^{-1}x_0$ would be the corresponding initial data for the transformed system (3.2.1) .We notice that the two equilibrium points of the two systems have the same qualitative properties .

Now ,we follow the procedure of sketching a phase –space analysis of a system.

First ,start with an initial value $y=y_0$,where

$$y_0 = \begin{bmatrix} y_0^{(1)} \\ y_0^{(2)} \end{bmatrix}$$

In the $y^{(1)}y^{(2)}$ – plane ,then we locate and follow the movement of the points y(1), y(2),y(3),... to draw the orbit $\{y(n,0,y_0):n=0,1,2,...\}$.The arrows in each resulting picture are used to signify increasing time *n* .

The above procedure is employed for each previous case :

Case(R)There are two real eigenvectors and the system takes the form :

$$y(n+1) = R(\lambda,\mu)\; y(n)$$

or

$$\begin{bmatrix} y_1(n+1) \\ y_2(n+1) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix}$$

Hence
$$y_1(n) = \lambda^n y_0^{(1)}$$
$$y_2(n) = \mu^n y_0^{(2)}$$

or
$$\frac{y_2(n)}{y_1(n)} = \left(\frac{\mu}{\lambda}\right)^n \left(\frac{y_0^{(2)}}{y_0^{(1)}}\right)$$

Now, If $\lambda>\mu$,we have $y_2(n)/y_1(n)\to 0$ as $n\to\infty$,and if $\lambda<\mu$, we get $y_2(n)/y_1(n)\to\infty$ as $n\to\infty$. (see fig,3.1a,3.1b,3.1c, 3.1d,3.1e)

Case(R_a) In this case ,we have

$$y_1(n)= \lambda^n y_0^{(1)}+n\lambda^{n-1} y_0^{(2)}$$

$$y_2(n) = \lambda^n y_0^{(2)}$$

Thus

$$\frac{y_2(n)}{y_1(n)} \to 0 \qquad as \quad n \to \infty$$

(see fig3.2a,3.2b)

Case(K) In this case ,the matrix A has two complex conjugate eigenvalues

$\lambda=\alpha+i\beta$ and $\mu=\alpha-i\beta$, $\beta > 0$

The eigenvector corresponding to $\lambda=\alpha+i\beta$ is given by

$$\xi_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and the solution may be given by

$$\begin{bmatrix} 1 \\ i \end{bmatrix}(\alpha + i\beta)^n = \begin{bmatrix} 1 \\ i \end{bmatrix}|\lambda|^n (\cos n\omega + i \sin n\omega)$$

$$= |\lambda|^n \begin{bmatrix} \cos n\omega \\ -\sin n\omega \end{bmatrix} + i|\lambda|^n \begin{bmatrix} \sin n\omega \\ \cos n\omega \end{bmatrix}$$

where $\omega=\tan^{-1}(\beta/\alpha)$

Ageneral solution is given by

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = |\lambda|^n \begin{bmatrix} c_1 \cos n\omega + c_2 \sin n\omega \\ -c_1 \sin n\omega + c_2 \cos n\omega \end{bmatrix}$$

Given the initial values $y_1(0)=y_0^{(1)}$ and $y_2(0)=y_0^{(2)}$,one may find $c_1=y_0^{(1)}$ and $c_2=y_0^{(2)}$

The solution is denoed by

$y_1(n)=|\lambda_1|^n(y_0^{(1)}\cos nw+y_0^{(2)}\sin nw)$

$y_2(n)=|\lambda_1|^n(-y_0^{(1)}\sin nw+y_0^{(2)}\cos nw)$

If we let $\cos\gamma=y_0^{(1)}/r_0$ and $\sin\gamma=y_0^{(2)}/r_0$,where $r_0 = \sqrt{\left(y_0^{(1)}\right)^2 + \left(y_0^{(2)}\right)^2}$,we have

$y_1(n)=|\lambda|^n r_0\cos(n\omega-\gamma)$ and $y_2(n)=|\lambda|^n r_0\sin(n\omega-\gamma)$.

Using polar coordinates we may

Now, write the solution as

$r(n) = r_0|\lambda|^n$, $\theta(n)=-(n\omega-\gamma)$

If $|\lambda|<1$,we have an asymptotically stable focus as illustrated by in fig 3.3a

If $|\lambda|>1$,we find an unstable focus as shown in fig3.3b

If $|\lambda|=1$,we obtain a center where orbits are circles with radii

$r_0 = \sqrt{\left(y_0^{(1)}\right)^2 + \left(y_0^{(2)}\right)^2}$ as shown in fig3.3c

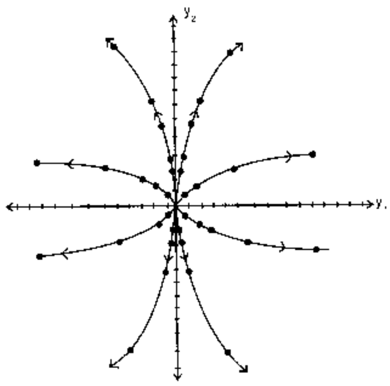


Figure 3.1a $\lambda > \mu > 1$,unstable node

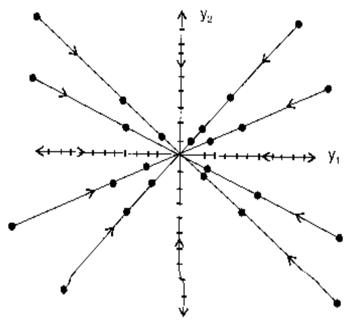


Figure 3.1b, $\lambda = \mu < 1$,asymptotically stable

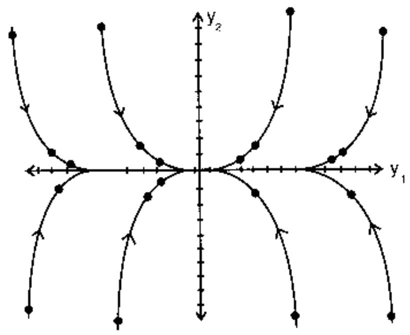


Figure 3.1c, λ< μ <1, asymptotically stable node

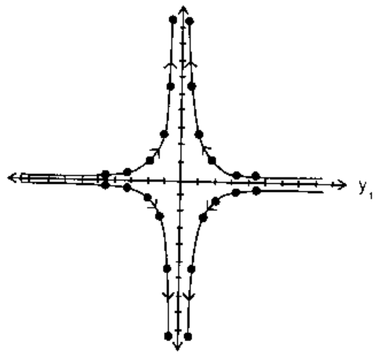


Figure 3.1d, λ<1, μ >1, saddle(unstable)

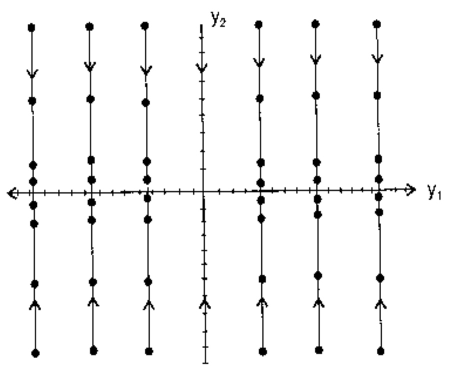


Figure 3.1e, λ=1, μ <λ ,degenerate node

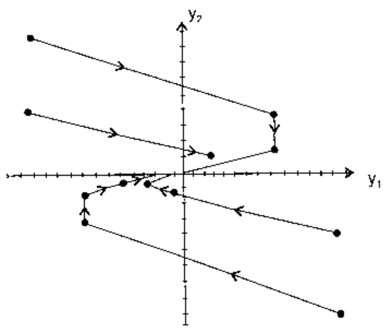


Figure 3.2a, λ<1 ,asymptotically stable nod

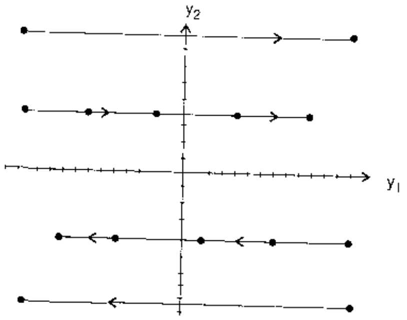


Figure 3.2a λ=1 ,degenerate case (unstable) .

