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Compactness and Lindelöfness of a topology with respect to another in bitopological spaces

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Declaration

I certify that the thesis, submitted for the degree of Master, is the result of my own research except where otherwise acknowledged, and the thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ………………..

Faten Diab Aqel Turkman

Date:

Dedication

To my father, my mother, my brothers, my fiancé and my friends for their help and support.

Acknowledgement

Thanks is given first to God.

I would like to express my thanks to my supervisor, Dr. Yousef Bdeir for his help and support during all phases of my graduate study.

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Abstract

In this thesis, two concepts are discussed, compactness and Lindelöfness of a topology with respect to another in bitopological spaces.

Also, other concepts in bitopological spaces are discussed, such as continuity, separation axioms, and their relations with compactness and Lindelöfness of a topology with respect to another in bitopological spaces.

Also, the hereditarity and productivity of these properties has been studied and some conditions has been considered to preserve them.

The existence of a countable inadequate family of members of a topology τ with respect to another topology σ with no maximal countable inadequate family of members of τ with respect to σ and contains it has been proved.

Finally, conversely Lindelöf nonempty subsets of $(\mathbb{R}, \ell, r)$ has been classified.

الملخص

في هذه الأطروحة تم بحث مفهومي التراص واللندلوف لتبولوجيا بالنسبة لأخرى في فضاءات التبولوجيا الثنائية.

كذلك تم بحث بعض المفاهيم الأخرى في فضاءات التبولوجيا الثنائية مثل الاتصال و فرضيات الفصل و علاقتهما بالتراص واللندلوف لتبولوجيا بالنسبة لأخرى في فضاءات التبولوجيا الثنائية.

و كذلك تم بحث خاصيتي الوراثة و الضرب لهذين المفهومين في فضاءات التبولوجيا الثنائية مع إضافة بعض الشروط لها.

ولقد تم برهان وجود عائلة ج ناقصة قابلة للعد من عناصر تبولوجيا ت بالنسبة لتبولوجيا أخرى خ مع عدم وجود عائلة ناقصة قابلة للعد عظمى من عناصر ت بالنسبة ل خ و تحتوي ج.

و أخيرا, تم تصنيف المجموعات الجزئية غير الخالية في التبولوجيا الثنائية (ح م,س) و التي تكون لندلوف بالاتجاهين.

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Introduction

In 1962, J.C. Kelly [9] has defined the concept of the bitopological spaces to be a nonempty set X on which two arbitrary topologies τ_1 and τ_2 are defined. This definition is denoted by the triple (X, τ_1, τ_2). Since this initiation, several authers have considered the problem of defining two concepts; compactness and Lindelöfness in bitopological spaces. And in this thesis the definitions of compactness and Lindelöfness in bitopological spaces were studied by Ian, E. Cooke and Ivan L. Reilly in [8] , Birsan in [4] , M.C. Datta in [5] , Adem Kilicman and Zabidin Salleh in [1] .

In fact these definitions are summarized into eight definitions, namely semi compact (s- compact), pairwise compact (p- compact), Birsan compact (conversely and B-compact), semi Lindelöf (s-Lindelöf), pairwise Lindelöf (p-Lindelöf) and Birsan Lindelöf (conversely and B- Lindelöf).

Whenever a bitopological space (X,τ_1,τ_2) is said to have a given topological property $\mathcal{P}$, it is ment that both (X,τ_1) and (X,τ_2) satisfy $\mathcal{P}$.

ℓ will stand for the left ray topology for $\mathbb{R}$, and r will stand for the right ray topology for $\mathbb{R}$.

Unless otherwise stated, i and j will stand for i , j $\in \{1,2\}$ and i $\neq$ j.

For a subset A of X, τ–cl(A) will stand for the closure of A in the topological space (X, τ).

Chapter one is divided into two sections. Section one discusses mappings in bitopological spaces. It begins with defining continuity, open functions and homeomorphism. Separation axioms in bitopological spaces are introduced in section two, and many useful results and conclusions concerning regularity and normality in bitopological spaces are deduced.

In section one of chapter two, definitions of four types of compactness in bitopological spaces are given (s-compactness, p-compactness, conversly compactess and B-compactness). The relations between them, and deduce the effect of pairwise Hausdorffness in comparison of topologies are studied. In section two, we define the notion of compactness of a topology with respect to another for a subset of a bitopological space, and its relations with closedness and openness. In section three, the effect of continuous and open functions on conversely (B-) compact bitopological spaces are studied. In section four, generalization of Alexander and Tychonoff theorems in bitopological spaces are made.

In section one of chapter three, four different definitions of Lindelöfness in bitopological spaces (s-Lindelöfness, p-Lindelöfness, conversely Lindelöfness and B- Lindelöfness) are given, and study the relations between them. And we deduce the effect of pairwise Hausdorffness in comparison of topologies. In section two, the notion of conversely Lindelöf of a subspace of a bitopological space is defined, and its relations with closedness and openness. Also we discuss the relations between conversely Lindelöf (conversely compact), p-regular and p_1-normal. In section three, the effect of continuous, open and surjective functions on conversely Lindelöf (B-Lindelöf) bitopological spaces are studied. In section four, productivity of conversely Lindelöf is studied and a condition is considered to preserve productivity. Also an example of a product of P-spaces that is not P-space, despite of a "theorem proved" in [3] is given. In section five, Tychonoff's Theorem for conversely Lindelöf bitopological spaces is studied, and the existence of a countable inadequate family of members of a topology τ with respect to another topology σ with no maximal countable inadequate family of members of τ with respect to σ and contains it. Conversely compact and conversely Lindelöf subsets in $(\mathbb{R},\ell,r)$ and the relations between them are introduced in section six. Finally, Conversely compact and conversely Lindelöf subsets in $(\mathbb{R},\ell,\mathcal{S})$ and the relations between them are introduced in section seven.

Chapter 1

Bitopological concepts

1.1 Mappings in bitopological spaces

1.1.1. Definition [11]:

Let (X,τ_1,τ_2) and (Y,σ_1, σ_2) be two bitopological spaces, and let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1, \sigma_2)$ be a function, then:

1) f is called i-continuous if the function $f: (X,\tau_i) \longrightarrow (Y,\sigma_i)$ is continuous. The function f is said to be continuous if it is i-continuous for each i=1,2.

2) f is called i-open (resp. i-closed) if the function $f: (X,\tau_i) \longrightarrow (Y,\sigma_i)$ is open (resp. closed). f is said to be open (resp. closed) if f is i-open (resp. i-closed) for each i=1,2.

3) f is called i-homeomorphism if the function $f: (X,\tau_i) \longrightarrow (Y,\sigma_i)$ is homeomorphism, or equivalently, if f is bijection, i-continuous and $f^{-1}: (Y,\sigma_1, \sigma_2) \longrightarrow (X,\tau_1,\tau_2)$ is i-continuous. The bitopological spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) are then called i-homeomorphic. A function $f: (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1, \sigma_2)$ is called homeomorphism if the function

$f:(X,\tau_i) \longrightarrow (Y,\sigma_i)$ is homeomorphism for each i=1,2 , or equivalently, if f is bijection, continuous and $f^{-1}: (Y,\sigma_1,\sigma_2) \longrightarrow (X,\tau_1,\tau_2)$ is continuous. The bitopological spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) are then called homeomorphic.

1.1.2. Example [2]:

Consider $X = \{a,b,c,d\}$ with τ_1 the discrete topology and topology $\tau_2 = \{\emptyset, \{a\}, \{a,b,c\}, X\}$ on X, and $Y = \{x, y, z, w\}$ with topology $\sigma_1 = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z,w\}, Y\}$ and $\sigma_2 = \{\emptyset, \{x\}, \{y, z, w\}, Y\}$ on Y. Define a function $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$, by $f(a) = y$, $f(b) = f(d) = z$, and $f(c) = w$. Observe that the functions $f:(X,\tau_1) \longrightarrow (Y,\sigma_1)$ and $f:(X,\tau_2) \longrightarrow (Y,\sigma_2)$ are continuous. Therefore the function $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is continuous. But the function f is not homeomorphism since it is not bijection. □

1.1.3. Example [2]:

Consider the bitopological spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) as in example (1.1.2). Define a function $g: (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ by $g(a) = g(b) = x$, $g(c) = z$ and $g(d) = w$. The function $g: (X,\tau_1) \longrightarrow (Y,\sigma_1)$ is continuous and $g: (X,\tau_2) \longrightarrow (Y,\sigma_2)$ is not continuous since $\{y, z, w\} \in \sigma_2$ but its inverse image $g^{-1}(\{y,z,w\}) = \{c,d\} \notin \tau_2$. Thus $g: (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is not continuous. □

1.1.4 Example [2]:

Consider the function $f: (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ as in example (1.1.2). Observe that the function $f: (X,\tau_2) \longrightarrow (Y,\sigma_2)$ is not open since $\{a\} \in \tau_2$ but $f(\{a\}) = \{y\} \notin \sigma_2$. Thus $f: (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is not open. □

Recall that, a property $\mathcal{P}$ on a topological space (X,τ) is called topological property if every topological space (Y,σ) homeomorphic to (X,τ) also satisfies the property $\mathcal{P}$.

In the case of bitopological space (X,τ_1,τ_2), a property $\mathcal{P}$ will be called i-topological property if whenever (X,τ_1,τ_2) has the property $\mathcal{P}$, then every space i-homeomorphic to (X,τ_1,τ_2) also has the property $\mathcal{P}$. If homeomorphism considered for the pairwise topology, we will call such property $\mathcal{P}$ as bitopological property.

1.2 Bitopological separation axioms

This definition is given before we start with separation axioms.

1.2.1. Definition:

Let (X, τ_1, τ_2) be a bitopological space. Then a set G is said to be τ_i–open (resp. τ_i –closed) if G is open (resp. closed) in the topology τ_i in X. And G is said to be open (resp. closed) if it is τ_i–open (resp. τ_i –closed) for each i=1, 2.

1.2.2. Definition [9]:

A bitopological space (X,τ_1,τ_2) is said to be pairwise Hausdorff (denoted p-Hausdorff) if for each pair of distinct points x and y in X there are disjoint open sets $U\in\tau_1$ and $V\in\tau_2$ such that $x \in U$ and $y \in V$.

Recall that a topological space (X,τ) is said to be regular if for each point $x \in X$ and each closed set P such that $x \notin P$, there are two disjoint open sets U and V such that $x \in U$ and $P \subseteq V$.

1.2.3 Definition [9]:

In a space (X,τ_1,τ_2), τ_1 is said to be regular with respect to τ_2 , if for each point $x \in X$ and each τ_1-closed set P such that $x \notin P$, there are a τ_1-open set U and a τ_2-open set V such that $x \in U$, $P \subseteq V$, and $U \cap V = \emptyset$.

(X, τ_1, τ_2) is pairwise regular (denoted p-regular) if τ_1 is regular with respect to τ_2 and vice versa.

1.2.4 Theorem [1]:

A bitopological space (X,τ_1,τ_2) is τ_i regular with respect to τ_j if and only if for each point $x \in X$ and τ_i-open set H containing x, there exists a τ_i-open set U such that $x \in U \subseteq \tau_j\text{-cl}(U) \subseteq H$.

Proof:

($\Rightarrow$) Suppose τ_i is regular with respect to τ_j. Let $x \in X$ and H be a τ_i-open set containing x. Then $G = X\backslash H$ is a τ_i-closed set for which $x \notin G$. Since τ_i is regular with respect to τ_j then there are τ_i-open set U and τ_j -open set V such that $x \in U$, $G \subseteq V$, and $U \cap V = \emptyset$. Since $U \subseteq X \backslash V$, then $\tau_j\text{-cl}(U) \subseteq \tau_j\text{-cl}(X\backslash V) = X\backslash V \subseteq X\backslash G = H$. Thus, $x \in U \subseteq \tau_j\text{-cl}(U) \subseteq H$ as desired.

($\Leftarrow$) Suppose that the condition holds. Let $x \in X$ and P be a τ_i-closed set such that $x \notin P$. Then $x \in X\backslash P$, and by the hypothesis, there exists a τ_i-open set U such that $x \in U \subseteq \tau_j\text{-cl}(U) \subseteq X\backslash P$. It follows that $x \in U$, $P \subseteq X\backslash (\tau_j\text{-cl}(U))$ and $U \cap (X\backslash \tau_j\text{-cl}(U)) = \emptyset$. This completes the proof. □

Theorem (1.2.4) stated that τ_i is regular with respect to τ_j, if and only if for each point $x \in X$, there is a τ_i-neighbourhood base of τ_j -closed sets containing x.

The following theorem shows that, pairwise regular spaces satisfy the hereditary property.

1.2.5 Theorem [1]:

Every subspace of a pairwise regular bitopological space is pairwise regular.

Proof:

Let (X,τ_1,τ_2) be a pairwise regular space and let $(Y, \tau_{1,Y}, \tau_{2,Y})$ be a subspace of (X,τ_1,τ_2). Furthermore, let F be a $\tau_{1,Y}$ -closed set in Y, then $F = A \cap Y$ where A is a τ_1-closed set in X. Now if $y \in Y$ and $y \notin F$, then $y \notin A$, so there are τ_1 -open set U and τ_2 -open set V such that $y \in U$, $A \subseteq V$, and $U \cap V = \emptyset$.

$U \cap Y$ and $V \cap Y$ are $\tau_{1,Y}$ -open set and $\tau_{2,Y}$ -open set in Y respectively. Also $y \in U \cap Y$, $F \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset$.

Similarly, let G be a $\tau_{2,Y}$ -closed set in Y, then $G = B \cap Y$ where B is a τ_2 -closed set in X. Now if $y \in Y$ and $y \notin G$, then $y \notin B$, so there are τ_2 -open set U and τ_1 -open set V such that $y \in U$, $B \subseteq V$, and $U \cap V = \emptyset$.

But $U \cap Y$ and $V \cap Y$ are $\tau_{2,Y}$ -open set and $\tau_{1,Y}$ -open set in Y respectively. Also $y \in U \cap Y$, $G \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. This completes the proof. □

Recall that a topological space (X,τ) is normal if given two disjoint closed sets A and B, there exist two disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

1.2.6 Definition [9]:

A bitopological space (X, τ_1, τ_2) is said to be p-normal if given a τ_1 -closed set A and a τ_2 -closed set B with $A \cap B = \emptyset$, there exist a τ_2 -open set U and a τ_1 -open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

1.2.7 Theorem [1]:

A bitopological space (X,τ_1,τ_2) is p-normal if and only if given a τ_j -closed set C and a τ_i-open set D such that $C \subseteq D$, there are a τ_i -open set G and a τ_j -closed set F such that $C \subseteq G \subseteq F \subseteq D$.

Proof:

($\Rightarrow$) Suppose (X,τ_1,τ_2) is p-normal. Let C be a τ_j -closed set and D a τ_i-open set such that $C \subseteq D$. Then $K = X\backslash D$ is a τ_i -closed set with $K \cap C = \emptyset$. Since (X,τ_1,τ_2) is p-normal, there exist a τ_j -open set U and a τ_i -open set G such that $K \subseteq U$, $C \subseteq G$, and $U \cap G = \emptyset$. Hence $G \subseteq X\backslash U \subseteq X\backslash K = D$. Thus $C \subseteq G \subseteq X\backslash U \subseteq D$ and the result follows by taking $X\backslash U = F$.

($\Leftarrow$) Suppose the condition holds. Let A be a τ_i -closed set and B be a τ_j-closed set with $A \cap B = \emptyset$. Then $D = X\backslash A$ is a τ_i-open set with $B \subseteq D$. By hypothesis, there are a τ_i-open set G and a τ_j -closed set F such that $B \subseteq G \subseteq F \subseteq D$.

It follows that $A = X\backslash D \subseteq X\backslash F$, $B \subseteq G$ and $(X\backslash F) \cap G = \emptyset$ where $X\backslash F$ is τ_j -open set and G is τ_i-open set. This completes the proof. □

Now we define a new weaker form of pairwise normal bitopological spaces.

1.2.8 Definition [1]:

A bitopological space (X, τ_1, τ_2) is said to be p_1-normal if given A and B are closed sets with $A \cap B = \emptyset$, there exist a τ_2 -open set U and a τ_1-open set V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

1.2.9 Theorem [1]:

A bitopological space (X,τ_1,τ_2) is p_1-normal if and only if given a closed set C and an open set D such that $C \subseteq D$, there are a τ_i -open set G and a τ_j -closed set F such that $C \subseteq G \subseteq F \subseteq D$.

Proof:

(⇒) Suppose (X,τ_1,τ_2) is p_1–normal. Let C be a closed set and D be an open set such that $C \subseteq D$. Then $K = X \backslash D$ is a closed set with $K \cap C = \emptyset$. Since (X, τ_1, τ_2) is p_1–normal, there exists a τ_j-open set U and a τ_i-open set G such that $K \subseteq U$, $C \subseteq G$, and $U \cap G = \emptyset$. Hence $G \subseteq X \backslash U \subseteq X \backslash K = D$. Thus $C \subseteq G \subseteq X \backslash U \subseteq D$ and the result follows by taking $X \backslash U = F$.

(⇐) Suppose the condition holds. Let A and B are closed sets with $A \cap B = \emptyset$. Then $D = X \backslash A$ is an open set with $B \subseteq D$. By hypothesis, there are a τ_i-open set G and a τ_j-closed set F such that $B \subseteq G \subseteq F \subseteq D$.

It follows that $A = X \backslash D \subseteq X \backslash F$, $B \subseteq G$ and $(X \backslash F) \cap G = \emptyset$ where $X \backslash F$ is τ_j-open set and G is τ_i-open set. This completes the proof. □

It is clear from the definition that every p-normal space is p_1-normal. The converse is not true in general as shown in the following counterexample.

