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Hypergeometric function representation of transport coefficients for drifting bi-Maxwellian plasmas

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We derive the momentum, parallel energy, and perpendicular energy collisional transport coefficients for drifting bi-Maxwellian plasmas by using the Boltzmann collision integral approach and present them in the form of triple hypergeometric functions. In the derivation, we write the drift velocity $\mathbf{u}$ of the bi-Maxwellian plasma in terms of parallel and perpendicular components (i.e., $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$), parallel and perpendicular with respect to the ambient magnetic field, and we consider the Coulomb collision interactions. We consider two special cases: first, when the drift velocity is parallel to the ambient magnetic field (i.e., $\mathbf{u} = \mathbf{u}_{\parallel}$), and second, when the drift velocity is perpendicular to the ambient magnetic field (i.e., $\mathbf{u} = \mathbf{u}_{\perp}$). For the first case, the transport equations and consequently the transport coefficients are derived and presented in the form of double hypergeometric functions; these results are consistent with the findings of Hellinger and Trávníček [Phys. Plasmas **16**(5), 054501 (2009)]. For the second case, the transport coefficients are obtained and found to be in the form of double hypergeometric functions. When we combine these two special cases, i.e., for general $\mathbf{u}$, the transport coefficients are shown to be in the form of triple hypergeometric functions. Also, we investigate the above problem by using another approach, i.e., Fokker Planck approximation. We obtain similar results for both approaches. *Published by AIP Publishing.* https://doi.org/10.1063/1.5000937

I. INTRODUCTION

Transport equations based on a bi-Maxwellian distribution function were first derived by Chew *et al.* (1956) for collisionless anisotropic plasma; their study was extended by several authors (Kennal and Green, 1966; Macmahon, 1965; Frieman *et al.*, 1966; Bowers and Haines, 1968; Oraevskii *et al.*, 1968; and Espedal, 1969) who derived the transport equations including transport phenomena such as collisionless plasma, viscosity, and heat flow.

All of these studies were dealing with collisionless anisotropic plasmas. Chodura and Pohl (1971) derived the transport equation for an arbitrary anisotropic plasma taking care of collisionless as well as Coulomb collision effect. Since then, Demars and Schunk (1979) have extended the work of Chodura and Pohl (1971) by deriving transport equations based on a bi-Maxwellian species distribution function for arbitrary anisotropic plasma (i.e., arbitrary temperature differences between the interacting gases and arbitrary temperature anisotropy). The relevant collision term has been calculated for the resonant charge exchange interaction between an ion and its neutral parent, inverse-power interaction potential that includes non-resonant ion-neutral (Maxwell molecule) and Coulomb collision, and constant cross-section (hard sphere) interaction.

The last two studies are valid just for small relative drift between the interacting gases. However, Barakat and Schunk (1981) removed this restriction and derived collision terms based on drift bi-Maxwellian gases that are valid for arbitrary drift velocity differences and for arbitrary temperature differences between the interacting gases as well as arbitrary temperature anisotropy.

These transport equations were all derived based on velocity moments of Boltzmann's equation, and the collision terms were all derived based on velocity moments of the Boltzmann collision integral.

Mitchener and Kruger (1973) and Hinton (1983) approximated the Boltzmann collision integral by the Fokker Planck equation under the assumption that small angle deflections dominate. Hellinger and Trávníček (2009) calculated collision terms for the bi-Maxwellian distribution function with drift along an ambient magnetic field by using the Fokker Planck equation and obtained similar results by using the Boltzmann collision integral method.

It is the purpose of this paper to extend the work of Hellinger and Trávníček (2009) by deriving transport coefficients based on the drift (in general, i.e., $u = u\| + u\perp$) bi-Maxwellian distribution function and taking into consideration the Coulomb interactions.

We are interested in the derivation of collisional transport coefficients for the drifting bi-Maxwellian velocity distribution function with respect to the background magnetic field because many applications in plasma physics are in special need to these coefficients. Usually, the differential velocity between different species is aligned with the ambient magnetic field. However, the drift velocity perpendicular to the ambient field is typically connected with the non-gyrotropic velocity distribution function and could also be related to plasma inhomogeneity. For example, different studies investigated the behavior of O^+ ions in the ionosphere under the

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effect of $\mathbf{E}\times\mathbf{B}$ drift, ion-ion Coulomb collision, and ion-neutral collisions (Barghouthi *et al.*, 1994; 2003; and Barghouthi, 2005); also, many studies investigated the ion outflow along "open geomagnetic" field lines (Ganguli, 1996; Barghouthi, 2008; and Nilsson *et al.*, 2013). In order to go forward in the above and similar studies, we need well established formulas for these collisional coefficients.

This paper is organized as follows: Theoretical formulation (Boltzmann equation, Boltzmann collision integral, Fokker Planck equation, and transport coefficients) is presented in Sec. II. In Sec. III, we presented transport coefficients for the drifting bi-Maxwellian velocity distribution function. Special cases (drift velocities perpendicular and parallel to the ambient magnetic field) are presented in Sec. IV. Our results and discussion are summarized in Sec. V.

II. THEORETICAL FORMULATION

In dealing with plasma or gas mixture, it is convenient to investigate the distribution of these particles or species; each species in the plasma is described by a separate velocity distribution function $f_s(\mathbf{r},\ \mathbf{v}_s,\ t)$ which defines such that $f_s(\mathbf{r},\mathbf{v}_s,t)d\mathbf{r}d\mathbf{v}_s$ represents the number of particles of species s which at time t have positions between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ and velocities between $\mathbf{v}_s$ and $\mathbf{v}_s+d\mathbf{v}_s$. The species distribution function changed with respect to time as a result of collisions and particle motions under the influence of external forces; this velocity distribution function is obtained by solving the following Boltzmann's equation:

$$\frac{\partial f_s}{\partial t}+\mathbf{v_s}\cdot\nabla f_s+\left[\mathbf{G}+\frac{q_s}{m_s}\left(\mathbf{E}+\frac{1}{c}\mathbf{v_s}\times\mathbf{B}\right)\right]\cdot\nabla_{\mathbf{v_s}}f_s=\frac{\delta f_s}{\delta t},\tag{1}$$

where q_s and m_s are the charge and mass of species s, $\mathbf{G}$ is the acceleration due to gravity, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, c is the speed of light, $\partial/\partial t$ is the time derivative, ∇ is the coordinate space gradient, $\nabla_{\mathbf{vs}}$ is the velocity space gradient, and the operator $\delta f_s/\delta t$ represents the rate of change of f_s due to the collisions, and this term is given in two forms: Boltzmann collision integral and Fokker Planck approximation.

A. Boltzmann collision integral

For Coulomb collision between s and t particles, the appropriate collision operator in the right hand side of Boltzmann's equation is the Boltzmann collision integral, which can be presented as

$$\frac{\delta f_s}{\delta t}=\sum_t\int d\mathbf{v_t}d\Omega g_{st}\sigma_{st}(g_{st},\theta)\left[f'_sf'_t-f_sf_t\right],\tag{2}$$

where $d\mathbf{v_t}$ is the velocity-space volume element of species t, g_{st} is the relative velocity of the colliding particles s and t, $d\Omega$ is an element of solid angle in the s particle reference frame, θ is the scattering angle, the primes denote quantities evaluated after a collision, and $\sigma(g_{st},\ \theta)$ is the differential scattering cross-section (Goldston and Rutherford, 1995 and Schunk and Nagy, 2009)

$$\sigma=\frac{q_s^2q_t^2}{64\pi^2\varepsilon_\circ^2\mu_{st}^2}\frac{1}{g^4\sin^4\theta},$$

where q_s and q_t are the charge of species and t species, respectively, $\mu_{st}=m_sm_t/(m_s+m_t)$ is the reduced mass, m_t is the mass of the t particle, and ε_o is the permittivity of free space.

