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Certain New Subclasses of Analytic Univalent Functions in the Unit Disk

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Dedication

I dedicate this thesis to my parents and brothers who led me through this darkness with their light of hope and support.

To my friends who touched my life with their love ,passion and support .

Declaration

I certify that this thesis submitted for the degree of master, is the result of my own research, except where otherwise acknowledged, and this study has been not submitted for a higher degree to any other university or institute .

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Abstract

The generalization of the families S, P and K are introduced for any $\alpha \in R$ such that

$0 \leq \alpha < 1$. Several properties are proved for these families . Also , the subclasses of analytic univalent functions with negative coefficients $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ for any $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$ are defined here and we derived some properties of these families .

The coefficient bounds of the functions in the families $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ are calculated . The integral mean inequality is investigated for the functions in the families $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$.

Introduction

Let f be a complex- valued function on a domain D . The function f is called a differentiable at a point $z_o \in D$ if the following limit exist

$$\lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o} .$$

A function f is called an analytic at a point $z_o \in D$ if f is differentiable in some neighborhood of z_o .

A function f is called univalent in a domain D if the function f does not take the same value twice , i.e if $z_1 \neq z_2$ in D , then $f(z_1) \neq f(z_2)$. [2]

A domain D is called a simply connected domain if any simple closed curve lies inside D can be shrunk to a point continuously in the domain .

The theory of univalent functions is an old subject , began to take shape around the beginning of the twentieth century , and it remains an active field of current researches .

Riemann proved that every simply connected domain can be mapped onto the unit disk

$\Delta = \{z: |z| < 1\}$ by an analytic univalent function f with $f(0) = 0$ and $f'(0) > 0$. [2]

From this theorem , to study the properties of a function f on a simply connected domain D , it is sufficient to study the properties of the functions on the unit disk Δ.

Therefore , the class of analytic univalent functions on Δ with normalization

$f(0) = 0$ and $f'(0) = 1$ are defined and called the class S. [2]

A function $f \in S$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \quad |z| \leq 1$$

A several properties of this family were studied in [2] , the coefficient estimate of $f \in S$ has been proved to be $|a_n| \leq en$, $n = 1,2,3, \dots\dots$.

A subclasses of the family S were studied in [3], here we are interested in the subclasses P and K.

A function $p \in P$ if f is analytic and $Re\, p(z) > 0$ $(|z| < 1)$ and $p(0) = 1$.

And $f \in K$ if $f \in S$ and

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \quad , \qquad |z| < 1$$

In chapter one the classes S , P and K are studied and several properties are derived .

Also , a generalization of the classes S , P and K are introduced with $0 \leq \alpha < 1$.

In chapter two a new subclasses $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ of univalent functions with negative coefficients are discussed with introduction and definitions . Coefficient bounds are estimated . Integral mean inequality for the univalent functions is illustrated .

Chapter one

Univalent Functions

The purpose of this chapter is to review and assemble for later reference some of general principles of complex analysis which underline the theory of univalent functions.

1.1 Basic properties of univalent functions

In this section we will present some of basic results and properties about univalent functions .

To do this , we first define what we mean by univalent function. But before that we will give a brief definition for analytic function.

Definition 1.1.1

A complex function is said to be analytic on a region R if it is complex differentiable at every point in it .

If $f(z)$ is analytic at a point z, then the derivative $f'(z)$ is continuous at z. And if $f(z)$ is analytic at a point z, then $f(z)$ has continuous derivatives of all order at the point z .

Definition 1.1.2

Let f be an analytic function in a domain D⊂ ₵ , f is said to be univalent in D if it never takes the same value twice , that is $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with

$z_1 \neq z_2$.

Definition 1.1.3

The function f is said to be locally univalent at a point $z_o \in$ D if it is univalent in some neighborhood of z_o .

Definition 1.1.4

Let a function f : D$\longrightarrow$ ₵ be analytic and univalent function in D, then we call it a conformal on D .

Now , we will introduce Noshiro-Warschawski Theorem that give us a simple condition for univalence .

Theorem 1.1.5 (Noshiro-Warschawski)

If f is analytic in a convex domain D and $Re\,\{f'(z)\} > 0, \;\; \forall z \in D$ then f is univalent in D .

For the proof see [2] .

The following examples explain what we mean by univalent function.

Example 1.1.6

Let $f(z) = -z$ and $g(z) = z^2$. Clearly f is univalent in Δ while g (z) = z^2 is not univalent in Δ . To see that if $f(z) = f(w)$ then $-z = -w$ and so $z = w$ which shows that f is univalent in Δ.

Also , $g(\frac{1}{2}) = g(-\frac{1}{2}) = \frac{1}{4}$ and this shows that g is not univalent in Δ .

Example 1.1.7

Let $f(z) = sin\,(z)$ .Then f is univalent in the disk D = $\left\{z \in ₵ : |Re\, z| < \frac{\pi}{2}\right\}$

Solution :

In order to use Theorem 1.1.5 , we should find $Re\ f'(z)$, but $f'(z) = cosz$.

If $z = x + iy$, then from [1] we can write $\cos z = cosx\ coshy - i\ sinx\ sinshy$.

Let $z \in D$, then $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and so $cosx > 0$.

also $coshy > 0$ for any $y \in R$. Therefore ,

$Re\ f'(z) = cosx\ coshy > 0 , \forall z = x + iy \in D$.

Which implies that f is univalent in$D$.

Example 1.1.8

Let $f(z) = e^z$. Then f is univalent in the strip

$$D = \left\{(x, y): x \in R \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$$

Solution :

Let $z = x + iy$. Then $f'(z) = e^z = e^{x+iy} = e^x cosy + ie^x siny$.

So $Re\big(f'(z)\big) = e^x cosy$. But $e^x > 0 , \forall x \in R$. And $cosy > 0$ if $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

So $Re\ f'(z) > 0 , \forall z \in D$.

Therefore $f(z) = e^z$ is univalent in D by Theorem 1.1.5 .

Example 1.1.9 :

Let $f(z) = z^2$, f is univalent in D= $\left\{z \in D : 0 < |z| < 1 ,\ 0 < \text{Arg}\, z < \frac{\pi}{2}\right\}$

Solution :

Write $z = re^{i\theta}$, where $r = |z|$ and $\theta = \text{Arg}\, z$. Then

$f'(z) = 2z = 2re^{i\theta} = 2rcos\theta + i2rsin\theta$.

Since $0 < \theta < \frac{\pi}{2}$, then $cos\,\theta > 0$ and so $Re\, f'(z) = 2rcos\theta > 0$.

Hence , using Theorem 1.1.5 we get $f(z) = z^2$ is univalent in D .

1.2 **Normalized univalent function .**

In this section we will normalize the univalent functions by some conditions and then classify it in different classes and study the main properties of each class. Next we recall an important theorem in the complex analysis.

Theorem 1.2.1 : (Riemann Mapping Theorem) [2]

let D ⊂ ₵ be a simply connected domain and let $z_o \in$ D be any given point . Then there is a unique function f which maps D conformally onto the unit disk Δ and has the properties that $f(z_0) = 0$ and $f'(z_0) > 0$.

According to Theorem 1.2.1 we will set some conditions if the function achieves it then we call it normalized univalent function and the normalization of univalent functions will be introduction to study some class of univalent functions .

The following definition introduces the class S of univalent analytic functions and some of its properties .

Definition 1.2.2 :

Let S denote the set of functions that are analytic and univalent in the unit disk Δ

normalized by the conditions $f(0) = 0$ and $f'(0) = 1$.

Thus each $f \in S$ has a Taylor series expansion of the form :

$$f(z) = z + a_2z^2 + a_3z^3 + \dots\dots\dots\dots\dots = z + \sum_{n=2}^{\infty} a_n z^n$$

In [2] it was proved that $|a_n| \leq n$. For n= 2 , 3 ,…… for any $f \in S$.

We now give some examples of normalized univalent function which is a member of S.

Example 1.2.3 :

Let $f(z) = z$. Clearly f is analytic in Δ and since $Re\, f'(z) > 0$, $\forall z \in \Delta$, then f is univalent in Δ . Also $f(0) = 0$ and $f'(0) = 1$.

Hence $f(z) \in S$.

Example 1.2.4 :

Let $f(z) = z - \frac{1}{2} z^2$, clearly f is analytic in Δ and $f'(z) = 1 - z$, $\forall\, z \in \Delta$.

Let $z = x + iy$. Since $z \in \mathit{\Delta}$, then $-1 < x < 1$ and so $1 - x > 0$,

but $Re\ f'(z) = 1 - x$

So $\mathrm{Re}\big(f'(z)\big) > 0$ and hence f is univalent in Δ .

Also $f(0) = 0$ and $f'(0) = 1$.

Hence $f(z) \in$ S.

Now we will talk about several properties of functions in S.

Fact :

Let f be a univalent function in Δ then :

1) $g(z) = f(z) + c$ is univalent, for any $c \in ₵$.

2) $h(z) = \lambda f(z)$ is univalent, for any $\lambda \in ₵$, $\lambda \neq 0$.

Proof :

Suppose f is a univalent function in Δ and c , $\lambda \in ₵$.

1) If $g(z) = g(w)$ then $f(z) + c = f(w) + c$ and then we get $f(z) = f(w)$. But f is univalent then $z = w$ and so $g(z)$ is univalent .

2) If $g(z) = g(w)$ then $\lambda f(z) = \lambda f(w)$. Since $\lambda \neq 0$ we get $f(z) = f(w)$. But f is univalent then $z = w$ and so $g(z)$ is univalent .

Theorem 1.2.5 :

Let $f \in S$ and $\phi : f(\Delta) \to ₵$ be an analytic and univalent function such that $\phi'(0) \neq 0$.

Let

$$g(z) = \frac{\phi o f(z) - \phi(0)}{\phi'(0)}, \qquad \text{then } g \in S.$$

Proof

Since $f \in S$ then f is analytic univalent , $f(0) = 0$ and $f'(0) = 1$.

Also , since ϕ is analytic and univalent , then $\phi o f$ is analytic and univalent in Δ.

And so $g(z)$ is analytic and univalent in D. For the normalization, we calculate $g(0)$and $g'(0)$.

$$g(0) = \frac{\phi(f(0)) - \phi(0)}{\phi'(0)} = \frac{\phi(0) - \phi(0)}{\phi'(0)} = 0.$$

Since $g'(z) = \frac{\phi'\big(f(z)\big)f'(z)}{\phi'(0)}$, then $g'(0) = \frac{\phi'\big(f(0)\big)f'(0)}{\phi'(0)} = \frac{\phi'(0).1}{\phi'(0)} = 1$

So g is univalent , $g(0) = 0$ and $g'(0) = 1$

Hence $g \in S$.