1.2.10 Example [1]:

Consider X = {a, b, c, d} with topologies $\tau_1 = \{\emptyset, \{a,b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c, d\}, X\}$ defined on X. Observe that τ_1-closed subsets of X are ∅, {c, d} and X, and τ_2-closed subsets of X are ∅,{b, c, d},{a} and X. Hence (X, τ_1, τ_2) is p_1-normal as we can check since the only closed sets of X are ∅ and X. However (X,τ_1,τ_2) is not p-normal since the τ_1-closed set A = {c, d} and τ_2 -closed set B ={a} satisfy $A \cap B = \emptyset$, but there is no τ_2 -open set U and τ_1-open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. □

1.2.11 Example:

Consider the bitopological space $(\mathbb{R}, \ell, r)$. It is clear that $(\mathbb{R}, \ell, r)$ is p-regular and p-normal, but it is not p-Hausdorff. □

1.2.12 Theorem:

Every closed subspace of a p–normal bitopological space is p-normal

Proof:

Let (X,τ_1,τ_2) be a p-normal bitopological space, and let (Y,τ_{1Y},τ_{2Y}) be a closed subspace of X. If A and B are disjoint subsets of Y such that A is τ_{1Y} -closed and B is τ_{2Y}–closed, then A is τ_1-closed in X and B is τ_2–closed in X, and since X is p-normal there are U which is τ_2–open set and V which is τ_1–open set such that $U \cap V = \emptyset$, where $A \subset U$ and $B \subset V$. Then

$U \cap Y$ and $V \cap Y$ are disjoint τ_{2Y} -open and τ_{1Y} –open sets respectively. Also $A \subset U \cap Y$ and $B \subset V \cap Y$. Thus Y is p-normal. □

The proof of the following theorem is similar to the proof of theorem (1.2.12).

1.2.13 Theorem:

Every closed subspace of a p_1–normal bitopological space is p_1-normal. □

The definitions of separation properties of two topologies τ_1 and τ_2 such as pairwise regularity, of course reduce to the usual separation properties of one topology τ_1, such as regularity, when we take $\tau_1 = \tau_2$, and the theorems quoted above then yield as corollaries of the classical results of which they are generalizations.

Chapter two

Compact topology with respect to another

2.1 Birsan and Conversely Compactness

In this chapter we consider some kinds of compactness in bitopological spaces, and the relations between them. Also, we deduce some related results and generalizations of some theorems in single topology.

Recall that a topological space $(X,\tau)$is compact if for every cover for X has a finite subcover.

2.1.1 Definition [10]:

A cover $\mathcal{V}$ of a bitopological space (X,τ_1,τ_2) is called $\tau_1\,\tau_2$ -open cover if $\mathcal{V} \subset \tau_1 \cup \tau_2$.

2.1.2 Definition [6]:

A $\tau_1\,\tau_2$-open cover $\mathcal{V}$ of a bitopological space (X, τ_1, τ_2) is called p-open cover if $\mathcal{V}$ contains at least one nonempty member of τ_1 and a nonempty member of τ_2.

2.1.3 Definition [4]:

We say that $\mathcal{V}_1 = \{V_i : i \in I\}$ is finer than $\mathcal{V} = \{U_\alpha : \alpha \in A\}$ if for each $i \in I$, there exists $\alpha \in A$ such that $V_i \subset U_\alpha$.

2.1.4 Definition [10]:

A bitopological space (X, τ_1, τ_2) is called semi compact (denoted s-campact) if every τ_1 τ_2 -open cover for X has a finite subcover.

Swart in [10] consider the above definition for compactness in bitopological spaces, and uses the term compact for s-compactness in (X, τ_1, τ_2).

We give in the next definition Fletcher, Holye and Patty definition of pairwise compactness in the bitopological space, denoted FHP–compactness.

2.1.5 Definition [6]:

A bitopological space (X, τ_1, τ_2) is called pairwise compact (denoted p-compact) if every p-open cover of X has a finite subcover.

The following definition of bitopological spaces is due to Birsan.

2.1.6 Definition [4]:

A bitopological space (X,τ_1, τ_2) is called τ_i-compact with respect to τ_j if for each τ_i-open cover $\mathcal{V}$ for X, there is a finite family of τ_j-open sets finer than $\mathcal{V}$ and covers X.
The space is called conversely compact if it is τ_1-compact with respect to τ_2 and is τ_2-compact with respect to τ_1.

2.1.7 Definition [4]:

A bitopological space (X,τ_1,τ_2) is called τ_i-compact within τ_j if for each τ_i-open cover $\mathcal{V}$ for X, has a finite subcover of τ_j-open sets for X. The space is called B-compact if it is τ_1-compact within τ_2 and is τ_2-compact within τ_1.

Ian E. Cook and Ivan E. Reilly, called the τ_i-compact within τ_j , τ_i-compact with respect to τ_j , and refer this definition to Birsan.

In fact, τ_i-compactness of (X, τ_1, τ_2) within τ_j implies τ_i-compactness of (X,τ_1,τ_2) with respect to τ_j , but the converse need not be true, as the following example shows.

2.1.8 Example [4]:

Let $X = [0,1]$, let

$\tau_1 = \{A \subset X: 0 \in A \text{ and } X\backslash A \text{ is finite}\} \cup \{A \subset (0,1) : (0,1)\backslash A \text{ is finite}\} \cup \{\emptyset\}$, and

$\tau_2 = \{A \subset X : 1 \in A \text{ and } X\backslash A \text{ is finite}\} \cup \{A \subset (0,1) : (0,1)\backslash A \text{ is finite}\} \cup \{\emptyset\}$. Then (X ,τ_1,τ_2) is a bitopological space which is τ_1-compact with respect to τ_2 but not τ_1-compact within τ_2 , because $\{[0,1]\backslash\{1/2\}, [0,1)\}$ is τ_1-open covering for X but has no finite τ_2-open subcovering. □

The following theorem illustrates the relation between s-compactness and p-compactness.

2.1.9 Theorem [8]:

The bitopological space (X,τ_1,τ_2) is s-compact if and only if it is p-compact, τ_1-compact and τ_2-compact.

Proof:

Assume that the bitopological space (X, τ_1, τ_2) is s-compact, and let $\mathcal{V}$ be any p-open cover of the space X, then $\mathcal{V}$ is $\tau_1\tau_2$-open cover for X. Since X is s-compact, then $\mathcal{V}$ has a finite subcover for X. Thus X is p-compact. Also, let $\mathcal{V}$ be any τ_i-open cover of X, (i =1,2), then $\mathcal{V} \subset \tau_1 \cup \tau_2$, which means that $\mathcal{V}$ is $\tau_1\tau_2$ -open cover for X. Since (X,τ_1,τ_2) is s-compact, then there is a finite subcover of $\mathcal{V}$ for X, which implies that X is τ_i-compact (i=1,2). Conversely, assume that (X,τ_1,τ_2) is p-compact, τ_1-compact and τ_2-compact. Let $\mathcal{V}$ be any $\tau_1\tau_2$-open cover for X, then $\mathcal{V} \subset \tau_1 \cup \tau_2$.

Case 1:

If $\mathcal{V}$ contains at least one nonempty member of τ_1, and at least one nonempty member of τ_2 , then $\mathcal{V}$ is p-open cover.Thus there is a finite subcover of $\mathcal{V}$ for X (as X is p- compact).

Case 2:

If $\mathcal{V}$ is contained entirely in τ_1 or τ_2, then $\mathcal{V}$ is either τ_1-open cover for X or τ_2-open cover for X. In either case, there is a finite subcover of $\mathcal{V}$ for X (as X is τ_1-compact and τ_2-compact). Hence X is s-compact. □

The following example shows that: "Not every p-compact bitopological space is s-compact ".

2.1.10 Example [7]:

Consider the bitopological space ($\mathbb{R}$,ℓ,r). Then ($\mathbb{R}$, ℓ,τ_2) is p-compact, but not s-compact.

To show this, let $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be a p-open cover for $\mathbb{R}$. Then there exist β , $\gamma \in \Delta$ such that $U_\beta \in \ell$, $U_\gamma \in r$, $U_\beta \neq \emptyset$ and $U_\gamma \neq \emptyset$. If $U_\beta = \mathbb{R}$ or $U_\gamma = \mathbb{R}$, then $\mathcal{V}$ has a finite subcover for $\mathbb{R}$, namely $\{\mathbb{R}\}$. Otherwise, let $U_\beta = (-\infty,x)$ and $U_\gamma = (y,\infty)$, for some x , y $\in \mathbb{R}$. If x > y, then $\{U_\beta , U_\gamma\}$ is a finite subcover of $\mathcal{V}$ for $\mathbb{R}$. If x = y, then there is some $\lambda \in \Delta$ such that $x \in U_\lambda$ and then $\{U_\beta , U_\gamma , U_\lambda\}$ is a finite subcover of $\mathcal{V}$ for $\mathbb{R}$.

Now, let x < y. Let $A = \{z \in [x,y]$: there is no $\alpha \in \Delta$ such that $z \in U_\alpha \in r\}$. If $A = \emptyset$, then $x \in U_\alpha \in r$ for some $\alpha \in \Delta$ and then $\{U_\beta , U_\alpha\}$ is a finite subcover of $\mathcal{V}$ for $\mathbb{R}$. If $A \neq \emptyset$, then A is bounded above and so, by completeness axiom for $\mathbb{R}$, it has a least upper bound, say t.

Then $x \leq t \leq y$.

Case 1: If t = x, then A = {x}. So there is no $\alpha \in \Delta$ such that $t \in U_\alpha \in r$, then there exists $\delta \in \Delta$ such that $t \in U_\delta \in \ell$. If $U_\delta = \mathbb{R}$, then $\mathcal{V}$ has a finite subcover for $\mathbb{R}$, namely $\{\mathbb{R}\}$. Otherwise $U_\delta = (-\infty,z)$ for some $z \in \mathbb{R}$. Then t < z. By definition of A and t, there exists $\lambda \in \Delta$ such that $z \in U_\lambda \in r$, and then $\mathcal{V}$ has $\{U_\delta ,U_\lambda\}$ as a finite subcover for $\mathbb{R}$.

Case 2: If t = y. Suppose now that there exists $\alpha \in \Delta$ such that $t \in U_\alpha \in r$. If $U_\alpha = \mathbb{R}$, then $\mathcal{V}$ has a finite subcover for $\mathbb{R}$, namely $\{\mathbb{R}\}$. Otherwise, $U_\alpha = (z,\infty)$ for some $z \in \mathbb{R}$ and z < t, and so there exists $w \in A$ such that z < w < t. It is clear that there exists $\lambda \in \Delta$ such that $w \in U_\lambda \in \ell$, and then $\{U_\alpha , U_\lambda\}$ is a finite subcover of $\mathcal{V}$ for $\mathbb{R}$. Suppose now that there exists no $\alpha \in \Delta$ such that $t \in U_\alpha \in r$, then there exists $\alpha \in \Delta$ such that $t \in U_\alpha \in \ell$. If $U_\alpha = \mathbb{R}$, then $\mathcal{V}$ has a finite subcover for $\mathbb{R}$, namely $\{\mathbb{R}\}$. Otherwise $U_\alpha = (-\infty,z)$ for some $z \in \mathbb{R}$, and then $\{U_\alpha , U_\beta\}$ is a finite subcover of $\mathcal{V}$ for $\mathbb{R}$.

Case 3: If x < t < y. Suppose that there exists $\alpha \in \Delta$ such that $t \in U_\alpha \in$. If $U_\alpha = \mathbb{R}$, then $\mathcal{V}$ has a finite subcover for $\mathbb{R}$, namely $\{\mathbb{R}\}$. Otherwise, $U_\alpha = (z,\infty)$ for some $z \in \mathbb{R}$ and z < t, and so there exists w∈A such that $z < w < t$. It is clear that there exists $\lambda \in \Delta$ such that $w \in U_\lambda \in \ell$, and then $\{U_\alpha, U_\lambda\}$ is a finite subcover of $\mathcal{V}$ for $\mathbb{R}$. Suppose now that there exists no $\alpha \in \Delta$ such that $t \in U_\alpha \in r$, then there exists $\alpha \in \Delta$ such that $t \in U_\alpha \in \ell$. If $U_\alpha = \mathbb{R}$, then $\mathcal{V}$ has a finite subcover for $\mathbb{R}$, namely $\{\mathbb{R}\}$. Otherwise $U_\alpha = (-\infty, z)$ for some $z \in \mathbb{R}$. Then $t < z$, and so there exists $w \in \mathbb{R}$ such that $t < w < z$. By definition of A and t, there exists $\lambda \in \Delta$ such that $w \in U_\lambda \in r$, and then $\mathcal{V}$ has $\{U_\alpha, U_\lambda\}$ as a finite subcover for $\mathbb{R}$. Hence, $(\mathbb{R}, \ell, r)$ is p-compact. However $(\mathbb{R}, \ell, r)$ is not s-compact, for $(\mathbb{R}, \ell)$ is not compact. □

The following example shows that if the bitopological space (X,τ_1,τ_2) is τ_i-compact, (i=1,2), then it is not necessarily that it is s-compact.

2.1.11 Example [10]:

Let $X=[0,1]$, $\tau_1=\{X, \emptyset\} \cup \{[0,b) : b \in X\}$, $\tau_2=\{X, \emptyset, \{1\}\}$. Every τ_1-open cover $\mathcal{V}$ for X must contain X, so (X,τ_1) is compact. Also, (X,τ_2) is compact as τ_2 is finite. However, (X,τ_1,τ_2) is not s-compact. Consider the following $\tau_1\tau_2$-open cover $\mathcal{V}$ for X, where $\mathcal{V} = \{ [0,b) \mid b \in X \} \cup \{\{1\}\}$. Suppose there exists a finite subfamily of $\mathcal{V}$ which covers X. This is equivalent to supposing that there is a subfamily $\{[0,b_i) \mid i=1,2,...,n\}$ of $\{ [0,b) \mid b \in X \}$ that covers [0,1). Now each b_i is in [0,1), so $m = \max\{b_1,b_2,\dots,b_n\}$ satisfies $0 < m < 1$, and so $m \notin \bigcup\{[0,b_i) \mid i=1,2,\dots,n\}$. Thus (X,τ_1,τ_2) is not s-compact.

Also, this implies that (X,τ_1,τ_2) is not p-compact. Hence, not every compact bitopological space (X,τ_1,τ_2) [i.e. (X,τ_1) and (X,τ_2) are compact] is p-compact. □

B-compactness and conversely compactness is independent of s-compactness and p-compactness, because any finite bitopological space is s-compact and p-compact but may not be B-compact as the following example shows.

2.1.12 Example [4]:

Let $X =\{a,b,c\}$, $\tau_1 =\{\emptyset,X,\{a,b\},\{c\}\}$, and $\tau_2 = \{\emptyset,X,\{a\},\{b,c\}\}$. Then (X,τ_1,τ_2) is s-compact and p-compact, but it is not τ_2-compact within τ_1 as $\{ \{a\},\{b,c\} \}$ is a τ_2-open cover of X which has no τ_1-open subcover. Also $\{ \{a\},\{b ,c\} \}$ is a τ_2-open cover of X which has no finite family of τ_1-open cover which is finer than this cover. Hence, (X ,τ_1,τ_2) is neither B-compact, nor conversely compact. □

The following example shows a bitopological space which is B-compact (and so conversely compact), but not p-compact (and so not s-compact).

2.1.13 Example [4]:

Let $X = [0,1]$, $\tau_1 =\{X ,\{0\} \} \cup \{ [0,a) : a \in X\}$ and $\tau_2=\{X,\{1\}\}\cup\{(a,1]: a \in X\}$. Then (X,τ_1,τ_2) is B-compact, for any τ_1-open cover of X or any τ_2-open cover for X must contain X as a member. However (X,τ_1,τ_2) is not p-compact (and so not s-compact), for the p-open cover $\{\{0\}\}\cup\{ (a,1] : a\in X, a\neq 0\}$ of X has no finite subcover. □

2.1.14 Theorem [4]:

If the bitopological space (X,τ_1,τ_2) is τ_i–compact with respect to τ_j (conversely compact) then (X,τ_1,τ_2) is τ_i–compact (compact).

Proof:

Let $\mathcal{V} = \{W_\alpha : \alpha \in \Delta\}$ be any τ_i-open cover for X. Since (X, τ_1, τ_2) is τ_i–compact with respect to τ_j, there is a finite τ_j-open cover $\mathcal{V}_1 = \{U_k : k = 1, \ldots, n\}$ for X, such that $\mathcal{V}_1$ is finer than $\mathcal{V}$. So, for each k =1,…,n, there exists $\alpha_k \in \Delta$ such that $U_k \subset W\alpha_k$. Consider the τ_i-open collection $\mathcal{V}_2 = \{ W\alpha_k : k = 1, \ldots, n\}$, then $\mathcal{V}_2$ covers X because $U_k \subset W\alpha_k$ for each k =1,2,…n, and $\mathcal{V}_1$ covers X. Since $\forall$ k =1,…,n, $W\alpha_k \in \mathcal{V}$, then $\mathcal{V}_2$ is the desired finite subfamily of $\mathcal{V}$ that covers X. Thus it means that (X,τ_i) is compact. □

We can replace conversely compact by B-compact in the above theorem because every B-compact space is conversely compact.

In example (2.1.12), (X, τ_1) and (X,τ_2) are compact, but the bitopological space (X,τ_1,τ_2) is neither B-compact, nor conversely compact, so the converse of the pervious theorem is not true.

2.1.15 Corollary:

Let (X,τ_1,τ_2) be a bitopological space, if X is conversely compact and p-compact, then (X, τ_1, τ_2) is s-compact.

Proof:

Since (X,τ_1,τ_2) is conversely compact then (X, τ_1) and (X, τ_2) are compact by theorem (2.1.14) and since (X, τ_1, τ_2) is p-compact, so by theorem (2.1.9), (X, τ_1, τ_2) is s-compact. □

The collection of closed sets plays an important role in B-compactness and conversely compactness.

2.1.16 Theorem [4]:

Let (X, τ_1, τ_2) be a bitopological space, then the following are equivalent:

a) X is τ_i-compact with respect to τ_j.

b) For any family $\{F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets which has empty intersection, there exists a finite family $\{G_k : k = 1,\dots,n\}$ of τ_j-closed sets with empty intersection and satisfies the condition that $\forall$ k=1,2,…,n , $\exists$ $\alpha_k \in \Delta$ such that $G_k \supset F_{\alpha_k}$.

c) For any family $\mathcal{V} = \{F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets with the property that every finite family $\{G_k : k = 1,\dots,n\}$ of τ_j -closed sets which satisfies the condition that $\forall$ k =1,2,…,n, $\exists$ $\alpha_k \in \Delta$ such that $G_k \supset F_{\alpha_k}$ has nonempty intersection, it results that $\mathcal{V}$ has nonempty intersection.

