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Fractional Derivatives

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Abstract

In this thesis we considered the main data for the concept of fractional derivatives and fractional calculus, and that was by presenting two different definitions for the development of the concept of fractional derivatives.

At first, we presented the development of the concept of the derivatives for the natural numbers then we generalized it to the real numbers, so that we got a group of special mathematical results respect to this development. These results were consistent to what was known about the derivatives of the natural numbers.

The other concept we presented was the concept of Rieman-Liouville definition, the one that was generalized to include the real numbers, not only the natural numbers.

After that we considered many properties of this definition that was generalized, we obtained the fractional derivatives of some elementary functions depending on this definition, that was generalized .We also proved the property that we got according to the first concept by employment of the second concept.

We also presented historical brief extract about the development of the concept of the fractional derivatives since the begging of 17th centaury until now to achieve the identification of the fractional derivatives.

We studied the special case of the definition of the fractional derivatives which introduced by Rieman-Liouville when $0 < \alpha < 1$ and that is by considering some of the relations of the fractional derivatives that where derived through the fractional calculus and we got similar results.

Finally, we searched through of many practical applications for the concept of the fractional derivatives in physics and statistical applications and we demonstrated the importance of the applications in the other areas.

Table of contents

Chapter One

A Short History of Fractional Derivatives

The idea of generalizing the concepts of differentiation and integration to non-integer (fractional) orders has a long mathematical history. It was first discussed in the correspondence of G.W. Leibniz around 1690. Over the centuries many famous mathematicians including Euler, Riemann, Liouville and Weyl have built up a body of mathematical knowledge on fractional integrals and derivatives that is known under the name of fractional calculus. Here we will give a brief history of this concept.

1.1 Leibniz 1690

The origin name of fractional derivatives is not certain at this time, but we do know that Leibniz, the inventor of the notation $d^n y / dx^n$, had "toyed" with the idea in the 1600's. In 1695 L'Hopital asked Leibniz: "what if n be $1/2$?" Surprisingly Leibniz, see [11] replied:

"… You can see by that, sir, which one can express by an infinite series a quantity such as $d^{1/2}xy$ or $d^{1:2}xy$.

Although infinite series and geometry are distant relations, infinite series admits only the use of exponents that are positive and negative integers, and does not, as yet, know the use of fractional exponents…" As with most great mathematicians, Leibniz had a unique insight into the unknown. He stumbled onto fractional derivatives realizing that one-day great things will come from his work. What they would be, he had no idea.

In the same letter he continued: "... thus it follows that $d^{1/2}x$ will be equal to $d^{1/2}xy$ or $d^{1:2}xy$.

This is an apparent paradox from which, one day, useful consequences will be drawn..."

Leibniz insight did not stop there. Three years latter in a letter to john Wallis, he discussed ways of using fractional derivatives in Wallis's infinite product for $1/2\pi$. He states, see [11]: "... differential calculus might have been used to achieve this result..." it should be evident that Leibniz did not have just a passing thought on fractional derivatives; he must have spent a considerable amount of time on the topic.

1.2 Euler 1738

Euler, another great mathematician, toyed with the idea of fractional derivatives.

43 years after Leibniz went public with his controversial ideas of fractional derivatives, Euler stated in his 1738 dissertation: "... when n is a positive integer, and if p should be a function of x, the ratio d^2p to dx^n (n is integer) can always be expressed algebraically, so that if $n=2$ and $p=x^3$, then d^2x^3 to dx^2 be $6x$ to 1. Now it is asked what kind of ratio can then be made if n be a fraction.

The difficulty in this case can easily be understood, for if n is a positive integer d^n can be found by continued differentiation, such a way, however, is not evident if n is a fraction. But yet with the interpolation which I have already explained in this dissertation, one may be able to expedite the matter..."

Searching through several books on this topic Miller only found one "hit" in 80 years after Euler's dissertation that "hit" was Laplace. In 1812 Laplace mentioned, in passing fractional derivatives by means of integrals. If he was around today I am sure he would be sorry he did not do more on the subject.

1.3 Lacroix 1819

In 1819 Lacroix wrote a 700-pages textbook on differential and integral calculus. He stumbled over fractional derivatives in a two-page exercise; he develops the *nth* derivative and then generalizes it with the gamma function $\Gamma(x)=(x-1)!$. He finishes the exercise with an example for when $y=x$ and $n=\frac{1}{2}$; he obtained:

$$\frac{d^{1/2}y}{dx^{1/2}}=\frac{2\sqrt{x}}{\sqrt{\pi}} \quad \text{......} \ (1.3.1)$$

1.4 Fourier 1822

Three years later Fourier wrote about derivatives of arbitrary number. He obtained:

$$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} p^u \cos\left[p(x-\alpha) + \tfrac{1}{2}u\pi\right] dp$$

He stated, see [16]: "... The number u that appears above will be regarded as any quantity whatever, positive or negative ...".

