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Approximate Solutions of Einstein Field Equations

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Dedication

Every challenging work needs self efforts as well as guidance of people especially those who were very close to our heart. My humble effort I dedicate to my sweet and loving

Mother, Father and Wife

whose affection, love, encouragement and prays of day and night make me able to get such success and honor.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Abstract

Einstein field equations (EFEs) play an important role in understanding the theory of general relativity and related phenomena such as gravitational waves. Since, in general, it is almost impossible to find analytical solutions of EFEs, it is necessary to solve these equations numerically (approximately).

In this work, we derive the Einstein field equations (EFEs) and the standard ADM (Arnowitt, Deser and Misner) equations form of EFEs.

The ADM form consists of constraint equations and evolution equations for the raw spatial metric and extrinsic curvature tensors. The corner stone in the derivation of this form is "3+1 formalism", where one splits spacetime into three-dimensional space on the one hand, and time on the other.

We study the BSSN (Baumgarte, Shapiro, Shibata and Nakamura) formulation. In this formulation the ADM equations were modified by factoring out the conformal factor and introducing three connections. The evolution equations can then be reduced to wave equations for the conformal metric components, which are coupled to evolution equations for the connection functions. Small amplitude gravitational waves were evolved and a direct comparison of the numerical performance of the modified ADM equations with the standard ADM equations was made. The results demonstrate that the standard implementation of the ADM system of equations, consisting of evolution equations for the bare metric and extrinsic curvature variables, is more susceptible to numerical instabilities than the modified form of the equations based on a conformal decomposition as suggested by Shibata and Nakamura.

Further, in this work, we consider the problem of specifying Cauchy initial data in the case 3+1 formalism. We also apply the Optimal Homotopy Asymptotic Method, OHAM, and solving the Einstein field equations corresponding to Schwarzschild geometry, i.e. we determine the Schwarzschild solution using OHAM.

Contents

Chapter 1

Introduction

The theory of general relativity (GR), proposed by Albert Einstein at the end of 1915, [13], [14], is the modern theory of gravitation. It is the cornerstone of modern cosmology, the physics of neutron stars and black holes, the generation of gravitational radiation, and countless other cosmic phenomena in which strong-field gravitation plays a dominant role. The general theory of relativity relates the energy-momentum content of the physical universe to the curvature of the model manifold through a set of partial differential equations.

The key insight of Einstein was what is known as the Einstein equivalence principle, the (local) equivalence of gravitation and inirtia. This led him to realization that gravity is best described and understood not as a physical external force like in Newotonian physics but rather as a manifistation of the geometry and curvature of spacetime itself. A massive object produces a distortion in the geometry of spacetime about it, and in turn this distortion affects the movement of physical objects. That is, GR explains gravitation as a consequence of the curvature of spacetime, while in turn spacetime curvature is the consequence of the presence of matter. Spacetime curvature affects the movement of matter, which reciprocally determines the geometric properties and evolution of spacetime. The theory of general relativity is covered in many textbooks, for example, Shutz, A First Course in Relativity [16]; Weinberg, Gravitation and Cosmology [5]; Misner, Thorne and Wheeler, Gravitation [6] and Wald, General Relativity [17].

The first solution presented to the Einstein field equations was published by Karl Schwarzschild in 1916, which has allowed researchers to make many physical predictions with increased precision. It is the unique solution for the field outside a static, spherically symmetric body. However, the Schwarzschild solution is not the only known solution to the Einstein field equations. Some of the famous solutions include the Kerr solution, for the spacetime surrounding a rotating mass, the Reissner-Nordström solution, for the spacetime surrounding a charged mass, and the Kerr-Newman solution, for the spacetime surrounding a charged and rotating mass, [9].

General Relativity of Einstein achieved a variety of other deep and subtle goals. It drops e.g. the assumption that the set M of all events should admit a bijection onto $\mathbb{R}^4$. It also explains why the (heavy) mass appearing in Newton's law of gravitation is the same as the (inertial) mass that appears in his first law of mechanics (the weak equivalence principle). This equivalence means that the trajectory of a freely falling body is completely determined by its initial position and velocity and it is independent of the object's mass or shape. In General Relativity this is explained by the fact that these preferred free-fall trajectories are a part of the structure of the spacetime M, [10].