B. Fokker-Planck equation

The collision operator can be represented by another equation called the Fokker-Planck equation; this equation can be derived directly from the Boltzmann collision integral [i.e., Eq. (2)] by taking the first order of Taylor expansion of it, and this expansion is valid for binary collisions, under the assumption that a series of consecutive weak (small-angle deflection) binary collisions is a valid representation for the Coulomb interactions

$$\frac{\delta f_s}{\delta t}=-\sum_t\nabla_v\cdot\frac{q_s^2q_t^2\ln\Lambda}{8\pi\varepsilon_\circ^2m_s}\int\frac{\mathbf{1}g^2-\mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial\mathbf{v_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial\mathbf{v_s}}\right)d^3\mathbf{v_t},\tag{3}$$

where $\mathbf{1}$ is the unity tensor, and $\ln\Lambda$ is the Coulomb logarithm, which is typically between 10 and 25 for space plasmas.

There are two approximations employed in the transformation of the Boltzmann collision integral, i.e., Eq. (2), to a Fokker–Planck equation. The first is to remove the scattering angle singularity by evaluating the total momentum transfer cross-section such that scattering angles not smaller than $\theta_{\min}$ are included. The angle $\theta_{\min}$ is defined in terms of the ratio of the Debye length, $\lambda_D=\sqrt{kT_b/4\pi e^2N}$, to a temperature-averaged impact parameter, $b_o=q_sq_t/3kT_b$, that is, $\sin^2(\theta_{\min}/2)=[1+\Lambda]^{-1}$, where $\Lambda=\lambda_D/b_o$. The impact parameter, b_o, is recognized as the impact parameter that corresponds to a deflection of $\theta=\pi/2$ (Shizgal, 2004 and Rosenbluth *et al.*, 1957).

The momentum transfer cross-section is obtained from the integration over the scattering solid angle in the Boltzmann collision integral, Eq. (2). The momentum transfer cross section, which occurs in the calculation of collisional energy transfer, is given by (Schunk and Nagy, 2009)

$$\begin{aligned}Q^{(1)}&=2\pi\int_{\theta_{\min}}^{2\pi}\sigma_{st}(g_{st},\theta)(1-\cos\theta)\sin\theta d\theta\\&=4\pi\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}g^2}\right)^2\ln\Lambda.\end{aligned}\tag{4}$$

The second approximation is to assume that collisions with large impact parameters that are small scattering angles dominate and the Boltzmann collision integral can be replaced with the differential Fokker–Planck equation. The details of these calculations are provided elsewhere and are important for the interpretation of the results of this paper (Mitchener and Kruger, 1973).

C. Transport coefficients

Transport coefficients represent the change in a transport property (momentum, energy, etc.) as a result of collisions, Coulomb collisions in our case.

Although it would be nice to know the individual velocity distribution functions of the different species, the mathematical difficulties associated with obtaining closed-form solutions to Boltzmann's equation preclude this approach for most flow situations. As a consequence, one is generally restricted to obtaining information on a limited number of low-order velocity moments of the species distribution function.

Burgers (1969) and Grad (1949, 1958) proposed the transport properties of a given species defined with respect to the average drift velocity of that species, $\mathbf{u}_s$, alternative to defining them with respect to the average as velocity, $\mathbf{v}_s$. This definition is more appropriate for large relative drifts between interacting species which can occur. In terms of the species average drift velocity, the random or thermal velocity is defined as

$$\mathbf{c_s} = \mathbf{v_s} - \mathbf{u_s}.$$

For most applications, the physically significant moments of the species distribution function are given by

$$\text{Species drift velocity:}\quad \mathbf{u_s} = \langle \mathbf{v_s} \rangle = (1/n_s)\int d\mathbf{v_s} f_s \mathbf{v_s}.$$

$$\text{Parallel temperature:}\quad T_{s\|} = m_s \langle c_{s\|}^2 \rangle / k = (1/n_s)\int d\mathbf{v_s} f_s m_s c_{s\|}^2 / k.$$

$$\text{Perpendicular temperature:}\quad T_{s\perp} = m_s \langle c_{s\perp}^2 \rangle / 2k == (1/n_s)\int d\mathbf{v_s} f_s m_s c_{s\perp}^2 / 2k.$$

Here, n_s is the number density of species s, k is Boltzmann's constant, and the symbols $\|$ and $\perp$ are used to identify quantities that are parallel and perpendicular to the magnetic field, respectively.

The starting point for the derivation of transport coefficients is the collision term in the right hand side of the Boltzmann equation. Moments of the Boltzmann collision integral are obtained by multiplying the right hand side of Boltzmann equation with an appropriate function of velocity $Q_s = Q_s(\mathbf{c_s})$ and integrating over all velocity space. The corresponding moment of the Boltzmann collision integral

$$\frac{\partial Q_s}{\partial t} = \int d^3c_s Q_s(c_s)\frac{\delta f_s}{\delta t} = \int\int\int d^3c_s d^3c_t d\Omega g_{st}\sigma_{st}(g_{st},\theta)\left[f'_s f'_t - f_s f_t\right] Q_s(c_s). \tag{5}$$

For $Q_s(c_s) = m_s\mathbf{c_s}$, $m_s c_{s\|}^2$, and $\frac{1}{2}m_s c_{s\perp}^2$, the obtained moments of the Boltzmann collision integral are the momentum, parallel energy, and perpendicular energy, which are symbolically written as $\delta M_s/\delta t$, $\frac{\delta E_{s\|}}{\delta t}$, and $\frac{\delta E_{s\perp}}{\delta t}$, respectively, for species s.

Due to the reversibility of elastic collisions, we can interchange primed and unprimed quantities in the expression on the right side of Eq. (5) without changing the result

$$\frac{\partial Q_s}{\partial t} = \sum_t \int\int d^3c_s d^3c_t g_{st} f_s f_t \int d\Omega \sigma_{st}(g_{st},\theta)\left[Q'_s - Q_s\right], \tag{6}$$

where Q'_s is the moment evaluated with the velocity found after the Coulomb collision. Integrals in Eq. (6) are called transfer integrals because of transfer of momentum and kinetic energy from one particle to the other particle due to the change in Q_s in a collision. Equation (6) is easier than Eq. (5) because they do not require the distribution functions after the collision.

The evaluation of the integral over $d\Omega$ in Eq. (6) has to be done using two steps. First, express $(Q'_s - Q_s)$ in terms of the center-of-mass velocity, $\mathbf{V_c}$, and the relative velocity, $\mathbf{g_{st}} = \mathbf{v_s} - \mathbf{v_t}$, while the second step in evaluating the collision integral is to integrate over solid angle $d\Omega = \sin\theta d\theta d\varphi$ by using the spherical coordinate system in the center of mass reference frame with relative velocity before the collision (Barakat and Schunk, 1981; Schunk and Nagy, 2009; Burgers, 1969; and Chapman and Cowling, 1970). The resulting system of transport coefficients is given by

Momentum

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \ln\Lambda \int\int d^3c_s d^3c_t g_{st}\mathbf{g_{st}} f_s f_t. \tag{7}$$

Parallel energy

$$\frac{\delta E_{s\|}}{\delta t} = \sum_t 8\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \left[\int\int\int d^3c_s d^3c_t g_{st}\mathbf{g_{st\|}} \times (\mathbf{V_c} - \mathbf{u_s})_\| + \int\int d^3c_s d^3c_t g_{st}\left(g_{st}^2 - 3g_{st\|}^2\right) f_s f_t\right]. \tag{8}$$

Perpendicular energy

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \left[\int\int\int d^3c_s d^3c_t g_{st}\mathbf{g_{st\perp}} \times (\mathbf{V_c} - \mathbf{u_s})_\perp f_s f_t + \frac{\mu_{st}}{2m_s}\int\int d^3c_s d^3c_t g_{st}\left(g_{st}^2 - 3g_{st\perp}^2\right) f_s f_t\right]. \tag{9}$$

Here, $\mathbf{V_c} = \frac{m_s\mathbf{v_s} + m_t\mathbf{v_t}}{m_s + m_t}$ is the center of velocity.