Proof: (a) $\Longrightarrow$ (b)

Assume (a) and let $\{F_\alpha : \alpha \in \Delta\}$ be any family of τ_i-closed sets which has empty intersection, then the family $\mathcal{V} = \{U_\alpha : U_\alpha = X \setminus F_\alpha , \alpha \in \Delta\}$ is a family of τ_i-open sets which covers X because $\bigcup_{\alpha\in\Delta} U_\alpha = \bigcup_{\alpha\in\Delta} X\setminus F_\alpha = X\setminus\bigcap_{\alpha\in\Delta} F_\alpha = X\setminus\emptyset = X$.

By the hypotheses of (a), there is a finite family $\mathcal{V}_1 = \{V_k : k=1,2,\dots,n\}$ of τ_j-open sets which covers X such that $\forall$ k =1,2,….,n, $\exists$ $\alpha_k \in \Delta$ with $V_k \subset U_{\alpha_k}$. Define $G_k = X\setminus V_k$, then for each k, G_k is τ_j–closed set and $G_k = X\setminus V_k \supset X\setminus U_{\alpha_k} = F_{\alpha_k}$, and $\bigcap_{k=1}^{n} G_k = \bigcap_{k=1}^{n}(X\setminus V_k) = X\setminus\bigcup_{k=1}^{n} V_k = X\setminus X = \emptyset$.

(b) $\Longrightarrow$ (a):

Assume (b), and let $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be any τ_i-open cover for X. Then the family $\{X \backslash U_\alpha : \alpha \in \Delta\}$ is a family of τ_i-closed sets such that $\bigcap_{\alpha\in\Delta} X\backslash U_\alpha = X\backslash \bigcup_{\alpha\in\Delta} U_\alpha = X\backslash X = \emptyset$, i.e. has empty intersection. Consequently, the hypotheses in (b) implies that there is a finite family $\{G_k : k = 1,\ldots,n\}$ of τ_j–closed sets such that $\forall k$, $\exists \alpha_k \in \Delta$ such that $G_k \supset X\backslash U_{\alpha_k}$ and $\bigcap_{k=1}^{n} G_k = \emptyset$. Consider $V_k = X\backslash G_k$, then $\forall k$, V_k is τ_j–open and $\bigcup_{k=1}^{n} V_k = \bigcup_{k=1}^{n} X\backslash G_k = X\backslash \bigcap_{k=1}^{n} G_k = X\backslash\emptyset = X$. Since $\forall k$, $V_k = X\backslash G_k \subset X\backslash(X\backslash U_{\alpha_k}) = U_{\alpha_k}$, then the finite family $\{V_k : k=1,2,\ldots,n\}$ of τ_j–open sets covers X and satisfies the desired condition. Hence (X,τ_1,τ_2) is τ_i-compact with respect to τ_j.

(b) $\Longrightarrow$ (c):

Assume (b), and let $\mathcal{V} = \{F_\alpha : \alpha\in\Delta\}$ of τ_i-closed sets with the property stated in (c). Suppose that $\bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$. By the hypotheses in (b), there is a finite family $\{G_k : k = 1,\ldots,n\}$ of τ_j–closed sets with empty intersection such that $\forall k$, $\exists \alpha_k\in\Delta$ with $G_k \supset F_{\alpha_k}$, and this contradicts the property of the family $\mathcal{V}$. Hence $\bigcap_{\alpha\in\Delta} F_\alpha \neq \emptyset$.

(c) $\Longrightarrow$ (b):

Assume (c), and let $\{F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets which has empty intersection. Suppose that there exists no finite family of the form $\{G_k : k = 1,\ldots,n\}$ of τ_j–closed sets with empty intersection and satisfies the condition that $\forall k$, $\exists\, \alpha_k \in \Delta$ with $G_k \supset F_{\alpha_k}$. This means that every finite family of the form $\{G_k : k = 1,\ldots,n\}$ of τ_j–closed sets which satisfies the condition $\forall k$, $\exists\, \alpha_k \in \Delta$ with $G_k \supset F_{\alpha_k}$ has nonempty intersection.

By (c), $\{F_\alpha : \alpha \in \Delta\}$ has nonempty intersection, and this contradict the assumption. So there exists a finite family $\{G_k : k = 1,\ldots,n\}$ of τ_j-closed sets with empy intersection and satisfies the condition that $\forall\, k = 1,2,\ldots,n$, $\exists \alpha_k\in\Delta$ such that $G_k \supset F_{\alpha_k}$. □

2.1.17 Theorem [4]:

Let (X,τ_1,τ_2) be a p-Hausdorff bitopological space and let (X,τ_1) be a compact topological space. Then $\tau_1 \subset \tau_2$.

Proof:

To prove this, it is sufficient to show that every τ_1-closed set is τ_2-closed set. Let A be τ_1–closed, then A is τ_1-compact. Let $x \notin A$. Since (X,τ_1,τ_2) is p-Hausdorff, then for each $a \in A$, there exist τ_1-open set V(a) and a τ_2-open set U(a) such that $a \in V(a)$, $x \in U(a)$, and $V(a) \cap U(a) = \emptyset$. The family $\{V(a) : a \in A\}$ forms a τ_1-open cover of A, and so by compactness of A, we find a finite subcover $\{V(a_1), V(a_2), \ldots, V(a_n)\}$ of $\{V(a) : a \in A\}$ for A. For each $V(a_k)$, $k=1,2,\ldots,n$, there is a corresponding τ_2-open sets $U(a_k)$, and hence $B=\bigcap_{k=1}^{n} U(a_k)$ is τ_2-open set containing x. Now $B \cap V(a_k) =\emptyset$ for each $k =1,2,\ldots,n$, for if this not true, then $B \cap V(a_i) \neq \emptyset$ for some $i =1,\ldots,n$, and then $U(a_i) \cap V(a_i) \neq \emptyset$ as $B \subset U(a_k)$ for each $k =1,2,\ldots,n$, and this is the contrary to the way $V(a_k)$ and $U(a_k)$ were chosen. Define $C=\bigcup_{k=1}^{n} V(a_k)$ which is τ_1-open, then we have $B \cap C = \emptyset$ and this implies that $B \cap A = \emptyset$. Therefore $x \in B \subset X \backslash A$ which means that A is τ_2-closed. □

2.1.18 Corollary [4]:

Let the bitopological space (X,τ_1,τ_2) be a p-Hausdorff:

(a) If the topologies τ_1 and τ_2 are compact, then $\tau_1 = \tau_2$.

(b) If (X,τ_1,τ_2) is τ_1-compact with respect to τ_2 , then $\tau_1 \subset \tau_2$.

(c) If (X,τ_1,τ_2) is conversely compact, then $\tau_1 = \tau_2$.

(d) If (X,τ_1,τ_2) is B-compact, then $\tau_1 = \tau_2$. □

2.1.19 Example [4]:

Let X = [0,1]. Let τ_1 be the usual topology on [0,1], and τ_2 be the discrete topology on [0,1]. Then (X,τ_1, τ_2) is p-Hausdorff bitopological space, and τ_1 is compact with respect to τ_2. But the topology τ_2 is not compact with respect to τ_1, and so τ_2 is not compact within τ_1. Consequently (X,τ_1,τ_2) is neither B–compact, nor conversely compact. □

2.1.20 Example [4]:

Let $X = [0,\infty)$, let τ_1 be the discrete topology, and τ_2 be the co-countable topology. (X,τ_1,τ_2) is p-Hausdorff, and p-normal. The topologies τ_1 and τ_2 are not compact and consequently (X, τ_1, τ_2) is neither B-compact nor conversely compact. To see that τ_2 is not compact consider the τ_2-open covering $\{(X\setminus\mathbb{N})\cup\{i\}: i\in\mathbb{N}\}$ for X which has no finite subcovering for X. □

2.1.21 Example [4]:

Let X = [0,1], τ_1 be the topology induced on X by the standard topology on $\mathbb{R}$, and τ_2 be the topology generated by the union of families of τ_1 and the families of sets whose complements are countable as a subbase. The bitopological space (X,τ_1,τ_2) is p-Hausdorff and τ_1-compact with respect to τ_2, (it is even τ_1-compact within τ_2), but it is not p-normal. □

2.1.22 Example [4]:

Let X={a,b,c}, τ_1={∅, {a},{a,c},{b,c},{c},X}, τ_2={∅,{b},{b,c},{a,b},{a},X}. Therefore in (X,τ_1,τ_2), $\tau_1 \neq \tau_2$. (X, τ_1, τ_2) is p-regular, p-normal and conversely compact. But it is not p–Hausdorff, as τ_1 and τ_2 are finite and so, they are compact. Since $\tau_1 \neq \tau_2$, then by corollary (2.1.19.a) it is not p-Hausdorff. □

2.1.23 Example [4]:

Let X = {a,b,c}, τ_1 = {∅,{a},X}, and τ_2 = {∅,{b},{b,c},X}. The bitopological space (X,τ_1,τ_2) is p-normal and B-compact but not p-regular.

The bitopological space (X,τ_1,τ_2) is :

1) p-normal, because {b,c} is the only nonempty proper τ_1-closed subset. And the only nonempty proper τ_2 -closed subset of X that is disjoint from {b,c} is {a}, and {b,c} is τ_2 -open, and {a} is τ_1-open.

2) B-compact, because each τ_1-open or τ_2-open cover for X must contain X as a member.

3) Not p-regular, because {a,c} is τ_2-closed and b∉{a,c}, the τ_2-open set that contains b is {b}, and the only τ_1-open set which contains {a,c} is X. So, τ_2 is not regular with respect to τ_1. □

2.1.24 Corolary:

Let (X,τ_1,τ_2) be a bitopological space, if X is conversely compact and p-Hausdorff, then (X, τ_1, τ_2) is p-regular and p-normal.

Proof:

By corollary (2.1.19) since (X, τ_1, τ_2) is conversely compact and p– Hausdorff, then $\tau_1 = \tau_2$, the result follows from the single topology theory. □

2.2 Conversely compactness of sets in bitopological spaces:

2.2.1 Definition [4]:

Let (X,τ_1,τ_2) be a bitopological space, and let $A \subset X$. We say that the set A is τ_i -compact with respect to τ_j [resp. conversely compact], if the bitopological subspace $(A, \tau_{1A}, \tau_{2A})$ is τ_{iA} - compact with respect to τ_{jA} [resp. conversely compact]; where $\tau_{1A} = \{A \cap U : U \in \tau_1\}$ and $\tau_{2A} = \{A \cap V : V \in \tau_2\}$.

2.2.2 Theorem [4]:

Let A be a set in a bitopological space (X,τ_1, τ_2). Then:

(a) A sufficient condition for the set A to be τ_i-compact with respect to τ_j is:

for every τ_i -open cover $\mathcal{V}$ of A, there is a finite τ_j-open cover $\mathcal{V}_1$ of A finer than $\mathcal{V}$.

(b) If the set A is τ_j -open, then a necessary condition for A to be τ_i -compact with respect to τ_j is: for every τ_i-open cover $\mathcal{V}$ of A, there is a finite τ_j –open cover $\mathcal{V}_1$of A finer than $\mathcal{V}$.

Proof: (a)

Let $\mathcal{V} = \{U_\alpha \cap A : \alpha \in \Delta\}$, where $U_\alpha \in \tau_i$ for each $\alpha \in \Delta$, be a τ_{iA}-open cover for A. Then, $\cup\{(U_\alpha \cap A): \alpha \in \Delta\} = A$. So, $\cup\{U_\alpha: \alpha \in \Delta\} \cap A = A$, and so, $\cup\{U_\alpha : \alpha \in \Delta\} \supset A$. i.e. $\mathcal{V}' = \{ U_j : \alpha \in \Delta\}$ is a τ_i -open cover for A. By the hypothesis, there is a finite τ_j -open cover for A; say $\mathcal{V}'_1 = \{W_k : k =1,2,\ldots,n\}$ finer than $\mathcal{V}'$. This means that $\forall$ k =1,2,….,n, there is $\alpha \in \Delta$ such that $W_k \subset U_\alpha$. This implies that $\forall$ k =1,2,…,n, $\exists$ $\alpha \in \Delta$ such that $(W_k \cap A) \subset (U_\alpha \cap A)$. Hence, the collection $\mathcal{V}_1 = \{W_k \cap A : k =1,2,\ldots,n\}$ is the desired finite τ_{jA}-open cover for A which is finer than $\mathcal{V}$.

Proof: (b)

Let A be a τ_j -open set that is τ_i–compact with respect to τ_j , and let the collection $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_i -open cover for A. Then $\mathcal{V}_1=\{ U_\alpha \cap A : \alpha \in \Delta\}$ is a τ_{iA}-open cover for A, so by the hypothesis, there is a finite family $\mathcal{V}_2$ of τ_{jA}-open sets finer than $\mathcal{V}_1$ that covers A, say $\mathcal{V}_2 = \{W_k \cap A : k=1,2,\ldots,n\}$ where $W_k \in \tau_j$, $\forall$ k =1,2,…,n. Since A is τ_j -open then for each k =1,2,…,n , $W_k \cap A$ is τ_j-open set, and so $\{W_k \cap A : k =1,2,\ldots,n\}$ is the desired finite family of τ_j-open sets which is finer than $\mathcal{V}$ and covers A. □

The following example shows that the converse of Theorem (2.2.2.a) is not necessarily true if A is not τ_j-open set.

2.2.3 Example [4]:

Let X ={a,b,c}, τ_1={∅,{a}, {a,c},{b,c},{c}, X}, and τ_2 ={∅, {b},{a,b},{a},X}. Let A={c}, and consider the τ_1-open cover {{b,c}}for A, then there is no τ_2-open cover for A finer than {{b,c}}. So A does not satisfy the condition in theorem (2.2.2.a) even though (A ,τ_{1A} ,τ_{2A}) is τ_1-compact with respect to τ_2. □

Even though, the union of finite family of compact subsets of a topological space is compact, but this result is not necirsserily true for τ_i-compact with respect to τ_j.

2.2.4 Theorem [4]:

Let A and B be τ_j-open sets, each of which is τ_i-compact with respect to τ_j , then there union (A∪B) is τ_i-compact with respect to τ_j .

Proof:

Let $\mathcal{V}$ = {U_α : α ∈ Δ} be a τ_i-open cover for A∪B, then $\mathcal{V}$ is τ_i-open cover for A and for B. By our hypothesis of A and B, and according to Theorem (2.2.2), there are two finite τ_j -open covers for A and B, say S_1 and S_2 respectively such that each of S_1 and S_2 is finer than $\mathcal{V}$. Therefore $S_1 \cup S_2$ is a finite τ_j -open cover for A∪B, and $S_1 \cup S_2$ is finer than $\mathcal{V}$. It follows that A∪B is τ_i -compact with respect to τ_j by Theorem (2.2.2). □

The following corollary follows by mathematical induction.

2.2.5 Corollary:

Let $\{A_1, A_2, \ldots, A_n\}$ be a finite family of τ_j-open sets, each of which is τ_i-compact with respect to τ_j , then $\bigcup_{i=1}^{n} A_i$ is τ_i-compact with respect to τ_j . □

The following example shows that the condition that A and B are τ_j-open sets in theorem (2.2.4) is essential.

2.2.6 Example [4]:

Let $X = \{a,b\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, X\}$, and $\tau_2=\{\emptyset, X\}$. The sets $\{a\}$, $\{b\}$ are τ_1-compact with respect to τ_2, but $\{a\}\cup\{b\} = X$ is not τ_1-compact with respect to τ_2. Note that $\{a\}$ and $\{b\}$ are not τ_2-open. □

2.2.7 Theorem [4]:

Let the bitopological space (X,τ_1,τ_2) be τ_i -compact with respect to τ_j [resp. conversely compact], and let the subset A of X be τ_i-closed [resp. closed]. Then A is τ_i -compact with respect to τ_j [resp. conversely compact].

Proof:

Assume that A is τ_i-closed and that (X,τ_1,τ_2) is τ_i-compact with respect to τ_j. Want to show that the subspace (A,τ_{1A},τ_{2A}) is τ_{iA}-compact with respect to τ_{jA}. Let $\mathcal{V} =\{U_\alpha : \alpha \in \Delta\}$ be any τ_{iA}-open cover of A, then for each $\alpha\in\Delta$, $U_\alpha = W_\alpha\cap A$; for some $W_\alpha \in \tau_i$. Since A is τ_i-closed, then $X\setminus A$ is τ_i-open, and so the collection $\mathcal{V}_1 = \{W_\alpha : \alpha \in \Delta\} \cup \{X\setminus A\}$ is a τ_i-open cover of X. By τ_i -compactness of X with respect to τ_j , there is a finite τ_j-open cover for X, say $\mathcal{V}_2$ such that $\mathcal{V}_2$ is finer than $\mathcal{V}_1$. Let the collection $\mathcal{V}_3$ be the set of all elements of $\mathcal{V}_2$ which are not subsets of X\A. Then $\mathcal{V}_3 = \{C_k : k =1,2,..,n\}$ is a family of τ_j-open sets which is finer than $\mathcal{V}_1$ and covers A. Consequently the collection $\mathcal{V}_4 = \{C_k \cap A : k =1,2,\dots,n \}$ is the desired τ_{jA}-open cover for A which is finite and finer than $\mathcal{V}$. This means that A is τ_i-compact with respect to τ_j.We use the same argument to complete the proof of the theorem.

□

2.3 Continuous (open) functions and conversely compactness in bitopological spaces

2.3.1 Theorem [4]:

If the bitopological space (X,τ_1,τ_2) is τ_i-compact with respect to τ_j , and if the function $f : (X ,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is i-continuous and j-open, then $f(X)$ is σ_i-compact with respect to σ_j.

Proof:

Let $\mathcal{V}' = \{U_\alpha : \alpha \in \Delta\}$ be a σ_i-open cover for $f(X)$ in (Y,σ_1,σ_2). Because f is i-continuous, then the collection $\mathcal{V} = \{ f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is τ_i -open cover for X, and therefore there exists a finite τ_j -open cover say $\{W_k: k=1,2,\dots,n\}$ for X finer than $\mathcal{V}$. That is to say that $\forall k, \exists\, \alpha_k \in \Delta$, such that $W_k \subset f^{-1}(U_{\alpha_k})$. Since the function f is j-open, then the collection $\{ f(W_k) : k =1,2,\dots,n\}$ is σ_j -open cover of $f(X)$ which is finite and finer than $\mathcal{V}'$ because $\forall\, k=1,2,\dots,n,\ \exists\, \alpha_k \in \Delta$, such that $f(W_k) \subset U_{\alpha_k}$ .This implies that $f(X)$ is σ_i-compact with respect to σ_j, by Theorem (2.2.2.a). □

The following corollary follows directely.