An important step in understanding general relativity is newotonian mechanics. When describing physical phenomena on Earth, it is natural to use a coordinate system with origin at the center of the Earth. This coordinate system is, however, not ideal for the description of the motion of the planets around the Sun. A coordinate system with origin at the center of the Sun is more natural. Since the Sun moves around the center of the galaxy, there is nothing special about a coordinate system with origin at the Sun's center. This argument can be continued ad infinitum.

The fundamental reference frame of Newton is called absolute space. The geometrical properties of this space are characterized by ordinary Euclidean geometry. This space can be covered by a Cartesian coordinate system. A non-rotating reference frame at rest, or moving uniformly in absolute space is called a Galilean reference frame. With chosen origin and orientation, the system is fixed. Newton also introduced a universal time which proceeds at the same rate at all positions in space. Relative to a Galilean reference frame, all mechanical systems behave according to Newton's three laws.

Newton's first law: Free particles move with constant velocity

$$u = \frac{dr}{dt} = constant \tag{1.1}$$

where r is a position vector.

Newton's second law: The acceleration $a = du/dt$ of a particle is proportional to the force F acting on it

$$F = m_i \frac{du}{dt} \tag{1.2}$$

where m_i is the inertial mass of the particle.

Newton's third law: If Particle 1 acts on Particle 2 with a force F_{12}, then Particle 2 acts on Particle 1 with a force

$$F_{21} = -F_{12}. \tag{1.3}$$

The first law can be considered as a special case of the second with $F = 0$. Alternatively, the first law can be thought of as restricting the reference frame to be non-accelerating. This is presupposed for the validity of Newton's second law. Such reference frames are called inertial frames.

Note that in this thesis the Greek indices take the values $0, 1, 2, 3$ while Latin indices take the values $1, 2, 3$. Further, during our work we use Einstein summation convention: Indices that appear twice in an expression as sub and super indices are understood to be summed over all their possible values.

Vectors and tensors are the most important mathematical tools used in both special and general relativities. A "$four-vector$" V^α is a quantity that undergoes the transformation

$$V^\alpha \to V^{\alpha\prime} = \Lambda^\alpha_\beta V^\beta \tag{1.4}$$

when the coordinate system is transformed by

$$x^\alpha \to x'^\alpha = \Lambda^\alpha_\beta x^\beta, \tag{1.5}$$

where Λ^{α}_{β} is constant, restricted by the conditions

$$\Lambda^{\alpha}_{\gamma}\Lambda^{\beta}_{\sigma}\eta_{\alpha\beta} = \eta_{\gamma\sigma}, \tag{1.6}$$

with

$$\eta_{\alpha\beta} = \begin{cases} +1: & \alpha = \beta = 1, 2, \ or\ 3 \\ -1: & \alpha = \beta = 0 \\ 0: & \alpha \neq \beta = 0 \end{cases} \tag{1.7}$$

More precisely, such a V^{α} should be called *contravariant* four-vector, to distinguish it from a *covariant* four-vector, defines as a quantity U_{α} whose transformation rule is

$$U_{\alpha} \to U'_{\alpha} = \Lambda^{\beta}_{\alpha} U_{\beta}, \tag{1.8}$$

where

$$\Lambda^{\beta}_{\alpha} \equiv \eta_{\alpha\gamma}\eta^{\beta\delta}\Lambda^{\gamma}_{\delta}. \tag{1.9}$$

The matrix $\eta^{\beta\delta}$ introduced here is numerically the same as $\eta_{\beta\delta}$, that is,

$$\eta^{\beta\delta} = \eta_{\beta\delta}. \tag{1.10}$$

Many physical quantities are not scalars or vectors, but more complicated objects called *tensors*. A tensor has several contravariant and/or covariant indices with corresponding Lorentz transformations properties, for example,

$$T^{\gamma}_{\alpha\beta} \to T'^{\gamma}_{\alpha\beta} = \Lambda^{\gamma}_{\delta}\Lambda^{\varepsilon}_{\alpha}\Lambda^{\zeta}_{\beta}T^{\delta}_{\varepsilon\zeta}. \tag{1.11}$$

A contravariant or covariant vector can be regarded as a tensor with one index, and a scalar is a tensor with no indices, [9].