Also these moments can be obtained by using the other form of collision term which is the Fokker Planck approximation by multiplying it also with an appropriate function of velocity $Q_s = Q_s(\mathbf{c_s})$ and integrating over all velocity space as follows:

$$\frac{\delta Q_s}{\delta t} = -\sum_t \nabla_v \cdot \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 m_s}\int \frac{1g^2 - \mathbf{gg}}{g^3} \times \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right) Q(c_s) d\mathbf{c_s} d\mathbf{c_t}. \tag{10}$$

After integration by parts, the corresponding transport coefficients can be expressed as

Momentum

$$\frac{\delta\mu_s}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 n_s}\int \frac{1g^2-\mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t}. \quad (11)$$

Parallel energy

$$\frac{\delta E_{s\|}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s}\int \mathbf{c}_{\mathbf{s}\|}\frac{1g^2-\mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t}. \quad (12)$$

Perpendicular energy

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s}\int \mathbf{c}_{\mathbf{s}\perp}\frac{1g^2-\mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t}. \quad (13)$$

In this study, we assume the distribution function to be drifting bi-Maxwellian function. This assumption will be used to evaluate Eqs. (7)–(9) and (11)–(13).

III. TRANSPORT COEFFICIENTS FOR THE DRIFTING BI-MAXWELLIAN VELOCITY DISTRIBUTION FUNCTION

We assume that all considered species in the plasma have the bi-Maxwellian velocity distribution functions with drift velocity parallel and perpendicular components with respect to the ambient magnetic field (i.e., $\mathbf{u}=\mathbf{u}_\|+\mathbf{u}_\perp$)

$$f_s=\frac{n_s}{\pi^{3/2}a_{s\|}a_{s\perp}^2}e^{-\frac{c_{s\|}^2}{a_{s\|}^2}-\frac{c_{s\perp}^2}{a_{s\perp}^2}}, \quad (14)$$

$$f_t=\frac{n_t}{\pi^{3/2}a_{t\|}a_{t\perp}^2}e^{-\frac{c_{t\|}^2}{a_{t\|}^2}-\frac{c_{t\perp}^2}{a_{t\perp}^2}}, \quad (15)$$

where $a_\|$ and $a_\perp$ are the average parallel and perpendicular thermal speeds of species s, which are equal to $(2kT_\|/m_s)^{1/2}$ and $(2kT_\perp/m_s)^{1/2}$, respectively.

In this section, we will derive the transport coefficients by using the two approaches, Boltzmann collision integral and Fokker Planck equation and verify that they are equivalent.

A. Boltzmann collision integral

The first step in calculating the momentum coefficient by using Boltzmann collision integral is multiplying f_s with f_t (f_sf_t) and writing it in the form

$$f_sf_t=\frac{n_sn_t}{\pi^3 a_{s\|}a_{s\perp}^2 a_{t\|}a_{t\perp}^2}\exp\left(-\frac{c_{s\|}^2}{a_{s\|}^2}-\frac{c_{s\perp}^2}{a_{s\perp}^2}-\frac{c_{t\|}^2}{a_{t\|}^2}-\frac{c_{t\perp}^2}{a_{t\perp}^2}\right). \quad (16)$$

The momentum coefficient according to Eq. (6) becomes

$$\frac{\delta\mathbf{M_s}}{\delta t}=-\sum_t 4\pi\mu_{st}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}g_{st}^2}\right)^2\ln\Lambda\frac{n_sn_t}{\pi^3 a_{s\|}a_{s\perp}^2 a_{t\|}a_{t\perp}^2}\times\int\int d^3c_s d^3c_t g_{st}\mathbf{g_{st}}exp\left(-\frac{c_{s\|}^2}{a_{s\|}^2}-\frac{c_{s\perp}^2}{a_{s\perp}^2}-\frac{c_{t\|}^2}{a_{t\|}^2}-\frac{c_{t\perp}^2}{a_{t\perp}^2}\right). \quad (17)$$

The integrations over d$\mathbf{c_s}$ and d$\mathbf{c_t}$ can be performed by changing the variables of integration from to (**h**, **l**) by using variables defined as follows:

$$\mathbf{c}_{\mathbf{s}\|}=\mathbf{h}_\|+\frac{a_{s\|}^2}{a_{s\|}^2+a_{t\|}^2}\mathbf{l}, \quad (18)$$

$$\mathbf{c}_{\mathbf{t}\|}=\mathbf{h}_\|-\frac{a_{t\|}^2}{a_{s\|}^2+a_{t\|}^2}\mathbf{l}, \quad (19)$$

$$\mathbf{c}_{\mathbf{s}\perp}=\mathbf{h}_\perp+\frac{a_{s\perp}^2}{a_{s\perp}^2+a_{t\perp}^2}\mathbf{l}, \quad (20)$$

$$\mathbf{c}_{\mathbf{t}\perp}=\mathbf{h}_\perp-\frac{a_{t\perp}^2}{a_{s\perp}^2+a_{t\perp}^2}\mathbf{l}. \quad (21)$$

Substituting Eqs. (18)–(21) into Eq. (17) and by using Jacobian transformation $d\mathbf{c_s}d\mathbf{c}_t=d\mathbf{h}d\mathbf{l}$, the expression for momentum transport coefficients therefore becomes

$$\frac{\delta\mathbf{M_s}}{\delta t}=-\sum_t\frac{\mu_{st}n_t}{\pi^3 a_{s\|}a_{s\perp}^2 a_{t\|}a_{t\perp}^2}\int\exp\left(-\frac{a_{s\|}^2h_\|^2}{a_{s\|}^2a_{t\|}^2}-\frac{a_{t\perp}^2h_\perp^2}{a_{s\perp}^2a_{t\perp}^2}\right)d\mathbf{h}\times\int g\mathbf{g}Q^{(1)}\exp\left(-l^2\left(\frac{1}{a_\|^2}+\frac{1}{a_\perp^2}\right)\right)a_\|a_\perp^2 d\mathbf{l}. \quad (22)$$

Because the first integral depends only on the variable **z**, it can be evaluated immediately by using a Gaussian integral technique, so Eq. (22) reduces to

$$\frac{\delta\mathbf{M_s}}{\delta t}=-\sum_t\frac{\mu_{st}n_t}{\pi^{3/2}a_\|a_\perp^2}\int g\mathbf{g}Q^{(1)}\exp\left(-y^2\left(\frac{1}{a_\|^2}+\frac{1}{a_\perp^2}\right)\right)d\mathbf{y}. \quad (23)$$