2.3.2 Corollary [4]:

If the bitopological space (X,τ_1,τ_2) is conversely compact, and if the function $f : (X ,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is continuous and open, then $f(X)$ is a conversely compact subset of the space (Y,σ_1,σ_2). □

2.3.3 Theorem:

If the bitopological space (X,τ_1,τ_2) is τ_i-compact within τ_j, and if the function $f : (X ,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is i-continuous and j-open, then $f(X)$ is σ_i-compact within σ_j.

Proof:

Let $\mathcal{V} = \{U_\alpha: \alpha\in\Delta\}$ be a $\sigma_{i f(X)}$-open cover for $f(X)$. Because f is i-continuous, then the collection $\mathcal{V}_1 = \{f^{-1}(U_\alpha):\alpha \in \Delta\}$ is τ_i -open cover for X, and therefore there exists a finite τ_j -open subcover of $\mathcal{V}_1$ say $\{f^{-1}(U_{\alpha_k}): k =1,2,\ldots,n\}$ for X.

The function f is j-open, so $f f^{-1}(U_{\alpha_k}) \in \sigma_j$ $\forall$ k=1,2,…,n. And since $f f^{-1}(U_{\alpha_k}) = U_{\alpha_k}$ $\forall$ k =1,2,…, n then the collection $\{ U_{\alpha_k} : k =1,2,\ldots,n\}$ is a finite $\sigma_{j f(X)}$ -open subcover of $\mathcal{V}$ for $f(X)$. Thus $f(X)$ is σ_i-compact within σ_j. □

2.3.4 Corollary [4]:

If we add to the hypothesis of corollary (2.3.2), the hypothesis that (Y,σ_1,σ_2) is p-Hausdorff, then $\sigma_1 = \sigma_2$ and $(f(X), \sigma_1 = \sigma_2)$ is a compact topological space.

Proof:

By corollary (2.3.2), (f(X) , σ_1 , σ_2) is conversely compact. Then by Corollary (2.1.19 -c) $\sigma_1 = \sigma_2$. Since (f(X),σ_1,σ_2) is conversely compact, then f(X) is σ_1-compact with respect to σ_2 , i.e. (f(X),σ_1) is a compact topological space, according to corollary (2.1.19). □

2.3.5 Corollary [4]:

In the bitopological space (X,τ_1,τ_2), the image of the τ_j -open (resp. open) subset A of X which is τ_i-compact with respect to τ_j (resp. conversely compact) by a function f : (X ,τ_1,τ_2) → (Y,σ_1,σ_2) which is i-continuous and j-open (resp. f is continuous and open) is σ_i-compact with respect to σ_j (resp. conversely compact).

Proof:

The proof is similar to the proof of Theorem (2.3.1), using Theorem (2.2.2). □

The following example proves that it is not sufficient to suppose that f is only continuous in Theorem (2.3.1).

2.3.6 Example [4]:

Let X={a , b , c}, $\tau_1 = \tau_2$ = the discrete topology. Let Y={1 , 2 , 3}, σ_1={∅, {1},{2,3},Y}, σ_2 = {∅,{1,2},{3},Y}. Define the function f by f(a) = 1, f(b) = 2, f(c) = 3. We observe that:

1) (X,τ_1,τ_2) is conversely compact (there is exactly one compact topological space).

2) f is continuous function, as τ_1 and τ_2 are the discrete topologies .

3) (Y,σ_1,σ_2) is neither σ_1-compact with respect to σ_2, nor σ_2 -compact with respect to σ_1.

Proof:

The proof of (1) and (2) are direct. To prove (3) we notice that $\mathcal{V}_1=\{\{1\},\{2,3\}\}$ is σ_1-open cover for Y, but there is no σ_2-open cover for Y that is finer than $\mathcal{V}_1$. Also, $\mathcal{V}_2 = \{\{1,2\},\{3\}\}$ is σ_2-open cover for Y, but there is no σ_1-open cover for Y that is finer than $\mathcal{V}_2$. □

2.4 Alexander's, Tychonoff's theorems and conversely compactness in bitopological spaces

In single topology we have , if $\{ (X_i , \tau_i) : i \in I \}$ is a family of topological spaces , then the product topology $(\prod_{i\in I} X_i , \rho)$ is the topology generated by the collection $\{ \pi_i^{-1}(U) : U \in \tau_i ; i\in I \}$ as a subbase, where π_i is the natural projection from $(\prod_{i\in I} X_i , \rho)$ onto (X_i , τ_i). In bitopological spaces we have the following analogous definition.

2.4.1 Definition [5]:

Let $\{(X_k, \tau^k_1 ,\tau^k_2) : k\in\Delta \}$ be a family of bitopological spaces. On the product set $X = \prod_{k\in\Delta} X_k$. We define a bitopological structure (ρ_1 , ρ_2) by taking ρ_1 as the product topology generated by the τ^k_1's, and ρ_2 as the product topology generated by the τ^k_2's . The resulting bitopological space (X, ρ_1, ρ_2) will be called the product bitopological space generated by the family $\{(X_k, \tau^k_1 ,\tau^k_2) : k\in\Delta \}$.

2.4.2 Theorem [10]:

Let $\{(X_k, \tau^k_1, \tau^k_2) : k \in \Delta\}$ be an arbitrary family of nonempty bitopological spaces. Then for each fixed k, the natural projection map, $\pi_k: (X, \rho_1, \rho_2) \longrightarrow (X_k, \tau^k_1, \tau^k_2)$ is continuous.

Proof: The result follows directely from single topology theory. □

2.4.3 Definition [4]:

A family $\mathcal{F}$ of τ_i-open sets in the bitopological space (X,τ_1,τ_2) is called τ_i -inadequate in (X, τ_1, τ_2), i=1,2 , if it fails to cover X. The family $\mathcal{F}$ of τ_i-open sets is called finitely τ_i-inadequate with respect to τ_j in X if and only if no finite family of τ_j-open sets which is finer than $\mathcal{F}$ covers X.

We can easily see that the bitopological space (X,τ_1,τ_2) is τ_i-compact with respect to τ_j if and only if each finitely τ_i -inadequate family with respect to τ_j in X, is τ_i -inadequate.

2.4.5 Lemma [4]:

If $\mathcal{F}$ is a finitely τ_i-inadequate family with respect to τ_j in the bitopological space (X,τ_1,τ_2), then there is a maximal finitely τ_i-inadequate family with respect to τ_j in (X,τ_1,τ_2), say $\mathcal{D}$, and $\mathcal{F} \subset \mathcal{D}$.

Proof:

Let ξ be the family of all finitely τ_i-inadequate families with respect to τ_j. $\mathcal{F}$ is finitely τ_i-inadequate family with respect to τ_j, so $\mathcal{F} \in \xi$.

Define a partial order $\leq$ on ξ, by $\forall\, C_1, C_2 \in \xi$, $C_1 \leq C_2$ iff $C_1 \subset C_2$.

$\{\mathcal{F}\}$ is a chain in ξ, then by Hausdorff maximal principle, there is a maximal chain $\mathcal{B}$ such that $\{\mathcal{F}\} \subset \mathcal{B}$.

Let $\mathcal{D} = \cup\mathcal{B}$. Each elment of $\mathcal{B}$ is finitely τ_i-inadequate family with respect to τ_j, then each element of $\mathcal{B}$ is a family of τ_i-open sets, so $\mathcal{D} = \cup\mathcal{B}$ is a family of τ_i-open sets .

Want to prove: i) $\mathcal{D}$ is finitely τ_i-inadequate family with respect to τ.

ii) $\mathcal{D}$ is maximal finitely τ_i-inadequate family with respect to τ_j, and $\mathcal{F} \subset \mathcal{D}$.

i) Suppose that $\mathcal{D}$ has a finite family of τ_j-open sets finer than $\mathcal{D}$ and covers X say $U = \{U_k : k = 1,2,\ldots n\}$. $\forall\, k = 1,2,\ldots n$, choose $V_k \in \mathcal{D}$ with $U_k \subset V_k$.

Then $\mathcal{D}' = \{V_k : k = 1,2,\ldots,n\} \subset \mathcal{D}$. $\mathcal{B}$ is a chain, so $\mathcal{D}' \subset E$ for some $E \in \mathcal{B}$. Since $\mathcal{D}'$ has a finite family of τ_j-open sets finer than $\mathcal{D}'$ and covers X, and $\mathcal{D}' \subset E$, then E has a finite family of τ_j-open sets finer than E and covers X, and this contradict the fact that E is a finitely τ_i-inadequate family with respect to τ_j.

Thus, $\mathcal{D}$ is a finitely τ_i-inadequate with respect to τ_j.

ii) Suppose that $\mathcal{D}$ is not maximal finitely τ_i-inadequate family with respect to τ_j, then there exists $G \in \tau_i$, such that $\mathcal{D} \cup \{G\}$ is still finitely τ_i-inadequate family with respect to τ_j, then $\mathcal{B} \cup \{\mathcal{D} \cup \{G\}\}$ is a chain contains $\mathcal{B}$ properly which contradicts the fact that $\mathcal{B}$ is maximal chain.

So, $\mathcal{D}$ is maximal finitely τ_i-inadequate family with respect to τ_j.

Since $\mathcal{D} = \cup\mathcal{B}$, and $\mathcal{F} \in \mathcal{B}$, then $\mathcal{F} \subset \mathcal{D}$. □

2.4.6 Lemma [4]:

Let (X,τ_1,τ_2) be a bitopological space. If $\mathcal{D}$ is a maximal finitely τ_1-inadequate family with respect to τ_2, and if some member of $\mathcal{D}$ contains $\bigcap_{i=1}^{n} G_i$, where each G_i is τ_1-open, then $G_k \in \mathcal{D}$ for some k in $\{1,2,\ldots,n\}$.

Proof:

First suppose that n = 2. Suppose that $G_1 \notin \mathcal{D}$ and $G_2 \notin \mathcal{D}$. Then by maximality of $\mathcal{D}$, $\mathcal{D} \cup \{G_1\}$ and $\mathcal{D} \cup \{G_2\}$ are not finitely τ_1-inadequate with respect to τ_2 , then for $\mathcal{D} \cup \{G_1\}$, $\exists$ A_1 , A_2 ,..., A_m , A, where A_i , A are τ_2-open sets, i=1,2,…,m , and $A \subset G_1$, and $A_i \subset A'_i$ for some $A'_i \in \mathcal{D}$,$\forall$ i=1,2,…m , such that $A_1 \cup A_2 \cup \ldots \cup A_m \cup A = X$.

And for $\mathcal{D} \cup \{G_2\}$, $\exists$ τ_2 -open sets $B_1, B_2, \ldots, B_t$, B, such that $B_1 \cup B_2 \cup \ldots \cup B_t \cup B = X$, where $B \subset G_2$ and $B_i \subset B'_i$ for some $B'_i \in \mathcal{D}$, $\forall$ j=1,2,…,t.

Claim: $(A \cap B) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_t = X$.

It is clear that $(A \cap B) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_t \subset X$.

Now, let $x \in X$. If either $x \in A_i$, for some i=1,2,…, m , or $x \in B_j$, for some j=1,2,…,t, then $x \in (A \cap B) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_t$. If not, then $x \in A$ and $x \in B$ and so $x \in (A \cap B)$. So $X \subset (A \cap B) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_t$. Then our claim is true.

Since $A \subset G_1$ and $B \subset G_2$, then $(A \cap B) \subset (G_1 \cap G_2)$. But $(G_1 \cap G_2)$ is contained in some element of $\mathcal{D}$,so $(A \cap B)$, $A_1 ,\ldots, A_m$, $B_1, \ldots, B_t$ is a finite family of τ_2-open sets that is finer than $\mathcal{D}$ and covers X, this contradicts that $\mathcal{D}$ is finitely τ_1-inadequate with respect to τ_2. So $G_1 \in \mathcal{D}$ or $G_2 \in \mathcal{D}$. So the result holds for n=2.

The result for arbitrary $n \in \mathbb{N}$ follows by mathematical induction. □

2.4.7 Theorem (Alexander) [4]:

Let (X,τ_1,τ_2) be a bitopological space, and assume that $\mathcal{S}$ is a subbase of the topology τ_i such that, for each τ_i-open cover $\mathcal{V}$ for X by members of $\mathcal{S}$, there is a finite family of τ_j-open sets finer than $\mathcal{V}$ that covers X, then (X,τ_1,τ_2) is τ_i-compact with respect to τ_j.

Proof:

Let $\mathcal{B}$ be a finitely τ_i-inadequate family with respect to τ_j, then by lemma (2.4.5) there is a maximal finitely τ_i-inadequate family with respect to τ_j, say $\mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$. If we prove that $\mathcal{D}$ is τ_i-inadequte, then $\mathcal{B}$ is also τ_i-inadequate.

Since $\mathcal{S}$ is a subbase of τ_i, and $\mathcal{D}$ is a family of τ_i -open sets, then $(\mathcal{S} \cap \mathcal{D})$ is a family of τ_i-open sets. Let $A \in \mathcal{D}$, then $A \in \tau_i$, and $\mathcal{S}$ is a subbase of τ_i, then there is a finite intersection of elements of $\mathcal{S}$ which is contained in A, then one of these elements of $\mathcal{S}$ is an element of $\mathcal{D}$. So $(\mathcal{S} \cap \mathcal{D})$ is a nonempty family of τ_i-open sets contained in $\mathcal{D}$, since $(\mathcal{S} \cap \mathcal{D}) \subset \mathcal{D}$, then $(\mathcal{S} \cap \mathcal{D})$ is a finitely τ_i-inadequate family with respect to τ_j. Which means that there is no finite family of τ_j-open sets finer than $(\mathcal{S} \cap \mathcal{D})$ and covers X. And since $(\mathcal{S} \cap \mathcal{D}) \subset \mathcal{S}$, $(\mathcal{S} \cap \mathcal{D})$ is τ_i-open family of $\mathcal{S}$ which does not cover X. Hence, $(\mathcal{S} \cap \mathcal{D})$ is τ_i-inadequate.

Want to prove that $\bigcup\{C : C \in \mathcal{D}\} = \bigcup\{C : C \in (\mathcal{S} \cap \mathcal{D})\}$.

Since $\mathcal{S} \cap \mathcal{D} \subset \mathcal{D}$, so $\bigcup\{C: C \in (\mathcal{S} \cap \mathcal{D})\} \subset \bigcup\{C : C \in \mathcal{D}\}$ ………………(1)

Let $x \in \bigcup\{C : C \in \mathcal{D}\}$; then $\exists\ A \in \mathcal{D}$ s.t. $x \in A$, since A is τ_i-open, then there is a finite intersection of elements of $\mathcal{S}$ containing x and contained in A. By maximality of $\mathcal{D}$, one of these elements of $\mathcal{S}$ is an element of $\mathcal{D}$, so $x \in \bigcup\{C: C \in (\mathcal{S} \cap \mathcal{D})\}$ ……………..(2)

Hence, $\bigcup\{C : C \in \mathcal{D}\} = \bigcup\{C : C \in (\mathcal{S} \cap \mathcal{D})\}$, from (1) and (2).

So, $\mathcal{D}$ is τ_i -inadequate, and so $\mathcal{B}$ is τ_i-inadequate.Therefore each finitely τ_i -inadequate family with respect to τ_j is τ_i-inadequate. So X is τ_i -compact with respect to τ_j . □

2.4.8 Theorem: (Tychonoff) [4]:

Let the bitopological space (X, ρ_1, ρ_2) be the product bitopological space of the family of bitopological spaces $\{(X_k, \tau^k{}_1, \tau^k{}_2) : k \in \Delta\}$. Then (X, ρ_1, ρ_2) is ρ_i -compact with respect to ρ_j (conversely compact), if and only if each factor space $(X_k, \tau^k{}_1, \tau^k{}_2)$ is $\tau^k{}_i$ -compact with respect to $\tau^k{}_j$ (conversely compact).

Proof:

$\Longrightarrow$) The natural projections are continuous and open, therefore theorem (2.3.1) and corollary (2.3.3) prove (i).

$\Longleftarrow$) Let $\mathcal{S} = \{\pi_k^{-1}(U_k) : U_k \in \tau_1{}^k,\ k \in \Delta\}$, where π_k is the natural projection into the k-th coordinate space X_k , then $\mathcal{S}$ is a subbase for the topology ρ_1. In view of theorem (2.4.7), the product bitopological space (X, ρ_1, ρ_2) will be ρ_i -compact with respect to ρ_j if each subfamily $\mathcal{A}$ of $\mathcal{S}$ which is finitely ρ_i-inadequate with respect to ρ_j in (X, ρ_1, ρ_2) is ρ_i-inadequate. For each index $k \in \Delta$, Let $\mathcal{B}_k$ be the family of all sets $U_k \in \tau_i{}^k$ such that $\pi_k^{-1}(U_k) \in \mathcal{A}$. Then $\mathcal{B}_k$ is finitely $\tau_i{}^k$ -inadequate with respect to $\tau_j{}^k$ in $(X_k , \tau_1{}^k , \tau_2{}^k)$. Since $(X_k , \tau_1{}^k , \tau_2{}^k)$ is $\tau_i{}^k$ -compact with respect to τ_j^k, then $\mathcal{B}_k$ is τ_i^k -inadequate in (X_k , τ_1^k , τ_2^k). So, there is $x_k \in X_k \setminus U_k$ for each $U_k \in \mathcal{B}_k$. Consider the point $x \in X$ whose k-th coordinate is x_k, then x belongs to no member of $\mathcal{A}$, and consequently, $\mathcal{A}$ is ρ_i-inadequate in (X, ρ_1 , ρ_2). Hence the product bitopological space (X, ρ_1, ρ_2) is ρ_i -compact with respect to ρ_j. □

Chapter Three

Lindelöfness of a topology with respect to another

3.1 Birsan and conversely Lindelöf

In this chapter, some kinds of Lindelöfness in bitopological spaces, and the relations between them are discussed.