Using the notion of multi-linear functions, f is a multi-linear provided it is linear in all its arguments, a tensor is defined as a multi-linear function that maps vectors and one-forms (linear functionals) into $\mathbb{R}$. We distinguish between covariant, contravariant and mixed tensors. A covariant tensor maps vectors only, a contravariant tensor maps one-forms only and a mixed tensor maps both vectors and one-forms.

The most important tensor that one can define on a manifold is the metric tensor, denoted by g (a basic block of Einstein field equations). It ia a bilinear functional that maps two vectors into the number that is their inner product, i.e.

$$g(U,V) \equiv U \cdot V. \tag{1.12}$$

Clearly the metric g is a symmetric second-rank tensor. Its covariant and contravariant components are given by

$$g_{\mu\nu} = g(e_\mu, e_\nu) = e_\mu \cdot e_\nu \quad and \quad g^{\mu\nu} = g(e^\mu, e^\nu) = e^\mu \cdot e^\nu, \tag{1.13}$$

where e_μ are basis vectors that span the vector space tangent to the corresponding manifold and e^μ are the 1−forms dual basis to the vector basis e^μ. The matrix $[g^{\mu\nu}]$ containing the contravariant components of the metric tensor is the inverse of the matrix $[g_{\mu\nu}]$ that contains its covariant components. The mixed components of g are given by

$$g(e^\nu, e_\mu) = g(e_\mu, e^\nu) = \delta^\nu_\mu, \tag{1.14}$$

where the last equality is a result of the reciprocity relation between basis vectors and their duals, [11].

The physics of compact objects is entering a particularly exciting phase, as new instruments can now yield unprecedented observations.

In order to learn from these observations, one has to predict the observed signal from theoretical modeling. The most promising candidates for detection by the gravitational wave laser interferometers are the coalescences of black hole and neutron star binaries. Simulating such mergers requires self-consistent, numerical solutions to Einstein field equations in 3 spatial dimensions, which is extremely challenging.

As stated before, according to Einstein gravitation is a manifestation of spacetime curvature. The relationship between the curvature of spacetime and its matter and energy content is encoded by Einstein field equations. Because of the symmetry, these equations composed of six, second order in time, second order in space, coupled, highly nonlinear, quasi-hyperbolic partial differential equations and four, second order in space, coupled, highly nonlinear elliptic partial differential equations. Thus the Einstein field equations are the set of ten equations in Albert Einstein's general theory of relativity that describes the fundamental interaction of gravitation as a result of spacetime being curved by matter and energy.

The concept of spacetime is an elegant concept; it can be used to describe physics in a different way. However, current numerical methods exist only for space (e.g. finite differencing methods or finite element methods) and time (e.g. Runge-Kutta methods). Therefore we want to decompose the spacetime into space and time, so that we can solve the Einstein equations more easily.

The Einstein field equations may be written in the form, see next chapter for the meaning of the involved variables and constants,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi}{c^4} G T_{\mu\nu}. \tag{1.15}$$

The structure of the spacetime is described by R, the Recci scalar and g, the metric tensor while T pertains to matter and energy affecting that structure. When Λ, the cosmological constant, is zero, the system of equations reduces to the original field equations of general relativity. If T is zero, the field equation describes empty space (the vacuum).

The cosmological constant (Λ) has the same effect as an intrinsic energy density of the vacuum, ρ_{vac}. In this context, it is commonly moved onto the right-hand side of the equation, and defined with a proportionality factor of κ, Einstein's constant, $\Lambda = \kappa\rho_{vac}$.