The calculation is further simplified by changing of variables and integrating over d**x** instead of d**l** where old and new variables are related by the following equations:

$$(x-\varepsilon)^2=l^2\left(\frac{1}{a_\|^2}+\frac{1}{a_\perp^2}\right), \quad (24)$$

$$a_\|=\sqrt{a_{s\|}^2+a_{t\|}^2}, \quad (25)$$

$$a_\perp=\sqrt{a_{s\perp}^2+a_{t\perp}^2}, \quad (26)$$

$$\mathbf{x}=\frac{\mathbf{g}_\|}{a_\|}+\frac{\mathbf{g}_\perp}{a_\perp}, \quad (27)$$

$$\boldsymbol{\varepsilon}=\frac{\Delta\mathbf{u}_\|}{a_\|}+\frac{\Delta\mathbf{u}_\perp}{a_\perp}, \quad (28)$$

$$\Delta u=\mathbf{u_s}-\mathbf{u_t}. \quad (29)$$

With these changes, the integral in Eq. (23) becomes

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 n_t}{4\pi^{3/2}\varepsilon_\circ^2 \mu_{st}} \ln\Lambda \int e^{-(x-\varepsilon)} 2\frac{\mathbf{g}}{g^3} dx \tag{30}$$

and the exponential in Eq. (30) can be simplified as follows:

$$(x-\varepsilon)^2 = \left(\frac{\mathbf{g}_\parallel}{a_\parallel} + \frac{\mathbf{g}_\perp}{a_\perp} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{a_\parallel} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{a_\perp}\right)^2, \tag{31}$$

$$(x-\varepsilon)^2 = \left(\frac{\mathbf{g}_\parallel^2}{a_\parallel^2} + \frac{\mathbf{g}_\perp^2}{a_\perp^2}\right) + \left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{a_\parallel^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{a_\perp^2}\right) + \left(\frac{2\mathbf{g}(\mathbf{u_t}-\mathbf{u_s})_\parallel \cos\theta}{a_\parallel^2} + \frac{2\mathbf{g}(\mathbf{u_t}-\mathbf{u_s})_\perp \cos\theta}{a_\perp^2}\right). \tag{32}$$

Also, we introduce

$$a_\parallel = \sqrt{2} v_{st} \quad \text{and} \quad a_\perp = \sqrt{2} v_{st\perp}, \tag{33}$$

where

$$v_{st\parallel} = \sqrt{\frac{v_{s\parallel}^2 + v_{t\parallel}^2}{2}} \quad \text{and} \quad v_{st\perp} = \sqrt{\frac{v_{s\perp}^2 + v_{t\perp}^2}{2}} \tag{34}$$

are the combined effective parallel and perpendicular velocities, respectively,

$$A_{st} = \frac{v_{st\perp}^2}{v_{st\parallel}^2} = \frac{m_t T_{s\perp} + m_s T_{t\perp}}{m_t T_{s\parallel} + m_s T_{t\parallel}} \tag{35}$$

is an effective temperature anisotropy.

$$\mathbf{g}_\parallel^2 = g^2 \cos^2\theta, \tag{36}$$

$$\mathbf{g}_\perp^2 = g^2 - g^2 \cos^2\theta. \tag{37}$$

Then

$$(x-\varepsilon)^2 = \left(\frac{g^2\cos^2\theta}{4v_{st\perp}^2}(A_{st}-1) + \frac{g^2}{4v_{st\perp}^2}\right) + \left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right) + \left(\frac{\sqrt{A_{st}}\, g(\mathbf{u_t}-\mathbf{u_s})_\parallel \cos\theta}{v_{st\parallel} v_{st\perp}} + \frac{g(\mathbf{u_t}-\mathbf{u_s})_\perp \cos\theta}{2\sqrt{A_{st}}\, v_{st\parallel} v_{st\perp}}\right). \tag{38}$$

We need also the substitution

$$v = \frac{g}{2v_{st\perp}}, \tag{39}$$

$$V = \frac{\sqrt{A_{st}}(\mathbf{u_t}-\mathbf{u_s})_\parallel}{v_{st\parallel}}, \tag{40}$$

$$w = \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{\sqrt{A_{st}}\, v_{st\perp}}, \tag{41}$$

$$A = (A_{st}-1), \tag{42}$$

$$\frac{\mathbf{g}}{g^3} d\mathbf{x} = \frac{1}{2v_{st\parallel}} \frac{\mathbf{v}}{v^3} d\mathbf{v}, \tag{43}$$

$$\frac{g_\parallel^2}{g^3} d\mathbf{x} = \sqrt{A_{st}} \cos^2\theta \frac{d\mathbf{v}}{v}, \tag{44}$$

$$\frac{g_\perp^2}{g^3} d\mathbf{x} = \sqrt{A_{st}} \sin^2\theta \frac{d\mathbf{v}}{v}. \tag{45}$$

So, Eq. (29) can be expressed as

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 n_t}{4\pi^{3/2}\varepsilon_\circ^2 \mu_{st}} \ln\Lambda \times \int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right) + \left(v^2 + Av^2\cos^2\theta + vV\cos\theta + vw\cos\theta\right)\right] \frac{1}{2v_{st\parallel}} \frac{\mathbf{v}}{v^3} d\mathbf{v}. \tag{46}$$

Because the term $e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}$ is constant with respect to variable x, we can get it out of the integration. The integration over all variables v using a spherical coordinate system in velocity space then becomes

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 n_t}{4\pi^{3/2}\varepsilon_\circ^2 \mu_{st}} \ln\Lambda e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} \times \frac{2\pi}{2v_{st\parallel}} \int_0^\pi \int_0^\infty e^{-(v^2 + Av^2\cos^2\theta + vV\cos\theta + vw\cos\theta)} \cos\theta \sin\theta\, dv\, d\theta. \tag{47}$$

In order to solve this integral, we used the technique Maclaurin series expansion for the exponential terms with $\cos\theta$ and finally wrote it in the triple hypergeometric function (Hellinger and Trávníček, 2009; Lebedev, 1965; and Koepf, 2014)

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2} \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} \times F^3\left(\begin{matrix} 1, 2, \frac{3}{2} \\ 3, \frac{3}{2}, \frac{3}{2} \end{matrix}, (1-A_{st}), \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} A_{st}, \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4A_{st} v_{st\perp}^2}\right), \tag{48}$$

where

$$\upsilon_{st} = \frac{q_s^2 q_t^2 n_t}{32\pi\varepsilon_\circ^2 \mu_{st} v_{st\parallel}^2 v_{st\perp}} \ln\Lambda \tag{49}$$

is a collision frequency of species s on species t.

Also, equations for the energy transport coefficients (7) and (8), $\delta Es/\delta t$, can be derived in a manner similar to that described above for $\delta\mu s/\delta t$. The first steps in evaluating these integrals are expressing them in terms of relative velocity and the variable (x)

$$\frac{\delta E_{s\|}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st}}\left[\frac{\mu_{st}}{m_t}4k_B\left(\frac{T_{t\|}-T_{s\|}}{2v_{s\|}^2}\right)\int e^{-(x-\varepsilon)2}\frac{\mathbf{g}}{g^3}d\mathbf{x}-2\mu_{st}\int e^{-(x-\varepsilon)2}\frac{g_\|^2}{g^3}d\mathbf{x}+\mu_{st}\int e^{-(x-\varepsilon)2}\frac{g_\perp^2}{g^3}d\mathbf{x}\right], \tag{50}$$

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}}\left[\frac{\mu_{st}}{m_t}2k_B\left(\frac{T_{t\perp}-T_{s\perp}}{2v_{s\perp}^2}\right)\int e^{-(x-\varepsilon)2}\frac{g_\perp^2}{g^3}d\mathbf{x}+\frac{2k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})}{\sqrt{2}v_{s\perp}}\right.$$
$$\left.\times\int e^{-(x-\varepsilon)2}\frac{\mathbf{g}}{g^3}dx+2\mu_{st}\int e^{-(x-\varepsilon)2}\frac{g_\|^2}{g^3}d\mathbf{x}-\frac{\mu_{st}}{2}\int e^{-(x-\varepsilon)2}\frac{g_\perp^2}{g^3}d\mathbf{x}\right]. \tag{51}$$