Theorem 1.2.6 : (Omitted-value transformation)

Let $z \in \Delta$, $\omega \in ₵$ and $f \in S$ such that $f(z) \neq w$. If

$$g(z) = \frac{wf(z)}{w - f(z)},$$

Then $g \in S$.

Proof:

To show $g \in S$, we must show that g is analytic and univalent in Δ and $g(0) = 0$, $g'(0) = 1$.

First , since f is analytic in Δ and $f(z) \neq w$, $\forall z \in \Delta$, then clearly g is analytic in Δ .

Also , since $f(0) = 0$, then

$g(0) = \dfrac{wf(0)}{w - f(0)} = 0$.

And $g'(z) = \dfrac{\big(w - f(z)\big)wf'(z) - wf(z)(-f'(z))}{(w - f(z))^2}$

So $g'(0) = \dfrac{w^2 f'(0) - 0}{w^2} = f'(0) = 1$.

Finally , to show g is univalent, suppose $g(a) = g(b)$, then

$$\frac{wf(a)}{w - f(a)} = \frac{wf(b)}{w - f(b)} \Rightarrow \big(w - f(a)\big)\big(wf(b)\big) = \big(w - f(b)\big)\big(wf(a)\big)$$

$$\Rightarrow \boldsymbol{w^2} f(a) - wf(a)f(b) = \boldsymbol{w^2} f(b) - wf(a)f(b)$$

$\Rightarrow \boldsymbol{w^2} f(a) = \boldsymbol{w^2} f(b)$. But $w \neq 0$, then $f(a) = f(b)$. Since f is univalent then $a = b$ and so g is univalent .

1.3 : Special families of univalent functions

In this section we will introduce some subfamilies of class S consisting of convex and starlike. Also we will investigate some classes of functions having a positive real part .

A set S in the complex space is called starlike domain if there exists x_o in S such that for all x in S the line segment from x_o to x is in S.

A function $f \in S$ is called a starlike function if $f(\Delta)$ is a starlike domain.

In [3], they proved that f is a starlike function if and only if

$$Re\left(\frac{zf^{'}(z)}{f(z)}\right) > 0\,, \quad |z| < 1$$

Now we can define the family S^* as follow :

Definition 1.4.1 :

The family S^* is defined as

$$S^* = \{ f \in S : Re\left(\frac{zf^{'}(z)}{f(z)}\right) > 0, \quad |z| < 1\}$$

In [3] it was proved that if $f \in S^*$ with $f(z) = z + a_2 z^2 + \cdots + a_n z^n,$ then

$$|a_n| \le n\,, \qquad n \ge 2$$

For example the function $z - \frac{1}{2}z \in S^*$

Definition 1.4.3 :

Let P denote the set of functions p that are analytic in Δ and satisfy $Re\, p(z) > 0$ and

$p(0) = 1, \quad |z| < 1$

In [3] , it was proved that if $p \in P$ with

$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \;\; , (|z| < 1)$

then $|p_n| \le 2 \qquad (n = 1 , 2 , 3 , \dots \dots . .)$

For example the function $(z + 1) - \frac{1}{2}z^2 \in P$

We now introduce the family of functions which maps Δ conformally onto convex domain .

Let K denote the subset of S consisting of functions f for which $f(\Delta)$ is a convex set .

Definition 1.4.3 :

Let $f \in S$ then we say $f \in K$ if

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \quad , \qquad |z| < 1$$

And we call it the class of convex functions .

For example the function $\frac{z}{2-z} + 1 \in K$

In [3]it was proved that if $f \in K$ and $f(z) = z + \sum_2^\infty a_n z^n$

then

$$|a_n| \le 1 \qquad (n = 2, 3, \ldots\ldots)$$

Chapter 2

Certain Subclasses of Analytic Univalent Functions

in the Unit Disk

After we talking about some families of univalent functions in the previous chapter we want to generalize it by defining new families of order α .

First we want to introduce new subclasses $K(w, \alpha , \lambda)$ and $H(w, \alpha , \lambda)$ of analytic functions with negative coefficients defined in the unit disk. And then we derive some properties of functions in these classes and obtain coefficient bounds and integral means inequality for the function $f(z)$ under these classes.

Section 2.1 : Introduction and Definitions

An analytic function f in a domain D containing z_o can be written by Taylor's Theorem as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

In this section we restrict the domain to the unit disk and impose some normalization on the functions .

Definition 2.1.1 :

Let $A(\omega)$ denote the class of functions of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k \ (z-\omega)^k$$

which are analytic in the unit disk Δ , and normalized with $f(\omega) = 0$ and $f'(\omega) = 1$,

w is a fixed point in Δ. Also, we can generalized the family S by the following definition

Definition 2.1.2 :

Let $S(\omega) = \{f \in A(\omega): f$ is univalent in $\Delta\}$.

If $w = 0$, then clearly $S(w) = S$.

Now we will define some classes of univalent functions of order α in Δ as follows:

Definition 2.1.3 :

Let $0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1$ and $z \in \Delta$. A function $f \in B(\omega, \alpha, \lambda)$ if :

1) $f \in S(\omega)$, and

2) $Re\left(\dfrac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)}\right) > \alpha.$

The following function is an example of the above family:

Example 2.1.4 :

Let $f(z) = (z-1) - \frac{1}{2}(z-1)^2$ with $w = 1, \alpha = \frac{1}{2}$ and $\lambda = 1$. Then $f \in B(1, \frac{1}{2}, 1)$

To see this , it is clearly that f is analytic in Δ ,$f(1) = 0$ and $f'(1) = 1$, since

$f'(z) = 2 - z$. If $z = x + iy$, then $Ref'(z) = 2 - x$, but $|x| < 1$, so $Ref'(z) > 0$ for all $z \in \Delta$. Hence f is univalent and so $f \in S(w)$.

Also , since $f''(z) = -1$, then

$$\frac{(z-1)^2 f''(z) + (z-1)f'(z)}{(z-1)f'(z)} = \frac{(z-1)^2(-1) + (z-1)(2-z)}{(z-1)(2-z)}$$

$$= \frac{(z-1)(-1)}{(2-z)} + 1$$

$$\frac{1-z}{2-z} + 1 = \frac{(1-z)(2-\overline{z})}{|z-2|^2} + 1 = \frac{2-\overline{z}-2z+|z|^2}{|z-2|^2} + 1$$

Now , if $z = x + iy$ where $x, y \in R$, then

$$\frac{(z-1)^2 f''(z) + (z-1)f'(z)}{(z-1)f'(z)} = \frac{2-\overline{z}-2z+|z|^2}{|z-2|^2} + 1$$

$$= \frac{2 - (\mathrm{x} - \mathrm{iy}) - 2(\mathrm{x} + \mathrm{iy}) + x^2 + y^2}{(2-x)^2 + y^2} + 1$$

$$= \frac{x^2 + \mathrm{y}^2 - \mathrm{x} + \mathrm{iy} - 2\mathrm{x} - 2\mathrm{iy} + 2}{(2-x)^2 + y^2} + 1$$

So , the real part is

$$\frac{x^2 + \mathrm{y}^2 - 3\mathrm{x} + 2}{(x-2)^2 + y^2} + 1 = \frac{(\mathrm{x}-1)(\mathrm{x}-2) + \mathrm{y}^2}{(x-2)^2 + y^2} + 1$$

But $(x-1)(x-2)+y^2 > (x-2)^2+y^2$, since $(x-2) < (x-1)$

Thus $\dfrac{(x-1)(x-2)+y^2}{(x-2)^2+y^2} > 1$

Hence $\dfrac{(x-1)(x-2)+y^2}{(x-2)^2+y^2} + 1 > \alpha = \dfrac{1}{2}$

Therefore Re $\left(\dfrac{(z-1)^2 f''(z) + (z-1)f'(z)}{(z-1)f'(z)}\right) > \dfrac{1}{2}$

And $f \in B\left(1, \frac{1}{2}, 1\right)$.

Definition 2.1.5:

Let $0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1$ and $z \in \Delta$. A function $f \in Q(\omega, \alpha, \lambda)$ if :

1) $f \in S(\omega)$, and

2) $Re\left(\dfrac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}\right) > \alpha$

The following example explains the definition :

Example 2.1.6:

Let $f(z) = (z-1) + \frac{1}{3}(z-1)^3$. Then $f \in Q(1, \frac{1}{3}, 1)$

To see this , clearly $f(1) = 0$, f is analytic in Δ and $f'(z) = 1 + (z-1)^2$ which implies that $f'(1) = 1$.

To show that f is univalent, let $z = x + iy$, then

$$f'(z) = (1 + (z-1)^2) = 1 + ((x-1) + iy)^2 = 1 + (x-1)^2 - y^2 + 2(x-1)yi\,.$$

Therefore , $Ref'(z) = 1 + (x-1)^2 - y^2 = (1-y^2) + (x-1)^2$.

Since $|x| < 1$ and $|y| < 1$, then

$1 - y^2 > 0$ and $(x-1)^2 > 0$. So $Ref'(z) > 0$. Thus by Theorem 1.1.5 , f is univalent in Δ and we show that $f \in S(1)$.

Also ,

$$\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega) f'(z)}{\lambda(z-\omega)^2 f''(z) + \lambda(z-\omega) f'(z)} =$$

$$= \frac{2(z-1)^3 + 6(z-1)^2(z-1) + (z-1)(1+(z-1)^2)}{2(z-1)^2\,(z-1) + (z-1)(1+(z-1)^2)}$$

$$= \frac{2(z-1)^3 + 6(z-1)^3 + (z-1) + (z-1)^3}{2(z-1)^3 + (z-1) + (z-1)^3} = \frac{9(z-1)^3 + (z-1)}{3(z-1)^3 + (z-1)}$$

$$= \frac{9(z-1)^2 + 1}{3(z-1)^2 + 1} = \frac{6(z-1)^2}{3(z-1)^2 + 1} + 1$$

$$= \frac{6(z-1)^2[3(\bar{z}-1)^2 + 1]}{|3(z-1)^2 + 1|^2} + 1$$

$$= \frac{18(z-1)^2(\bar{z}-1)^2 + 6(z-1)^2}{|3(z-1)^2 + 1|^2} + 1$$

$$= \frac{18|z-1|^2 + 6(z-1)^2}{|3(z-1)^2 + 1|^2} + 1 = \mathrm{A}$$

Now , if $z = x + iy$, then

$$Re(A) = \frac{18((x-1)^2 + y^2) \; + 6((x-1)^2 - y^2)}{|3(z-1)^2 + 1|^2} + 1$$

$$= \frac{18(x-1)^2 + 18y^2 + 6(x-1)^2 - 6y^2}{|3(z-1)^2 + 1|^2} + 1$$

$$= \frac{24(x-1)^2 + 12y^2}{|3(z-1)^2 + 1|^2} + 1 \geq 1 > \frac{1}{3}$$

Hence $f(z) \in Q\left(1, \frac{1}{3}, 1\right)$.