$R_{\mu\nu}$, the Ricci curvature tensor, is the part of the curvature of space-time that determines the degree to which matter will tend to converge or diverge in time (via the Raychaudhuri equation). It is related to the matter content of the universe by means of the equation, Ricci scalar: $R = R^{\mu\nu} g_{\mu\nu}$.

Note that the metric tensor is a central object in general relativity that describes the local geometry of spacetime (as a result of solving the Einstein field equations). Using the weak-field approximation, the metric can also be thought of as representing the "gravitational potential". The metric tensor is often just called "the metric" and is used to generate the connections that are used to construct the geodesic equations of motion and the Riemann curvature tensor.

Solutions to Einstein field equations, except for a few idealized cases characterised by high degrees of symmetry, have not been obtained as yet for many of the important dynamical scenarios though to occur in nature. With the development of computers, it is now possible to tackle these complicated equations numerically and explore these scenarios in detail. That is the main goal of numerical relativity, the art and science of developing algorithms to solve Einstein's equations for astrophisically realistic, high-velocity, strong field systems, see [7]. Many numerical codes to solve Einstein's equations of general relativity in 3+1 dimensional spacetimes employ the standard 3+1 dimensional Arnwowitt-Deser-Misner ADM form of the field equations. This form involves evolution equations for the raw spatial metric and extrinsic curvature tensors, [1].

The goal of numerical relativity is to study spacetimes that cannot be studied by analytic means. The focus is therefore primarily on dynamical systems. Numerical relativity has been applied in many areas: cosmological models, critical phenomena, perturbed black holes and neutron stars, and the coalescence of black holes and neutron stars, etc. In any of these cases, Einstein's equations can be formulated in several ways that allow us to evolve the dynamics. While ADM methods have received a majority of the attention, characteristic and Reggi calculus based methods have also been used. All of these methods begin with a snapshot of the gravitational fields evolve prescribed data on some hypersurface, the initial data to neighboring hypersurfaces.

Before Einstein's field equations can be solved numerically, they have to be cast into a suitable initial value form. Most commonly, this is done via the standard 3+1 decomposition of Arnowitt, Deser and Misner (ADM). In this formulation, the gravitational fields are described in terms of spatial quantities (the spatial metric and the extrinsic curvature), which satisfy some initial constraints and can then be integrated forward in time.

The standard ADM form, derived in Chapter 2, involves evolution equations for extrinsic curvature tensors and the raw spatial metric, as follows

$$\frac{d}{dt}K_{ij} = -D_iD_j\alpha + \alpha(R_{ij} - 2K_{il}K_i^l + KK_{ij} - M_{ij}). \tag{1.16}$$

Here γ_{ij} is the raw spatial 3-metric, R_{ij} contains the second derivatives of γ_{ij}, K_{ij} is the extrinsic curvature tensors, D_i denotes a spatial, covariant derivative with respect to the coordinate x_i, β^i is the shift, and α is the lapse.

Note also that

$$\gamma_{ij} = g_{ij} + n_i n_j, \quad K_{ij} = -D_i n_j, \quad n_i = \frac{D_i t}{|D_i t|}. \tag{1.17}$$

Further the mass of this spacetime, as measured by a distant static observer in the vacuum exterior, is M, and n^i is the unit normal to the slices. These variables, if known everywhere, describe the whole spacetime. 3-metric and extrinsic curvature describe the hypersurfaces themselves, lapse and shift describe the relation between hypersurfaces, [7], [12]. Standard ADM can be modified to get better stability, the modified version, derived in Chapter 3, reads

$$\frac{d}{dt}\phi = \frac{-1}{6}\alpha K \tag{1.18}$$

$$\frac{d}{dt}K = -\gamma^{ij} D_j D_i \alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + \frac{1}{2}\alpha(\rho + S), \tag{1.19}$$

where γ_{ij} is the metric, and K_{ij} is the extrinsic curvature [1].