The next step in evaluating the energy collision integrals is the substitution of Eq. (38) in Eqs. (50) and (51)

$$\frac{\delta E_{s\|}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st}}\left[\frac{\mu_{st}}{m_t}4k_B\left(\frac{T_{t\|}-T_{s\|}}{2v_{s\|}^2}\right)\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]\right.$$
$$\times\frac{1}{2v_{st\|}}\frac{\mathbf{v}}{v^3}d\mathbf{v}-\frac{2k_B T_{s\|}(\mathbf{u_t}-\mathbf{u_s})_\|}{2v_{s\perp}^2}\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]$$
$$\times\frac{1}{2v_{st\|}}\frac{\mathbf{v}}{v^3}d\mathbf{v}\quad-2\mu_{st}\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]\sqrt{A_{st}}\cos^2\theta\frac{d\mathbf{v}}{v}$$
$$\left.+\mu_{st}\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v}\right], \tag{52}$$

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}}\left[\frac{\mu_{st}}{m_t}2k_B\left(\frac{T_{t\perp}-T_{s\perp}}{2v_{s\perp}^2}\right)\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)\right.\right.$$
$$+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)]\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v}+\frac{2k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})}{\sqrt{2}v_{s\perp}}$$
$$\times\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]\frac{1}{2v_{st\|}}\frac{\mathbf{v}}{v^3}d\mathbf{v}$$
$$+2\mu_{st}\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]\sqrt{A_{st}}\cos^2\theta\frac{d\mathbf{v}}{v}$$
$$\left.-\frac{\mu_{st}}{2}\int\exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)\right]\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v}\right]. \tag{53}$$

By taking the integration over φ, the last two equations become

$$\frac{\delta E_{s\|}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st}}\left[\frac{\mu_{st}}{m_t}4k_B\left(\frac{T_{t\|}-T_{s\|}}{2v_{s\|}^2}\right)e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)}\cos\theta\sin\theta dv d\theta\right.$$
$$-\frac{2k_B T_{s\|}(\mathbf{u_t}-\mathbf{u_s})_\|}{2v_{s\perp}^2}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)}\cos\theta\sin\theta dv d\theta$$
$$-2\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)}v\cos^2\theta\sin\theta dv d\theta$$
$$\left.+2\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\|^2}{4v_{st\|}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta)}v\sin^3\theta dv d\theta\right], \tag{54}$$

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}} \left[\frac{\mu_{st}}{m_t} 2k_B \left(\frac{T_{t\perp} - T_{s\perp}}{2v_{s\perp}^2}\right) e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} 2\pi\sqrt{A_{st}} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta + vV\cos\theta + vw\cos\theta)} v\sin^3\theta dv d\theta \right.$$
$$+ \frac{2k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})_\perp}{\sqrt{2}v_{s\perp}} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta + vV\cos\theta + vw\cos\theta)} \cos\theta\sin\theta dv d\theta$$
$$+2\mu_{st} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} 2\pi\sqrt{A_{st}} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta + vV\cos\theta + vw\cos\theta)} v\cos^2\theta\sin\theta dv d\theta - \frac{\mu_{st}}{2} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}$$
$$\left. \times 2\pi\sqrt{A_{st}} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta + vV\cos\theta + vw\cos\theta)} v\sin^3\theta dv d\theta \right]. \tag{55}$$

And finally, write them in the triple hypergeometric function

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2} \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2} F^{(st)}_{12\frac{333}{22}}, \tag{56}$$

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \upsilon_{st} k_B T_{s\parallel} \left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\parallel}}{T_{s\parallel}} - 1\right) F^{(st)}_{1\frac{3115}{2222}} - \frac{3\sqrt{\pi}}{4} \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2v_{st\perp}} F^{(st)}_{123\frac{335}{222}} - 2\left(F^{(st)}_{1\frac{1113}{2222}} - F^{(st)}_{1\frac{3115}{2222}}\right)\right], \tag{57}$$

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{\upsilon_{st}}{A_{st}} k_B T_{s\perp} \left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\perp}}{T_{s\perp}} - 1\right) F^{(st)}_{1\frac{1115}{2222}} - \frac{3\sqrt{\pi}}{4} \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2v_{st\parallel}} \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2} F^{(st)}_{123\frac{333}{222}} + \left(F^{(st)}_{1\frac{3115}{2222}} - F^{(st)}_{1\frac{1115}{2222}}\right)\right]. \tag{58}$$

Here, $F^{(st)}_{abcd}$ are defined through triple hypergeometric functions

$$F^{(st)}_{abcd} = e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} F^3\left(\begin{matrix} a, b \\ c, c, d \end{matrix}, (1-A_{st}), \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} A_{st}, \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4A_{st}v_{st\perp}^2}\right). \tag{59}$$

B. Fokker Planck equation

The first step in evaluating the transport coefficients by using the Fokker Planck equation is the derivation of f_s and f_t with respect to $\mathbf{c_s}$ and $\mathbf{c_t}$, respectively, as follows:

$$\frac{\partial f_s}{\partial \mathbf{c_s}} = -2f_s\left(\frac{\mathbf{c_{s\parallel}}}{a_{s\parallel}^2} + \frac{\mathbf{c_{s\perp}}}{a_{s\perp}^2}\right), \tag{60}$$

$$\frac{\partial f_t}{\partial \mathbf{c_t}} = -2f_t\left(\frac{\mathbf{c}_{t\parallel}}{a_{t\parallel}^2} + \frac{\mathbf{c}_{t\perp}}{a_{t\perp}^2}\right). \tag{61}$$

Using Eqs. (60) and (61), the integration in Eqs. (9)–(11) may be simplified by using matrix technique as follows:

$$\frac{1g^2 - \mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right) = \frac{1}{g^3}\begin{bmatrix} \mathbf{g}_\perp^2 & -\mathbf{g}_\parallel\mathbf{g}_\perp \\ -\mathbf{g}_\perp\mathbf{g}_\parallel & \mathbf{g}_\parallel^2 \end{bmatrix}\cdot\frac{-2f_sf_t}{m_sm_t}\begin{bmatrix} \frac{m_s\mathbf{c}_{t\parallel}}{a_{t\parallel}^2} - \frac{m_t\mathbf{c}_{s\parallel}}{a_{s\parallel}^2} \\ \frac{m_s\mathbf{c}_{t\perp}}{a_{t\perp}^2} - \frac{m_t\mathbf{c}_{s\perp}}{a_{s\perp}^2} \end{bmatrix}$$
$$= \frac{-2f_sf_t}{m_sm_tg^3}\begin{bmatrix} (m_s+m_t)\mathbf{g}_\parallel \\ \frac{(m_s+m_t)\mathbf{g}_\perp}{2} \end{bmatrix} = \frac{-2f_sf_t}{g^3}\frac{(m_s+m_t)}{m_sm_t}\begin{bmatrix} \mathbf{g}_\parallel \\ \frac{\mathbf{g}_\perp}{2} \end{bmatrix} = \frac{-2f_sf_t}{\mu_{st}g^3}\left(\mathbf{g} - \frac{\mathbf{g}_\perp}{2}\right), \tag{62}$$

$$\mathbf{c}_{\mathbf{s}\parallel}\frac{1g^2-\mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)=\frac{\mathbf{c}_{\mathbf{s}\parallel}}{g^3}\begin{bmatrix}\mathbf{g}_\perp^2 & -\mathbf{g}_\parallel\mathbf{g}_\perp\\ -\mathbf{g}_\perp\mathbf{g}_\parallel & \mathbf{g}_\parallel^2\end{bmatrix}\cdot\frac{-2f_sf_t}{m_sm_t}\begin{bmatrix}\frac{m_s\mathbf{c}_{t\parallel}}{a_{t\parallel}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\parallel}}{a_{s\parallel}^2}\\ \frac{m_s\mathbf{c}_{t\perp}}{a_{t\perp}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\perp}}{a_{s\perp}^2}\end{bmatrix}$$