Recall that a topological space (X,τ) is Lindelöf if every open cover for X has a countable subcover.

3.1.1 Definition [3]:

A bitopological space $(X,\tau_1.\tau_2)$ is called semi Lindelöf (s-Lindelöf) if every $\tau_1\tau_2$-open cover for X has a countable subcover.

3.1.2 Definition [3]:

A bitopological space (X,τ_1,τ_2) is called pairwise Lindelöf (denoted p-Lindelöf) if every p-open cover of X has a countable subcover.

3.1.3 Definition [4]:

A bitopological space (X,τ_1,τ_2) is called τ_i-Lindelöf with respect to τ_j if for each τ_i-open cover $\mathcal{V}$ for X, there is a countable family of τ_j-open sets finer than $\mathcal{V}$ and covers X.

The space is called conversely Lindelöf if it is τ_1-Lindelöf with respect to τ_2 and is τ_2 -Lindelöf with respect to τ_1.

3.1.4 Definition [3]:

A bitopological space (X, τ_1, τ_2) is called τ_i-Lindelöf within τ_j if for each τ_i-open cover $\mathcal{V}$ for X, has a countable subcover of τ_j open sets for X. The space is called B-Lindelöf if it is τ_1-Lindelöf within τ_2 and is τ_2 -Lindelöf within τ_1.

In fact, τ_i-Lindelöfness of (X,τ_1,τ_2) within τ_j implies τ_i-Lindelöfness of (X,τ_1,τ_2) with respect to τ_j , that is every B-Lindelöf is conversely Lindelöf but the converse need not be true.

As in example (2.1.8), since X is τ_1-compact with respect to τ_2 then it is τ_1-Lindelöf with respect to τ_2 . But it is not τ_1-Lindelöf within τ_2 , since $\{ [0,1]\backslash\{1/2\} , [0,1) \}$ is τ_1 –open cover which has no countable τ_2 –open subcover.

3.1.5 Note:

Let (X,τ_1,τ_2) be a bitopological space , then :

i) If X is compact, then it is Lindelöf.

ii) If X is s-compact, then it is s-Lindelöf.

iii) If X is p-compact, then it is p-Lindelöf.

iv) If X is τ_i-compact with respect to τ_j , then it is τ_i-Lindelöf with respect to τ_j .

v) If X is τ_i-compact within τ_j , then it is τ_i-Lindelöf within τ_j . □

It is knowing from single topology theory that if (X,τ) is a second countable space, then (X,τ) is Lindelöf.

Then the following corollary follows directely.

3.1.6 Theorem [1]:

If (X ,τ_1,τ_2) is second countable space, then (X ,τ_1 ,τ_2) is Lindelöf. □

The following theorem illustrates the relation between s-Lindelöfness and p-Lindelöfness.

3.1.7 Theorem:

The bitopological space (X,τ_1,τ_2) is s-Lindelöf if and only if it is p-Lindelöf, and Lindelöf.

Proof:

Assume that the bitopological space (X,τ_1,τ_2) is s-Lindelöf, and let $\mathcal{V}$ be any p-open cover of the space X, then $\mathcal{V}$ is $\tau_1\tau_2$-open cover for X. Since X is s-Lindelöf, then $\mathcal{V}$ has a countable subcover for X. Thus X is p-Lindelöf. Also, let $\mathcal{V}$ be any τ_i-open cover of X, where $i \in \{1,2\}$, then $\mathcal{V} \subset \tau_1 \cup \tau_2$, which means that $\mathcal{V}$ is $\tau_1\tau_2$-open cover of X. Since (X,τ_1,τ_2) is s-Lindelöf, then there is a countable subcover of $\mathcal{V}$ for X, which implies that X is τ_i-Lindelöf for each $i=1,2$. Conversely, assume that (X, τ_1, τ_2) is p-Lindelöf, τ_1-Lindelöf and τ_2 -Lindelöf. Let $\mathcal{V}$ be any $\tau_1\tau_2$-open cover for X, then $\mathcal{V} \subset \tau_1 \cup \tau_2$.

Case1:

If $\mathcal{V}$ contains at least one nonempty member of τ_1, and at least one nonempty member of τ_2, then $\mathcal{V}$ is p-open.Thus there is a countable subcover of $\mathcal{V}$ for X (as X is p-Lindelöf).

Case 2:

If $\mathcal{V}$ is contained entirely in τ_1 or τ_2, then $\mathcal{V}$ is either τ_1-open cover for X or τ_2 -open cover for X. In either case, there is a countable subcover of $\mathcal{V}$ for X (as X is Lindelöf).

Hence X is s-Lindelöf. □

3.1.8 Theorem:

If the bitopological space (X,τ_1,τ_2) is τ_i-Lindelöf with respect to τ_j then (X,τ_i) is Lindelöf.

Proof:

Let $\mathcal{V} = \{ W_\alpha : \alpha \in \Delta\}$ be any τ_i -open cover for X. Since (X,τ_1,τ_2) is τ_i-Lindelöf with respect to τ_j, there is a countable τ_j-open cover $\mathcal{V}_1=\{U_k : k \in \mathbb{N}\}$ for X, such that $\mathcal{V}_1$ is finer than $\mathcal{V}$. So, for each $k \in \mathbb{N}$, there exists $\alpha_k \in \Delta$ such that $U_k \subset W\alpha_k$, Consider the τ_i-open collection $\mathcal{V}_2 = \{ W\alpha_k : k \in \mathbb{N}\}$. Then $\mathcal{V}_2$ covers X because $U_k \subset W\alpha_k$ for each $k \in \mathbb{N}$ and $\mathcal{V}_1$covers X. Since $\forall\, k \in \mathbb{N}$, $W\alpha_k \in \mathcal{V}$, then $\mathcal{V}_2$ is the desired countable subfamily of $\mathcal{V}$ that covers X, which means that (X,τ_i) is Lindelöf. □

3.1.9 Corollary:

If the bitopological space (X,τ_1,τ_2) is conversely Lindelöf, then (X,τ_1,τ_2) is Lindelöf. □

The following example shows that the converse of corollary (3.1.9) is not true.

3.1.10 Example:

Consider the bitopological space $(\mathbb{R}, \ell, r)$. Then $(\mathbb{R}, \ell, r)$ is second countable as $\mathcal{B}_1=\{(-\infty,a) : a\in\mathbb{Q}\}$ is a countable base for the left ray topology on $\mathbb{R}$, and $\mathcal{B}_2=\{(b,\infty) : b\in\mathbb{Q}\}$ is a countable base for the right ray topology on $\mathbb{R}$, so $(\mathbb{R}, \ell, r)$ is Lindelöf.

But not every ℓ -open cover of $\mathbb{R}$ has a countable family of r-open sets finer than ℓ -open cover and covers $\mathbb{R}$, such as $\{(-\infty,n) : n \in \mathbb{N}\}$.

Hence, $(\mathbb{R}, \ell,)$ is not ℓ -Lindelöf with respect to r, and so it is not conversely Lindelöf.

So, not every second countable bitopological space is conversely Lindelöf.

Also, being Lindelöf bitopological space doesn't imply being conversely Lindelöf and so doesn't imply being B-Lindelöf. □

3.1.11 Theorem:

Let (X, τ_1, τ_2) be a bitopological space. If X is conversely Lindelöf and p-Lindelöf, then (X, τ_1, τ_2) is s-Lindelöf.

Proof:

Since (X,τ_1,τ_2) is conversely Lindelöf, then (X,τ_1) and (X,τ_2) are Lindelöf by corollary (3.1.9), and by p-Lindelöfness, (X, τ_1, τ_2) is s-Lindelöf. □

The following example shows that the converse of theorem (3.1.11) is not true.

3.1.12 Example:

Let $X =\{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b,c\}\}$, $\tau_2 = \{\emptyset , X, \{a,b\} ,\{c\}\}$. Then (X,τ_1,τ_2) is Lindelöf, s-Lindelöf and p-Lindelöf, but it is not τ_1-Lindelöf with respect to τ_2, as $\{ \{a\},\{b,c\}\}$ is a τ_1-open cover for X which has no countable family of τ_2-open sets finer than it and covers X. Also, (X,τ_1,τ_2) is not τ_2-Lindelöf with respect to τ_1, as $\{ \{a,b\}, \{c\} \}$ is a τ_2-open cover of X which has no countable family τ_1-open finer than it and covers X. Hence, (X,τ_1,τ_2) is neither B-Lindelöf nor conversely Lindelöf. □

3.1.13 Theorem:

Let (X, τ_1, τ_2) be a bitopological space, then the following are equivalent:

a) X is τ_i-Lindelöf with respect to τ_j.

b) For any family $\{F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets which has empty intersection, there exists a countable family $\{G_k : k \in \mathbb{N}\}$ of τ_j–closed sets with empty intersection and satisfies the condition that $\forall k \in \mathbb{N}$, $\exists\ \alpha_k \in \Delta$ such that $G_k \supset F_{\alpha_k}$.

c) For any family $\mathcal{V} = \{ F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets with the property that every countable family $\{G_k : k \in \mathbb{N}\}$ of τ_j–closed sets which satisfies the condition that $\forall k \in \mathbb{N}$, $\exists\ \alpha_k \in \Delta$ such that $G_k \supset F_{\alpha_k}$ has nonempty intersection, it results that $\mathcal{V}$ has nonempty intersection.

Proof: (a) $\Longrightarrow$ (b)

Assume (a) and let $\{F_\alpha : \alpha \in \Delta\}$ be any family of τ_i-closed sets which has empty intersection, then the family $\mathcal{V} = \{U_\alpha : U_\alpha = X \setminus F_\alpha, \alpha \in \Delta\}$ is a family of τ_i–open sets which covers X because $\bigcup_{\alpha\in\Delta} U_\alpha = \bigcup_{\alpha\in\Delta} X\setminus F_\alpha = X \setminus \bigcap_{\alpha\in\Delta} F_\alpha = X\setminus\emptyset = X$.

By the hypotheses of (a), there is a countable family $\mathcal{V}_1 = \{V_k : k \in \mathbb{N}\}$ of τ_j-open sets which covers X such that $\forall k \in \mathbb{N}$, $\exists\ \alpha_k \in \Delta$ with $V_k \subset U_{\alpha_k}$. Define $G_k = X\setminus V_k$, then for each k, G_k is τ_j–closed set and $G_k = X\setminus V_k \supset X\setminus U_{\alpha_k} = F_{\alpha_k}$, and $\bigcap_{k\in\mathbb{N}} G_k = \bigcap_{k\in\mathbb{N}} X\setminus V_k = X\setminus \bigcup_{k\in\mathbb{N}} V_k = X\setminus X = \emptyset$.

(b) $\Longrightarrow$ (a):

Assume (b), and let $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be any τ_i–open cover of X. Then the family $\{X\setminus U_\alpha : \alpha \in \Delta\}$ is a family of τ_i-closed sets such that $\bigcap_{\alpha\in\Delta} X\setminus U_\alpha = X\setminus \bigcup_{\alpha\in\Delta} U_\alpha = X\setminus X = \emptyset$, i.e. has empty intersection. Consequently, the hypotheses in (b) implies that there is a countable family $\{G_k : k \in \mathbb{N}\}$ of τ_j-closed sets such that $\forall\ k$, $\exists\ \alpha_k \in \Delta$ such that $G_k \supset X\setminus U_{\alpha_k}$ and $\bigcap_{k=1}^{n} G_k = \emptyset$. Consider $V_k = X\setminus G_k$. Then $\forall k$, V_k is τ_j–open and $\bigcup_{k=1}^{\infty} V_k = \bigcup_{k=1}^{\infty} X\setminus G_k = X\setminus\bigcap_{k=1}^{\infty} G_k = X\setminus\emptyset = X$. Since $\forall k$, $V_k = X\setminus G_k \subset X\setminus(X\setminus U_{\alpha_k}) = U_{\alpha_k}$, then the countable family $\{V_k : k \in \mathbb{N}\}$ of τ_j–open sets covers X and satisfies the desired condition. Hence (X,τ_1,τ_2) is τ_i-Lindelöf with respect to τ_j.

(b) $\Longrightarrow$ (c):

Assume (b), and let $\mathcal{V} = \{ F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets with the property stated in (c). Suppose that $\bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$. By the hypotheses in (b), there is a countable family $\{G_k : k \in \mathbb{N}\}$ of τ_j–closed sets with empty intersection such that $\forall k$, $\exists\ \alpha_k \in \Delta$ with $G_k \supset F_{\alpha_k}$. And this contradicts the property of the family $\mathcal{V}$. Hence, $\bigcap_{\alpha\in\Delta} F_\alpha \neq \emptyset$.

(c) $\Longrightarrow$ (b):

Assume (c), and let $\{F_\alpha : \alpha \in \Delta\}$ of τ_i-closed sets which has empty intersection. Suppose that there exists no countable family of the form $\{G_k : k \in \mathbb{N}\}$ of τ_j –closed sets with empty intersection and satisfies the condition that $\forall k$, $\exists\ \alpha_k \in \Delta$ with $G_k \supset F_{\alpha_k}$ .This means that every countable family of the form $\{G_k : k \in \mathbb{N}\}$ of τ_j–closed sets which satisfies the condition $\forall k$, $\exists\ \alpha_k \in \Delta$ with $G_k \supset F_{\alpha_k}$ has nonempty intersection.

By (c), $\{F_\alpha : \alpha \in \Delta\}$ has nonempty intersection, and this contradict the assumption. □

We introduce the following definition before proving theorem (3.1.15).

3.1.14 Definition [3]:

A bitopological space (X,τ_1,τ_2) is said to be i-P-space if countable intersection of i-open sets in X is i-open. X is said P-space if it is i-P-space for each i = 1; 2.

3.1.15 Theorem:

Let (X,τ_1,τ_2) be a p-Hausdorff, τ_j-P-space bitopological space and let (X,τ_i) be a Lindelöf topological space. Then $\tau_i \subset \tau_j$.

Proof:

To prove this, it is sufficient to show that every τ_i -closed set is τ_j-closed set. Let A be τ_i–closed, then A is τ_i–Lindelöf. Let $x \notin A$. Since (X,τ_1,τ_2) is p-Hausdorff, then for each $a \in A$, there exist τ_i-open set V(a) and a τ_j-open set U(a) such that $a \in V(a)$, $x \in U(a)$, and $V(a) \cap U(a) = \emptyset$. The family $\{V(a) : a \in A\}$ forms a τ_i-open cover for A, and so by Lindelöfness of A, there is a countable subcover $\{V(a_k) : k \in \mathbb{N}\}$ of $\{V(a): a \in A\}$ for A. For each $V(a_k)$, $k \in \mathbb{N}$, there is a corresponding τ_j-open sets $U(a_k)$. Then $B=\bigcap_{k=1}^{\infty} U(a_k)$ is τ_j -open set containing x since X is τ_j-P-space. Now $B \cap V(a_k) = \emptyset$ for each $k \in \mathbb{N}$, for if this not true, then $B \cap V(a_n) \neq \emptyset$ for some $n \in \mathbb{N}$, and then $U(a_n) \cap V(a_n) \neq \emptyset$ as $B \subset U(a_k)$ for each $k \in \mathbb{N}$, and this is the contrary to the way $V(a_k)$ and $U(a_k)$ were chosen. Define $C = \bigcup_{k=1}^{\infty} V(a_k)$ which is τ_i –open, then we have $B \cap C = \emptyset$ and this implies that $B \cap A = \emptyset$. Therefore $x \in B \subset X \setminus A$ which means that A is τ_j -closed. □

3.1.16 Corollary:

Let the bitopological space (X,τ_1,τ_2) be a p-Hausdorff. Then:

(a) If the topologies τ_1 and τ_2 are Lindelöf and P-spaces, then $\tau_1 = \tau_2$.

(b) If (X, τ_1, τ_2) is τ_i-Lindelöf with respect to τ_j and τ_j -P-space, then $\tau_i \subset \tau_j$.

(c) If (X, τ_1, τ_2) is conversely Lindelöf and P-space, then $\tau_1 = \tau_2$.

(d) If (X,τ_1,τ_2) is B-Lindelöf and P-space, then $\tau_1 = \tau_2$. □

3.2 Conversely Lindelöfness of sets in bitopological spaces

3.2.1 Definition:

Let (X,τ_1,τ_2) be a bitopological space, and let $A\subset X$. We say that the set A is τ_i- Lindelöf with respect to τ_j [resp. conversely Lindelöf], if the bitopological subspace $(A, \tau_{1A}, \tau_{2A})$ is τ_{iA}-Lindelöf with respect to τ_{jA} [resp. conversely Lindelöf]; where $\tau_{1A} = \{A \cap U : U \in \tau_1\}$ and $\tau_{2A} = \{A \cap V : V \in \tau_2\}$. □

3.2.2 Theorem:

Let A be a set in a bitopological space (X, τ_1, τ_2). Then:

a) A sufficient condition for the set A to be τ_i -Lindelöf with respect to τ_j is: for every τ_i -open cover $\mathcal{V}$ of A, there is a countable τ_j -open cover $\mathcal{V}_1$ of A finer than $\mathcal{V}$.

b) If the set A is τ_j –open set, then a necessary condition for A to be τ_i-Lindelöf with respect to τ_j is: for every τ_i -open cover $\mathcal{V}$ of A, there is a countable τ_j -open cover $\mathcal{V}_1$ for A finer than $\mathcal{V}$.

Proof: (a)

Let $\mathcal{V} = \{U_\alpha \cap A : \alpha \in \Delta\}$, where $U_\alpha \in \tau_i$ for each $\alpha \in \Delta$, be a τ_i-open cover for A. Then, $\bigcup\{(U_\alpha \cap A): \alpha \in \Delta\} = A$. So, $\bigcup\{U_\alpha: \alpha \in \Delta\} \cap A = A$, and so $\bigcup\{U_\alpha: \alpha \in \Delta\} \supset A$. i.e. $\mathcal{V}' = \{ U_\alpha : \alpha \in \Delta\}$ is a τ_i-open cover for A. By the hypothesis, there is a countable τ_j -open cover for A; say $\mathcal{V}'_1 = \{W_k : k \in \mathbb{N}\}$ finer than $\mathcal{V}'$. This means that $\forall\, k \in \mathbb{N}$, there is $\alpha \in \Delta$ such that $W_k \subset U_\alpha$. This implies that $\forall\, k \in \mathbb{N}$, $\exists\, \alpha \in \Delta$ such that $(W_k \cap A) \subset (U_\alpha \cap A)$. Hence, the collection $\mathcal{V}_1 = \{W_k \cap A: k \in \mathbb{N}\}$ is the desired countable τ_{jA}-open cover for A which is finer than $\mathcal{V}$.