Since ADM formulation treated Einstein field equations as an initial value problem, initial data are the starting point for any numerical simulation. In the case of numerical relativity, Einstein's equations constrain the choices of these initial data. Several formalisms are used for specifying Cauchy initial data in the 3+1 decomposition of Einstein's equations, see [8]. The focus of these formalisms is on the initial data needed for Cauchy evolutions of Einstein's equations. These initial data cannot be freely specified in their entirety. Rather they are subject to certain constraints which must be satisfied. Because of the nonlinearity of Einstein's equations, there is no unique way of choosing which pieces of the initial data can be freely specified and which are constrained.

The outline of the thesis is as follows. In the next chapter we derive Einstein fields equation. Chapter Three is devoted to the derivation of ADM and modified ADM methods, further in this chapter we present some numerical tests. In chapter Four we discuss several formalisms of constructing initial data of numerical relativity.

Finally, in Chapter Five, we present the semi-analytic method, Optimal Homotopy Asymptotic Method, and apply it to find the Schwarzschild solution.

Chapter 2

Einstein Field Equations (EFEs)

Intoduction

This chapter is devoted to the derivation of Einstein field equations. As suggested by Einstein himself, gravity is a manifestation of space time curvature induced by the presence of matter. Thus the expected set of equations must describe quantitatively how the curvature of spacetime at any event is related to the matter distribution at that event. These equations will be the Einstein's field equations. They relate the spacetime curvature to its source, the energy-momentum of matter. In fact general relativity explains gravitation as a consequence of the curvature of spacetime, while in turn spacetime curvature is a consequence of the presence of matter, which reciprocally determines the geometric properties of spacetime. This situation is described by John Wheeler as "spacetime tells matter how to move and matter tells spacetime how to curve"[15].

Mathematically, these equations are a set of second order nonlinear partial differential equations for the metric coefficients of spacetime, $g_{\mu\nu}$. The nonlinearity in these equations representing the effect of graviation on itself.

Einstein field equations take the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.1}$$

where $R_{\mu\nu}$ is Ricci tensor, R is Ricci scalar, $g_{\mu\nu}$ is metric tensor, Λ is cosmological constant, G is Newton's gravitational constant, c is the speed of light, $T_{\mu\nu}$ is the stress energy-momentum tensor and μ, ν take the values $0, 1, 2, 3$.

In order to derive the set of equations (2.1), a background knowledge of basics of differential geometry is needed. In the following sections we briefly present metric tensor, Christoffel symbols (affine connections), Ricci tensor, and the stress-energy momentum tensor. At the end of the chapter, we derive Einstein field equations.

2.1 Metric Tensor, $g_{\mu\nu}$

In the mathematical field of differential geometry, a metric tensor is a type of function which takes as input a pair of tangent vectors v and w at a point of a surface (or higher dimensional differentiable manifold) and produces a real number $g(v, w)$ in a way that generalizes many of the familiar properties of the dot product of vectors in Euclidean space. In the same way as a dot product, metric tensors are used to define the length of and angle between tangent vectors. In general relativity (GR) we are concerned with a particular class of differentiable manifolds known as Riemannian manifolds. A Riemannian manifold is a differentiable manifold on which a distance, or metric, has been defined. By manifold we mean any set that can be continuously parameterised. A manifold is continuous if, in the neighbourhood of every point P, there are other points whose coordinates differ infinitesimally from those of P. A manifold is differentiable if it is possible to define a scalar field at each point of the manifold that can be differentiated everywhere. The most important tensor that one can define on a manifold is the metric tensor g. This defines a linear map of two vectors into the number that is their inner product, i.e. $g(u, v) = u \cdot v$. A rank-2 tensor (tensor of order 2) of particular importance is the metric tensor $g_{\mu\nu} = e_\mu \cdot e_\nu$, where e_ν are four basis vectors span the vector space tangent to the spacetime manifold, the metric tensor is associated with the line element as $ds^2 = \sum_{\mu,\nu=0}^{4} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu$. The metric tensor is symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$, non singular and satisfies the usual rank-2 tensor transformation law

$$g'_{\mu\nu} = \frac{\partial \xi^\alpha \partial \xi^\beta}{\partial x^\mu \partial x^\nu} g_{\alpha\beta}. \tag{2.2}$$