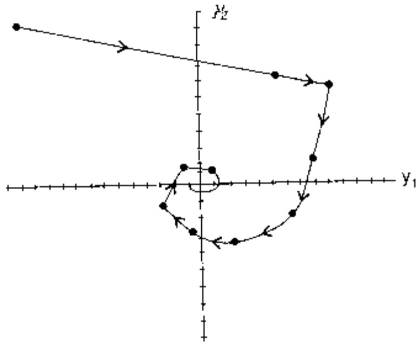


Figure 3.3a |λ|< 1,asymptotically stable focus

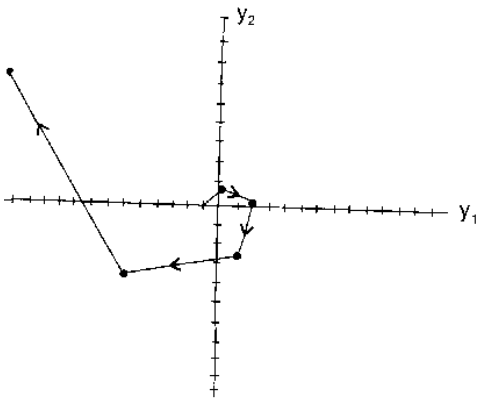


Figure3.3b, |λ|> 1 ,unstable focus

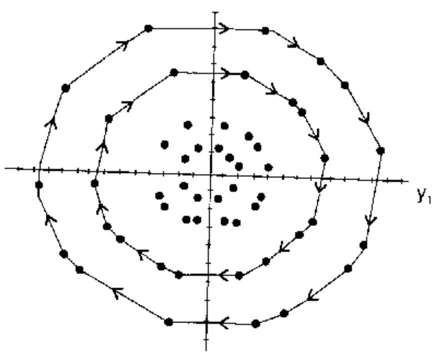


Figure 3.3c, |λ|=1,center(stable)

3.3 Linearized Stability

This section discusses the linearization method in which a linear system is used to approximate the behavior of a nonlinear system .Since most problems that arise in the real world are not linear ,and in most cases ,nonlinear systems can not be "solved" there is typically no method for deriving a solution to the equations .This method was initiated by the two mathematicians Lyapunov and Perron ,each with his own approach.

Consider the system of autonomous nonlinear first order difference equations

$$x(n+1) = f(x(n)) \tag{3.3.1}$$

where $f : \Re^{k} \to \Re^{k}$

and the initial value of the k-dimensional state variable , $x(n_0) = x_0$, is given by

$$\begin{aligned} x_1(n+1) &= f^{(1)}(x_1(n), x_2(n), \cdots, x_k(n)) \\ x_2(n+1) &= f^{(2)}(x_1(n), x_2(n), \cdots, x_k(n)) \\ \vdots \qquad & \qquad \vdots \\ x_k(n+1) &= f^{(k)}(x_1(n), x_2(n), \cdots, x_k(n)) \end{aligned} \tag{3.3.2}$$

The system (3.3.2) can be approximated linearly by using Taylar theorem .

To approximate $f^{(i)}(x)$, around the equilibrium point , $x^* = (x_1^*, x_2^*, \cdots, x_k^*)$,we write

$$f^{(i)}(x) \approx f^{(i)}(x^*) + \sum_{j=1}^{k} f_j^{(i)}(x^*)(x_j - x^*) \tag{3.3.3}$$

where $f_j^{(i)}(x^*) = \dfrac{\partial f^{(i)}}{\partial x_j}(x^*)$

This is sometimes called tangent plane approximation .

Now ,we can find the linearization of a discrete system at the equilibrium point $x^* = (x_1^*, x_2^*, \cdots, x_k^*)$.In this case , replacing $f^{(i)}(x)$ with its tangent plane approximation at $x^* = (x_1^*, x_2^*, \cdots, x_k^*)$,then the i^th^ equation in (3.3.2) becomes

$$x_i(n+1) = f^{(i)}(x^*) + \sum_{j=1}^{k} f_j^{(i)}(x^*)(x_j - x_j^*)$$

Now,let $u_i(n) = (x_i(n) - x^*)$ $\quad ,i = 1,2,\cdots,k$ .then equation (3.3.3) becomes

$$u_i(n+1) = \sum_{j=1}^{k} f_j^{(i)}(x^*)u_j(n)$$

and system (3.3.3) becomes

$$\begin{bmatrix} u_1(n+1) \\ u_2(n+1) \\ \vdots \\ u_k(n+1) \end{bmatrix} = \begin{bmatrix} f_1^{(1)}(x^*) & f_2^{(1)}(x^*) & \cdots & f_k^{(1)}(x^*) \\ f_1^{(2)}(x^*) & f_2^{(2)}(x^*) & \cdots & f_k^{(2)}(x^*) \\ \vdots & \vdots & & \vdots \\ f_1^{(k)}(x^*) & f_2^{(k)}(x^*) & \cdots & f_k^{(k)}(x^*) \end{bmatrix} \begin{bmatrix} u_1(n) \\ u_2(n) \\ \vdots \\ u_k(n) \end{bmatrix}$$

This is the linearization of (3.3.3) at $x^* = (x_1^*, x_2^*, \cdots, x_k^*)$.

Now, let $U(n) = [u_1(n), u_2(n), \cdots, u_{k(}n)]^T$,we can write the system in matrix form as

$$U(n+1) = D_x f\big|_{x^*} U(n) \tag{3.3.4}$$

where

$$D_x f\big|_{x^*} \equiv \begin{bmatrix} f_1^{(1)}(x^*) & f_2^{(1)}(x^*) & \cdots & f_k^{(1)}(x^*) \\ f_1^{(2)}(x^*) & f_2^{(2)}(x^*) & \cdots & f_k^{(2)}(x^*) \\ \vdots & \vdots & & \vdots \\ f_1^{(k)}(x^*) & f_2^{(k)}(x^*) & \cdots & f_k^{(k)}(x^*) \end{bmatrix}$$

is the Jacobian matrix of $f(x(n))$ evaluated at x^* .

If neither eigenvalue has magnitude equal to 1 ,then the behavior of the system (3.3.1)near (x^*,y^*) is qualitatively the same as the behavior of the linear approximation (3.3.4) .The classification of the equilibrium point of the nonlinear system is the same as the classification of the origin in the linearization which is confirmed by the following theorem. .

Theorem (3.3.1):[15]

Suppose all of the eigenvalues of $D_x f\,(x^*)$ have modulus less than one .Then the equilibrium solution x=x* of the nonlinear system(3.3.1) is asymptotically stable.

Proof :see[15]

Example (3.3.2)

Consider the system

$$\begin{aligned} x(n+1) &= (\tfrac{1}{2} - x(n))x(n) + by(n) \\ y(n+1) &= \frac{y(n)}{3} + 2x(n) \end{aligned} \qquad (3.3.5)$$

where b is a constant .This is a system of the form shown in (3.4.1),with

$f^{(1)}(x,y) = (\tfrac{1}{2} - x)x + by$ and $f^{(2)}(x,y) = \dfrac{y}{3} + 2x$

The equilibrium points are (0,0),(3b-0.5,9b-1.5)

The Jacobian matrix at an equilibrium point (x*,y*) is

$$D_x f\big|_{x^*} = \begin{bmatrix} f^{(1)}{}_x(x^*,y^*) & f^{(1)}{}_y(x^*,y^*) \\ f^{(2)}{}_x(x^*,y^*) & f^{(2)}{}_y(x^*,y^*) \end{bmatrix} = \begin{bmatrix} 1-2x^* & b \\ 1 & \tfrac{1}{2} \end{bmatrix}$$

At the equilibrium point (0,0) ,we find

$$D_x f\big|_{x^*} = \begin{bmatrix} \tfrac{1}{2} & b \\ 1 & \tfrac{1}{3} \end{bmatrix}$$

The eigenvalues of the Jacobian are

$$\lambda_1 = \frac{5}{12} - \frac{1}{2}\sqrt{\frac{1}{36} + 4b} \quad ,\lambda_2 = \frac{5}{12} + \frac{1}{2}\sqrt{\frac{1}{36} + 4b} \qquad (3.3.6)$$

The first thing to determine is whether the eigenvalues are complex or real .The eigenvalues are complex if

$$\frac{1}{36} + 4b < 0 \Rightarrow b < -\frac{1}{144}$$

So we have complex eigenvalues if $b < -\frac{1}{144}$ and real eigenvalues if $b \geq -\frac{1}{144}$.