Proof: (b)

Let A be τ_j -open, and let the collection $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_i -open cover for A. Then $\mathcal{V}_1=\{ U_\alpha \cap A : \alpha \in \Delta\}$ is a τ_{iA}-open cover for A, so by the hypothesis, there is a countable family $\mathcal{V}_2$ of τ_{jA}-open sets finer than $\mathcal{V}_1$ that covers A, say $\mathcal{V}_2=\{W_k \cap A : k \in \mathbb{N} \}$, where $W_k \in \tau_j$ $\forall k \in \mathbb{N}$. Since A is τ_j -open then for each $k \in \mathbb{N}$, $W_k \cap A$ is τ_j -open, and so $\{W_k \cap A : k \in \mathbb{N}\}$ is the desired countable family of τ_j -open sets which is finer than $\mathcal{V}$ and covers A. □

The following example shows that the converse of theorem (3.2.2.a) is not necessarily true if A is not τ_j -open.

3.2.3 Example:

Let $X =\{a,b,c\}$, $\tau_1=\{\emptyset,\{a\},\{a,c\},\{b,c\},\{c\},X \}$, and $\tau_2=\{\emptyset,\{b\},\{a,b\},\{a\}, X\}$. Let $A=\{c\}$, and consider the τ_1-open cover $\{ \{b,c\} \}$ for A, then there is no τ_2-open cover for A finer than $\{ \{b,c\} \}$. So A does not satisfy the condition in theorem (3.2.2.a) even though $(A, \tau_{1A}, \tau_{2A})$ is τ_1-Lindelöf with respect to τ_2. □

3.2.4 Theorem:

Let A and B be τ_j-open sets, each of which is τ_i-Lindelöf with respect to τ_j, then there union $(A \cup B)$ is τ_i -Lindelöf with respect to τ_j.

Proof:

Let $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_i -open cover for A∪B, then $\mathcal{V}$ is τ_i -open cover for A and for B. By our hypothesis of A and B, and according to theorem (3.2.2.b), there are two countable τ_j -open covers for A and B, say S_1 and S_2 respectively such that each of S_1 and S_2 is finer than $\mathcal{V}$. Therefore $S_1 \cup S_2$ is a countable τ_j -open cover for A∪B, and $S_1 \cup S_2$ is finer than $\mathcal{V}$. It follows that A∪B is τ_i -Lindelöf with respect to τ_j, by theorem (3.2.2.a). □

3.2.5 Theorem:

Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of τ_j -open sets, each of which is τ_i -Lindelöf with respect to τ_j, then $\bigcup_{n=1}^{\infty} A_n$ is τ_i -Lindelöf with respect to τ_j .

Proof:

Let $\mathcal{V} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_i -open cover for $\bigcup_{n=1}^{\infty} A_n$, then $\mathcal{V}$ is τ_i -open cover for A_n, $\forall\, n \in \mathbb{N}$. By our hypothesis of A_n, $\forall\, n \in \mathbb{N}$, and according to theorem (3.2.2.b), for each A_n there is a countable τ_j -open cover S_n, such that each of S_n is finer than $\mathcal{V}$, $\forall\, n \in \mathbb{N}$. Therefore $\bigcup_{n=1}^{\infty} S_n$ is a countable τ_j -open cover for $\bigcup_{n=1}^{\infty} A_n$, and $\bigcup_{n=1}^{\infty} S_n$ is finer than $\mathcal{V}$. It follows that $\bigcup_{n=1}^{\infty} A_n$ is τ_i -Lindelöf with respect to τ_j, by theorem (3.2.2.a). □

The following example shows that the condition that A and B are τ_j –open in theorem (3.2.4) is essential.

3.2.6 Example:

Let $X = \{a,b\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, X\}$, and $\tau_2 = \{\emptyset, X\}$. The sets $\{a\}$, $\{b\}$ are τ_1-Lindelöf with respect to τ_2, but $\{a\} \cup \{b\} = X$ is not τ_1-Lindelöf with respect to τ_2. Note that $\{a\}$ and $\{b\}$ are not τ_2-open. □

3.2.7 Theorem:

Let the bitopological space (X,τ_1,τ_2) be τ_i-Lindelöf with respect to τ_j, and let the subset A of X be τ_i-closed. Then every τ_i–open cover $\mathcal{V}$ for A has a countable τ_j-open cover for A finer than $\mathcal{V}$.

Proof:

Assume that A is τ_i-closed and that (X,τ_1,τ_2) is τ_i-Lindelöf with respect to τ_j. Let $\mathcal{V} = \{W_\alpha : \alpha \in \Delta\}$ be any τ_i-open cover of A. Since A is τ_i-closed, then $X \setminus A$ is τ_i-open, and so the collection $\mathcal{V}_1 = \{W_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is a τ_i-open cover of X. By τ_i-Lindelöfness of X with respect to τ_j, there is a countable τ_j-open cover for X, say $\mathcal{V}_2$ such that $\mathcal{V}_2$ is finer than $\mathcal{V}_1$. Let the collection $\mathcal{V}_3$ be the set of all elements of $\mathcal{V}_2$ which are not subsets of $X \setminus A$. Then $\mathcal{V}_3 = \{C_k : k \in \mathbb{N}\}$ is the desired countable family of τ_j-open sets which is finer than $\mathcal{V}$ and covers A. □

The following corollary follows directly from theorem (3.2.7) and theorem (3.2.2).

3.2.8 Theorem [1]:

Let the bitopological space (X,τ_1,τ_2) be τ_i -Lindelöf with respect to τ_j [resp. conversely Lindelöf], and let the subset A of X be τ_i-closed [resp. closed]. Then A is τ_i -Lindelöf with respect to τ_j [resp. conversely Lindelöf].

3.2.9 Theorem [1]:

Every pairwise regular and conversely Lindelöf bitopological space (X,τ_1,τ_2) is p_1 -normal.

Proof:

Let A and B be closed sets with $A \cap B = \emptyset$ in X. Then A and B are both τ_1 -closed and τ_2 -closed set in X. Since (X,τ_1,τ_2) is pairwise regular, then by theorem (1.2.6), for each x in B, for the τ_1-open set X\A that contains x, there is a τ_1-open set P_x such that $x \in P_x \subseteq \tau_2\text{-cl}(P_x) \subseteq X \setminus A$, i.e. $\tau_2\text{-cl}(P_x) \cap A=\emptyset$. The collection $\{P_x : x \in B\}$ forms a τ_1-open cover for B. Since (X,τ_1,τ_2) is conversely Lindelöf, and B is τ_1-closed subset of X. So, by theorem (3.2.7), there is a countable τ_2-open cover for B and finer than $\{P_x : x \in B\}$, which we denote by $\{P'_i : i \in \mathbb{N}\}$.

Similarly, for each y in A, for the τ_2-open set X\B contains y, there is a τ_2-open set Q_y such that $y \in Q_y \subseteq \tau_1\text{-cl}(Q_y) \subseteq X \setminus B$, i.e. $\tau_1\text{-cl}(Q_y) \cap B = \emptyset$. The collection $\{Q_y : y \in A\}$ forms a τ_2-open covering of A. Since (X,τ_1,τ_2) is conversely Lindelöf, and A is τ_2-closedsubset of X. So, by theorem (3.2.7), there is a countable τ_1-open cover for A finer than $\{Q_y : y \in A\}$, which we denote by $\{Q'_i : i \in \mathbb{N}\}$.

Let $U_n = Q_n \setminus \bigcup\{\tau_2\text{-cl}(P_i) : i \leq n\}$ and $V_n = P_n \setminus \bigcup\{\tau_1\text{-cl}(Q_i) : i \leq n\}$.

Since $U_n \cap \tau_2$-cl $(P_m) = \emptyset \ \forall\ m \leq n$, then $U_n \cap P_m = \emptyset \ \forall\ m \leq n$, it follows that $U_n \cap V_m = \emptyset$ for $m \leq n$.

Similarly, $V_m \cap \tau_1$-cl$(Q_n) = \emptyset$ for each $n \leq m$, then $V_m \cap Q_n = \emptyset \ \forall\ n \leq m$. It follows that $V_m \cap U_n = \emptyset \ \forall n \leq m$. Thus $U_n \cap V_m = \emptyset$ for all m and n, and consequently $U = \bigcup\{U_n : n \in \mathbb{N}\}$ is disjoint from $V = \bigcup\{V_n : n \in \mathbb{N}\}$. Finally, τ_2-cl$(P_i) \cap A$ and τ_1-cl$(Q_i) \cap B$ are empty set for all i and hence the set U contains A and is τ_2-open set, while the set V contains B and is τ_1-open. The proof is complete. □

3.2.10 Corollary:

Let (X, τ_1, τ_2) be a bitopological space, if X is conversely compact and p-regular, then (X, τ_1, τ_2) is p_1-normal. □

3.3 Mappings on conversely and Birsan Lindelöf bitopological spaces

It is known from single topology theory that the continuous image of Lindelöf topological space is Lindelöf. In this section we study mappings on conversely Lindelöf and Birsan Lindelöf bitopological spaces.

The following corollary follows directly from single topology theory.

3.3.1 Cororllary [2]:

Let f : $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be an i-continuous and surjective function. If (X,τ_1,τ_2) is τ_i -Lindelöf, then (Y,σ_1,σ_2) is σ_i -Lindelöf. □

3.3.2 Corollary [2]:

The Lindelöf property is both topological property and bitopological property. □

3.3.3 Theorem:

Let f: $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be an i-continuous, surjective and j-open function. If (X,τ_1,τ_2) is τ_i -Lindelöf with respect to τ_j, then (Y,σ_1,σ_2) is σ_i -Lindelöf with respect to σ_j.

Proof:

Let $\{G_k : k \in \Delta\}$ be a σ_i –open cover for Y. Since f is i-continuous, then $f^{-1}(G_k) \in \tau_i$ for each $k \in \Delta$, and $X = f^{-1}(Y) = f^{-1}(\bigcup_{k\in\Delta} G_k) = \bigcup_{k\in\Delta} f^{-1}(G_k)$.

Hence $\{f^{-1}(G_k) : k \in \Delta\}$ is a τ_i –open cover for X. Since X is τ_i -Lindelöf with respect to τ_j, there exists a countable family of τ_j –open sets finer than $\{f^{-1}(G_k) : k \in \Delta\}$ and covers X, say $\{V_\alpha : \alpha \in \mathbb{N}\}$. Since f is j-open and $V_\alpha \in \tau_j$, $\forall \alpha \in \mathbb{N}$, then $f(V_\alpha) \in \sigma_j$, $\forall\, \alpha \in \mathbb{N}$.

Since f is surjective, $Y = f(X) = f(\bigcup_{\alpha\in\mathbb{N}} V_\alpha) = \bigcup_{\alpha\in\mathbb{N}} f(V_\alpha)$.

And since $\forall\, \alpha \in \mathbb{N}$, $\exists\, k \in \Delta$ such that $V_\alpha \subset f^{-1}(G_k)$, then $\forall\, \alpha \in \mathbb{N}$, $\exists\, k \in \Delta$ such that

$f(V_\alpha) \subset f f^{-1}(G_k) = G_k$.

Hence $\{f(V_\alpha): \alpha \in \mathbb{N}\}$ is a countable σ_j-open cover for Y and finer than $\{G_k: k \in \Delta\}$.

Thus, Y is σ_i-Lindelöf with respect to σ_j. □

3.3.4 Corollary:

Let f : $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a continuous, surjective and open function. If (X,τ_1,τ_2) is conversely Lindelöf, then (Y,σ_1,σ_2) is conversely Lindelöf. □

3.3.5 Theorem [2]:

Let f : $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be an i-continuous, surjective and j-open function. If (X,τ_1,τ_2) is τ_i-Lindelöf within τ_j, then (Y,σ_1,σ_2) is σ_i-Lindelöf within σ_j.

Proof:

Let $\{G_k : k \in \Delta\}$ be a σ_i –open cover for Y. Since f is i-continuous, then $f^{-1}(G_k) \in \tau_i$ $\forall$ $k \in \Delta$, and $X = f^{-1}(Y) = f^{-1}(\bigcup_{k\in\Delta} G_k)=\bigcup_{k\in\Delta} f^{-1}(G_k)$.

Hence $\{f^{-1}(G_k): k \in \Delta\}$ is a τ_i –open cover for X. Since X is τ_i -Lindelöf within τ_j, there exists a countable subfamily of τ_j –open sets of $\{f^{-1}(G_k): k \in \Delta\}$ and covers X, say $\{f^{-1}(G_{k\alpha}) : k_\alpha \in \mathbb{N}\}$. Since f is j-open and $f^{-1}(G_{k\alpha}) \in \tau_j$, $\forall$ $k_\alpha \in \mathbb{N}$, $f(f^{-1}(G_{k\alpha})) \in \sigma_j$, $\forall$ $k_\alpha \in \mathbb{N}$. Since f is surjective, since $\forall$ $k_\alpha \in \mathbb{N}$, such that $f f^{-1}(G_{k\alpha}) = G_{k\alpha}$, then $Y=f(X) =f(\bigcup_{k\alpha\in\mathbb{N}} f^{-1}(G_{k\alpha})) = \bigcup_{k\alpha\in\mathbb{N}} f(f^{-1}(G_{k\alpha})) =\bigcup_{\alpha\in\mathbb{N}} G_{k_\alpha}$. And Hence $\{G_{k\alpha} : k_\alpha \in \mathbb{N}\}$ is a countable subcover of σ_j–open sets of $\{G_k: k\in\Delta\}$ for Y. Thus, Y is σ_i -Lindelöf within σ_j.□

3.3.6 Corollary [2]:

Let f : $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a continuous, surjective and open function. If (X,τ_1,τ_2) is B-Lindelöf, then (Y,σ_1,σ_2) is B-Lindelöf. □

3.3.7 Corollary [2]:

Being conversely Lindelöf and B-Lindelöf are bitopological properties. □

3.3.8 Theorem:

Let f :$(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be an i-continuous, surjective function. If (X,τ_1,τ_2) is τ_i-Lindelöf with respect to τ_j , then (Y,σ_1,σ_2) is σ_i-Lindelöf.

Proof:

(X,τ_1,τ_2) is τ_i -Lindelöf with respect to τ_j, so (X,τ_i) is Lindelöf. By i-continuity of f, (Y,σ_i) is Lindelöf, and so (Y,σ_1 ,σ_2) is σ_i -Lindelöf. □

3.3.9 Corollary:

Let f : $(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a continuous and surjective function. If (X,τ_1,τ_2) is conversely Lindelöf, then (Y,σ_1,σ_2) is Lindelöf. □

3.3.10 Theorem:

Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be i-continuous and j-open function. And let A be τ_j-open set and τ_i-Lindelöf subset of X with respect to τ_j, then f (A) is σ_i -Lindelöf with respect to σ_j.

Proof:

The proof is similar to the proof of theorem (2.3.5). □

3.4 Product of conversely and Birsan Lindelöf bitopological spaces

Before studying productivity of conversely Lindelöf bitopological spaces, we will study some properties of P-spaces.

3.4.1 Lemma:

The bitopological space (X,τ_1,τ_2) is τ_i-P-space if and only if any countable intersection of basic τ_i-open sets is τ_i-open.

Proof:

$\Rightarrow$) It is obvios since every basic τ_i-open set is τ_i-open.

$\Leftarrow$) Let $\{U_n : n \in \mathbb{N}\}$ be any countable collection of τ_i-open sets of X. Want to prove that $\bigcap_{n\in\mathbb{N}} U_n$ is a τ_i-open set of X.

Let $x \in \bigcap_{n\in\mathbb{N}} U_n$, then $x \in U_n$ $\forall$ $n\in\mathbb{N}$. Since $x \in U_n \in \tau_i$ $\forall$ $n\in\mathbb{N}$, there exists a basic τ_i-open set B_n such that $x \in B_n \subset U_n$, $\forall$ $n \in \mathbb{N}$. So $x \in \bigcap_{n\in\mathbb{N}} B_n$ and $\bigcap_{n\in\mathbb{N}} B_n$ is a τ_i-open set in X since it is the intersection of a countable collection of basic τ_i-open sets. Thus $\bigcap_{n\in\mathbb{N}} U_n$ is a union of τ_i-open sets. Hence $\bigcap_{n\in\mathbb{N}} U_n$ is a τ_i-open set in X.

So X is τ_i-P-space. □

3.4.2 Lemma [3]:

Let (X,τ_1,τ_2) be τ_i-P-space and (Y,σ_1,σ_2) be σ_i-P-space. Then $(X\times Y, \rho_1,\rho_2)$ is ρ_i-P-space where ρ_i is the product topology.

Proof:

By Lemma (3.4.1), we will restrict our attention to the collection of basic ρ_i–open sets in X×Y.

Let $\{V_n \times W_n : n \in \mathbb{N}\}$ be a countable collection of basic ρ_i –open sets in X×Y. Where V_n and W_n are τ_i–open sets and σ_i-open sets of X and Y respectively, $\forall\, n \in \mathbb{N}$.

Now, $\bigcap_{n\in\mathbb{N}}(V_n \times W_n) = (\bigcap_{n\in\mathbb{N}} V_n) \times (\bigcap_{n\in\mathbb{N}} W_n)$ is a ρ_i–open set, since X is τ_i-P-space and Y is σ_i-P-spaces. So X×Y is ρ_i-P-space. □

The following corollary follows by mathematical induction.

3.4.3 Corollary [3]:

Let $\{ (X_k, \tau_1^k, \tau_2^k) : k = 1,2,\ldots,n\}$ be a collection of τ_i^k-P-spaces. Then $(\prod_{k=1}^{n} X_k, \rho_1, \rho_2)$ is ρ_i-P-space, where ρ_i is the product topology. □

Adem Kilicman and Zabidin Salleh claim, in proposition (3.2) in [3], that the product of arbitrary family of P-spaces is P-space, and gave a "proof" for that. Despite of this we give here a counter example to show that this result is not true.