Let $T(\omega)$ denote subclass of $S(\omega)$ whose elements can be expressed in the form

$$f(z) = (z - \omega) - \sum_{k=2}^{\infty} a_k \; (z - \omega)^k \; .$$

Where $a_k \geq 0$, $\forall k = 2 , 3 , ...$

We denote by $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ the subfamilies of $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ obtained by

$$H(\omega, \alpha, \lambda) = B(\omega, \alpha, \lambda) \cap T(\omega)$$

and

$$K(\omega, \alpha, \lambda) = Q(\omega, \alpha, \lambda) \cap T(\omega) \; .$$

Definition 2.1.7 :

Let $P(\omega)$ denote the class of functions of the form

$$P_\omega(z) = 1 + \sum_{k=1}^{\infty} B_k \ (z-\omega)^k .$$

with $Re\ P_\omega(z) > 0$ and

$$|B_k| \le \frac{2}{(1+d)(1-d)^k}, \qquad k \ge 1, \qquad d = |\omega|$$

Example 2.1.8 :

Take $w = \frac{1}{2}$, then the function $f(z) = 1 + \frac{1}{2}\left(z - \frac{1}{2}\right)^2$ is in $P(\frac{1}{2})$

To see this ,

It is clearly that $P\left(\frac{1}{2}\right) = 1$ and if $z = x + iy$, then

$$f(z) = 1 + \frac{1}{2}\left[\left(x - \frac{1}{2}\right) + iy\right]^2$$

$$= 1 + \frac{1}{2}\left[\left(x - \frac{1}{2}\right)^2 - y^2 + 2\left(x - \frac{1}{2}\right) iy\right] \text{, then}$$

$$Re\ f(z) = 1 + \frac{1}{2}\left[\left(x - \frac{1}{2}\right)^2 - y^2\right]$$

Since $y^2 < 1$ then $-y^2 > -1$ and $1 + \frac{1}{2}\left(x - \frac{1}{2}\right)^2 \ge 1$, then

$$1+\frac{1}{2}\left(x-\frac{1}{2}\right)^2 - y^2 > 1-1=0$$

Hence $Re\, f(z) = 1+\frac{1}{2}\left[\left(x-\frac{1}{2}\right)^2 - y^2\right]$ always greater than 0 .

Now , $B_1 = 0$ and $\frac{2}{(1+d)(1-d)} = \frac{2}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)} > 0$ since $d = \frac{1}{2}$,

Then $|B_1| \leq \frac{2}{(1+d)(1-d)}$. Also ,

$B_2 = \frac{1}{2}$ and $\frac{2}{(1+d)(1-d)^2} = \frac{2}{\left(\frac{3}{2}\right)\left(\frac{1}{4}\right)} = \frac{16}{3} > \frac{1}{2}$

So , $|B_2| \leq \frac{2}{(1+d)(1-d)^2}$.

Hence , $f \in P\left(\frac{1}{2}\right)$.

Note that :

1) When $(\omega, \alpha$ and $\lambda = 0)$ then $B(\omega, \alpha, \lambda)$ is the same as family S^*.

2) When $(\omega, \alpha$ and $\lambda = 0)$ then $Q(\omega, \alpha, \lambda)$ is the same as family K.

3) When $(\omega = 0)$ then $P(\omega)$ is the same as family P .

Where S^*, K and P are illustrated in chapter 1 .

Section 2.2 : Coefficient Estimates

In this section we want to estimate and obtain coefficient bounds , and for our main result we first derive the following :

Lemma 2.2.1 :

A function $f(z) \in T(\omega)$ is in the class $H(\omega, \alpha, \lambda)$ if and only if

$$\sum_{k=2}^{\infty}(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\alpha \quad \ldots\ldots\ldots\ldots\ldots(2.1)$$

Where $0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1$ and $d = |w|$

Proof :

Assume that the inequality 2.1 holds and let $|z-w| = 1-d$. Since $f \in T(w)$, then to show that $f \in H(\omega, \alpha, \lambda)$ we must show $f \in B(\omega, \alpha, \lambda)$, but $T(w) \subseteq S(w)$, so $f \in S(w)$. For the second condition , we first simplify the following :

$$\left|\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)} - 1\right|$$

Since $f(z) \in T(\omega)$ then $f(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k\ (z-\omega)^k$. Therefore

$f'(z) = 1 - \sum_2^{\infty} k a_k (z-w)^{k-1}$

$f''(z) = -\sum_2^{\infty} k(k-1)a_k(z-w)^{k-2}$. Then

$\lambda(z-w)^2 f''(z) + (z-\omega)f'(z) =$

$-\lambda(z-w)^2 \sum_2^{\infty} k(k-1)a_k(z-w)^{k-2} + (z-w)[1-\sum_2^{\infty} k a_k (z-w)^{k-1}]$

$$= (z-w)\sum_{k=2}^{\infty}(-\lambda)k(k-1)a_k(z-w)^{k-1} + (z-w) - (z-w)\sum_{2}^{\infty} ka_k(z-w)^{k-1}$$

$$= (z-w)[1 + \sum_{k=2}^{\infty}(-\lambda k(k-1) - k)\ a_k(z-w)^{k-1}]$$

$$= (z-w)[1 + \sum_{k=2}^{\infty}(-\lambda k^2 + \lambda k - k)\ a_k(z-w)^{k-1}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2)$$

And ,

$$(1-\lambda)f(z) + \lambda(z-\omega)f'(z) \ =$$

$$(1-\lambda)(z-w)[1 - \sum_{2}^{\infty} a_k(z-w)^{k-1}] + \lambda(z-w)[1 - \sum_{2}^{\infty} a_k k(z-w)^{k-1}]$$

$$= (z-w)[(1-\lambda) - \sum_{2}^{\infty}(1-\lambda)a_k(z-w)^{k-1} + \lambda - \sum_{2}^{\infty} \lambda k a_k(z-w)^{k-1}]$$

$$= (z-w)[1 - \sum_{2}^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots (2.3)$$

Divide (2.2) and (2.3) and subtract 1 then we get

$$\frac{1 + \sum_{k=2}^{\infty}(-\lambda k^2 + \lambda k - k)\ a_k(z-w)^{k-1}}{1 - \sum_{2}^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}} - 1$$

$$= \frac{1 + \sum_{n=2}^{\infty}(-\lambda k^2 + \lambda k - k)\ a_k(z-w)^{k-1} - 1 + \sum_{2}^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}}{1 - \sum_{2}^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}}$$

$$= \frac{\sum_{k=2}^{\infty}(-\lambda k^2 + \lambda k - k + 1 - \lambda + \lambda k)\ a_k(z-w)^{k-1}}{1 - \sum_{2}^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}}$$

$$= \frac{\sum_{k=2}^{\infty} -(k-1)(1-\lambda+\lambda k)\ a_k(z-w)^{k-1}}{1 - \sum_{2}^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}}$$

Take the modulus then we have,

$$\left|\frac{\sum_{k=2}^{\infty}(k-1)(1-\lambda+\lambda k)\,a_k(z-w)^{k-1}}{1-\sum_2^{\infty}(1-\lambda+\lambda k)a_k(z-w)^{k-1}}\right| \leq \frac{\sum_2^{\infty}|(k-1)a_k(z-w)^{k-1}(1-\lambda+\lambda k)|}{1-\sum_2^{\infty}|a_k(z-w)^{k-1}(1-\lambda+k\lambda|}$$

$= \dfrac{\sum_2^{\infty}(k-1)a_k|z-w|^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^{\infty}a_k|z-w|^{k-1}(1-\lambda+\lambda k)}$. Replacing $|z-w|$ by $(1-d)$ then we get

$$= \frac{\sum_2^{\infty}(k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^{\infty}a_k(1-d)^{k-1}(1-\lambda+\lambda k)}$$

Since $(k-1)<(k-\alpha)$, then by (2.1) ,

$$\sum_2^{\infty}(k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k) < \sum_2^{\infty}(k-\alpha)a_k(1-d)^{k-1}(1-\lambda+\lambda k)$$

$$\leq 1-\alpha$$

Let $A_k = a_k(1-d)^{k-1}(1-\lambda+\lambda k)$, then $\sum_2^{\infty}(k-\alpha)A_k \leq 1-\alpha$ by (2.1),

which implies that

$$\sum_2^{\infty}kA_k - \alpha\sum_2^{\infty}A_k \leq 1-\alpha$$

$$\Rightarrow \sum_2^{\infty}kA_k \leq 1-\alpha+\alpha\sum_2^{\infty}A_k = 1-\alpha(1-\sum_2^{\infty}A_k)$$

Adding $-\sum_2^{\infty}A_k$ to both sides , we get

$$\sum_2^{\infty}kA_k - \sum_2^{\infty}A_k \leq 1-\alpha(1-\sum_2^{\infty}A_k) - \sum_2^{\infty}A_k$$

Which implies

$$\sum_2^{\infty}(k-1)A_k \leq (1-\sum_2^{\infty}A_k) - \alpha(1-\sum_2^{\infty}A_k)$$

Now , dividing both sides by $1-\sum_2^{\infty}A_k$, we get

$$\frac{\sum_2^\infty (k-1)A_k}{1-\sum_2^\infty A_k} \le \frac{(1-\sum_2^\infty A_k)(1-\alpha)}{(1-\sum_2^\infty A_k)} = 1-\alpha$$

Hence , we get

$$\frac{\sum_2^\infty (k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k(1-d)^{k-1}(1-\lambda+\lambda k)} \le 1-\alpha$$

This shows that the values of $\frac{\lambda(z-\omega)^2 f''(z)+(z-\omega)f'(z)}{(1-\lambda)f(z)+\ \lambda(z-\omega)f'(z)}$

lie in the circle whose radius is $1-\alpha$ and the center is 1. Hence the real part is lies between α and $2-\alpha$. So the real part of $f(z) > \alpha$

Hence $f(z)$ is in the class $H(w,\lambda,\alpha)$.