Now, when $b < -\frac{1}{144}$,the eigenvalues are

$$\lambda_1 = \frac{5}{12} - \frac{i}{2}\sqrt{-\frac{1}{36} - 4b} \quad ,\lambda_2 = \frac{5}{12} + \frac{i}{2}\sqrt{-\frac{1}{36} - 4b}$$

To classify the equilibrium point and to determine its stability ,we must determine whether the magnitude of the eigenvalues are greater than or less than one .To do this ,

we will find the values of b ,if any ,where $|\lambda|=1$,we have

$1=|\lambda|^2$

$$1 = \left(\frac{5}{12}\right)^2 + \left(\frac{1}{2}\sqrt{-\frac{1}{36} - 4b}\right)^2 = \frac{24}{144} - b$$

which gives $b = -\frac{5}{6}$ .So we have the following

(i) If $b < -\frac{5}{6}$,then $|\lambda|>1$, and (0,0) is unstable.

(ii) If $-\frac{5}{6} < b < -\frac{1}{144}$,then $|\lambda|<1$, and (0,0) is stable.

Next we consider $b > -\frac{1}{144}$,where the eigenvalues are real.From (3.3.5) we observe

$\lambda_1<5/12$,and $\lambda_2>5/12$,so we only need to determine if $\lambda_1<-1$ or $\lambda_2>1$.First consider

$\lambda_1= -1$

$$\frac{5}{12} - \frac{1}{2}\sqrt{\frac{1}{36} + 4b} = -1$$

$$\frac{1}{144} + b = \frac{289}{144}$$

b=2

So $\lambda_1<-1$ if b>2

Finally ,consider

λ_2= 1,then we have

$$\frac{5}{12}+\frac{1}{2}\sqrt{\frac{1}{36}+4b}=1$$

$$\frac{1}{144}+b=\frac{49}{144}$$

$$b=\frac{1}{3}$$

So λ_2<1.if $-\frac{1}{144}<b<\frac{1}{3}$ .Thus , we have

(i) If $-\frac{1}{144}<b<\frac{1}{3}$,we have $-1<\lambda_1<1$ and $5/6<\lambda_2<1$,so (0,0) is stable.

(ii) If $\frac{1}{3}<b<2$,then $-1<\lambda_1<1$ but $\lambda_2>1$, so (0,0) is unstable .

(iii) If $b>\frac{1}{3}$,then $\lambda_1<-1$ and $\lambda_2>1$, so (0,0) is unstable.

At the equilibrium point (3b-0.5,9b-1.5),we find

$$D_x f\big|_{x*}=\begin{bmatrix} -\frac{1}{2}-6b & b \\ 1 & \frac{1}{3} \end{bmatrix}$$

The eigenvalues of the Jacobian are

$$\lambda_1=-\frac{1}{12}-3b-\frac{1}{2}\sqrt{(\tfrac{1}{6}+6b)^2+8b+\tfrac{2}{3}} \quad ,\lambda_2=-\frac{21}{12}-3b+\frac{1}{2}\sqrt{(\tfrac{1}{6}+6b)^2+8b+\tfrac{2}{3}} \tag{3.3.5}$$

The eigenvalues are complex if $b<-\frac{10}{72}$ and real eigenvalues if $b\geq-\frac{10}{72}$.

When the eigenvalues are complex,we have

If $\frac{-9-\sqrt{58}}{36}<b<-\frac{10}{72}$ then (3b-0.5,9b-1.5) is stable .

When the eigenvalues are real,we have

If $-\frac{10}{72} < b < \frac{96}{1116}$ then (3b-0.5,9b-1.5) is stable

3.4 Lyapunov Theory of Stability

In this section we introduce an alternative technique to investigate the stability of system (3.3.1),$x(n+1) = f(x(n))$

where f: :G$\rightarrow\Re$,G$\subset \Re^k$ is continous.

This technique is due to Lyapunov [3].In the first part of this section ,we need to state some definitions and theorems that is needed in the sequel.

Now,let V: $\Re^k \rightarrow \Re$ be defined as a real valued function and we define

$\Delta V(x)= V(f(x))-V(x)$ and

$\Delta V(x)= V(f(x))-V(x) = V(x(n+1)-V(x(n))$.

Definition(3.4.1) :(Lyapunov function):[3,11]

The function V is said to be Lyapunov function on a subset S of $\Re^k$ if

1)V is continuous on S and

2)$\Delta V(x)\leq 0$ whenever x and $f(x)\varepsilon$ S.

Definition(3.4.2) :(Locally Positive definite):[3,11]

A continuous function V: $\Re^k \rightarrow \Re$ is a locally positive definite function if

(1)$V(x^*)=0$ and

(2)$V(x)>0$ $\forall x \in B(x^*, r)$, for some r > 0

Where $B(x,r)=\{y\in \Re^k \mid \|x-y\|<r\}$.

Definition(3.4.3) :(Positive definite):[3,11,15]

A continuous function V: $\Re^k \to\Re$ is a positive definite function if it satisfies the conditions of definition (3..4.2) and ,additionally , $V(x)\to \infty$ as $\|x\|\to \infty$.

Theorem(3.4.4) :(Stability Theorem of Lyapunov):[3,11,15]

Let V be a Lyapunov function for Equation (3.3.1) on a neighborhood S of the equilibrium point x^* and V is a locally positive definite with respect to x^* ,then

(1)x^* is stable .

2)If $\Delta V(x)<0$,whenever $x,f(x)\varepsilon$ S and $x\neq x^*$.,then $x^*$is asymptotically stable.

(3)If V is a positive definite with respect to x^* ,then $x^*$is globally a symptotically stable.

(4)If $\Delta V(x) \le -\alpha V$ and $V(x)\ge b\|x\|^\beta$ in S (α,b,β are positive constants) ,then $x^*$is an exponentially stable .

Proof (1) ,(2) and (3) see[3].

Proof (4)

we have $\Delta V(x(n)) \le -\alpha V(n)$ and thus ,by Gronwall lemma, $V(x(n))\le V(x(n_0))e^{-\alpha(n-n0)}$

Since $b\|x\|^\beta \le V(x)$,we obtain

$$\|x(n)\| \le \frac{V(x(n_0))}{b} Exp\left(-\tfrac{\alpha}{\beta}(n-n_0)\right)$$

So x^* is an exponentially stable .

Example(3..4.5)

Consider the stability of the following system

$x_1(n+1)=ax_2(n)/(1+x_1^2(n))$

$x_2(n+1)=bx_1(n)/(1+x_2^2(n))$

To find conditions on a and b if the zero solution is stable or the zero solution is asymptotically stable .First , it is clear that the equilibrium point is (0,0)

Now, we choose Lyapunov function to be $V(x_1,x_2)=x_1^2+x_2^2$,this is clearly a continuous and positive function on $\Re^2$

$$\Delta V(x)=\frac{a^2 x_2^{\,2}}{\left(1+x_1^{\,2}\right)^2}+\frac{b^2 x_2^{\,2}}{\left(1+x_2^{\,2}\right)^2}-x_1^{\,2}-x_2^{\,2}$$

$$.\qquad =\frac{-\left((1-a^2)x_2^{\,2}+2x_1^{\,2}x_2^{\,2}+x_1^{\,4}\right)}{\left(1+x_1^{\,2}\right)^2}+\frac{-\left((1-b^2)x_1^{\,2}+2x_1^{\,2}x_2^{\,2}+x_2^{\,4}\right)}{\left(1+x_2^{\,2}\right)^2}$$

Hence ,by theorem (3.4.4) the zero solution is stable if $a\leq 1$,$b\leq 1$,where $\Delta V(x) \leq 0$ and the zero solution is a symptotically stable if $a<1$,$b<1$, where $\Delta V(x) < 0$.

Theorem(3..4.6) : :(Instability Theorem of Lyapunov): [3,11]

Let V be a Lyapunov function (not necessarily positive definite) for Equation (3.3.1) on a neighborhood S of the equilibrium point x^* which satisfies

1) $\lim_{\|x\|\to 0} V(x)=0$

2) $\Delta V(x)>0$ if $x \varepsilon S\backslash\{ x^*\}$

3) $V(x)$ takes positive values in each sufficiently small neighborhood of the equilibrium point x^* ,then x^* is unstable.

Proof

Let the zero solution is stable .by assumption (3) there exist $n_i \to \infty$,let $x_0 = n_i$ with $V(x_0)>0$

and $\|x_0\|<\delta$,by assumption (1) ,we have lim $_{n\to\infty} x(n)$ and lim $_{n\to\infty} V(x(n))=0$

By assumption (2), we have $V(x_0)< V(x_1)< V(x_2)<\ldots<$ lim $_{n\to\infty} V(x(n))=0$

This is a contradiction . So the zero solution is unstable

3.5: Stability of Some Difference Equations

This section is devoted to study some illustrations on stability theory by considering the volterra equations and homogeneous difference equations.