3.4.4 Example:

Let $A_k = \{ k, k+\sqrt{2} \}$, $k \in \mathbb{N}$, and τ_1^k, τ_2^k be the discrete topology for A_k. $(A_k, \tau_1^k, \tau_2^k)$ is a P-space, $\forall k \in \mathbb{N}$.

Let $A = \prod_{k\in\mathbb{N}} A_k$, and ρ_i be the product topology. Take $B_k = \pi_k^{-1}(\{k\}) \in \rho_i$, $\forall k \in \mathbb{N}$.

$\bigcap_{k\in\mathbb{N}} B_k = \prod_{k\in\mathbb{N}}\{k\} \notin \rho_i$, even though $B_k \in \rho_i$, $\forall k\in\mathbb{N}$. Hence (A, ρ_1, ρ_2) is not P-space. □

3.4.5 Definition [3]:

A bitopological space X is said to be (i, j)-P-space if every countable intersection of i-open sets in X is j-open. X is said to be B-P-space if it is (1, 2)-P-space and (2,1)-P-space. Note that if (X,τ_1,τ_2) is B-P-space, then $\tau_1 = \tau_2$.

The proof of the following Lemma is similar to the the proof of lemma (3.4.1).

3.4.6 Lemma:

The bitopological space (X,τ_1,τ_2) is (i,j)- P-space if and only if any countable intersection of τ_i-basic open sets is τ_j-open. □

3.4.7 Lemma [3]:

Let (X,τ_1,τ_2) be a (τ_i,τ_j) –P-space and (Y,σ_1,σ_2) be a (σ_i,σ_j)–P-space. Then $(X\times Y, \rho_1, \rho_2)$ is (ρ_i,ρ_j)-P-space, where ρ_i is the product topology , i=1,2 .

Proof: Similar to the proof of lemma (3.4.2). □

3.4.8 Corollary [3]:

Let $\{ (X_k,\tau_1^k,\tau_2^k): k=1,2,\ldots,n\}$ be a collection of (τ_i^k,τ_j^k) – P-spaces. Then $(\prod_{k=1}^{n} X_k, \rho_1, \rho_2)$ is (ρ_i,ρ_j)–P-space, where ρ_i is the product topology .

Proof: Follows by induction on k. □

The previous corollary is not true for arbitrary collection of bitopological spaces. Take example (3.4.4).

3.4.9 Theorem:

A bitopological space (X, τ_1, τ_2) is τ_i –Lindelöf with respect to τ_j if and only if every cover $\mathcal{F}$ of basic τ_i–open sets for X has a countable family of τ_j–open sets finer than $\mathcal{F}$ and covers X.

Proof:

⇒) It is obvious, as every basic τ_i –open set is τ_i –open.

⇐) Let $\{ U_\gamma : \gamma \in \Delta\}$ be a τ_i –open cover for X, and let $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda \}$ be a τ_i –base, then each U_γ is a union of members of $\mathcal{B}$.

Let $\mathcal{B}_1 =\{ B_t : t \in \Lambda \text{ and } B_t \subset U_\gamma \text{ for some } \alpha \in \Delta\} = \{ B_t : t \in \Lambda_1\}$, then $\Lambda_1 \subset \Lambda$.

Then $\bigcup_{t\in\Lambda 1} B_t = \bigcup_{\alpha\in\Delta} U_\alpha = X$. So $\{B_t : t \in \Lambda_1 \}$ is a τ_i –open cover for X consisting of elements from the base of τ_i. By the assumption, there exists a countable family $\mathfrak{S}$ of τ_j –open sets finer than $\{B_t : t \in V_1\}$ and covers X, say $\mathfrak{S} = \{ W_n : n \in \mathbb{N} \}$. Then $\forall\, n \in \mathbb{N}$, $\exists\, t \in \Lambda$ such that $W_n \subset B_t$. But $B_t \subset U_\gamma$ for some $\gamma \in \Delta$, so $W_n \subset U_\gamma$ for some $\gamma \in \Delta$. Then $\{W_n: n \in \mathbb{N}\}$ is a countable family of τ_j –open sets finer than $\{U_\gamma: \gamma \in \Delta\}$ and covers X.

Hence (X,τ_1, τ_2) is τ_i –Lindelöf with respect to τ_j. □

3.4.10 Theorem:

Let (X,τ_1,τ_2) be a τ_i –Lindelöf with respect to τ_j , and (Y,σ_1,σ_2) is σ_i -compact with respect to σ_j . Then $(X\times Y, \rho_1, \rho_2)$ is ρ_i-Lindelöf with respect to ρ_j , where ρ_i is the product topology.

Proof:

We will restrict our attention to the ρ_i –open cover $\{ V_\alpha \times W_\alpha : \alpha \in \Delta \}$ consisting of basic ρ_i – open sets by theorem (3.4.9).

Fix $x \in X$. $\forall\, y \in Y$, $\exists\, x,\alpha_y \in \Delta$ such that $(x,y) \in V_{x,\alpha_y} \times W_{x,\alpha_y}$, where $V_{x,\alpha_y} \in \tau_i$ and $W_{x,\alpha_y} \in \sigma_i$.

The family $\{W_{x,\alpha_y} : y \in Y\}$ is σ_i –open cover of Y, and since Y is σ_i -compact with respect to σ_j , there exists a finite family of σ_j –open sets covers Y and finer than $\{W_{x,\alpha_y} : y \in Y\}$, say $\{ W'_{x,\alpha_{y1}}, W'_{x,\alpha_{y2}}, \ldots\ldots, W'_{x,\alpha_{yn_x}} \}$.

Let $T_x = \bigcap_{k=1}^{n} V_{x,\alpha_{yk}}$ .Then $T_x \in \tau_i$, since each $V_{x,\alpha_{yk}} \in \tau_i$ for each $k = 1, 2, \ldots, n$.

$\{T_x : x \in X\}$ is a τ_i –open cover for X, and since X is τ_i –Lindelöf with respect to τ_j, then there exists a countable family of τ_j, say $\{T'_{x_m} : m \in \mathbb{N}\}$ finer than $\{T_x : x \in X\}$ and covers X.

Then $\{ T'_{x_m} \times W'_{x_m,\alpha_{yk}} : k = 1,\ldots,n_{x_m} , m \in \mathbb{N} \}$ is a countable ρ_j –open cover for $X\times Y$ and finer than $\{V_\alpha \times W_\alpha : \alpha \in \Delta\}$.

Hence, $X\times Y$ is ρ_i -Lindelöf with respect to ρ_j . □

3.4.11 Corollary:

Let (X,τ_1,τ_2) be conversely Lindelöf, and (Y,σ_1,σ_2) is conversely compact. Then $(X\times Y, \rho_1, \rho_2)$ is conversely Lindelöf, where ρ_i is the product topology. □

3.4.12 Corollary:

Let $\{ (X_\alpha ,\tau_1^\alpha ,\tau_2^\alpha) : \alpha\in\Delta\}$ be a collection of τ_i^α –compact with respect to τ_j^α (conversely compact),but for some $\beta \in\Delta$, $(X_\beta ,\tau_1^\beta ,\tau_2^\beta)$ is τ_i^β –Lindelöf with respect to τ_j^β (conversely Lindelöf). Then $(\prod_{\alpha\in\Delta} X_\alpha , \rho_1 , \rho_2)$ is ρ_i-Lindelöf with respect to ρ_j (conversely Lindelöf), where ρ_i is the product topology. □

3.4.13 Example [3]:

Let $\mathcal{B}_1 = \{\mathbb{R}\} \cup \{\{x\} : x \in \mathbb{R} \setminus \{0\} \}$ and $\mathcal{B}_2 = \{\mathbb{R}\} \cup \{\{x\} : x \in \mathbb{R} \setminus \{1\} \}$. Let τ_1 and τ_2 be the topologies on $\mathbb{R}$ generated by $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively as bases.

Then $(\mathbb{R},\tau_1,\tau_2)$ is B-Lindelöf and conversely Lindelöf , for any τ_i-open cover of $\mathbb{R}$ must contain $\mathbb{R}$ as a member. We see that $(\mathbb{R} \times \mathbb{R}, \tau_1\times \tau_1 , \tau_2\times \tau_2)$ is B–Lindelöf and conversely Lindelöf, since any $(\tau_i\times \tau_i)$ – open cover of $\mathbb{R} \times \mathbb{R}$ must contain $\mathbb{R} \times \mathbb{R}$ as a member.

Actually $(\mathbb{R} \times \mathbb{R}, \tau_1\times \tau_1 , \tau_2\times \tau_2)$ is B–compact and so is conversely compact. □

The product of two τ_1 – Lindelöf with respect to τ_2 spaces is not necessarily $\tau_1\times \tau_1$- Lindelöf with respect to $\tau_2\times \tau_2$.

3.4.14 Example:

Let τ_s denote the Sorgenfrey topology on $\mathbb{R}$, and τ_d denote the discrete topology on $\mathbb{R}$, then the bitopological space $(\mathbb{R}, \tau_s, \tau_d)$ is τ_s-Lindelöf with respect to τ_d (τ_s-Lindelöf within τ_d) . However $(\mathbb{R}\times\mathbb{R}, \tau_s\times \tau_s, \tau_d\times \tau_d)$ is not $\tau_s\times \tau_s$ –Lindelöf with respect to $\tau_d\times \tau_d$ (and so not $\tau_s\times \tau_s$–Lindelöf within $\tau_d\times \tau_d$), since $(\mathbb{R}\times\mathbb{R}, \tau_s\times \tau_s)$ is not Lindelöf, as the closed subset $\mathcal{L} = \{ (x,-x) : x \in \mathbb{R} \}$ which is uncountable set with the discrete topology is not Lindelöf. □

3.4.15 Theorem:

Let (X,τ_1,τ_2) be a τ_i–Lindelöf with respect to τ_j and τ_i-P-space, and (Y,σ_1,σ_2) be σ_i -Lindelöf with respect to σ_j. Then $(X\times Y, \rho_1, \rho_2)$ is ρ_i-Lindelöf with respect to ρ_j, where ρ_i is the product topology.

Proof:

We will restrict our attention to the ρ_i –open cover $\{V_\alpha \times W_\alpha : \alpha\in\Delta\}$ consisting of basic ρ_i–open sets, by theorem (3.4.9) .

Fix $x \in X$. $\forall\, y \in Y$, $\exists\, x,\alpha_y \in \Delta$ such that $(x,y) \in V_{x,\alpha_y} \times W_{x,\alpha_y}$, where $V_{x,\alpha_y} \in \tau_i$ and $W_{x,\alpha_y} \in \sigma_i$.

So the family $\{ W_{x,\alpha_y} : y \in Y\}$ is σ_i –open cover for Y, and since Y is σ_i -Lindelöf with respect to σ_j , then there exists a countable family of σ_j –open sets cover Y and finer than $\{W_{x,\alpha_y} : y \in Y\}$, say $\{W'_{x,\alpha_{yn}} : n \in \mathbb{N}\}$.

Let $H_x = \bigcap_{n=1}^{\infty} V_{x,\alpha_{y_n}}$ .Then $H_x \in \tau_i$, since each $V_{x,\alpha_{yk}} \in \tau_i$ and X is τ_i –P-space .

$\{H_x : x \in X\}$ is a τ_i –open cover for X, and since X is τ_i –Lindelöf with respect to τ_j, this implies that there exists a countable family of τ_j –open sets, say $\{ H'_{x_m} : m \in \mathbb{N} \}$ finer than $\{ H_x : x \in X\}$ and covers X .

Then $\{ H'_{x_m} \times W'_{x_m, \alpha y_n} : n, m \in \mathbb{N}\}$ is a countable ρ_j –open cover for X×Y and finer than $\{ V_\alpha \times W_\alpha : \alpha \in \Delta \}$.

Hence, X×Y is ρ_i-Lindelöf with respect to ρ_j . □

Example (3.4.14) shows that being τ_i –P-space is essential as $(\mathbb{R}, \tau_s, \tau_d)$ is τ_s-Lindelöf with respect to τ_d, but $(\mathbb{R} \times \mathbb{R}, \tau_s\times \tau_s, \tau_d\times \tau_d)$ is not $\tau_s\times \tau_s$ –Lindelöf with respect to $\tau_d\times \tau_d$. Note that $(\mathbb{R}, \tau_s, \tau_d)$ is not τ_s –P-space, as $\bigcap_{n\in\mathbb{N}} [2 - \frac{1}{n}, 2 + \frac{1}{n}) = \{2\} \notin \tau_s$ even though $[2- \frac{1}{n}, 2+ \frac{1}{n}) \in \tau_s \ \forall n \in \mathbb{N}$.

3.4.16 Corollary:

Let (X,τ_1,τ_2) be a conversely Lindelöf and P-space, and (Y,σ_1,σ_2) is conversely Lindelöf. Then $(X\times Y, \rho_1, \rho_2)$ is conversely Lindelöf, where ρ_i is the product topology. □

3.4.17 Corollary:

Let (X, τ_1, τ_2) be a conversely Lindelöf and τ_i -P-space, and (Y,σ_1,σ_2) is conversely Lindelöf and σ_j-P-space. Then $(X\times Y, \rho_1, \rho_2)$ is conversely Lindelöf, where ρ_i is the product topology. □

By mathematical induction the following corollary follows.

3.3.18 Corollary:

Let { $(X_k, \tau_1^k, \tau_2^k)$: k =1,2,…,n} be a collection of τ_i^k –Lindelöf with respect to τ_j^k (conversely Lindelöf) and τ_i^k –P-space, but for some $\beta \in \{1,..,n\}$, $(X_\beta, \tau_1^\beta, \tau_2^\beta)$ is τ_i^β –Lindelöf with respect to τ_j^β (conversely Lindelöf). Then $(\prod_{k=1}^{n} X_k, \rho_1, \rho_2)$ is ρ_i -Lindelöf with respect to ρ_j (conversely Lindelöf), where ρ_i is the product topology. □

The proof of the following theoerem is similar to the proof of theorem (3.4.15).

3.4.19 Theorem:

Let (X, τ_1, τ_2) be a τ_j –Lindelöf with respect to τ_i and (τ_i, τ_j) -P-space, and (Y, σ_1, σ_2) is σ_i -Lindelöf with respect to σ_j . Then $(X\times Y, \rho_1, \rho_2)$ is ρ_i-Lindelöf with respect to ρ_j , where ρ_i is the product topology. □

3.4.20 Corollary:

Let (X, τ_1, τ_2) be a conversely Lindelöf and B-P-space, and (Y, σ_1, σ_2) is conversely Lindelöf. Then $(X\times Y, \rho_1, \rho_2)$ is conversely Lindelöf, where ρ_i is the product topology. □

3.4.21 Corollary:

Let { $(X_k, \tau_1^k, \tau_2^k)$: k =1,2,…,n} be a collection of τ_j^k – Lindelöf with respect to τ_i^k and (τ_i^k, τ_j^k) –P-space ,but for some $\beta \in \{1,..,n\}$, $(X_\beta, \tau_1^\beta, \tau_2^\beta)$ is τ_i^β – Lindelöf with respect to τ_j^β . Then $(\prod_{k=1}^{n} X_k, \rho_1, \rho_2)$ is ρ_i -Lindelöf with respect to ρ_j, where ρ_i is the product topology. □

3.4.22 Theorem:

Let (X, τ_1, τ_2) be a τ_i –Lindelöf with respect to τ_j, and (Y, σ_1, σ_2) is σ_i–P-space. Then the projection $\pi_y : (X \times Y, \rho_1, \rho_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is i –closed, where ρ_i is the product topology.

Proof:

Let U be a ρ_i –closed set in $X \times Y$, and let $y_o \notin \pi_y(U)$. Clearly $X \times \{ y_o\} \cap U = \emptyset$. So $\forall\, x \in X$, the point $(x, y_o) \notin U$ has a ρ_i –basic neighborhood $V_x \times W_{x,yo}$ disjoint from U, where V_x is τ_i –open set in X containing x, and $W_{x,yo}$ is σ_i –open set in Y containing y_o. Now $\{V_x \times W_{x,yo} : x \in X\}$ forms a ρ_i –open cover of $X \times \{ y_o\}$ by ρ_i –open sets in $X \times Y$, $\{V_x : x \in X\}$ is a τ_i –open cover for X, and since X is τ_i –Lindelöf with respect to τ_j, then there exist a countable family of τ_j –open sets $\{V'_{x_k} : k \in \mathbb{N} \}$ finer than $\{V_x : x \in X \}$ and covers X.

Let $W = \bigcap_{n\in\mathbb{N}} W_{x_n, y_o}$. Since Y is σ_i-P-space, W is σ_i –open set in Y and a σ_i –open neighborhood of y_o. We need to prove that $W \cap \pi_y(U) = \emptyset$.

Suppose that $W \cap \pi_y(U) \neq \emptyset$, then there exist $y_1 \in (W \cap \pi_y(U))$. $y_1 \in W$ then $y_1 \in W_{xn,yo}$ $\forall\, n \in \mathbb{N}$. $y_1 \in \pi_y(U)$ means for some $x_o \in X$, $(x_o, y_1) \in U$. Since $\{V'_{x_k} : k \in \mathbb{N} \}$ is a cover for X, then $x_o \in V'_{x_k}$ for some $k \in \mathbb{N}$, which implies $(x_o, y_1) \in (V'_{x_k} \times W) \subset (V_{x_k} \times W_{xn,yo})$ for some $n \in \mathbb{N}$, and this is a contradiction since $(V_{x_n} \times W_{x_n,y_o}) \cap U = \emptyset$, $\forall\, n \in \mathbb{N}$. Hence $W \cap \pi_y(U) = \emptyset$. So, W is σ_i-open neighborhood of y_o disjoint from $\pi_y(U)$. So $\pi_y(U)$ is σ_i –closed set in Y. Hence the projection $\pi_y : (X \times Y, \rho_1, \rho_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is i–closed. □

3.5 Tychonoff Theorem for Conversely Lindelöf Bitopological Spaces

3.5.1 Definition:

The family $\mathcal{F}$ of τ_i-open sets is called countably τ_i -inadequate with respect to τ_j in X if no countable family of τ_j -open sets which is finer than $\mathcal{F}$ covers X.