Conversely assume that

$$Re\left(\frac{\lambda(z-\omega)^2 f''(z)+(z-\omega)f'(z)}{(1-\lambda)f(z)+\ \lambda(z-\omega)f'(z)}\right) > \alpha \text{ then}$$

Divide (2.2) and (2.3) we get :

$$\frac{(z-w)[1-\sum_2^\infty(\lambda k(k-1)+k)a_k(z-w)^{k-1}]}{(z-w)[1-\sum_2^\infty a_k(z-w)^{k-1}(1-\lambda+k\lambda)]}$$

$$=\frac{1-\sum_2^\infty ka_k(z-w)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k(z-w)^{k-1}(1-\lambda+k\lambda)}$$

Since $Re\left(\frac{\lambda(z-\omega)^2 f''(z)+(z-\omega)f'(z)}{(1-\lambda)f(z)+\ \lambda(z-\omega)f'(z)}\right) > \alpha$ then

$$Re\left(\frac{1-\sum_2^\infty ka_k(z-w)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k(z-w)^{k-1}(1-\lambda+k\lambda)}\right) > \ \alpha$$

Choose values of z on the real axis so that

$$\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)}$$

is real.

Upon clearing the denominator and letting z → 1^- through real values then we have

$$\frac{\lambda(1-w)^2 f''(z) + (1-w)f'(z)}{(1-\lambda)f(z) + \lambda(1-w)f'(z)} \text{ which is equal to } \frac{1-\sum_2^\infty k a_k (1-w)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k (1-w)^{k-1}(1-\lambda+k\lambda)}$$

But $|1-w| \geq 1-|w| = 1-d$

Hence we have

$$\frac{1-\sum_2^\infty k a_k (1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k (1-d)^{k-1}(1-\lambda+k\lambda)} \geq \alpha$$

Then we get ,

$$\alpha(1-\sum_2^\infty a_k (1-d)^{k-1}(1-\lambda+\lambda k) \leq 1-\sum_2^\infty k(1-\lambda+\lambda k) a_k (1-d)^{k-1}$$

$$\alpha - \alpha \sum_2^\infty a_k (1-d)^{k-1}(1-\ \lambda+\lambda k) \leq 1-\sum_2^\infty k a_k (1-d)^{k-1}(1-\ \lambda+\lambda k)$$

$$\sum_2^\infty k a_k (1-d)^{k-1}(1-\ \lambda+\lambda k) - \alpha \sum_2^\infty a_k (1-d)^{k-1}(1-\ \lambda+\lambda k) \leq 1-\alpha$$

Hence we get

$$= \sum_2^\infty (k-a)(1-d)^{k-1}\,(1-\ \lambda+\lambda k)\, a_k \leq 1-\alpha$$

Theorem 2.2.2 :

Let $f(z) \in T(\omega)$ be in the class $H(\omega, \alpha, \lambda)$. Then we have

$$a_k \leq \frac{1-\alpha}{(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}$$

Proof :

From lemma 2.2.1 we have :

$\sum_{k=2}^{\infty}(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\ \alpha$ and since

$(k-\alpha), (1-d)^{k-1}, (1-\lambda+\lambda k)$ and a_k are positive terms then each term in the summation must be less than $1-\ \alpha$.

Hence , $(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\ \alpha$

And then

$$a_k \leq \frac{1-\alpha}{(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}$$

Lemma 2.2.3 :

A function $f(z) \in T(\omega)$ is in the class $K(\omega, \alpha, \lambda)$ if and only if

$\sum_{k=2}^{\infty} k(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\ \alpha$ ……………..(2.4)

Proof :

Assume that the inequality (2.4) holds and let $|z-w| = 1-d$.

Since $f \in T(w)$, then to show that $f \in K(\omega,\alpha,\lambda)$ we must show $f \in Q(\omega,\alpha,\lambda)$, but $T(w) \subseteq S(w)$, so $f \in S(w)$.

For the second condition , we first simplify the following :

$$\left|\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-w)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-w)^2 f''(z) + (z-\omega)f'(z)} - 1\right|$$

$$= \left|\frac{\lambda(z-\omega)^3 f'''(z) + (z-w)^2 f''(z) + 2\lambda(z-w)^2 f''(z) - \lambda(z-w)^2 f''(z)}{\lambda(z-w)^2 f''(z) + (z-\omega)f'(z)}\right|$$

$$= \left|\frac{\lambda(z-\omega)^3 f'''(z) + (z-w)^2 f''(z) + \lambda(z-w)^2 f''(z)}{\lambda(z-w)^2 f''(z) + (z-\omega)f'(z)}\right|$$

$$= \left|\frac{\lambda(z-\omega)^3 f'''(z) + (z-w)^2 f''(z)(1+\lambda)}{\lambda(z-w)^2 f''(z) + (z-\omega)f'(z)}\right|$$

Since $f(z) \in T(\omega)$ then $f(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k \ (z-\omega)^k$. Therefore

$f'(z) = 1 - \sum_2^\infty a_k k (z-w)^{k-1}$

$f''(z) = -\sum_2^\infty k(k-1) a_k (z-w)^{k-2}$. And ,

$f'''(z) = -\sum_2^\infty k(k-1)(k-2) a_k (z-w)^{k-3}$

$\lambda(z-\omega)^3 f'''(z) = \lambda(z-\omega)^3 (-\sum_2^\infty k(k-1)(k-2) a_k (z-w)^{k-3})$

$$= -\sum_2^\infty \lambda k(k-1)(k-2) a_k (z-w)^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5)$$

$$(z-w)^2(1+\lambda)f''(z) = (z-w)^2(1+\lambda)(-\sum_2^\infty k(k-1)a_k(z-w)^{k-2})$$

$$= -\sum_2^\infty(1+\lambda)k(k-1)a_k(z-w)^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.6)$$

$$\lambda(z-w)^2f''(z) = \lambda(z-w)^2(-\sum_2^\infty k(k-1)a_k(z-w)^{k-2})$$

$$= -\sum_2^\infty \lambda k(k-1)a_k(z-w)^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.7)$$

$$(z-\omega)f'(z) = (z-\omega)(1-\sum_2^\infty ka_k(z-w)^{k-1})$$

$$= (z-\omega) - \sum_2^\infty ka_k(z-w)^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.8)$$

Adding (2.5) and (2.6) then we get

$$-\sum_2^\infty \lambda k(k-1)(k-2)a_k(z-w)^k - (1+\lambda)\sum_2^\infty k(k-1)a_k(z-w)^k$$

$$= -\sum_2^\infty(1-\lambda+\lambda k)k(k-1)a_k(z-w)^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.9)$$

Also adding (2.7) and (2.8) we have

$$\sum_2^\infty -\lambda k(k-1)a_k(z-w)^k + (z-w) - \sum_2^\infty ka_k(z-w)^k$$

$$= \sum_2^\infty -ka_k\,(z-w)^k[\lambda(k-1)+1] + (z-w)$$

$$= (z-w)[1-\sum_2^\infty ka_k(z-w)^{k-1}(1-\lambda+\lambda k)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.10)$$

Divide (2.9) and (2.10) and take the modulus then we have

$$= \left|\frac{\sum_2^\infty k(k-1)a_k(z-w)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty ka_k(z-w)^{k-1}(1-\lambda+\lambda k)}\right|$$

$$\leq \frac{\sum_2^\infty k(k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty ka_k(1-d)^{k-1}(1-\lambda+\lambda k)}$$

Let $A_k = a_k(1-d)^{k-1}(1-\lambda+\lambda k)$, since $\sum_2^\infty k(k-\alpha)A_k \leq 1-\alpha$ by (2.4) ,

then

$$\sum_2^\infty k^2 A_k - \alpha \sum_2^\infty kA_k \leq 1-\alpha$$

$$\Rightarrow \sum_2^\infty k^2 A_k \leq 1-\alpha+\alpha \sum_2^\infty kA_k = 1-\alpha(1-\sum_2^\infty kA_k)$$

Adding $-\sum_2^\infty kA_k$ to both sides , we get

$$\sum_2^\infty k^2 A_k - \sum_2^\infty kA_k \leq 1-\alpha(1-\sum_2^\infty kA_k) - \sum_2^\infty kA_k$$

Which implies

$$\sum_2^\infty k(k-1)A_k \leq (1-\sum_2^\infty kA_k) - \alpha(1-\sum_2^\infty kA_k)$$

Now , dividing both sides by $1-\sum_2^\infty kA_k$, we get

$$\frac{\sum_2^\infty k(k-1)A_k}{1-\sum_2^\infty kA_k} \leq \frac{(1-\sum_2^\infty kA_k)(1-\alpha)}{1-\sum_2^\infty kA_k} = 1-\alpha$$

Hence , we get

$$\frac{\sum_2^\infty k(k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty ka_k(1-d)^{k-1}(1-\lambda+\lambda k)} \leq 1-\alpha$$

This shows that the values of

$$\frac{\lambda(z-\omega)^3 f'''(z)+(1+2\lambda)(z-w)^2 f''(z)+(z-\omega)f'(z)}{\lambda(z-w)^2 f''(z)+(z-\omega)f'(z)}$$

lie in the circle whose radius is $1-\alpha$ and centered at 1.

Hence the real part is lies between α and $2-\alpha$.

So $f(z)$ in the class $K(w,\lambda,\alpha)$.