Illustration (i): the Volterra equations

We recall the Volterra system (2.3.5) of convolution type

$$x(n+1) = Ax(n) + \sum_{j=0}^{n} B(n-j)x(j) \qquad (3.5.1)$$

where $A=(a_{ij})$ is a kxk real matrix and B(n) is a kxk real matrix defined on Z^+ such that $B(n)\in L_1$.

Before we start the stability analysis of Volterra system we need some theorems.

Theorem(3..5.1):[3,7]

A necessary and sufficient condition for the uniform asymptotic stability is

$$\det(zI - A - \widetilde{B}(z_0)) \neq 0 \quad \text{for all , } |z| \geq 1 \qquad (3.5.2)$$

Lemma(3..5.2):[3]

Let $G=(g_{ij})$ be a kxk matrix if z_0 is an eigenvalue of G ,then

(1) $\left|z_0 - g_{ii}\right|\left|z_0 - g_{jj}\right| \le \left(\sum_r^t \left|g_{ir}\right|\right)\left(\sum_r^t \left|g_{jr}\right|\right)$ for some $i,j.i \neq j$

(2) $\left|z_0 - g_{ii}\right|\left|z_0 - g_{jj}\right| \le \left(\sum_r^t \left|g_{ri}\right|\right)\left(\sum_r^t \left|g_{rj}\right|\right)$ for some $i,j.i \neq j$

where

$$\sum_r^t g_{ir} = \sum_{r=1}^k g_{ir} - g_{ii}$$

Theorem(3.5.3):[3,7]

The zero solution of equation(3.5.1) is uniformly asymptotically stable if either one of the following conditions hold

(1) $\sum_{j=1}^{k}(\left|a_{ij}\right| + \beta_{ij}) < 1$ for each $i, 1 \le i, j \le k$ or

(2) $\sum_{i=1}^{k}(\left|a_{ij}\right| + \beta_{ij}) < 1$ for each $j, 1 \le j \le k$

where

$$\beta_{ij} = \sum_{n=0}^{\infty}\left|b_{ij}(n)\right| \qquad 1 \le j \le k \ .$$

Proof

(1) To prove uniform asymptotic stability under condition (1) we need to show that condition (3.5.2) holds .So assume the contrary ,that is

$\det(z_0 I - A - \tilde{B}(z_0)) = 0$ for some z_0 with $\left|z_0\right| \ge 1$

Then z_0 is an eigenvalue of the matrix $.A+\widetilde{B}(z_0)$ Hence by condition (1) in Lemma (3.5.2)

,we have

$$\left|z_0 - a_{ii} - \widetilde{b}_{ii}(z_0)\right| \left|z_0 - a_{jj} - \widetilde{b}_{jj}(z_0)\right| \le \sum_{r}^{l} \left|a_{ir} - \widetilde{b}_{ir}(z_0)\right| \sum_{r}^{l} \left|a_{jr} - \widetilde{b}_{jr}(z_0)\right| \qquad (3.5.3)$$

But

$$\left|z_0 - a_{ii} - \widetilde{b}_{ii}(z_0)\right| \ge |z_0| - |a_{ii}| - \left|\widetilde{b}_{ii}(z_0)\right|$$

$$\ge 1 - |a_{ii}| - \left|\widetilde{b}_{ii}(z_0)\right|$$

$$> \sum_{r}^{l} (|a_{ir}| + |\beta_{ir}| \qquad \text{(by condition (1))}$$

Similarly

$$\left|z_0 - a_{jj} - \widetilde{b}_{jj}(z_0)\right| > \sum_{r}^{l} (|a_{jr}| + |\beta_{jr}|$$

Combining both inequalities,we get

$$\left|z_0 - a_{ii} - \widetilde{b}_{ii}(z_0)\right| \left|z_0 - a_{jj} - \widetilde{b}_{jj}(z_0)\right| > \sum_{r}^{l} (|a_{ir}| + |\beta_{ir}|) \sum_{r}^{l} (|a_{jr}| + |\beta_{jr}|)$$

It is clear that this contradicts inequality (3.5.3)

The second part of the theorem is proved in a similar way.

.

Theorem (3..5.4) :[3]

The zero solution of equation(3.5.1) is uniformly stable if

$$\sum_{i=1}^{k} (|aij| + \beta_{ij}) < 1 \qquad \text{for all } j=1,2,3,...k \qquad (3.5.4).$$

Proof

Define the Lyapunov function

$$V(x) = \sum_{i=1}^{k}\left[|x_i(n)| + \sum_{j=1}^{k}\sum_{r=0}^{n-1}\sum_{s=n}^{\infty} |b_{ij}(s-r)||x_j(r)| \right]$$

Then

$$\Delta V(x) \le \sum_{i=1}^{k}\left[\sum_{j=1}^{k} |a_{ij}||x_j(n)| - |x_i(n)| + \sum_{j=1}^{k}\sum_{s=0}^{\infty} |b_{ij}(s-n)||x_j(n)| \right] \qquad (3.5.5)$$

Observe that

$$\sum_{i=1}^{k}\sum_{j=1}^{k} |a_{ij}| \left| x_j(n) = \sum_{j=1}^{k}\sum_{i=1}^{k} |b_{ji}||x_i(n)| \right| \quad ,\text{and}$$

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{s=n}^{\infty} |b_{ij}(s-n)||x_j(r)| = \sum_{i=1}^{k}\sum_{j=1}^{n-1}\sum_{s=n}^{\infty} |b_{ji}(s-n)||x_i(r)|$$

Hence ,Inequaliy(3.5.5) becomes

$$\Delta V(x) \le \sum_{i=1}^{k}\left[\sum_{j=1}^{k} |a_{ji}| + b_{ji} - 1 \right] x_i(n) \le 0$$

This implies that

$$|x(n)| \le V(x) \le \sum_{i=1}^{k} |x_i(0)| = \|x(0)\|$$

which proves uniform stability

Example (3.5.5)

Consider the stability of the system

$$y(n+1) = Ay(n) + \sum_{j=0}^{n} B(n-j)y(j)$$

where

$$A = \begin{bmatrix} 0 & 1/10 \\ 1/3 & 1/4 \end{bmatrix} \qquad B(n) = \begin{bmatrix} 4^{-n-1} & 0 \\ 0 & 3^{-n-1} \end{bmatrix}$$

We use the previous theorem

$$\beta_{ij} = \sum_{n=0}^{\infty} |b_{ij}(n)|$$

$$\beta_{11} = \sum_{n=0}^{\infty} |4^{-n-1}| = \frac{1}{3}$$

$$\beta_{22} = \sum_{n=0}^{\infty} |3^{-n-1}| = \frac{1}{2}$$

Now we show that

$$\sum_{i=1}^{2} \left(|a_{ij}| + \beta_{ij} \right) < 1$$

Now at j=1 we get ,

$$a_{11}+\beta_{11}+a_{21}+\beta_{21}=0+1/3+1/3+0=2/3<1$$

and at j=2 we get ,

$$a_{12}+\beta_{12}+a_{22}+\beta_{22}=1/10+0+1/3+1/2=56/60<1$$

So the zero solution $x^*=0$ is uniformly stable.

Example (3.5.6)

Consider the stability of the system

$$y(n+1) = \sum_{j=0}^{n} B(n-j)y(j)$$

where

$$B(n) = \begin{bmatrix} 3^{-n-1} & e^{-n-1} \\ 0 & 5^{-n-1} \end{bmatrix}$$

We use the previous theorem

$$\beta_{ij} = \sum_{n=0}^{\infty} \left| b_{ij}(n) \right|$$

$$\beta_{11} = \sum_{n=0}^{\infty} \left| 2^{-n-1} \right| = \frac{1}{2}$$

$$\beta_{12} = \sum_{n=0}^{\infty} \left| 5^{-n-1} \right| = \frac{1}{4}$$

$$\beta_{22} = \sum_{n=0}^{\infty} \left| e^{-n-1} \right| = \frac{1}{(e-1)}$$

Now , we show that

$$\sum_{i=1}^{2} \left(\left| a_{ij} \right| + \beta_j \right) < 1$$

Now at j=1 we get ,

$$a_{11}+\beta_{11}+a_{21}+\beta_{21}=0+1/2+1/3+0=5/6<1$$

and at $j=2$ we get ,

$$a_{12}+\beta_{12}+a_{22}+\beta_{22}=0+1/4+0+1/(e-1)<1$$

So the zero solution $x^*=0$ is uniformly stable

Illustration (ii): the homogeneous difference equations

Let us rewrite the difference equation (1.2.2) as

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+2) \end{bmatrix} = \begin{bmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} \qquad x(n_0)=x_0 \qquad (3.5.6)$$