1.5 Abel 1823

Up to this point in time mathematicians only "played" with the nation of fractional derivatives. One year after Fourier, Abel, see [19] took the proverbial ball and ran with it. While Abel was "toying" with the tautochrone problem (The problem of finding the curve down which a bead placed anywhere will fall to the bottom in the same amount of time) he stumbled over the solution by using fractional calculus. Without going too far into the solution, Abel's general integral equation for k is given as follows:

$$k = \int (x-t)^{\frac{1}{2}} f(t) dt$$

where k is a known constant for the amount of time it takes for a frictionless mass to slide down a curve no matter where the mass starts.

Abel "played" with general integral equations until he came up with the following:

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} k = \sqrt{\pi} f(x)$$

where k is a known constant.

Abel used Fourier's integral formulas to solve his problem but never gave him credit for the solution.

1.6 Liouville 1832

Nine years after Abel's solution, in 1832, the famous mathematician Liouville published three memories, which were the fruit of the first major study is fractional derivatives. Shortly after his three memoirs, Liouville published several papers on theoretical application using fractional derivatives in the solutions. Liouville begun with a well known formula of his time:

$$D^m e^{ax} = a^m e^{ax} \quad \text{Where } D \text{ is a derivative}$$

He then let v be a derivative with arbitrary order, which yielded:

$$D^v e^{ax} = a^v e^{ax}$$

He "played" with it in an intuitive way with derivatives of arbitrary order and expanded the formula in a series until he came up with:

$$f(x) = \sum_{n=0}^{\infty} c_n e^{\alpha_n x} \quad , \quad \operatorname{Re} \alpha_n > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

This yielded:

$$D^v f(x) = \sum_{n=0}^{\infty} c_n \alpha_n^v e^{\alpha_n x}$$

The above formula is sometimes known as Lioville's first formula of fractional derivatives, which is an intuitive approach of arbitrary order v, where Liouville allowed v being any number; rational, irrational, or complex. It should be easy to see that Liouville's first formula is applicable to functions only in the form of (a).

Liouville may or may not have been aware of the narrowness of his first formula for fractional derivatives, but he came up with his second formula of fractional derivatives. He started with a definite integral:

$$I = \int_0^{\infty} u^{\alpha-1} e^{-xu} du, \qquad \alpha > 0,\ x > 0$$

He "played" with the formula by changing variable and operating on both sides with D^v to obtain his second formula of fractional derivatives:

$$D^v x^{-\alpha} = \frac{(-1)^v \Gamma(\alpha + v)}{\Gamma(\alpha)} x^{-\alpha-v}, \qquad \alpha > 0 \ldots\ldots\ldots\ldots\ldots (b)$$

where v is any number rational, irrational, or complex.

Although Liouville was the first to try solving fractional differential equations, he was not totally correct. He realized that his first and second formulas for fractional derivatives needed too narrow restrictions to be of much use. His first formula was only good for the class of (a), and his second formula was only good for functions in the form $x^{-\alpha}$ with $\alpha > 0$.

It is clear that Liouville was aware of this fact since in one of his memories of 1834 he says: "... The ordinary differential equation $d^n y/dx^n = 0$ has the complementary solution $y_c = c_0 + c_1 x + c_2 x^2 + ... + c_{n-1}x^{n-1}$. Thus $d^u y/dx^u = 0$ has (u arbitrary) should have a corresponding complementary solution...". While Liouville did come up with a corresponding complementary solution it became the center of controversy during his time. One would wish that he had gone further in developing fractional derivative.

1.7 The Fight of the Decade

From 1833 to 1848 several mathematicians ended up fighting over the work of Lacroix, Abel, and Liouville. In 1833 Peacock supported Lacroix's formula as in equation 1.3.1, while holding Liouville's formulas as being useless except for a few special cases. Peacock made several errors while trying to support Lacroix's formula; one of his biggest errors was misapplication of symbolic operations, where he believed that the principles of symbolic algebra (+, -, x, /) would hold true for derivatives.

On the other hand Kelland supported Liouville on two separate occasions in 1839 and the other time in 1846 when he believed that Liouville's second formula had useful implications in the form of $x^{-\alpha}$.

In 1840 De Morgan writes (referring to Lacroix formula and Liouville's second formula): "... Both these systems may very possibly be part of a more general system, but at present I incline to the conclusion that neither system has any claim to be considered as given the form $D^n x^m$, though either my be a form ...".

Even De Morgan, one of the great mathematicians of all times, could not make up his mind on this matter. In 1848, William Center could not make up his mind either.

He stated, see [9]: "... according to Liouville's system, by letting $\alpha = 0$ the fractional derivative of unity equals zero because $\Gamma(0) = \infty$... the whole question is plainly reduced to what is $d^u x^0/dx^u$. For when this is determined we shall determine at the same time which is the correct system..."

Well, who was right? It turns out that De Morgan was correct for both Lacroix formula and Liouville's second formula were incorporated into a more general formula year later.