Conversely assume that

$$Re\left(\frac{\lambda(z-\omega)^3 f'''(z)+(1+2\lambda)(z-w)^2 f''(z)+(z-\omega)f'(z)}{\lambda(z-w)^2 f''(z)+\ (z-\omega)f'(z)}\right)>\alpha$$

then

$\lambda(z-\omega)^3 f'''(z)+(1+2\lambda)(z-w)^2 f''(z)+(z-\omega)f'(z)=$

$\lambda(-\sum_2^\infty k(k-1)(k-2)a_k(z-w)^{k-1})+(1+2\lambda)(-\sum_0^\infty k(k-1)a_k(z-w)^{k-1})+$

$(z-w)[1-\sum_2^\infty ka_k(z-w)^{k-1}]$

$=(z-w)[1-\sum_2^\infty ka_k(z-w)^{k-1}[1+\lambda(k-1)(k-2)+(1+2\lambda)(k-1)]]$

$=(z-w)[1-\sum_2^\infty ka_k(z-w)^{k-1}(1+\lambda k^2-3\lambda k+2\lambda+k-1+2\lambda k-2\lambda]$

$=(z-w)[1-\sum_2^\infty ka_k(z-w)^{k-1}k(\lambda k-\lambda+1]$ ....(2.11)

Divide (2.11) by (2.10) we get

$\frac{(z-w)[1-\sum_2^\infty ka_k(z-w)^{k-1}k(\lambda k-\lambda+1]}{(z-w)[1-\sum_2^\infty a_k k(z-w)^{k-1}(1-\lambda+k\lambda)]}$. Hence,

$$Re\left(\frac{\lambda(z-\omega)^3 f'''(z)+(1+2\lambda)(z-w)^2 f''(z)+(z-\omega)f'(z)}{\lambda(z-w)^2 f''(z)+\ (z-\omega)f'(z)}\right)$$

$$=Re(\frac{1-\sum_2^\infty k^2 a_k(z-w)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k k(z-w)^{k-1}(1-\lambda+k\lambda)}>\alpha$$

Choose values of z on the real axis so that

$$\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-w)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-w)^2 f''(z) + (z-\omega)f'(z)}$$

is real.

Upon clearing the denominater and letting z → 1 − through real values we have

$$\frac{\lambda(1-\omega)^3 f'''(z) + (1+2\lambda)(1-w)^2 f''(z) + (1-\omega)f'(z)}{\lambda(z-w)^2 f''(z) + (z-\omega)f'(z)}$$

$$= \frac{1-\sum_2^\infty k^2 a_k (1-w)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k k(1-w)^{k-1}(1-\lambda+k\lambda)}$$

But $|1-w| \geq 1-|w| = 1-d$

Hence we have

$$\frac{1-\sum_2^\infty k^2 a_k (1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_2^\infty a_k k(1-d)^{k-1}(1-\lambda+k\lambda)} \geq \alpha$$

Then we get ,

$$\alpha(1-\sum_2^\infty k a_k (1-d)^{k-1}(1-\lambda+\lambda k) \leq 1-\sum_2^\infty k^2(1-\lambda+\lambda k) a_k (1-d)^{k-1}$$

$$\Rightarrow \alpha - \alpha\sum_2^\infty k a_k (1-d)^{k-1}(1-\lambda+\lambda k) \leq 1-\sum_2^\infty k^2 a_k (1-d)^{k-1}(1-\lambda+\lambda k)$$

$$\Rightarrow \sum_2^\infty k^2 a_k (1-d)^{k-1}(1-\lambda+\lambda k) - \alpha\sum_2^\infty k a_k (1-d)^{k-1}(1-\lambda+\lambda k) \leq 1-\alpha$$

$$\Rightarrow \sum_{k=2}^\infty k(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k) a_k \leq 1-\alpha$$

Corollary 2.2.4 :

Let $f(z) \in T(\omega)$ be in the class $K(\omega, \alpha, \lambda)$, then we have

$$a_k \leq \frac{1-\alpha}{k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}$$

Proof :

From lemma 2.2.3 we have :

$\sum_{k=2}^{\infty} k(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\alpha$. Since

$k, (k-\alpha), (1-d)^{k-1}, (1-\lambda+\lambda k)$ and a_k are positive terms , then each term in the summation must be less than $1-\alpha$.

Hence ,

$$k(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\alpha$$

And then

$$a_k \leq \frac{1-\alpha}{k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}$$

Theorem 2.2.5 :

Let $f(z) \in H(\omega, \alpha, \lambda)$ and $f(z) = (z-w) - a_2(z-w)^2 - \dots ..$

for $0 \leq \alpha < 1$ and w is a fixed point in Δ . Then

$$|a_2| \leq \frac{2(1-\alpha)}{(1+\lambda)(1-d^2)}$$

$$|a_3| \leq \frac{1}{(1+2\lambda)}\left[\frac{1-\alpha}{(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{(1-d^2)^2}\right]$$

$$|a_4| \leq \frac{1}{(1+3\lambda)}\left[\frac{2(1-\alpha)}{3(1+d)(1-d)^3} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{3(1-d^2)^3}\right]$$

Proof :

Let $g(z) = \frac{\lambda(z-w)^2 f''(z) + (z-w)f'(z)}{(1-\lambda)f(z) + \lambda(z-w)f'(z)}$. Since $f(z) \in H(\omega, \alpha, \lambda)$

Then $Re\ g(z) > \alpha$. But $f(z) = (z-w) - \sum_2^\infty a_k (z-w)^k$, then we can write the denominator of $g(z)$ as :

$(1-\lambda)f(z) + \lambda(z-w)f'(z) =$

$(1-\lambda)(z-w)[1 - \sum_2^\infty a_k (z-w)^{k-1}] + \lambda(z-w)f'(z)$

$= (z-w)[(1-\lambda) - \sum_2^\infty a_k (1-\lambda)(z-w)^{k-1} + \lambda f'(z)]$

Dividing $(z-w)$ from numerator and denominator , we have

$$g(z) = \frac{\lambda(z-w)f''(z) + f'(z)}{(1-\lambda) - \sum_2^\infty a_k (1-\lambda)(z-w)^{k-1} + \lambda f'(z)}$$

So $g(w) = \frac{f'(w)}{1-\lambda+\lambda f'(w)}$, but $f'(w) = 1$

Hence , $g(w) = \frac{1}{(1-\lambda)+\lambda} = 1.$

Now , define $P_w(z) = \dfrac{g(z) - \alpha}{1 - \alpha}$. We want to show that $P_w \in P(w)$

To do this , $P_w(w) = \dfrac{g(w) - \alpha}{1 - \alpha} = \dfrac{1 - \alpha}{1 - \alpha} = 1 \qquad 0 \leq \alpha < 1$

And $Re\, P_w(z) = Re \dfrac{g(z) - \alpha}{1 - \alpha} > \dfrac{\alpha - \alpha}{1 - \alpha} = 0.$ Hence , $P_w \in P(w)$

So we can write $g(z)$ as

$$\frac{\lambda(z-w)^2 f''(z) + (z-w)f'(z)}{(1-\lambda)f(z) + \lambda(z-w)f'(z)} = \alpha + (1-\alpha)P_w(z) \ldots\ldots..(2.12)$$

Since $P_w(z) = 1 + \sum_{k=1}^{\infty} B_k (z-w)^k$, then

$\alpha + (1-\alpha)P_w(z) = 1 + (1-\alpha)\sum_{k=1}^{\infty} B_k (z-w)^k$

Then if we multiply $g(z)$ by $(1-\lambda)f(z) + \lambda(z-w)f'(z)$ we get the left hand side

$\lambda(z-w)^2 f''(z) + (z-w)f'(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots(2.13)$

Which is equal the right hand side

$[(1-\lambda)f(z) + \lambda(z-w)f'(z)][1 + (1-\alpha)\sum_{k=1}^{\infty} B_k (z-w)^k] \ldots\ldots$ (2.14)

Now we will compare the coefficient of $(z-w)^2$, $(z-w)^3$ and $(z-w)^4$ in both sides,

to do this , let

$f(z) = (z-w) - a_2(z-w)^2 - a_3(z-w)^3 - a_4(z-w)^4 - a_5(z-w)^5 - \cdots$, then

$f'(z) = 1 - 2a_2(z-w) - 3a_3(z-w)^2 - 4a_4(z-w)^3 - 5a_5(z-w)^4 \; \ldots\ldots,$

and

$f''(z) = -2a_2 - 6a_3(z-w) - 12a_4(z-w)^2 - 20a_5(z-w)^3 \; \ldots\ldots\ldots$

Substitute in (2.13) then we have

$\lambda(z-w)^2[-2a_2 - 6a_3(z-w) - 12a_4(z-w)^2 - 20a_5((z-w)^3 - \cdots]$

$+(z-w)[1 - 2a_2(z-w) - 3a_3(z-w)^2 - 4a_4(z-w)^3 - 5a_5(z-w)^4 \; \ldots]$

So ,the coefficient of $(z-w)^2 = -2a_2\lambda - 2a_2 = -2a_2(\lambda+1)$,

the coefficient of $(z-w)^3 = -6a_3\lambda - 3a_3 = -3a_3(2\lambda+1)$,

the coefficient of $(z-w)^4 = -12a_4\lambda - 4a_4 = -4a_4(3\lambda+1)$,

and the coefficient of $(z-w)^5 = -20a_5\lambda - 5a_5 = -5a_5(4\lambda+1)$.

Also , substitute in (2.14) , we get

$$(1-\lambda)f(z) = (1-\lambda)[(z-w) - a_2(z-w)^2 - a_3(z-w)^3 - 4a_4(z-w)^4 - a_5(z-w)^5 \ldots.] \ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad (2.15)$$

$\lambda(z-w)f'(z) = \lambda(z-w)[1 - 2a_2(z-w) - 3a_3(z-w)^2 - 4a_4(z-w)^3 - \ldots..]$

$$= \lambda[(z-w) - 2a_2(z-w)^2 - 3a_3(z-w)^3 - 4a_4(z-w)^4 - \cdots] \ldots\ldots\ldots\ldots\ldots. \quad (2.16)$$

Adding (2.15) and (2.16) we get

$(1-\lambda)f(z)+\lambda(z-w)f'(z)=$

$(z-w)-(1+\lambda)a_2(z-w)^2-(1+2\lambda)a_3(z-w)^3+(1+3\lambda)a_4(z-w)^4)+$

$(1+4\lambda)a_5(z-w)^5 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots\dots\dots\dots..(2.17)$

Also ,

$1+(1-\alpha)\sum_1^\infty B_k(z-w)^k=$

$1+(1-\alpha)[B_1(z-w)+B_2(z-w)^2+B_3(z-w)^3+\ B_4(z-w)^4+\cdots]\dots\dots.(2.18)$

Multiply (2.17) and (2.18) we get (2.14)

So the coefficient of $(z-w)^2$:

$(1-\alpha)B_1-(1+\lambda)a_2$

The coefficient of $(z-w)^3$:

$(1-\alpha)B_2-B_1(1-\alpha)(1+\lambda)a_2-\ (1+2\lambda)a_3$

The coefficient of $(z-w)^4$:

$(1-\alpha)B_3-B_2(1-\alpha)(1+\lambda)a_2-B_1(1-\alpha)(1+2\lambda)a_3-(1+3\lambda)a_4$

The coefficient of $(z-w)^5$:

$(1-\alpha)B_4-B_3(1-\alpha)(1+\lambda)a_2-B_2(1-\alpha)(1+2\lambda)a_3-B_1(1-\alpha)(1+3\lambda)a_4$

$-(1+4\lambda)a_5$

Now we want to compare the coefficient of $(z-w)^2$ in both sides (2.13) and (2.14) to get the upper bound of a_2

$-2a_2(\lambda+1) = (1-\alpha)B_1 - (1+\lambda)a_2$

$-2a_2(\lambda+1) + (1+\lambda)a_2 = (1-\alpha)B_1$

$-a_2(\lambda+1) = (1-\alpha)B_1$

So $-a_2 = \dfrac{(1-\alpha)B_1}{(1+\lambda)}$, take the modulus both sides , we get

$|a_2| = \dfrac{(1-\alpha)|B_1|}{(1+\lambda)}$. But from definition 2.1.5 we know that $|B_1| \leq \dfrac{2}{(1+d)(1-d)}$

So ,

$$|a_2| \leq \frac{2(1-\alpha)}{(1+d)(1-d)(1+\lambda)} = \frac{2(1-\alpha)}{(1+\lambda)(1-d^2)}$$

Then from comparison the coefficients of $(z-w)^3$ in both sides we get the following

$-3a_3(2\lambda+1) = (1-\alpha)B_2 - B_1(1-\alpha)(1+\lambda)a_2 - (1+2\lambda)a_3$

$a_3(-3(2\lambda+1) + (1+2\lambda)) = (1-\alpha)B_2 - B_1(1-\alpha)(1+\lambda)a_2$

Take the modulus both sides , we get

$|-2a_3(2\lambda+1)| \leq |(1-\alpha)B_2| + |B_1(1-\alpha)(1+\lambda)a_2|.$

$\Rightarrow 2|a_3|(2\lambda+1) \leq (1-\alpha)|B_2| + (1-\alpha)(1+\lambda)|a_2||B_1|$

Since

$$|B_2| \le \frac{2}{(1+d)(1-d)^2}, |B_1| \le \frac{2}{(1+d)(1-d)} \text{ and } |a_2| \le \frac{2(1-\alpha)}{(1+\lambda)(1-d^2)}$$

Then

$$2|a_3|(2\lambda+1) \le \frac{2(1-\alpha)}{(1-d^2)(1-d)} + \frac{4(1-\alpha)^2}{(1-d^2)^2}$$

$$|a_3| \le \frac{1}{(1+2\lambda)}\left[\frac{1-\alpha}{(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{(1-d^2)^2}\right]$$

Now for the upper bound of a_4 we repeat the above steps as the following :

$$-4a_4(3\lambda+1) = B_3(1-\alpha) - B_2(1-\alpha)(1+\lambda)a_2 - B_1(1-\alpha)(1+2\lambda)a_3 -$$

$$(1+3\lambda)a_4$$

Then

$$-4a_4(3\lambda+1) + (1+3\lambda)a_4 = B_3(1-\alpha) - B_2(1-\alpha)(1+\lambda)a_2$$

$$-B_1(1-\alpha)(1+2\lambda)a_3$$

Take the modulus both sides , we get

$$|-3a_4(3\lambda+1)| \le |B_3(1-\alpha)| + |B_2(1-\alpha)(1+\lambda)a_2| + |B_1(1-\alpha)(1+2\lambda)a_3|$$

$$3|a_4|(3\lambda+1) \le (1-\alpha)|B_3| + (1-\alpha)(1+\lambda)|B_2||a_2| + (1-\alpha)(1+2\lambda)|B_1||a_3|$$

Since

$$|B_3| \le \frac{2}{(1+d)(1-d)^3}\ , |B_2| \le \frac{2}{(1+d)(1-d)^2}, |B_1| \le \frac{2}{(1+d)(1-d)},$$

$$|a_3| \le \frac{1}{(1+2\lambda)}\left[\frac{1-\alpha}{(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{(1-d^2)^2}\right] \quad \text{and} \quad |a_2| \le \frac{2(1-\alpha)}{(1+\lambda)(1-d^2)}$$

Then ,

$$3|a_4|(3\lambda+1) \le \frac{2(1-\alpha)}{(1-d)^3(1+d)} + \frac{4(1-\alpha)^2}{(1-d^2)(1-d^2)(1-d)} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{(1-d^2)^3}$$

That is

$$3|a_4|(3\lambda+1) \le \frac{2(1-\alpha)}{(1-d)^3(1+d)} + \frac{6(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{(1-d^2)^3}$$

$$|a_4| \le \frac{1}{(1+3\lambda)}\left[\frac{2(1-\alpha)}{3(1+d)(1-d)^3} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{3(1-d^2)^3}\right]$$

Continue the comparing process we get :

$$|a_5| \le \frac{1}{4(1+4\lambda)}\left[|B_4|(1-\alpha) + |B_3|(1-\alpha)(1+\lambda)|a_2| + |B_2|(1-\alpha)(1+2\lambda)|a_3| + |B_1|(1-\alpha)(1+3\lambda)|a_4|\right]$$

And

$$|a_6| \le \frac{1}{5(1+5\lambda)}\left[|B_5|(1-\alpha) + |B_4|(1-\alpha)(1+\lambda)|a_2| + |B_3|(1-\alpha)(1+2\lambda)|a_3| + |B_2|(1-\alpha)(1+3\lambda)|a_4| + |B_1|(1-\alpha)(1+4\lambda)|a_5|\right]$$

We note from the proof that :

$$|a_2| = \frac{1}{1+\lambda}[(1-\alpha)|B_1|]$$

$$|a_3| \le \frac{1}{2(1+2\lambda)}[|B_2|(1-\alpha) + |B_1|(1-\alpha)(1+\lambda)|a_2|]$$

$$|a_4| \le \frac{1}{3(1+3\lambda)}[|B_3|(1-\alpha) + |B_2|(1-\alpha)(1+\lambda)|a_2| + |B_1|(1-\alpha)(1+2\lambda)|a_3|]$$

$$|a_5| \le \frac{1}{4(1+4\lambda)}\begin{bmatrix}|B_4|(1-\alpha) + |B_3|(1-\alpha)(1+\lambda)|a_2| + |B_2|(1-\alpha)(1+2\lambda)|a_3| + \\ |B_1|(1-\alpha)(1+3\lambda)|a_4|\end{bmatrix}$$

And

$$|a_6| \le \frac{1}{5(1+5\lambda)}\begin{bmatrix}|B_5|(1-\alpha) + |B_4|(1-\alpha)(1+\lambda)|a_2| + |B_3|(1-\alpha)(1+2\lambda)|a_3| + \\ |B_2|(1-\alpha)(1+3\lambda)|a_4| + |B_1|(1-\alpha)(1+4\lambda)|a_5|\end{bmatrix}$$

If we continue this process , we can show that

$$|a_k| \le \frac{1}{(k-1)[1+(k-1)\lambda]}\begin{bmatrix}|B_{k-1}|(1-\alpha) + |B_{k-2}|(1-\alpha)(1+\lambda)|a_2| + |B_{k-3}|(1-\alpha)(1+2\lambda)|a_3| \\ +|B_{k-4}|(1-\alpha)(1+3\lambda)|a_4| + \cdots\end{bmatrix}$$

Theorem 2.2.6 :

Let $f(z) \in k(\omega, \alpha, \lambda)$ and $f(z) = (z-w) - a_2(z-w)^2 - \ldots..$

for $0 \le \alpha < 1$ and w is a fixed point in Δ . Then

$$|a_2| \le \frac{(1-\alpha)}{(1+\lambda)(1-d^2)}$$

$$|a_3| \le \frac{1}{(1+2\lambda)}\left[\frac{1-\alpha}{3(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{3(1-d^2)^2}\right]$$

$$|a_4| \le \frac{1}{(1+3\lambda)}\left[\frac{(1-\alpha)}{6(1+d)(1-d)^3} + \frac{(1-\alpha)^2}{2(1-d)(1-d^2)^2} + \frac{(1-\alpha)^3}{3(1-d^2)^3}\right]$$

Proof :

Let $g(z) = \dfrac{\lambda(z-w)^3 f'''(z) + (1+2\lambda)(z-w)^2 f''(z) + (z-w)f'(z)}{\lambda(z-w)^2 f''(z) + (z-w)f'(z)}$. Since $f(z)$

$\in K(\omega, \alpha, \lambda)$

Then $Re\ g(z) > \alpha$.

$$g(z) = \frac{(z-w)[\lambda(z-w)^2 f'''(z) + (1+2\lambda)(z-w)f''(z) + f'(z)]}{(z-w)[\lambda(z-w)f''(z) + f'(z)]}$$

Dividing $(z-w)$ from numerator and denominator , we have

$$g(z) = \frac{\lambda(z-w)^2 f'''(z) + (1+2\lambda)(z-w)f''(z) + f'(z)}{\lambda(z-w)f''(z) + f'(z)}$$

So $g(w) = \dfrac{f'(w)}{f'(w)}$, but $f'(w) = 1$

Hence $g(w) = \dfrac{1}{1} = 1$

Now , define $P_w(z) = \dfrac{g(z)-\alpha}{1-\alpha}$. We want to show that $P_w \in P(w)$

To do this , $P_w(w) = \dfrac{g(w)-\alpha}{1-\alpha} = \dfrac{1-\alpha}{1-\alpha} = 1 \qquad 0 \le \alpha < 1$

And $Re\, P_w(z) = Re\dfrac{g(z)-\alpha}{1-\alpha} > \dfrac{\alpha-\alpha}{1-\alpha} = 0.$ Hence $P_w \in P(w)$.