First,let us introduce some definitions and notations that are needed in the following theorems relevant to the stability of equation (3.5.6)

Now,let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad , x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad a_{11}, a_{12}, a_{21}.a_{22} \in \Re$$

and D=D(A) and the spectral norm ||A||of A is defined as follows

$D = D(A) = a_{11}a_{22} - a_{12}a_{21}$

$$\| x \| = \sqrt{x_1^{\,2} + x_2^{\,2}} \qquad , \| A \| = \max_{\|x\|=1} \| Ax \| \tag{3.5.7}$$

Further , assume that

$$\prod_{n=0}^{\infty} \max(\| A_n \|, 1) = M \quad , M \text{ is finite} \tag{3.5.8}$$

Theorem(3.5.6) :[8]

Assume that $\prod_{n=0}^{\infty} \max(\| A(n) \|, 1) = M$, M is finite . Then the difference equation (3.5.6)

,x(n+1)=A(n)x(n) ,x(n_0)=x_0

has a solution tending to zero if and only if

$$\prod_{n=0}^{\infty} | \det A(n) | = 0 \tag{3.5.9}$$

Proof :[8]

Theorem(3.5.7) :[8]

Suppose (3.5.8)holds and

$$\prod_{n=0}^{\infty} \| A_n \| = 0 \tag{3.5.10}$$

Then every solution of (3.5.6) tends to zero.

Proof :[8]

Theorem(3.5.8) :[3]

Assuming that A is any kxk matrix ,then lim $_{n\to\infty}$ $A^n = 0$ if and only if $|\lambda| < 1$ for all eigenvalues λ of A .

Comments:

To have a stable economy in Example (1.2.14),we assume that the sum of the domestic consumption $d_j(n+1)$ and double of the imports $2m_j(n+1)$in period $(n+1)$ must be less than the national income $y_j(n)$ in period (n): that is

$$d_j(n+1)+ 2m_j(n+1)< y_j(n), \qquad j =1,2,3$$

or

$$\alpha_1 + 2\beta_1 <1$$

$$\alpha_2 + 2\beta_2 <1$$

$$\alpha_3 + 2\beta_3 <1$$

From theorem (3.1.3) we have all the eigenvalues λ of A , $|\lambda|<1$.

This implies from theorem (3.5.8) that $A^n \to 0$ as $n \to \infty$

and so $\lim_{n\to\infty} \sum_{r=0}^{n-1} A^r = \sum_{r=0}^{\infty} A^r = (I - A)^{-1}$

It follows from formula (1.2.23) that

$$\lim_{n\to\infty} y(n) = (I - A)^{-1} i$$

This equation says that the national income of countries 1,2 and 3 approach equilibrium values independent of the initial values of the national incomes $y_1(0)$,$y_2(0)$, $y_3(0)$.

Chapter Four

Local Bifurcations Behavior

In this chapter we investigate the local stability of the system (3.4.1): $x(n+1) = f(x(n))$ that has nonhyperbolic equilibrium points.In particular,we will concentrate on system (3.4.1) that depends on the vector parameter ,μ, and has nonhyperbolic equilibrium points.Such systems undergo bifurcation phenomena.

4.1 Center Manifolds for Discrete Systems

This section includes the basic theorems which help in transforming an equilibrium point in $\Re^n$ which depends on a vector parameter $\mu \in \Re^p$ into the origin in $\Re^1 x \Re^1$.

First ,consider the following systems

$$\begin{aligned} x(n+1) &= Ax(n) + f(x(n), y(n)) \\ y(n+1) &= By(n) + g(x(n), y(n)), \qquad (x, y) \in \Re^c x \Re^s \end{aligned} \tag{4.1.1}$$

where

$$f(0,0) = 0, \qquad Df(0,0) = 0$$
$$g(0,0) = 0, \qquad Dg(0,0) = 0$$

and *f* and g are C^r ($r \geq 2$) functions in some neighborhood of the origin ,A is a cxc matrix with eigenvalues of modulus one,and B is an sxs matrix with eigenvalues of modulus less than one.

Definition (4.1.1) :[15,9]

An invariant manifold will be called a center manifold for (4.1.1) if it can locally be represented as follows

$$W_{loc}^{c} = \{(x, y) \in \Re^{c} \times \Re^{s} \left| y = h(x), \right| x |< \delta, h(0) = 0, Dh(0) = 0\} \qquad (4.1.2)$$

for δ sufficiently small ,where $h(x)$ is a function from $\Re^c$ into $\Re^s$.

Theorem(4.1.2) :[15]

There exists a C^r center manifold for(4.1.1) .The dynamics of (4.1.1) restricted to the center manifold is, for *u* sufficiently small ,given by the c-dimensional system

$$u(n+1) = Au(n) + f(u(n), h(u(n))), \qquad u \in \Re^{c} \qquad (4.1.3)$$

Proof :[15]

Theorem(4.1.3) :[11,15]

1) Suppose the zero solution of (4.1.3) is stable (asymptotic stable) (unstable),then the zero solution of (4.1.1) is stable (asymptotic stable) (unstable),respectively.

2) Suppose the zero solution of (4.1.3) is stable then ,if $(x(n), y(n))$ is a solution of (4.1.1) with (x(0),y(0)) is sufficiently small , there is a solution u(n) of (4.1.3) such that $\left|x(n) - u(n)\right| \leq k\beta^{n}$ and $\left|x(n) - h(u(n))\right| \leq k\beta^{n}$ for all n where k and β are positive constants with β<1.

Proof :[15]

Now we want to compute the center manifold .From the existence theorem for center manifolds,locally,we have

$$W_{loc}^{c} = \{(x, y) \in \Re^{c} x \Re^{s} \left| y = h(x), \right| x |< \delta, h(0) = 0, Dh(0) = 0\}$$

for δ sufficiently small .

Using invariance of W_{loc}^{c} under the dynamics of (4.1.1) ,we derive a nonlinear functional equation that the graph of h(x) must satisfy .

In this case we have

$$x(n+1) = Ax(n) + f(x(n), h(x(n)))$$

$$y(n+1) = h(x(n+1)) = Bh(x(n)) + g(x(n), h(x(n))) \qquad (4.1.4)$$

or

$$N(n+1) \equiv h(Ax(n) + f(x(n), h(x(n)))) - Bh(x(n)) - g(x(n), h(x(n))) = 0 \qquad (4.1.5)$$

Theorem(4.1.4) :[11]

Let $f : \Re^{c} \to \Re^{s}$ be a C^1 system with $f(0) = 0, f'(0) = 0$ and $N(f(x)) = O(|x|^{q})$ as x→0 for some q>1 .Then

$$h(x) = f(x) + O(|x|^{q}). \qquad \text{as } x\to 0 \qquad (4.1.6)$$

Proof :[11]

Now ,if the system (4.1.1) depends on a parameter ,say μ the system(4.1.1) becomes

$$\begin{aligned} x(n+1) &= Ax(n) + f(x(n), y(n), \mu(n)) \\ y(n+1) &= By(n) + g(x(n), y(n), \mu(n)) \qquad (x, y, \mu) \in \Re^{c} x \Re^{s} x \Re^{p} \end{aligned} \qquad (4.1.7)$$

where

$$f(0,0,0) = 0, \qquad Df(0,0,0) = 0$$

$$g(0,0,0) = 0, \qquad Dg(0,0,0) = 0$$

The previous system can be written as

$$x(n+1) = Ax(n) + f(x(n), y(n), \mu(n))$$

$$\mu(n+1) = \mu(n) \qquad (4.1.8)$$

$$y(n+1) = By(n) + g(x(n), y(n), \mu(n)) \qquad (x, y, \mu) \in \Re^c x \Re^s x \Re^p$$

The previous theorems are applied to center manifolds depending on parameters.

Now we want to compute the center manifold .From the existence theorem for center manifolds,locally,we have

$$W_{loc}{}^c = \{(x, y) \in \Re^c x \Re^s \,|\, y = h(x), |\, x \,| < \delta, h(0) = 0, Dh(0) = 0\}$$

for δ sufficiently small .

Using invariance of $W_{loc}{}^c$ under the dynamics of (4.1.7) ,we derive a nonlinear functional equation that the graph of h(x) must satisfy .