$$=\frac{-2f_sf_t}{g^3}\begin{bmatrix}\mathbf{g}\cdot\left(\frac{\mu_{st}}{m_s}\frac{\mathbf{g}_\perp}{2}\right) & \frac{\mathbf{g}}{m_s+m_t}\cdot(m_s\mathbf{c}_{\mathbf{s}\parallel}+m_t\mathbf{c}_{\mathbf{t}\parallel})\\ \frac{\mathbf{g}}{m_s+m_t}\cdot\left(-\frac{m_t\mathbf{g}_\parallel}{2}+\frac{(m_s+m_t)\mathbf{c}_{\mathbf{s}\parallel}}{2}-\frac{m_t\mathbf{c}_{\mathbf{t}\parallel}}{2}-\frac{m_sa_{s\parallel}^2\mathbf{c}_{t\parallel}}{2a_{t\parallel}^2}+\frac{m_t(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel})}{2}\right) & 0\end{bmatrix}$$

$$=\frac{-2f_sf_t}{g^3}\mathbf{g}\cdot\left(\frac{2k\left(T_{\mathbf{s}\parallel}-T_{\mathbf{t}\parallel}\right)}{(m_s+m_t)a_\parallel^2}(\mathbf{c}_{\mathbf{s}\parallel}-\mathbf{c}_{\mathbf{t}\parallel})+\frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel-\frac{\mathbf{g}_\perp}{2}\right)-\frac{\mu_{st}(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel})}{m_s}\frac{m_s\mathbf{c}_{s\parallel}}{2(m_s+m_t)}\right.$$
$$\left.-\frac{m_sa_{s\parallel}^2\mathbf{c}_{t\parallel}}{(m_s+m_t)a_{t\parallel}^2}+\frac{m_t\mathbf{c}_{s\parallel}}{(m_s+m_t)}+\frac{m_s\mathbf{c}_{\mathbf{t}\parallel}}{(m_s+m_t)}-\frac{\mu_{st}\left(m_sa_{t\parallel}^2-m_ta_{s\parallel}^2\right)}{(m_s+m_t)a_\parallel^2}(\mathbf{c}_{\mathbf{s}\parallel}-\mathbf{c}_{\mathbf{t}\parallel})\right),\tag{63}$$

$$\mathbf{c}_{\mathbf{s}\perp}\frac{1g^2-\mathbf{gg}}{g^3}\cdot\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)=\frac{\mathbf{c}_{\mathbf{s}\perp}}{g^3}\begin{bmatrix}\mathbf{g}_\perp^2 & -\mathbf{g}_\parallel\mathbf{g}_\perp\\ -\mathbf{g}_\perp\mathbf{g}_\parallel & \mathbf{g}_\parallel^2\end{bmatrix}\cdot\frac{-2f_sf_t}{m_sm_t}\begin{bmatrix}\frac{m_s\mathbf{c}_{t\parallel}}{a_{t\parallel}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\parallel}}{a_{s\parallel}^2}\\ \frac{m_s\mathbf{c}_{t\perp}}{a_{t\perp}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\perp}}{a_{s\perp}^2}\end{bmatrix},$$

$$=\frac{-2f_sf_t}{g^3}\begin{bmatrix}\mathbf{g}\cdot\left(\frac{\mu_{st}}{m_s}\frac{\mathbf{g}_\perp}{2}\right) & \frac{\mathbf{g}}{m_s+m_t}\cdot(m_s\mathbf{c}_{\mathbf{s}\parallel}+m_t\mathbf{c}_{\mathbf{t}\parallel})\\ \frac{\mathbf{g}}{m_s+m_t}\cdot\left(-\frac{m_t\mathbf{g}_\parallel}{2}+\frac{(m_s+m_t)\mathbf{c}_{\mathbf{s}\parallel}}{2}-\frac{m_t\mathbf{c}_{\mathbf{t}\parallel}}{2}-\frac{m_sa_{s\parallel}^2\mathbf{c}_{t\parallel}}{2a_{t\parallel}^2}+\frac{m_t(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel})}{2}\right) & 0\end{bmatrix},$$

$$=\frac{-f_sf_t\mathbf{g}}{g^3}\cdot\left(\frac{2k(T_{\mathbf{s}\perp}-T_{\mathbf{t}\perp})}{(m_s+m_t)a_\perp^2}(\mathbf{c}_{\mathbf{s}\perp}-\mathbf{c}_{\mathbf{t}\perp})-\frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel-\frac{\mathbf{g}_\perp}{2}\right)-\frac{\mu_{st}(\mathbf{u}_{\mathbf{s}\perp}-\mathbf{u}_{\mathbf{t}\perp})}{m_s}\frac{m_s\mathbf{c}_{s\perp}}{2}+m_t\mathbf{c}_{s\perp}+m_s\mathbf{c}_\perp\right.$$
$$\left.-\frac{\mu_{st}\left(m_sa_{t\perp}^2-m_ta_{s\perp}^2\right)}{a_\parallel^2}(\mathbf{c}_{\mathbf{s}\perp}-\mathbf{c}_{\mathbf{t}\perp})\right).\tag{64}$$

The transport coefficients reduced to

$$\frac{\delta\mu_s}{\delta t}=-\sum_t\frac{q_s^2q_t^2\ln\Lambda}{4\pi\varepsilon_\circ^2\mu_{st}n_s}\int\frac{\mathbf{g}}{g^3}f_sf_td\mathbf{c_s}d\mathbf{c_t}+\sum_t\frac{q_s^2q_t^2\ln\Lambda}{8\pi\varepsilon_\circ^2\mu_{st}n_s}\int\frac{\mathbf{g}_\perp}{g^3}f_sf_td\mathbf{c_s}d\mathbf{c_t},\tag{65}$$

$$\frac{\delta E_{s\parallel}}{\delta t}=-\sum_t\frac{q_s^2q_t^2\ln\Lambda}{2\pi\varepsilon_\circ^2\mu_{st}n_s}\int f_sf_t\frac{\mathbf{g}}{g^3}d\mathbf{c_s}d\mathbf{c_t}\cdot\left(\frac{2k\left(T_{\mathbf{s}\parallel}-T_{\mathbf{t}\parallel}\right)}{(m_s+m_t)a_\parallel^2}(\mathbf{c}_{\mathbf{s}\parallel}-\mathbf{c}_{\mathbf{t}\parallel})+\frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel-\frac{\mathbf{g}_\perp}{2}\right)-\frac{\mu_{st}(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel})}{m_s}\right)$$
$$-\sum_t\frac{q_s^2q_t^2\ln\Lambda}{2\pi\varepsilon_\circ^2\mu_{st}n_s}\int\frac{f_sf_t}{(m_s+m_t)}\frac{\mathbf{g}}{g^3}d\mathbf{c_s}d\mathbf{c_t}\cdot\left(\frac{m_s\mathbf{c}_{s\parallel}}{2}-\frac{m_sa_{s\parallel}^2\mathbf{c}_{\mathbf{t}\parallel}}{a_{t\parallel}^2}+m_t\mathbf{c}_{s\parallel}+m_s\mathbf{c}_{\mathbf{t}\parallel}\frac{\mu_{st}\left(m_sa_{t\parallel}^2-m_ta_{s\parallel}^2\right)}{a_\parallel^2}(\mathbf{c}_{\mathbf{s}\parallel}-\mathbf{c}_{\mathbf{t}\parallel})\right),\tag{66}$$