So we can write $g(z)$ as

$$\frac{\lambda(z-w)^3 f'''(z) + (1+2\lambda)(z-w)^2 f''(z) + (z-w)f'(z)}{\lambda(z-w)^2 f''(z) + (z-w)f'(z)} = \alpha + (1-\alpha)P_w(z)$$

Since $P_w(z) = 1 + \sum_{k=1}^{\infty} B_k (z-w)^k$, then

$\alpha + (1-\alpha)P_w(z) = 1 + (1-\alpha)\sum_{k=1}^{\infty} B_k (z-w)^k$

Then if we multiply $g(z)$ by $\lambda(z-w)^2 f''(z) + (z-w)f'(z)$ we get the left hand side

$\lambda(z-w)^3 f'''(z) + (1+2\lambda)(z-w)^2 f''(z) + (z-w)f'(z) \ldots\ldots\ldots\ldots\ldots\ldots$ (2.19)

Which is equal the right hand side

$[\lambda(z-w)^2 f''(z) + (z-w)f'(z)][1 + (1-\alpha)\sum_{k=1}^{\infty} B_k (z-w)^k] \ldots\ldots$ (2.20)

Now we will compare the coefficient of $(z-w)^2$, $(z-w)^3$ and $(z-w)^4$ in both sides, to do this , let

$f(z) = (z-w) - a_2(z-w)^2 - a_3(z-w)^3 - \ldots\ldots\ldots$,

then

$f'(z) = 1 - 2a_2(z-w) - 3a_3(z-w)^2 - 4a_4(z-w)^3 - \ldots\ldots$,

$f''(z) = -2a_2 - 6a_3(z-w) - 12a_4(z-w)^2 - 20(z-w)^3 - \ldots\ldots\ldots$, and

$f'''(z) = -6a_3 - 24a_4(z-w) - 60a_5(z-w)^2 - \dots\dots\dots$

Substitute in (2.19) then we have

$\lambda(z-w)^3[-6a_3 - 24a_4(z-w) - 60a_5(z-w)^2 - \dots]$

$+(1+2\lambda)(z-w)^2[-2a_2 - 6a_3(z-w) - 12a_4(z-w)^2 - \cdots]$

$+(z-w)[1 - 2a_2(z-w) - 3a_3(z-w)^2 - 4a_4(z-w)^3 - \dots]$

So , the coefficient of $(z-w)^2 = -2a_2(1+2\lambda) - 2a_2 = -4a_2(1+\lambda)$,

the coefficient of $(z-w)^3 = -6a_3\lambda + -6a_3(1+2\lambda) - 3a_3 = -9a_3(1+2\lambda)$

And the coefficient of $(z-w)^4 = -24a_4\lambda - 12(1+2\lambda)a_4 - 4a_4 = -16a_4(3\lambda+1)$

Also substitute in (2.20) we get

$\lambda(z-w)^2 f''(z) =$

$\lambda(z-w)^2[-2a_2 - 6a_3(z-w) - 12a_4(z-w)^2 - 20(z-w)^3 - \dots] \dots\dots\dots\dots. (2.21)$

$(z-w)f'(z) = (z-w)[1 - 2a_2(z-w) - 3a_3(z-w)^2 - 4a_4(z-w)^3 - ..] \dots (2.22)$

Adding (2.21) and (2.22) we get

$\lambda(z-w)^2 f''(z) + (z-w)f'(z) =$

$(z-w) - 2a_2(1+\lambda)(z-w)^2 - 3a_3(1+2\lambda)(z-w)^3 -$

$4a_4(1+3\lambda)(z-w)^4 - \cdots) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\dots\dots\dots\dots\dots\dots.\dots\dots\dots\dots.\dots\dots (2.23)$

Also ,

$1 + (1-\alpha)\sum_1^{\infty} B_k(z-w)^k =$

$1 + (1-\alpha)[B_1(z-w) + B_2(z-w)^2 + B_3(z-w)^3 + \ B_4(z-w)^4 + \cdots] \ldots \ldots (2.24)$

Multiply (2.23) and (2.24) we get (2.20)

So the coefficient of $(z-w)^2$ is

$(1-\alpha)B_1 - 2(1+\lambda)a_2$

The coefficient of $(z-w)^3$ is

$B_2(1-\alpha) - 2B_1(1-\alpha)(1+\lambda)a_2 - 3(1+2\lambda)a_3$

The coefficient of $(z-w)^4$ is

$(1-\alpha)B_3 - 2B_2(1-\alpha)(1+\lambda)a_2 - 3B_1(1-\alpha)(1+2\lambda)a_3 - 4(1+3\lambda)a_4$

Now we want to compare the coefficient of $(z-w)^2$ in both sides (in 2.19 and 2.20) to get the upper bound of a_2

$-4a_2(\lambda+1) = (1-\alpha)B_1 - 2(1+\lambda)a_2$

$-4a_2(\lambda+1) + 2(1+\lambda)a_2 = (1-\alpha)B_1$

$-2a_2(\lambda+1) = (1-\alpha)B_1$, then

$-a_2 = \dfrac{(1-\alpha)B_1}{2(1+\lambda)}$, take the modulus both sides we get

$|a_2| = \dfrac{(1-\alpha)|B_1|}{2(1+\lambda)}$. But from definition 2.1.5 we know that $|B_1| \leq \dfrac{2}{(1+d)(1-d)}$

So ,

$$|a_2| \leq \frac{(1-\alpha)}{(1+\lambda)(1-d^2)}$$

Then from comparison the coefficients of $(z-w)^3$ in both sides we get the following

$$-9a_3(2\lambda+1) = (1-\alpha)B_2 - 2B_1(1-\alpha)(1+\lambda)a_2 - 3(1+2\lambda)a_3$$

$$-9a_3(2\lambda+1) + 3\alpha(1+2\lambda) = (1-\alpha)B_2 - 2B_1(1-\alpha)(1+\lambda)a_2$$

Take the modulus of both sides we get

$$|-6a_3(2\lambda+1)| \leq |(1-\alpha)B_2| + |2B_1(1-\alpha)(1+\lambda)a_2|.$$

$$\Rightarrow 6|a_2|(2\lambda+1) \leq (1-\alpha)|B_2| + 2(1-\alpha)(1+\lambda)|a_2||B_1|$$

Since

$$|B_2| \leq \frac{2}{(1+d)(1-d)^2}, |B_1| \leq \frac{2}{(1+d)(1-d)} \text{ and } |a_2| \leq \frac{(1-\alpha)}{(1+\lambda)(1-d^2)}$$

Then

$$6|a_3|(2\lambda+1) \leq \frac{2(1-\alpha)}{(1-d^2)(1-d)} + \frac{4(1-\alpha)^2}{(1-d^2)^2}$$

$$|a_3| \leq \frac{1}{(1+2\lambda)}\left[\frac{1-\alpha}{3(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{3(1-d^2)^2}\right]$$

Now for the upper bound of a_4 we repeat the above steps as the following :

$$-16a_4(3\lambda+1) = B_3(1-\alpha) - 2B_2(1-\alpha)(1+\lambda)a_2 - 3B_1(1-\alpha)(1+2\lambda)a_3$$
$$-4(1+3\lambda)a_4$$

Then ,

$$-16a_4(3\lambda+1) + 4\alpha(1+3\lambda)a_4$$
$$= B_3(1-\alpha) - 2B_2(1-\alpha)(1+\lambda)a_2 - 3B_1(1-\alpha)(1+2\lambda)a_3$$

Take the modulus of both sides we get

$$|-12a_4(3\lambda+1)| \le |B_3(1-\alpha)| + |2B_2(1-\alpha)(1+\lambda)a_2| + |3B_1(1-\alpha)(1+2\lambda)a_3|$$

Hence ,

$$12|a_4|(3\lambda+1) \le (1-\alpha)|B_3| + 2(1-\alpha)(1+\lambda)|B_2||a_2| + 3(1-\alpha)(1+2\lambda)|B_1||a_3|$$

Since

$$|B_3| \le \frac{2}{(1+d)(1-d)^3}\ , |B_2| \le \frac{2}{(1+d)(1-d)^2}, |B_1| \le \frac{2}{(1+d)(1-d)},$$

$$|a_3| \le \frac{1}{(1+2\lambda)}\left[\frac{1-\alpha}{3(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{3(1-d^2)^2}\right] \text{ and } \quad |a_2| \le \frac{(1-\alpha)}{(1+\lambda)(1-d^2)}$$

$$12|a_4|(3\lambda+1) \le \frac{2(1-\alpha)}{(1-d)^3(1+d)} + \frac{4(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2}$$
$$+\frac{4(1-\alpha)^3}{(1-d^2)^3}$$

That is

$$12|a_4|(3\lambda+1) \le \frac{2(1-\alpha)}{(1-d^3)(1+d)} + \frac{6(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{(1-d^2)^3}$$

$$|a_4| \leq \frac{1}{(1+3\lambda)}\left[\frac{(1-\alpha)}{6(1+d)(1-d)^3} + \frac{(1-\alpha)^2}{2(1-d)(1-d^2)^2} + \frac{(1-\alpha)^3}{3(1-d^2)^3}\right]$$

Continue the comparing process we get :

$$|a_5| \leq \frac{1}{20(1+4\lambda)}\left[\begin{matrix} |B_4|(1-\alpha) + 2\,|B_3|(1-\alpha)(1+\lambda)|a_2| + 3|B_2|(1-\alpha)(1+2\lambda)|a_3| + \\ 4\,|\,B_1|(1-\alpha)(1+3\lambda)|a_4| \end{matrix}\right]$$

And

$$|a_6| \leq \frac{1}{30(1+5\lambda)}\left[\begin{matrix} |B_5|(1-\alpha) + \ 2|B_4|(1-\alpha)(1+\lambda)|a_2| + 3|B_3|(1-\alpha)(1+2\lambda)|a_3| + \\ 4|B_2|(1-\alpha)(1+3\lambda)|a_4| + 5|B_1|(1-\alpha)(1+4\lambda)|a_5| \end{matrix}\right]$$

We note from the proof that :

$$|a_2| = \frac{1}{1+\lambda}[(1-\alpha)|B_1|]$$

$$|a_3| \leq \frac{1}{6(1+2\lambda)}[|B_2|(1-\alpha) + \ 2|B_1|(1-\alpha)(1+\lambda)|a_2|]$$

$$|a_4| \leq$$

$$\frac{1}{12(1+3\lambda)}[|B_3|(1-\alpha) + 2|B_2|(1-\alpha)(1+\lambda)|a_2| + 3|B_1|(1-\alpha)(1+2\lambda)|a_3|\,]$$

$$|a_5| \leq \frac{1}{20(1+4\lambda)}\left[\begin{matrix} |B_4|(1-\alpha) + \ 2|B_3|(1-\alpha)(1+\lambda)|a_2| + 3|B_2|(1-\alpha)(1+2\lambda)|a_3| + \\ 4\,|\,B_1|(1-\alpha)(1+3\lambda)|a_4| \end{matrix}\right]$$

And

$$|a_6| \leq \frac{1}{30(1+5\lambda)}\left[\begin{matrix} |B_5|(1-\alpha) + \ 2|B_4|(1-\alpha)(1+\lambda)|a_2| + 3|B_3|(1-\alpha)(1+2\lambda)|a_3| + \\ 4|B_2|(1-\alpha)(1+3\lambda)|a_4| + 5|B_1|(1-\alpha)(1+4\lambda)|a_5| \end{matrix}\right]$$

If we continue this process , we can show that

$|a_k| \leq$

$$\frac{1}{k(k-1)[1+(k-1)\lambda]}\begin{bmatrix} |B_{k-1}|(1-\alpha) + \ 2|B_{k-2}|(1-\alpha)(1+\lambda)|a_2| + 3|B_{k-3}|(1-\alpha) \\ (1+2\lambda)|a_3| + 4|B_{k-4}|(1-\alpha)(1+3\lambda)|a_4| + \cdots \end{bmatrix}$$

2.3 Integral Mean Inequality [6]

In this section we want to recall the concept of subordination between analytic functions .