In this case we have

$$x(n+1) = Ax(n) + f(x(n), h(x(n)), \mu(n))$$

$$y(n+1) = h(x(n+1), \mu(n)) = Bh(x(n), \mu(n)) + g(x(n), h(x(n)), \mu(n)) \qquad (4.1.9)$$

or

$$N(n+1) \equiv h(Ax(n) + f(x(n), h(x(n))), \mu(n)) - Bh(x(n)) - g(x(n), h(x(n))) = 0$$

(4.1.10)

4.2 Local Bifurcation for Discrete Systems

In this section we study local bifurcations of systems of difference equations .By the term "local "we mean bifurcations occurring in a neighborhood of an equilibrium point .The term "bifurcation of a equilibrium point " is a sudden change in the number or nature of the equilibrium and periodic points of the system.

Consider a *p*-parameter system of equations :

$$x(n+1) = f(x(n);\mu) \tag{4.2.1}$$

with $x \in D \subset \Re^n$, $\mu \in \Re^p$ and $f \in C^r$ for some r ≥2.Suppose that (4.1.1) has an equilibrium point at $(x;\mu) = (x_0;\lambda_0)$ i.e

$f(x_0;\mu_0) = x_0$.

Assume also that (x*,0) is a bifurcation point of (4.2.1) ,which means that

$f(x^*,0) = x^*$

$D_x f(x^*,0) \equiv A$

where A has n_0 nonnegative eigenvalues of modulus 1 .For simplicity ,we assume that all other eigenvalues λ of A have |λ|<1 .

To investigate how the stability or instability is affected as the parameter μ varies ,an examination of the a ssociated linearized system is the place where one starts in order to study this local behavior.The associated linear system is given by the mapping

$x(n+1) = D_x f(x_0,\mu_0)\, y(n), y \in \Re^n$.

Now ,if the equilibrium point is hyperbolic (i.e none of the eigenvalues of the matrix $D_x f(x_0,\mu_0)$ have unit modulus) then stability (instability) in the linear approximation implies stability (instability)of the equilibrium point of the nonlinear system.

The following are the cases in which an equilibrium point of a system can be nonhyperbolic

(1) $D_x f(x_0,\mu_0)$ has a single eigenvalue equal to 1 and all other eigenvalues have moduli not equal to 1

(2) $D_x f(x_0,\mu_0)$ has a single eigenvalue equal to -1 and all other eigenvalues have moduli not equal to 1

(3) $D_x f(x_0, \mu_0)$ has two complex conjugate eigenvalues having modulus 1 and all other eigenvalues having moduli not equal to 1

Now ,we begin with the first case

Case(1)An eigenvalue of modulus 1

Here , we have three types of behavior

(i) The saddle node bifurcation

Suppose that the system (4.1.1) on the center manifold is given by

$$x(n+1) = f(x(n);\mu), x \in \Re^1, \mu \in \Re^1 \tag{4.2.2}$$

such that

$f(0;0) = 0$

$\frac{\partial f}{\partial x}(0,0) = 1$

The equilibrium points of(4.2.1) are given by

$f(x(n);\mu) - x \equiv h(x;\mu) = 0$

The curve of these equilibrium points satisfied two properties

a) It was tangent to the line μ=0 at x=0.

b)It lay entirely to one side of μ=0.

If $\frac{\partial h}{\partial \mu}(0,0) = \frac{\partial f}{\partial \mu}(0,0) \neq 0$ then ,by the implicit function theorem ,there exists a unique function μ=μ(x) ,μ(0)=0 defined for *x* sufficiently small such that

$h(x;\mu(x)) = 0$.

In order for the curve of equilibrium points to satisfy the above mentioned characteristic,it is sufficient to have

$\frac{d\mu}{dx}(0) = 0$

$$\frac{d^2\mu}{d^2x}(0) \neq 0$$

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial h}{\partial x}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)} = -\frac{\left(\frac{\partial f}{\partial x}(0,0) - 1\right)}{\frac{\partial f}{\partial \mu}(0,0)} = 0$$

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial h}{\partial x}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)} = -\frac{\left(\frac{\partial f}{\partial x}(0,0) - 1\right)}{\frac{\partial f}{\partial \mu}(0,0)} = 0$$

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial^2 h}{\partial x^2}(0,0)}{\frac{\partial h}{\partial \mu}(0,0)} = -\frac{\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)} = 0$$

(ii) The Transcritical Bifurcation

Suppose that the system on the center manifold is given by

$$x(n+1) = f(x(n);\mu), x \in \Re^1, \mu \in \Re^1 \qquad (4.2.3)$$

such that

$$f(0;0) = 0$$

$$\frac{\partial f}{\partial x}(0,0) = 1$$

The equilibrium points of (4.2.3) are given by

$$f(x(n);\mu) - x \equiv h(x;\mu) = 0 \qquad (4.2.4)$$

The curves of these equilibrium points satisfied the following properties

a) Two curves of these equilibrium points passed through $(x;\mu) = (0,0)$,one given by

x=μ,the other by x=0.

b) Both curves of these equilibrium points existed on both sides of μ=0.

c) The stability along each curve of these equilibrium points changed on passing through μ=0.

In order to have more than one curve of equilibrium points passing through (x,μ)=(0,0) we must have

$$\frac{\partial h}{\partial \mu}(0,0) = \frac{\partial f}{\partial \mu}(0,0) = 0$$

Since one of these curves of equilibrium points to be given by

X=0

So ,the system (4.2.4) can be written as

$$h(x,\mu) = xH(x,\mu) = x(F(x,\mu) - 1)$$

where

$$F(x,\mu) = \begin{Bmatrix} \frac{f(x,\mu}{x}, x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu), x = 0 \end{Bmatrix} \qquad (4.2.5)$$

and ,hence

$$H(x,\mu) = \begin{Bmatrix} \frac{h(x,\mu}{x}, x \neq 0 \\ \frac{\partial h}{\partial x}(0,\mu), x = 0 \end{Bmatrix}$$

Since we want only one additional curve of equilibrium points to pass through (x,μ)=(0,0), we require

$$\frac{\partial H}{\partial \mu}(0,0) = \frac{\partial F}{\partial \mu}(0,0) \neq 0 \qquad (4.2.6)$$

Using (4.2.5) and (4.2.6) we get

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0.$$

by the implicit function theorem ,there exists a unique function μ=μ(x) ,μ(0)=0 defined for x sufficiently small such that

$$H(x,\mu)=F(x,\mu)-1=0$$

Now ,we simply require that

$$\frac{d\mu}{dx}(0)\neq 0$$

$$\frac{d\mu}{dx}(0)=-\frac{\frac{\partial H}{\partial x}(0,0)}{\frac{\partial H}{\partial \mu}(0,0)}=-\frac{\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)}$$

$$\frac{d\mu}{dx}(0)=-\frac{\frac{\partial h}{\partial x}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)}=-\frac{\left(\frac{\partial f}{\partial x}(0,0)-1\right)}{\frac{\partial f}{\partial \mu}(0,0)}=0$$

$$\frac{d\mu}{dx}(0)=-\frac{\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x\partial \mu}(0,0)}$$

(iii) The Pitchfork Bifurcation

Suppose that the system on the center manifold is given by

$$x(n+1)=f(x(n);\mu),x\in\Re^1,\mu\in\Re^1 \qquad (4.2.7)$$

such that

$$f(0;0)=0\ , \qquad \frac{\partial f}{\partial x}(0,0)=1$$

The equilibrium points of (4.2.7) are given by

$$f(x(n);\mu)-x\equiv h(x;\mu)=0$$

The curve of these equilibrium points satisfied the following properties

a) Two curves of these equilibrium points passed through $(x;\mu)=(0,0)$,one given by

$\mu=\mu$ (x),the other by x=0.

b) The curve x=0 existed on both sides of $\mu=0$,the other curve $\mu=\mu$ (x) existed on one side of $\mu=0$.

c) The equilibrium points on the curve x=0 had different stability types on opposite sides of $\mu=0$. The equilibrium points on the other curve $\mu=\mu$ (x),all had the same stability type.