$$\frac{\delta E_{s\perp}}{\delta t}=-\sum_t\frac{q_s^2q_t^2\ln\Lambda}{4\pi\varepsilon_\circ^2\mu_{st}n_s}\int f_sf_t\frac{\mathbf{g}}{g^3}d\mathbf{c_s}d\mathbf{c_t}\cdot\left(\frac{2k(T_{s\perp}-T_{\mathbf{t}\perp})}{(m_s+m_t)a_\perp^2}(\mathbf{c}_{\mathbf{s}\perp}-\mathbf{c}_{\mathbf{t}\perp})-\frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel-\frac{\mathbf{g}_\perp}{2}\right)-\frac{\mu_{st}(\mathbf{u}_{\mathbf{s}\perp}-\mathbf{u}_{\mathbf{t}\perp})}{m_s}\right)$$
$$-\sum_t\frac{q_s^2q_t^2\ln\Lambda}{4\pi\varepsilon_\circ^2\mu_{st}n_s}\int\frac{f_sf_t}{(m_s+m_t)}\frac{\mathbf{g}}{g^3}d\mathbf{c_s}d\mathbf{c_t}\cdot\left(\frac{m_s\mathbf{c}_{s\perp}}{2}+m_t\mathbf{c}_{s\perp}+m_s\mathbf{c}_\perp-\frac{\mu_{st}\left(m_sa_{t\perp}^2-m_ta_{s\perp}^2\right)}{a_\parallel^2}(\mathbf{c}_{\mathbf{s}\perp}-\mathbf{c}_{\mathbf{t}\perp})\right).\tag{67}$$

For momentum, the first term is the same as we got from the Boltzmann collision integral, and the second integral vanishes when integrating over the solid angle Ω. For parallel and perpendicular energy, the first integral is the same as that obtained from the Boltzmann collision integral, and the second integral reduces to (n_sn_t/g) and $[((m_s+2m_t)n_ta_{s\perp}^2)/(2g^3(m_s+m_t))+m_sn_sa_{t\perp}^2/(g^3(m_s+m_t))-n_sn_t(m_sa_{t\perp}^2-m_ta_{s\perp}^2)/(a_\parallel^2(m_s+m_t))]$, respectively.

The transport coefficients are summarized as follows:

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2} \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2} F^{st}_{12\frac{33}{22}3}, \tag{68}$$

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \upsilon_{st} k_B T_{s\parallel} \left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\parallel}}{T_{s\parallel}}-1\right) F^{(st)}_{1\frac{3115}{2222}} - \frac{3\sqrt{\pi}}{4} \frac{(\mathbf{u_t}-\mathbf{u_s})^2_\parallel}{4v^2_{st\parallel}} \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2v_{st\perp}} F^{(st)}_{123\frac{335}{222}} - 2\left(F^{(st)}_{1\frac{1113}{2222}} - F^{(st)}_{1\frac{3115}{2222}}\right)\right] + \sum_t \upsilon_{st} k_B T_{s\parallel} \frac{n_s n_t}{g}, \tag{69}$$

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{\upsilon_{st}}{A_{st}} k_B T_{s\perp} \left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\perp}}{T_{s\perp}}-1\right) F^{(st)}_{1\frac{1115}{2222}} - \frac{3\sqrt{\pi}}{4} \frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2v_{st\parallel}} \frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}} F^{(st)}_{123\frac{333}{222}} + \left(F^{(st)}_{1\frac{3115}{2222}} - F^{(st)}_{1\frac{1115}{2222}}\right)\right] + \sum_t \frac{\upsilon_{st}}{A_{st}} k_B T_{s\perp} \left[\frac{m_s+2m_t}{2g^3} n_t a^2_{s\perp} + \frac{m_s n_s a^2_{t\perp}}{g^3} - \frac{n_s n_t \left(m_s a^2_{t\perp} - m_t a^2_{s\perp}\right)}{a^2_\parallel} \right]. \tag{70}$$

Because of the Fokker Planck derivation from expanding the Boltzmann collision integral and taking first terms in the Taylor series, the transport coefficients by using the Fokker Planck equation for drifting bi-Maxwellian distribution functions with velocities parallel and perpendicular to the ambient magnetic field give approximately similar results when compared to the result of Boltzmann collision integral.

IV. SPECIAL CASES

A. $\mathbf{u}_\parallel = 0$, i.e., $\mathbf{u} = \mathbf{u}_\perp$

No drift velocity component is parallel to the ambient magnetic field, and the drift velocity is perpendicular with respect to the ambient magnetic field; the integrals in Eqs. (47), (54), and (55) reduce to

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 n_t}{4\pi^{3/2}\varepsilon_\circ^2 \mu_{st}} \ln\Lambda e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \frac{2\pi}{2v_{st\parallel}} \int_0^\pi \int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} \cos\theta \sin\theta dv d\theta, \tag{71}$$

$$\begin{aligned} \frac{\delta E_{s\parallel}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v^2_{st\perp}} \Bigg[& \frac{\mu_{st}}{m_t} 4k_B \left(\frac{T_{t\parallel}-T_{s\parallel}}{2v^2_{st\parallel}}\right) 2\pi\sqrt{A_{st}} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} v\cos^2\theta\sin\theta dv d\theta \\ & -4\pi\sqrt{A_{st}}\mu_{st} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} v\cos^2\theta\sin\theta dv d\theta \\ & +4\pi\mu_{st}\sqrt{A_{st}} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} \sin^3\theta dv d\theta \Bigg]. \end{aligned} \tag{72}$$

$$\begin{aligned} \frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}} \Bigg[& \frac{\mu_{st}}{m_t} 2k_B \left(\frac{T_{t\perp}-T_{s\perp}}{2v^2_{s\perp}}\right) e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} 2\pi\sqrt{A_{st}} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} \sin^3\theta dv d\theta \\ & + 2k_B T_{s\perp} \frac{(\mathbf{u_t}-\mathbf{u}_s)_\perp}{\sqrt{2}v_{s\perp}} \frac{2\pi}{2v_{st\parallel}} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} \cos\theta\sin\theta dv d\theta \\ & + 2\pi\mu_{st} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} \sqrt{A_{st}} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} v\cos^2\theta\sin\theta dv d\theta \\ & - \frac{\mu_{st}}{2} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}\right)} 2\pi\sqrt{A_{st}} \int_0^\pi\int_0^\infty e^{-(v^2+Av^2\cos^2\theta+vw\cos\theta)} \sin^3\theta dv d\theta \Bigg]. \end{aligned} \tag{73}$$

The coefficients may be evaluated by expanding exponential terms with $\cos\theta$ into infinite sums, integrating the resulting terms, and then writing the results in the form of double hypergeometric functions. The transport coefficients are summarized as follows:

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{u_{t\perp} - u_{s\perp}}{2} G^{(st)}_{1\frac{3}{2}\frac{5}{2}}, \tag{74}$$

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \upsilon_{st} A_{st} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\parallel} - T_{s\parallel}\right) G^{(st)}_{2\frac{1}{2}\frac{5}{2}} + 2\mu_{st} v_{st}^2 \left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}} - G^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right], \tag{75}$$

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \upsilon_{st} \left[\frac{\mu_{st}}{m_t} k_B (T_{t\perp} - T_{s\perp}) G^{(st)}_{1\frac{1}{2}\frac{5}{2}} + \frac{k_B T_{s\perp} (\mathbf{u_t} - \mathbf{u_s})^2_\perp}{2 v^2_{st\perp}} G^{(st)}_{1\frac{3}{2}\frac{5}{2}} - \mu_{st} v^2_{st\perp} \left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}} - G^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right], \tag{76}$$

where

$$\upsilon_{st} = \frac{q_s^2 q_t^2 n_t \ln\Lambda}{12\pi^{3/2} \varepsilon_\circ^2 \mu_{st} v_{st\parallel} v^2_{st\perp}} \tag{77}$$

is a collision frequency of species s on species t, and

$$G^{(st)}_{abc} = e^{-\frac{(\mathbf{u_t}-\mathbf{u_s})^2_\perp}{4v^2_{st\perp}}} F^{2\cdot\cdot}_{1.1} \begin{pmatrix} a & b \\ c & b \end{pmatrix}, 1 - A_{st}, \frac{(u_t - u_s)^2_\perp}{4 v^2_{st\perp}} \Bigg) \tag{78}$$

is generalized double hypergeometric or Kampe̍ de Fériet functions.