Given two functions $f(z)$ and $g(z)$, which are analytic in Δ , the function $f(z)$ is said to be subordinate to $g(z)$ in Δ if there exists a function $h(z)$ analytic in Δ with $h(0) = 0$ and $|h(z)| < 1$, such that $f(z) = g(h(z))$. We denote this subordination by

$f(z) \prec g(z)$.

Now , we will introduce the relation between subordination and the integral mean inequality by Littlewood Theorem .

Lemma 2.3.1 (littlewood) :

If $f(z)$ and $g(z)$ are analytic in Δ with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}$

$0 < r < 1$

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \ \leq \ \int_0^{2\pi} |g(z)|^{\mu} d\theta$$

For proof see [4].

We want to show that the integral mean inequality can be applied to functions in the both subclasses $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$.

Theorem 2.3.2 :

Let $\delta > 0$. If $f(z) \in H(\omega, \alpha, \lambda)$ then we have

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\delta} d\theta$$

With $z = re^{i\theta}$, $\omega = de^{i\theta}$ and $0 \leq d < r < 1$

Where

$$g(z) = (z - w) - \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)}(z-w)^2$$

Proof :

Since $f(z) \in H(\omega, \alpha, \lambda)$ then $f(z) = (z - w) - \sum_2^\infty a_k (z-w)^{k-1}$

If we write f and g as ,

$$f(z) = (z - w)(1 - \textstyle\sum_2^\infty a_k (z-w)^{k-1}) \quad \text{and}$$

$$g(z) = (z - w)\left[1 - \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)}(z-w)\right]$$

Then we want to show that

$$f_w(z) = 1 - \sum_2^\infty a_k (z-w)^{k-1}$$

is subordinate to

$$g_w(z) = 1 - \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)}(z-w)$$

To do this , define

$$h_w(z) = (z-w) + \frac{(2-\alpha)(1+\lambda)}{(3-\alpha)(1+2\lambda)(1-\mathrm{d})}(z-w)^2 + \cdots$$

which is equal to

$$\sum_{n=2}^{\infty} \frac{(2-\alpha)(1+\lambda)(1-\mathrm{d})}{(1-\alpha)} a_k (z-w)^{k-1}$$

Clearly $h_w(w) = 0$ and

$$|h_w(z)| = \left| \sum_{n=2}^{\infty} \frac{(2-\alpha)(1+\lambda)(1-\mathrm{d})}{(1-\alpha)} a_k (z-w)^{k-1} \right|$$

Then ,

$$|h_w(z)| \leq |z-w| \sum_{2}^{\infty} \frac{(2-\alpha)(1+\lambda)(1-\mathrm{d})^{\mathrm{k}-1}}{(1-\alpha)} a_k$$

Since $k \geq 2$, then $2-\alpha \leq k-\alpha$, and

$|1+k\lambda-\lambda| \geq |1+\lambda|$, so

$$|h_w(z)| \leq |z-w| \sum_{2}^{\infty} \frac{(k-\alpha)(1+k\lambda+\lambda)(1-\mathrm{d})^{\mathrm{k}-1}}{(1-\alpha)} a_k$$

But we know from lemma 2.2.1 that $\sum_2^\infty (k-\alpha)(1+\lambda\mathrm{k}-\lambda)(1-\mathrm{d})^{\mathrm{k}-1} \leq 1-\alpha$

Hence

$$|h_w(z)| \leq |z-w| \sum_{2}^{\infty} \frac{(k-\alpha)(1+\lambda\mathrm{k}-\lambda)(1-\mathrm{d})^{\mathrm{k}-1}}{(1-\alpha)} a_k \leq |z-w| \frac{(1-\alpha)}{(1-\alpha)}$$

$$\leq |z-w| < 1$$

Also

$$g_w\big(h_w(z)\big) = 1 - \frac{1-\alpha}{(2-\alpha)(1+\lambda)(1-\mathrm{d})} \sum_{2}^{\infty} \frac{(2-\alpha)(1+\lambda)(1-\mathrm{d})}{(1-\alpha)} a_k (z-w)^{k-1}$$

$= f_w(z)$.

Hence $f_w \prec g_w$. But $f(z) = (z-w) f_w(z)$ and $g(z) = (z-w) g_w(z)$

Then $f(z) = g\big(h_w(z)\big)$, i.e $f \prec g$.

Then , by Littlewood theorem we get that :

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \;\leq\; \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\delta} d\theta$$

Theorem 2.3.3 :

Let $\delta > 0$. If $f(z) \in K(\omega, \alpha, \lambda)$, then

$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \;\leq\; \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\delta} d\theta$

With $z = re^{i\theta}$, $\omega = de^{i\theta}$ and $0 \leq d < r < 1$

Where

$$g(z) = (z-w) - \frac{1-\alpha}{2(2-\alpha)(1-d)(1+\lambda)} (z-w)^2$$

Proof :

Since $f(z) \in k(\omega, \alpha, \lambda)$ then $f(z) = (z-w) - \sum_2^\infty a_k (z-w)^{k-1}$

If we write f and g as ,

$$f(z) = (z-w)(1 - \sum_2^\infty a_k (z-w)^{k-1}) \quad \text{and}$$

$$g(z) = (z-w)\left[1 - \frac{1-\alpha}{2(2-\alpha)(1-d)(1+\lambda)}(z-w)\right]$$

Then we want to show that

$$f_w(z) = 1 - \sum_2^\infty a_k (z-w)^{k-1})$$

is subordinate to

$$g_w(z) = 1 - \frac{1-\alpha}{2(2-\alpha)(1-d)(1+\lambda)}(z-w)$$

To do this , define

$$h_w(z) = (z-w) + \frac{2(2-\alpha)(1+\lambda)}{(3-\alpha)(1+2\lambda)(1-\mathrm{d})}(z-w)^2 + \cdots$$

which is equal to

$$\sum_{n=2}^\infty \frac{2(2-\alpha)(1+\lambda)(1-\mathrm{d})}{(1-\alpha)} a_k (z-w)^{k-1}$$

Clearly $h_w(w) = 0$ and

$$|h_w(z)| = \left|\sum_{n=2}^{\infty} \frac{2(2-\alpha)(1+\lambda)(1-\mathrm{d})}{(1-\alpha)} a_k (z-w)^{k-1}\right|$$

Then ,

$$|h_w(z)| \leq |z-w| \sum_{2}^{\infty} \frac{2(2-\alpha)(1+\lambda)(1-\mathrm{d})^{\mathrm{k}-1}}{(1-\alpha)} a_k$$

Since $k \geq 2$, then $2(2-\alpha) \leq k(k-\alpha)$, and

$|1+k\lambda-\lambda| \geq |1+\lambda|$, so

$$|h_w(z)| \leq |z-w| \sum_{2}^{\infty} \frac{k(k-\alpha)(1+k\lambda+\lambda)(1-\mathrm{d})^{\mathrm{k}-1}}{(1-\alpha)} a_k$$

But we know from lemma 2.2.1 that $\sum_2^\infty k(k-\alpha)(1+\lambda \mathrm{k}-\lambda)(1-\mathrm{d})^{\mathrm{k}-1} \leq 1-\alpha$

Hence

$$|h_w(z)| \leq |z-w| \sum_{2}^{\infty} \frac{k(k-\alpha)(1+\lambda\mathrm{k}-\lambda)(1-\mathrm{d})^{\mathrm{k}-1}}{(1-\alpha)} a_k \leq |z-w| \frac{(1-\alpha)}{(1-\alpha)}$$

$$\leq |z-w| < 1$$

Hence $f_w \prec g_w$. But $f(z) = (z-w)f_w(z)$ and $g(z) = (z-w)g_w(z)$

Then $f(z) = g\big(h_w(z)\big)$, i.e $f \prec g$.

Then , by Littlewood theorem we get that :

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\delta d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^\delta d\theta$$

Conclusion

In this thesis I studied the analytic univalent functions and some of main related theorems such as Riemann Mapping Theorem which leads us to define the normalized univalent functions .

Moreover I used the normalization of univalent functions to define the class S of univalent functions and other subclasses of S with specified conditions such as P and K .

In 2013 the researchers "B. Srutha Keerthi & M.Revathi " in [4] defined new subclasses of analytic univalent functions with negative coefficients $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ and they discussed the coefficients bound for the functions in both classes.

Here we analyzed the main concepts of the paper and proved many relations that I used it later to prove some theorems in the thesis .

In addition , I used the subordinate concept between analytic functions in order to prove the integral main inequality for the functions in both classes $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$.

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ملخص

في هذه الدراسة تم تعميم الصفوف S و P وK لأي عدد حقيقي α يقع بين الصفر والواحد . كما تم إثبات العديد من خصائص هذه الصفوف . أيضا تم التعرف على أنواع جديدة من الصفوف الجزئية التي تحوي الاقترانات الواحد لواحد وقابلة للتحليل في مجالها وهي $H(\omega,\alpha,\lambda)$ و $K(\omega,\alpha,\lambda)$ ، حيث λ عدد حقيقي يقع ما بين الصفر الواحد وجميع معاملات هذه الاقترانات تكون أعدادا سالبة. أيضا تم إثبات بعض خصائص هذه الصفوف الجزئية ، بالإضافة إلى أنه تم حساب الحد الأعلى للقيمة المطلقة لمعاملات الاقترانات في هذه الصفوف الفرعية حسب متسلسلة تايلور، وحساب متباينة التكامل لها .