In order to have more than one curve of equilibrium points passing through $(x,\mu)=(0,0)$ we must have

$$\frac{\partial h}{\partial x}(0,0)=\frac{\partial f}{\partial \mu}(0,0)=0$$

Since one of these curves of equilibrium points to be given by

x=0

So ,the system (4.2.7) can be written as

$$h(x,\mu)=xH(x,\mu)=x(F(x,\mu)-1)$$

where

$$F(x,\mu)=\begin{Bmatrix} \dfrac{f(x,\mu)}{x}, x\neq 0 \\ \dfrac{\partial f}{\partial x}(0,\mu), x=0 \end{Bmatrix} \qquad (4.2.8)$$

and ,hence

$$H(x,\mu)=\begin{Bmatrix} \dfrac{h(x,\mu)}{x}, x\neq 0 \\ \dfrac{\partial h}{\partial x}(0,\mu), x=0 \end{Bmatrix}$$

Since we want only one additional curve of equilibrium points to pass through $(x,\mu)=(0,0)$, we require

$$\frac{\partial H}{\partial \mu}(0,0)=\frac{\partial F}{\partial \mu}(0,0)\neq 0 \qquad (4.2.9)$$

Using (4.2.8) and (4.2.9) we get

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0.$$

by the implicit function theorem ,there exists a unique function μ=μ(x) ,μ(0)=0 defined for *x* sufficiently small such that

$$H(x,\mu) = F(x,\mu) - 1 = 0$$

Now ,we simply require that

$$\frac{d\mu}{dx}(0) = 0$$

$$\frac{d^2\mu}{d^2x}(0) \neq 0$$

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial H}{\partial x}(0,0)}{\frac{\partial H}{\partial \mu}(0,0)} = -\frac{\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)}$$

$$\frac{d^2\mu}{dx^2}(0) = -\frac{\frac{\partial^2 H}{\partial x^2}(0,0)}{\frac{\partial H}{\partial \mu}(0,0)} = -\frac{\frac{\partial^2 F}{\partial x^2}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)}$$

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)}$$

$$\frac{d^2\mu}{dx^2}(0) = -\frac{\frac{\partial^3 f}{\partial x^3}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)}$$

Case(2) An eigenvalue of -1

Suppose that the system on the center manifold is given by

$$x(n+1) = f(x(n);\mu), x \in \Re^1, \mu \in \Re^1 \qquad (4.2.10)$$

such that

$$f(0;0)=0 \quad , \qquad \frac{\partial f}{\partial x}(0,0)=-1$$

$$f^2(0;0)=0 \quad , \qquad \frac{\partial f^2}{\partial x}(0,0)=1$$

$$\frac{\partial f^2}{\partial \mu}(0,0)=0 \ , \qquad \frac{\partial^2 f^2}{\partial x^2}(0,0)=0$$

$$\frac{\partial^2 f^2}{\partial x \partial \mu}(0,0)\neq 0 \, , \qquad \frac{\partial^3 f^2}{\partial x^3}(0,0)\neq 0$$

Case(3) A pair of eigenvalues of modulus 1

Suppose that the map on the center manifold is given by

$$x(n+1)=f(x(n);\mu), x\in \Re^{2,} \mu \in \Re^1 \qquad (4.2.11)$$

such that

$$f(0;0)=0$$

with the matrix $D_x f(0,0)$ having two complex conjugate eigenvalues ,denoted $\lambda(0), \bar{\lambda}(0)$**,**

with $|\lambda(0)|=1$and $\lambda''(0)\neq 0$

The normal form is

$$z(n+1)=\lambda(\mu(n))z(n)+c(\mu(n))z^2(n)\bar{z}(n)+O(4), \quad z\in C, \mu \in R^1 \qquad (4.2.12)$$

To transform (4.2.12) into polar coordinate we let

$$z=re^{2\pi i\theta},$$

and we get

$$r\to |\lambda(\mu)|\left(r+\mathrm{Re}\left(\frac{c(\mu)}{\lambda(\mu)}\right)r^2+O(r^4)\right),$$

$$\theta \to \theta + \phi(\mu) + \frac{1}{2\pi}\left(\operatorname{Im}\left(\frac{c(\mu)}{\lambda(\mu)} \right) r^2 + O(r^3) \right), \tag{4.2.13}$$

where

$$\phi(\mu) \equiv \frac{1}{2\pi}\tan^{-1}\frac{\beta(\mu)}{\alpha(\mu)}$$

and

$$c(\mu) = \alpha(\mu) + i\beta(\mu)$$

$$r \to \left(1 + \frac{d}{d\mu}\left|\lambda(\mu)\right|\Big|_{\mu=0} \mu \right) r + \left(\operatorname{Re}\left(\frac{c(0)}{\lambda(0)} \right) r^3 \right) + O(\mu^3 r, \mu r^3, r^4)$$

$$\theta \to \theta + \phi(0) + \frac{d}{d\mu}(\phi(\mu))\Big|_{\mu=0} \mu + \frac{1}{2\pi}\operatorname{Im}\left(\frac{c(0)}{\lambda(0)} \right) r^2 + O(\mu^2, \mu r^2, r^3) \tag{4.2.14}$$

$$k \equiv \frac{d}{d\mu}\left|\lambda(\mu)\right|\Big|_{\mu=0}$$

$$a \equiv \operatorname{Re}\left(\frac{c(0)}{\lambda(0)} \right)$$

$$\phi_0 = \phi(0)$$

$$\phi_1 \equiv \frac{d}{d\mu}\left|\phi(\mu)\right|\Big|_{\mu=0}$$

$$b \equiv \frac{1}{2\pi}\operatorname{Im}\left(\frac{c(0)}{\lambda(0)} \right)$$

Hence ,(4.2.14) becomes

$$r(n+1) = r(n) + \left(k\mu(n) + ar^2(n)\right) r(n) + O(\mu^2(n) r(n), \mu(n) r^3(n), r^4(n))$$

$$\theta(n+1) = \theta(n) + \phi_0 + \phi_1 \mu(n) + br^2(n) + O(\mu^2(n) r(n), \mu(n) r^3(n), r^3(n)) \tag{4.2.15}$$

The truncated normal form is given by

$$r(n+1) = r(n) + \left(k\mu(n) + ar^2(n)\right)r(n)$$

$$\theta(n+1) = \theta(n) + \phi_0 + \phi_1\mu(n) + br^2(n) \qquad (4.2.16)$$

Lemma (4.2.1) :[15]

$\left\{(r,\theta) \in \Re^+ xS^1 \middle| r = \sqrt{\frac{-\mu k}{a}}\right\}$ is a circle which is invariant under the dynamics generated by (4.2.15)

Lemma (4.2.2) :[15]

The invariant circle is asymptotically stable for $a<0$ and unstable for $a>0$.

Example (4.2.3)

Consider the stability by the following system

$$x(n+1) = -x(n) - \mu x(n) - xy$$

$$y(n+1) = \tfrac{1}{2}y(n) - \mu y(n) - x^2$$

this system can be written as

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} f(x(n), y(n), \mu(n)) \\ g(x(n), y(n), \mu(n)) \end{bmatrix}$$

where

$$f(x(n), y(n), \mu(n)) = -\mu x - xy$$

$$g(x(n), y(n), \mu(n)) = -x^2$$

Let $\quad h(x,\mu) = ax^2 + bx\mu + c\mu^2$

using formula (4.1.10) to get

$$h(x,\mu) = -2x^2 + O(3)$$

The orbit structure near (0,0) is determined by the associated center manifold which we write as

$$x(n+1) = F(x(n), \mu(n)) = -x(n) - \mu x(n) + 2x^3(n)$$

such that

$$F(0;0) = 0 \ , \qquad \frac{\partial F}{\partial x}(0,0) = -1$$

$$F^2(x(n), \mu(n)) = x + \mu(2+\mu)x - 2x^3 + O(4)$$

$$F^2(0;0) = 0 \ , \qquad \frac{\partial F^2}{\partial x}(0,0) = 1$$

$$\frac{\partial F^2}{\partial \mu}(0,0) = 0 , \qquad \frac{\partial^2 F^2}{\partial x^2}(0,0) = 0$$

$$\frac{\partial^2 F^2}{\partial x \partial \mu}(0,0) \neq 0 , \qquad \frac{\partial^3 F^2}{\partial x^3}(0,0) \neq 0$$