The transport coefficients can be also calculated from the Fokker Planck equation. This calculation also leads to transport coefficients in the form of double hypergeometric function which is nearly the same transport coefficients (74)–(76) as obtained from the Boltzmann collision integral

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{u_{t\perp} - u_{s\perp}}{2} G^{(st)}_{1\frac{3}{2}\frac{5}{2}}, \tag{79}$$

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \upsilon_{st} A_{st} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\parallel} - T_{s\parallel}\right) G^{(st)}_{2\frac{1}{2}\frac{5}{2}} + 2\mu_{st} v_{st}^2 \left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}} - G^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right] + \sum_t \upsilon_{st} k_B T_{s\parallel} \frac{n_s n_t}{g}, \tag{80}$$

$$\begin{aligned} \frac{\delta E_{s\perp}}{\delta t} = \sum_t \upsilon_{st} \left[\frac{\mu_{st}}{m_t} k_B (T_{t\perp} - T_{s\perp}) G^{(st)}_{1\frac{1}{2}\frac{5}{2}} + \frac{k_B T_{s\perp} (\mathbf{u_t} - \mathbf{u_s})^2_\perp}{2 v^2_{st\perp}} G^{(st)}_{1\frac{3}{2}\frac{5}{2}} - \mu_{st} v^2_{st\perp} \left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}} - G^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right] \\ + \sum_t \frac{\upsilon_{st}}{A_{st}} k_B T_{s\perp} \left[\frac{m_s + 2m_t}{2g^3} n_t a^2_{s\perp} + \frac{m_s n_s a^2_{t\perp}}{g^3} - \frac{n_s n_t \left(m_s a^2_{t\perp} - m_t a^2_{s\perp} \right)}{a^2_\parallel} \right]. \end{aligned} \tag{81}$$

B. $\mathbf{u}_\perp = 0$, i.e., $\mathbf{u} = \mathbf{u}_\parallel$

No drift velocity component is perpendicular to the ambient magnetic field, and the drift velocity is parallel with respect to the ambient magnetic field; the transport coefficients take the form

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{u_{t\parallel} - u_{s\parallel}}{2} H^{(st)}_{1\frac{3}{2}\frac{5}{2}}, \tag{82}$$

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \upsilon_{st} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\parallel} - T_{s\parallel}\right) H^{(st)}_{2\frac{1}{2}\frac{5}{2}} + k_B T_{s\parallel} \frac{(u_t - u_s)^2_\parallel}{2 v^2_{st\parallel}} H^{(st)}_{2\frac{3}{2}\frac{5}{2}} \right] + \sum_t \mu_{st} \upsilon_{st} \nu^2_{st\parallel} \left(H^{(st)}_{1\frac{1}{2}\frac{5}{2}} - H^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right), \tag{83}$$

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{\upsilon_{st}}{A_{st}} \left[\frac{\mu_{st}}{m_t} k_B (T_{t\perp} - T_{s\perp}) H^{(st)}_{1\frac{1}{2}\frac{5}{2}} - \mu_{st} v^2_{st\perp} \left(H^{(st)}_{1\frac{1}{2}\frac{5}{2}} - H^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right], \tag{84}$$

where $\upsilon_{st} = \frac{q_s^2 q_t^2 n_t \ln\Lambda}{12\pi^{3/2} \varepsilon_\circ^2 \mu_{st} v^3_{st\parallel}}$ is a collision frequency of species s on species t, and $H^{(st)}_{abc} = e^{-\frac{(u_t-u_s)^2_\parallel}{4v^2_{st\parallel}}} F^{2\cdot\cdot}_{1.1} (\begin{matrix} a & b \\ c & b \end{matrix}, 1 - A_{st}, A_{st} \frac{(u_t-u_s)^2_\parallel}{4v^2_{st\parallel}})$ is generalized double hypergeometric or Kampe̍ de Fériet functions.

These results agree with the results of Hellinger and Trávníček (2009).

V. RESULTS AND DISCUSSION

Coulomb collisions play a very important role in the kinetics of the inner magnetosphere, plasmasphere, and ionosphere coupling processes. They are responsible for the plasma production in these regions as well as for the energy and momentum transfer between the different plasma species as a result of collisions. The mathematical description of the change in a transport property (momentum, energy, etc.) is obtained as a result of collisions called the transport coefficients which depend on the form of velocity distribution function of colliding species.

For temperature anisotropic plasmas (i.e., unequal species temperatures parallel and perpendicular to the ambient magnetic field, with the degree of the anisotropy given by the parallel to perpendicular temperature ratio), we obtained the transport coefficients (momentum, parallel energy, and perpendicular energy) based on bi-Maxwellian velocity distribution functions with drift velocity $\mathbf{u}$ (parallel and perpendicular) with respect to the ambient magnetic field (i.e., $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$) by using Boltzmann collision integral and Fokker Planck approximation. The final results are presented in the form of triple hypergeometric functions. The two approaches give nearly the same results and are valid for arbitrary temperature anisotropies, arbitrary temperature differences between interacting gases, and arbitrary relative drift velocities both parallel and perpendicular to the magnetic field.

We also calculated the transport coefficients by using Boltzmann collision integral for two special cases where the relative drift is either parallel or perpendicular to the magnetic field, which are the two most common cases in astronomy and space physics. Then, we investigated the previous problem by using another approach, Fokker Planck approximation, and we obtained nearly similar results. The transport coefficients are in the form of double hypergeometric functions. These results can be further generalized to an inverse power force interaction.

It should be noted that significant temperature anisotropies occur in plasma at all levels of ionization. The temperature anisotropy in the solar wind measured typically varies between a factor of 2 to 4 at the orbit of the Earth (cf. Brandt, 1970 and Hundhausen, 1972) and developed in a region of the flow where only Coulomb collisions are important (i.e., the flow is effectively fully ionized), while in the terrestrial polar wind proton initial theoretical calculations indicate that the temperature anisotropy is about a factor of 20 at a distance of eight Earth radii (Holzer *et al.*, 1971) and developed in a region of flow where Coulomb collisions and non-resonant ion-neutral interaction occur (i.e., the flow is partially ionized).

To sum up, we extended the work of Hellinger and Trávníček (2009) and calculated the transport coefficients for drifting bi-Maxwellian plasmas. Hellinger and Trávníček (2009) considered that the plasma drift is along the ambient magnetic field, but in our study, we have considered general drift $(\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp})$ and investigated two special cases $(\mathbf{u} = \mathbf{u}_{\parallel}, \mathbf{u}_{\perp} = 0$, and $\mathbf{u} = \mathbf{u}_{\perp}, \mathbf{u}_{\parallel} = 0)$. We have reproduced the results of Hellinger and Trávníček (2009) for case $(\mathbf{u} = \mathbf{u}_{\parallel}, \mathbf{u}_{\perp} = 0)$. In our study, we showed in detail derivation for transport coefficients by using two approaches, Boltzmann collision integral and Fokker Planck equation.

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