We can easily see that the bitopological space (X,τ_1,τ_2) is τ_i -Lindelöf with respect to τ_j if and only if each countably τ_i -inadequate family with respect to τ_j in X is τ_i -inadequate.

3.5.2 Lemma:

Let (X,τ_1,τ_2) be a bitopological space. If $\mathcal{D}$ is a maximal countably τ_i-inadequate family with respect to τ_j, and if some member of $\mathcal{D}$ contains $\bigcap_{i=1}^{n} G_i$, where each G_i is τ_i -open , then $G_k \in \mathcal{D}$ for some k in $\{1,2,\ldots,n\}$.

Proof:

First suppose that n = 2. Suppose that $G_1 \notin \mathcal{D}$ and $G_2 \notin \mathcal{D}$. Then by maximality of $\mathcal{D}$, $\mathcal{D}\cup\{G_1\}$ and $\mathcal{D}\cup\{G_2\}$ are not countably τ_i -inadequate with respect to τ_j. Then for $\mathcal{D}\cup\{G_1\}$, $\exists$ A, A_1 , A_2 ,…., A_k ,… , where A, A_k are τ_j -open sets $\forall k\in\mathbb{N}$, $A\subset G_1$, and $A_k \subset A'_k$ for some $A'_k \in \mathcal{D}$, $\forall$ $k\in\mathbb{N}$, such that $A \cup (\bigcup_{k\in\mathbb{N}} A_k) = X$.

And for $\mathcal{D}\cup\{G_2\}$, $\exists$ τ_j -open sets B, B_1, B_2,…, B_n,…, such that $B \cup (\bigcup_{n\in\mathbb{N}} B_n) = X$, where $B \subset G_2$ and $B_n \subset B'_n$ for some $B'_n \in \mathcal{D}$, $\forall$ $n\in\mathbb{N}$.

Claim: $(A\cap B) \cup (\bigcup_{k\in\mathbb{N}} A_k) \cup (\bigcup_{n\in\mathbb{N}} B_n) = X$.

It is clear that: $(A\cap B) \cup (\bigcup_{k\in\mathbb{N}} A_k) \cup (\bigcup_{n\in\mathbb{N}} B_n) \subset X$.

Now, let $x\in X$. If either $x \in A_k$, for some $k \in \mathbb{N}$, or $x \in B_n$, for some $n \in \mathbb{N}$, then

$x \in (A \cap B) \cup (\bigcup_{k\in\mathbb{N}} A_k) \cup (\bigcup_{n\in\mathbb{N}} B_n)$. If not, then $x \in A$ and $x \in B$ and so $x \in (A\cap B)$.

So, $X \subset (A\cap B) \cup (\bigcup_{k\in\mathbb{N}} A_k) \cup (\bigcup_{n\in\mathbb{N}} B_n)$. This completes the proof of the claim.

Since $A \subset G_1$ and $B \subset G_2$, then $A\cap B \subset G_1\cap G_2$. But $G_1\cap G_2$ is contained in some element of $\mathcal{D}$, so $(A\cap B) \cup \{A_k : k\in\mathbb{N} \} \cup \{B_n : n\in\mathbb{N}\}$ is a countable family of τ_j-open sets that is finer than $\mathcal{D}$ and covers X, this contradicts that $\mathcal{D}$ is countably τ_i -inadequate with respect to τ_j.

So $G_1 \in \mathcal{D}$ or $G_2 \in \mathcal{D}$. So the result holds for n = 2.

The result for arbitrary $n\in\mathbb{N}$ follows by mathematical induction. □

3.5.3 Theorem (Alexander):

If (X,τ_1,τ_2) be a bitopological space in which every countably τ_i -inadequate family with respect to τ_j , say $\mathcal{B}$, there is a maximal countably τ_i -inadequate family with respect to τ_j in (X,τ_1,τ_2), say $\mathcal{D}$, and that $\mathcal{B} \subset \mathcal{D}$, and if $\mathcal{S}$ is a subbase of the topology τ_i such that, for each τ_i -open cover $\mathcal{V}$ for X by members of $\mathcal{S}$, there is a countable family of τ_j -open sets finer than $\mathcal{V}$ that covers X, then (X,τ_1,τ_2) is τ_i -Lindelöf with respect to τ_j.

Proof:

Let $\mathcal{B}$ be a cuontably τ_i -inadequate family with respect to τ_j, then there is a maximal countably τ_i -inadequate family with respect to τ_j, say $\mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$. If we prove that $\mathcal{D}$ is τ_i -inadequte, then $\mathcal{B}$ is also τ_i -inadequate.

$\mathcal{S}$ is a subbase of τ_i, and since $\mathcal{D}$ is a family of τ_i -open sets, then $(\mathcal{S}\cap\mathcal{D})$ is a family of τ_i -open sets. Let $A \in \mathcal{D}$, then $A \in \tau_i$, and $\mathcal{S}$ is a subbase of τ_i , then there is a finite intersection of elements of $\mathcal{S}$ which is contained in A, then one of these elements of $\mathcal{S}$ is an element of $\mathcal{D}$.

So $(\mathcal{S}\cap\mathcal{D})$ is a nonempty family of τ_i -open sets contained in $\mathcal{D}$, since $(\mathcal{S}\cap\mathcal{D}) \subset \mathcal{D}$, then $(\mathcal{S}\cap\mathcal{D})$ is a countably τ_i -inadequate family with respect to τ_j. Which means that there is no countable family of τ_j -open sets finer than $(\mathcal{S}\cap\mathcal{D})$ and covers X. And since $(\mathcal{S}\cap\mathcal{D}) \subset \mathcal{S}$. So $(\mathcal{S}\cap \mathcal{D})$ is τ_i -open family of $\mathcal{S}$ which does not cover X. Hence, $(\mathcal{S} \cap\mathcal{D})$ is τ_i -inadequate.

Want to prove that $\bigcup\{C : C \in \mathcal{D}\} = \bigcup\{C : C \in (\mathcal{S}\cap\mathcal{D})\}$.

Since $(\mathcal{S}\cap\mathcal{D}) \subset \mathcal{D}$, so $\bigcup\{C: C \in (\mathcal{S}\cap\mathcal{D})\} \subset \bigcup\{C : C \in \mathcal{D}\}$. …………(1)

Let $x \in\bigcup\{C : C \in \mathcal{D}\}$; then $\exists$ $A \in \mathcal{D}$ such that $x \in A$. Since A is τ_i -open, then there is a finite intersection of elements of $\mathcal{S}$ containing x and contained in A. By maximality of $\mathcal{D}$, one of these elements of $\mathcal{S}$ is an element of $\mathcal{D}$, so

$x \in \bigcup\{C: C \in (\mathcal{S}\cap\mathcal{D})\}$ …………(2)

Hence, $\bigcup\{C: C \in \mathcal{D}\} = \bigcup\{C: C \in (\mathcal{S}\cap\mathcal{D})\}$, from (1) and (2).

So, $\mathcal{D}$ is τ_i -inadequate, and so $\mathcal{B}$ is τ_i -inadequate. Therefore each countably τ_i -inadequate family with respect to τ_j is τ_i -inadequate. So X is τ_i -Lindelöf with respect to τ_j . □

3.5.4 Theorem (Tychonoff):

Let the bitopological space (X,τ,τ') be the product bitopological space of the family of bitopological spaces $\{(X_i,\tau_i,\tau_i'): i\in I\}$. Then

i.) If (X,τ,τ') is τ -Lindelöf with respect to τ'(conversely Lindelöf), then each factor space (X_i,τ_i,τ_i') is τ_i -Lindelöf with respect to τ_i' (conversely Lindelöf).

ii.) If for every countably τ_i -inadequate family with respect to τ_j , say $\mathcal{B}$, in the product bitopological space (X,τ,τ'), there is a maximal countably τ_i -inadequate family with respect to τ_j in (X,τ,τ'), say $\mathcal{D}$, and $\mathcal{B} \subset \mathcal{D}$, then the converse of (i) is true. (X,τ,τ') is τ-Lindelöf with respect to τ'(conversely Lindelöf), if for every i∈ I, the bitopological space (X_i, τ_i, τ_i') is τ_i -Lindelöf with respect to τ_i'(conversely Lindelöf).

Proof:

(i) The natural projections are continuous, surjective and open, then each component (X_i,τ_i,τ_i') is τ_i -Lindelöf with respect to τ_j'(conversely Lindelöf).

(ii) Let $\mathcal{S}$ ={ π_i^{-1}(U_i) : U_i∈τ_i , i∈I }, where π_i is the natural projection into the i-th coordinate space X_i , then $\mathcal{S}$ is a subbase for the topology τ. In view of Theorem (3.5.3), the product bitopological space (X,τ,τ') will be τ-Lindelöf with respect to τ' if each subfamily $\mathcal{A}$ of $\mathcal{S}$ which is countably τ-inadequate with respect to τ' in (X,τ,τ') is τ-inadequate. For each index i∈I, Let $\mathcal{B}_i$ be the family of all sets $U_i \in \tau_i$ such that π_i^{-1}(U_i) ∈ $\mathcal{A}$. Then $\mathcal{B}_i$ is countably τ_i -inadequate with respect to τ'_i in (X_i, τ_i, τ_i'). Since (X_i, τ_i, τ_i') is τ_i -Lindelöf with respect to τ_i', then $\mathcal{B}_i$ is τ_i -inadequate in (X_i, τ_i, τ_i'). So, there is $x_i \in X_i \setminus U_i$ for each $U_i \in \mathcal{B}_i$. Consider the point x ∈ X whose i-th coordinate is x_i , then x belongs to no member of $\mathcal{A}$, and consequently, $\mathcal{A}$ is τ-inadequate in (X,τ,τ'). Hence the product bitopological space (X, τ, τ') is τ-Lindelöf with respect to τ'. □

3.5.5 Example:

In example (3.4.13), (ℝ ,τ_s,τ_d) is τ_s-Lindelöf with respect to τ_d . However (ℝ × ℝ, $\tau_s \times \tau_s$,$\tau_d \times \tau_d$) is not $\tau_s \times \tau_s$ –Lindelöf with respect to $\tau_d \times \tau_d$.

By theorem (3.5.4), there exists a countably $\tau_s \times \tau_s$ -inadequate family with respect to $\tau_d \times \tau_d$, say $\mathcal{B}$, which has no maximal countably $\tau_s \times \tau_s$-inadequate family with respect to $\tau_d \times \tau_d$ in $(\mathbb{R} \times \mathbb{R}, \tau_s \times \tau_s, \tau_d \times \tau_d)$, say $\mathcal{D}$, and that $\mathcal{B} \subset \mathcal{D}$. □

3.6 Conversely compact and conversely Lindelöf Subsets of $(\mathbb{R}, \boldsymbol{\ell}, \boldsymbol{r})$

In this section, compactness and Lindelöfness of subsets in the bitopological space $(\mathbb{R},\ell,r)$ are studied.

We note that $(\mathbb{R}, \boldsymbol{\ell}, \boldsymbol{r})$ is neither $\boldsymbol{\ell}$-compact(Lindelöf) with respect to $\boldsymbol{r}$ nor $\boldsymbol{r}$-compact (Lindelöf) with respect to $\boldsymbol{\ell}$.

3.6.1 Theorem [4]:

A nonempty subset A of $(\mathbb{R}, \boldsymbol{\ell}, \boldsymbol{r})$ is $\boldsymbol{\ell}$-compact with respect to $\boldsymbol{r}$ if and only if A is bounded above and contains its supremum.

Proof:

$\Longrightarrow$) Suppose that A is not bounded above, we can find $\{x_n : n \in \mathbb{N}\} \subset A$ such that $x_1 < x_2 < \ldots < x_n < \ldots$, and $n < x_n$,$\forall\, n \in \mathbb{N}$.

Then there exists $\{\alpha_n : n \in \mathbb{N}\}$ such that $x_1 < \alpha_1 < x_2 < \alpha_2 < \ldots < x_n < \alpha_n \ldots$

$\{ (-\infty, \alpha_n) : n \in \mathbb{N} \}$ is an $\boldsymbol{\ell}$-open cover of $\mathbb{R}$, so $\mathcal{V} = \{ (-\infty, \alpha_n) \cap A : n \in \mathbb{N} \}$ is an $\boldsymbol{\ell}_A$-open cover for A, and the only $\boldsymbol{r}_A$-open set that is contained in any element of $\mathcal{V}$ is $\emptyset$. Thus $\mathcal{V}$ does not have a finite $\boldsymbol{r}_A$-open cover for A finer than $\mathcal{V}$.

Hence, A is not ℓ-compact with respect to r, and this is a contradiction. So A is bounded above, and it has a supremum, say t.

Suppose that t ∉ A, then ∀n∈ ℕ there exist x_n ∈ A such that $t-\frac{1}{n} < x_n < t$.

$\mathcal{V} = \{(-\infty, t-\frac{1}{n}) \cap A : n\in\mathbb{N}\}$ is an ℓ_A-open cover for A. If U ∈ r, U ∩ A ≠ ∅, then t ∈ U.

$U \cap A \not\subset (-\infty, t-\frac{1}{n}) \cap A$, ∀ n∈ℕ. So A is not ℓ-compact with respect to r, and this is a contradiction.

⟸) Suppose that A is bounded above and contains its supremum, say t.

Let $\mathcal{V}$ = { (-∞,α) ∩ A : α ∈ Δ } be any ℓ_A–open cover for A. t ∈ (-∞,α) for some α ∈ Δ, then (-∞,α) ∩ A = A ∈ r_A .So {A} is the r_A–open cover for A which is finer than $\mathcal{V}$.

Hence A is ℓ-compact with respect to r. □

The following theorems are proved similarly.

3.6.2 Theorem:

A nonempty subset A of (ℝ , ℓ, r) is ℓ-Lindelöf with respect to r if and only if A is bounded above and contains its supremum. □

3.6.3 Theorem [4]:

A nonempty subset A of (ℝ, ℓ, r) is r-compact (Lindelöf) with respect to ℓ if and only if A is bounded below and contains its infimum. □

3.6.4 Corollary:

For arbitrary nonempty subset A of (ℝ , $\boldsymbol{\ell}$, $\boldsymbol{r}$), the following are equivalent:

i) A is bounded and contains its infimum and its supremum.

ii) A is conversely compact.

iii) A is conversely Lindelöf. □

3.7 Conversely compact and conversely Lindelöf Subsets of (ℝ , $\boldsymbol{\ell}$, $\boldsymbol{\mathcal{S}}$)

Conversely compact and conversely Lindelöf Subsets of (ℝ , $\boldsymbol{\ell}$, $\boldsymbol{\mathcal{S}}$) are studied, where ℝ is the set of real numbers, $\boldsymbol{\ell}$ is the left ray topology, $\mathcal{S}$ is the standard topology.

It is clear that $\boldsymbol{\ell} \subset \mathcal{S}$, and since $\boldsymbol{\ell}$ is Lindelöf, then ℝ is $\boldsymbol{\ell}$ Lindelöf with respect to $\boldsymbol{\mathcal{S}}$.

Also, every subset of ℝ is $\boldsymbol{\ell}$-Lindelöf, and then every subset of ℝ is $\boldsymbol{\ell}$ Lindelöf with respect to $\boldsymbol{\mathcal{S}}$. But not every subset of ℝ is $\boldsymbol{\mathcal{S}}$ Lindelöf with respect to $\boldsymbol{\ell}$. To show this, take any subset U of ℝ and suppose that x and y are any two distinct points of U, such that x < y. Let $\mathcal{V}$= {(x,∞)∩U, (-∞,y)∩U} be an $\boldsymbol{\mathcal{S}}_U$ –open cover for U then there is no $\boldsymbol{\ell}_U$ open set finer than $\mathcal{V}$ contains y. Hence U is not $\boldsymbol{\mathcal{S}}$ Lindelöf with respect to $\boldsymbol{\ell}$, and therefore not $\boldsymbol{\mathcal{S}}$ compact with respect to $\boldsymbol{\ell}$.

3.7.1 Theorem:

Every nonempty subset A of ℝ is $\boldsymbol{\ell}$-compact with respect to $\boldsymbol{\mathcal{S}}$ if and only if it is bounded above and contains its supremum.

Proof:

$\Longrightarrow$) Suppose that A is not bounded above, we can find $\{x_n: n\in\mathbb{N}\} \subset A$ such that $x_1 < x_2 < \ldots < x_n < \ldots$, and $n < x_n$,$\forall$ $n\in\mathbb{N}$.

Then there exists $\{\alpha_n : n\in\mathbb{N}\}$ such that $x_1 < \alpha_1 < x_2 < \alpha_2 < \ldots < x_n < \alpha_n \ldots$

$\{ (-\infty,\alpha_n): n\in\mathbb{N} \}$ is an $\boldsymbol{\ell}$-open cover of $\mathbb{R}$, so $\mathcal{V} = \{ (-\infty,\alpha_n) \cap A : n\in\mathbb{N} \}$ is an $\boldsymbol{\ell}_A$-open cover for A which has no finite subcover. So A is not $\boldsymbol{\ell}$-compact and therefore is not $\boldsymbol{\ell}$-compact with respect to $\boldsymbol{S}$ which is a contradiction. Hence A is bounded above, and so has a supremum say t.

Suppose that $t \notin A$, then $\forall n\in\mathbb{N}$ there exist $x_n \in A$ such that $t-\frac{1}{n} < x_n < t$.

$\mathcal{V} = \{(-\infty, t-\frac{1}{n}) \cap A : n\in\mathbb{N} \}$ is an $\boldsymbol{\ell}_A$-open cover for A which has no finite subcover. So A is not $\boldsymbol{\ell}$-compact, and therefore A is not $\boldsymbol{\ell}$-compact with respect to $\boldsymbol{S}$ which is a contradiction.

$\Longleftarrow$) Suppose that A is bounded above and contains its supremum, say t.

Let $\mathcal{V} = \{ (-\infty,\alpha) \cap A : \alpha \in \Delta \}$ be any $\boldsymbol{\ell}_A$–open cover for A. $t \in (-\infty,\alpha)$ for some $\alpha \in \Delta$, then $(-\infty,\alpha) \cap A = A \in \boldsymbol{S}_A$ .So $\{A\}$ is the $\boldsymbol{S}_A$–open cover for A which is finer than $\mathcal{V}$.

Hence A is $\boldsymbol{\ell}$-compact with respect to $\boldsymbol{S}$.

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