The two curves of equilibrium points are

$x = 0$ and $x^2 = \dfrac{2+\mu}{2}$

$x = 0$ is stable whenever $-2 < \mu < 0$

$x = 0$ is unstable whenever $\mu > 0$ and $\mu \leq -2$

$x^2 = \dfrac{2+\mu}{2}$ is unstable $\mu \geq -2$

since all of the three equilibrium points for $\mu > 0$ are unstable we compute $F^2(x(n), \mu(n))$

$$F^2(x(n), \mu(n)) = x(n) + \mu(2+\mu)x(n) - 2x^3(n)$$

The two curves of equilibrium points are

$x = 0, \quad x^2 = \dfrac{\mu(2+\mu)}{4}$

$x = 0$ is stable whenever $-2 < \mu < 0$

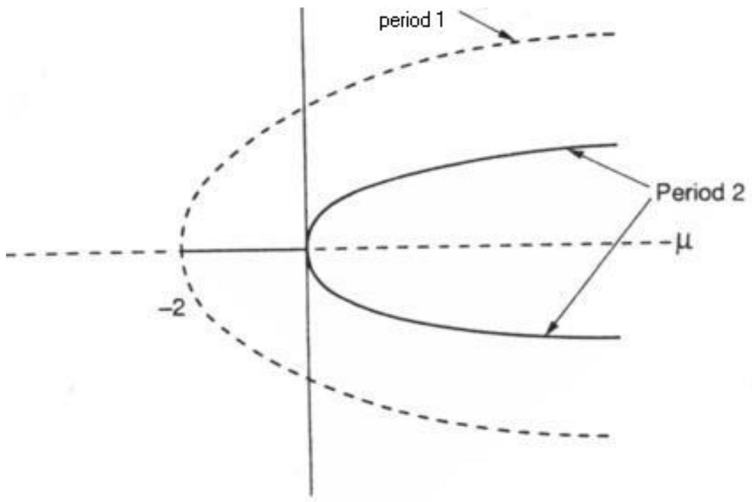
$x = 0$ is unstable whenever $\mu > 0$ and $\mu \leq -2$

$x^2 = \frac{\mu(2+\mu)}{4}$ is stable whenever $0 < \mu < -1+\sqrt{2}$

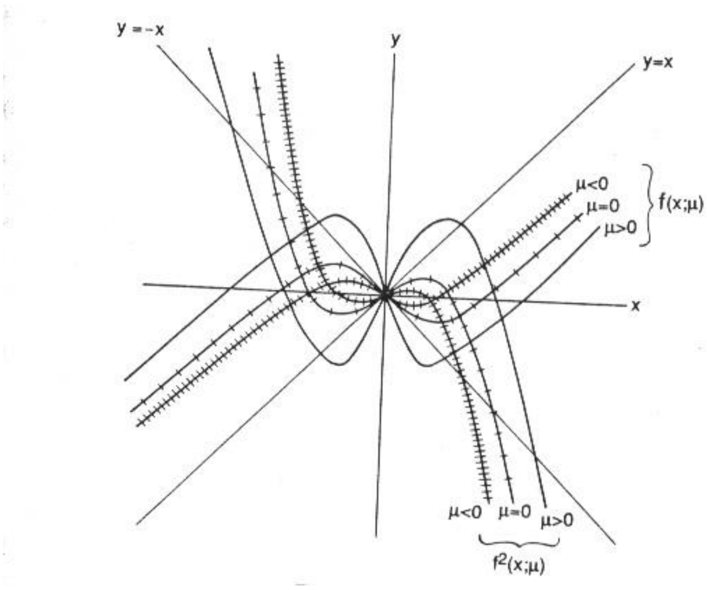
The figure (4.2.1) illustrates the two curves of equilibrium points for

$F^2(x(n), \mu(n)), \quad F(x(n), \mu(n))$

Fiqure (4.2.2) illustrate period –doubling bifurcation for different values of μ .



Figure(4.2.1)



Figure(4.2.2)

References

[1] Nils Berglund,Geometrical Theory of Dynamical Systems ,Lecture Notes ,2001.

[2] R.V Churchill and J.W Brown ,Complex Variables and Applications,Mc-Grow Hill ,New York ,1990

[3] Saber N . Elaydi ,An Introduction to Difference Equations ,San Antonio ,1995

[4] Saber Elaydi ,Tahseen Mughrabi , et al ,Proceedings of the mathematics Conference ,Birzeit University ,Palestine , World Scientific publishing Co.Pte.Ltd, 2000.

[5] S.Elaydi,"Global Stability of Difference Equations".Proceedings of the First World Congress of Nonlinear Analysis ,Florida,1992,Kluwer.

[6] S.Elaydi ,Dannon,Ponomarenko,Stability of hyperbolic and non-hyperbolic equilibrium points of one-dimentional map,,J.Diff.Eq. and Appl. 9(2003),449-457.

[7] S.Elaydi ,Dannon,Ponomarenko,Asymptotic stability of linear difference equations of advanced type ,J.Computational Anal .Appl .,Vol.6,No.2,423-428,2004.

[8] A' Elbert ,Stability in Difference Equations , Proceedings of the Second International Conference on Difference Equations

[9] Richard A. Holmgren ,A First Course in Discrete Dynamics Systems ,Spring Verlag NewYork Berlin Heidel Berg ,1994.

[10] Samuel Goldberg ,Introduction to Difference Equations ,John Wiley and Sons.Inc,1958.

[11] Oded Galor ,Introduction to Stability Analysis of Discrete Dynamical System,JEL,2003.

[12] R.k Miller and A.N Michael ,Ordinary Differential Equations ,Academic , New York ,1982.

[13] Peter Lancaster ,Miron Tismenetsky,The Theory of Matrices Second Edition with Applications ,Harcourt Brace Jovanovich Publishers ,1985.

[14] A.N Shorkovsky ,Yu .L Maistrenko and E.Yu .Romanenko Difference equations and their applications ,kluwer Academic puplishers ,1993.

[15]S.Wiggins ,Introduction to Applied Nonlinear Dynamical Systems and chaos ,springer-verlag,NewYork.Inc,1990.

[16] Kosaku Yosida ,Functional Analysis ,New York Berlin Heidelberg Berlin ,1980 .

Number	Summation	Definite sum
1)	$\sum_{k=0}^{n} k$	$\frac{n(n+1)}{2}$
2)	$\sum_{k=0}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
3)	$\sum_{k=0}^{n} k^3$	$\left[\frac{n(n+1)}{2}\right]^2$
4)	$\sum_{k=0}^{n} k^4$	$\frac{n(6n^4+15n^3+10n^2-1)}{30}$
5)	$\sum_{k=0}^{n} a^k$	$\begin{cases}(a^{n+1}-1)/a-1 & if\ a\neq 1\\ n+1 & if\ a=1\end{cases}$
6)	$\sum_{k=0}^{n} ka^k \quad (a\neq 1)$	$\frac{(a-1)(n+1)a^{n+1}-a^{n+2}+a}{(a-1)^2}$

Table(1)

No	$x(n)$ for n=0,1,2,... $x(n)$=0 for n=-1,-2,-3,...	$\tilde{x}(z)=\sum_{n=0}^{\infty}x(n)z^{-n}$
1	1	z/(z-1)
2	a^n	z /(z-a)
3	a^{n-1}	1/(z-a)
4	n	$z/(z-1)^2$
5	n^2	$z(z+1)/(z-1)^3$
6	n^3	$Z(z^2+4z+1)/(z-1)^4$
7	n^k	$(-1)^k D^k(z/(z-1))$,D=z(d/dz)
8	$n\, a^n$	$az/(z-a)^2$
9	$n^2 a^n$	$Az(z+a)/(z-a)^3$
10	$n^3 a^n$	$Az(z^2+4az+a^2)/(z-a)^4$
11	$n^k a^n$	$(-1)^k D^k(z/(z-a))$,D=z(d/dz)
12	Sin $n\omega$	$z\sin\omega/(z^2-2z\cos\omega+1)$
13	Cos $n\omega$	$z(z-\cos w)/(z^2-2z\cos\omega+1)$
14	a^n Sin $n\omega$	$az\sin\omega/(z^2-2az\cos\omega+a^2)$
15	a^n cos $n\omega$	$z(z-a\cos\omega)/(z^2-2az\cos\omega+a^2))$
16	$a^n/n!$	$e^{a/z}$
17	Sinh $n\omega$	$z\sinh\omega/(z^2-2z\cosh\omega+1)$
18	Cosh $n\omega$	$z(z-\cosh w)/(z^2-2z\cosh\omega+1)$
19	$1/n, n>0$	ln(z/z-1)
20	$e^{-n\omega}$ x(n)	$\tilde{x}(e^{\omega}z)$
21	$n^{(2)}=n(n-1)$	$2z/(z-1)^3$
22	$n^{(3)}=n(n-1)(n-2)$	$3!\, z/(z-1)^4$
23	$n^{(k)}=n(n-1)...(n-k+1)$	$k!\, z/(z-1)^{k+1}$
24	$x(n-k)$	$z^{-k}\tilde{x}(z)$
25	$x(n+k)$	$z^{-k}\tilde{x}(z)-\sum_{r=0}^{k-1}x(r)z^{k-r}$

Table(2)