1.8 Riemann (late 1800's)

Exactly when Riemann worked on fractional derivatives no one knows, for he never publicized any of it. But Miller and Ross know that Riemann does his work in his student years. Riemann tried to find the general solution of fractional derivative by way of the Taylor series and letting $\Psi(x)$ be the complementary function, which yielded:

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)dt + \Psi(x) \text{..........(c)}$$

No one is sure that Riemann knew exactly what the outcome of a complementary function would be, for he used it to provide a "measure of the deviation". In 1880 Cayley, see [4] stated: "... The greatest difficulty in Riemann's theory, it appears to me, is the question of the meaning a complementary function containing infinity of arbitrary constants ... Any satisfactory definition of a fractional operation will demand that this difficulty be removed ..." Later in his paper; Cayley says: "... Riemann was hopelessly entangled in his version of a complementary function ..."

All too many times, when we become to close to a project that we are working on, we can not see the trees through the proverbial forest. It appears to me that Riemann had this same problem. Riemann did little more with this topic, but we will see that he had tremendous insight, and several mathematicians built on his work.

1.9 Laurent 1884

Two mathematicians, Sonin and Letnikov, developed the prelude to the idea of fractional derivatives for modern mathematicians. In 1869 Sonin wrote a paper, "On Differentiation with arbitrary index", and Letnikov wrote four papers between 1868 and 1872 on the same topic. Both mathematicians started their work with Cauchy's integral formula:

$$D^n f(z) = \frac{n!}{2\pi i} \int_c \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \text{ .... (d)}$$

where c represents a closed contour going around once counter clockwise. Sonin and Letnikov were off to a great start since it was permitted to generalize $n!$. Both knew about the gamma function and how $v! = \Gamma(v+1)$, when v takes on arbitrary values of integers.

They knew when n was an integer they would obtain a simple pole in the contour of the close circuits. They saw when n was not an integer they would no longer have a simply pole but a branch cut. Sonin and Letnikov realized the problem but did not provide a solution.

Unfortunately for Sonin and Letnikov for, 12 years later, in 1884, Laurent solved the problem Laurent, as well, started with Cauchy's integral formula (d). He used the rules of transformation and his contour was an open path on Riemann surface. He produced his definition for differentiation for arbitrary order:

$$ {}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)dt, \qquad \operatorname{Re} v > 0 \ldots\ldots\ldots\ldots \text{(e)} $$

Do you notice what happens if we let $x > c$ (c any real or complex number) in Laurent's definition (e)? You should see that it is Riemann's definition (c) without his complementary function $\Psi(x)$. It is important to note that when $c = 0$ Laurent obtained:

$$ {}_0D_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \qquad v > 0 $$

This version is the most commonly used, and is named the Riemann-Liouville fractional integral. I believe that it should be named the Liouville-Riemann fractional integral, since Liouville tried to solve the problem first. In any event he finally received recognition for his work. John M. Beach stated "I wish I were around today to witness the fruits of his labor"., see [18].

1.10 Heaviside 1892

Oliver Heaviside, a genius in his time, has become one of Miller and Ross, see [17].pp.13 "heroes", although he was an untrained scientist, as they stated, and not a mathematician. I look at things the same way as he did. I prefer to use an intuitive approach when looking at problems. This was much more common in previous centuries than now; in 1892 he published several papers on liner functional operators,

المشتقات الكسرية

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الخلاصة

لقد درسنا في هذه الرسالة المعلومات الأساسية لمفهوم المشتقات الكسرية والتفاضل الكسري و التكامل الكسري وذلك عن طريق تقديم تعريفين مختلفين لمفهوم المشتقات الكسرية ففي البداية قدمنا مفهوم تطوير المشتقات للاعداد الطبيعية وبعدها تم تعميمه على جميع الاعداد الحقيقية وحصلنا على العديد من النتائج الرياضية الخاصة بذلك والتي كانت مطابقة لما كان معروفا للاعداد الطبيعية. والمفهوم الثاني الذي تم تقديمه فكان مفهوم المشتقات الكسرية حسب تعريف ريمان- لوفيل والذي تم تعميمه ليشمل الاعداد الحقيقية أيضا وليس الاعداد الصحيحه فقط, وبعد ذلك بحثنا في العديد من خصائص هذا التعريف , وحصلنا على المشتقات الكسرية للعديد من الاقترانات الأولية بالاعتماد على هذا التعريف الذي تم تعميمه، كذلك أثبتنا الخصائص التي حصلنا عليها حسب المفهوم الاول باستخدام المفهوم الثاني.

ولكي يتم التعرف على المشتقات الكسرية قدمنا نبذة تاريخية عن تطور مفهوم المشتقات الكسرية منذ بداية القرن السابع عشر ولغاية يومنا هذا.

لقد قمنا بدراسة الحالة الخاصة من المشتقات الكسرية لريمان- لوفيل عندما يكون الكسر ما بين 0 و 1 وذلك بدراسة بعض علاقات المشتقات الكسرية التي تم اشتقاقها من خلال التفاضل والتكامل الكسري وحصلنا على نتائج مشابهة لما حصلنا عليه سابقا .

وفي الختام بحثنا في العديد من التطبيقات العملية لمفهوم المشتقات الكسرية في الفيزياء والاحصاء الرياضي وبينا مدى أهمية تطبيقها في العلوم الاخرى.