Further as matrix, $g_{\mu\nu}$ has four eigenvalues have signs $(-,+,+,+)$, that is, one negative eigenvalue associated with the time dimension, and three positive eigenvalues associated with the spatial dimensions. In special relativity, the metric tensor reduces to the so-called Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu$, which corresponds to a flat space-time. The contravariant metric tensor $g^{\mu\nu}$ is defined by the requirement $g_{\mu\beta} g^{\beta\nu} = \delta_\mu^\nu$ where δ_μ^ν is Kronecker delta

$$\delta_\mu^\nu = \delta_{\mu\nu} = \delta^{\mu\nu} = \begin{cases} 1, & \text{if } \mu = \nu \\ 0, & \text{if } \mu \neq \nu \end{cases}$$

Thus $g^{\mu\nu}$ and $g_{\mu\nu}$ are matrix inverses. We may use contraction with the metric tensor to raise and lower tensor indices. For example: $A^\mu = g^{\mu\nu} A_\nu$. Thus, the scalar product of vectors may also be expressed as $A \cdot B = g_{\mu\nu} A^\mu B^\nu = A_\mu B^\nu$. The Lorentz transformations imply that the interval ds^2 has the same value as measured by any observer [5]. This is a direct consequence of the postulate of the invariance of the speed of light. Note that due to the presence of a negative eigenvalue, the invariant distance is not positive definite. In fact, from the metric one can distinguish events related to each other in three different ways:

$$\begin{aligned} ds^2 &> 0, && \text{spacelike separation,} \\ ds^2 &< 0, && \text{timelike separation,} \\ ds^2 &= 0, && \text{null separation.} \end{aligned}$$

2.2 Christoffel Symbols, $\Gamma^\mu_{\nu\sigma}$

In mathematics and physics, the Christoffel symbols are an array of numbers describing an affine connection.

They are defined as $\Gamma^\mu_{\nu\sigma} = \dfrac{\partial x^\mu}{\partial \zeta^\alpha} \dfrac{\partial^2 \zeta^\alpha}{\partial x^\nu \partial x^\sigma}$, where $\zeta^\alpha(x)$ is the locally inertial coordinate system [5]. Christoffel symbol is a nontensor quantity and associated with the spacetime metric $g_{\mu\nu}$. In tensor calculation, the Christoffel symbol is related to partial derivatives of the metric, and it is given by, see [5] for the proof

$$\Gamma^\mu_{\nu\sigma} = g^{\mu\kappa} \Gamma_{\kappa\nu\sigma} = \frac{1}{2} g^{\mu\kappa} \left(\frac{\partial g_{\kappa\nu}}{\partial x^\sigma} + \frac{\partial g_{\kappa\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\kappa} \right)$$

or

$$\Gamma_{\kappa\nu\sigma} = \frac{1}{2}\left(\frac{\partial g_{\kappa\nu}}{\partial x^{\sigma}} + \frac{\partial g_{\kappa\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\kappa}}\right).$$

A useful usage of Christoffel symbol is in the context of covariant derivatives, $\nabla_\lambda V^\mu = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\nu} V^\nu$ and $\nabla_\lambda V_\mu = \frac{\partial V_\mu}{\partial x^\lambda} - \Gamma^\sigma_{\lambda\mu} V_\sigma$. These covariant derivatives can be extended to general tensors in a natural way. Also Christoffel symbols appear in the definition of the Riemann curvature tensor that used to distinguish between spaces that are not flat .

2.3 Ricci Tensor

Riemann curvature tensor measures the change of a vector as it is transported around a closed circuit while keeping it always parallel to itself "parallel transport". On a flat space, the vector does not change when this is done, while on a curved space it does change. The Ricci tensor, contracted from the Riemann curvature tensor, is the part of the curvature of space-time that determines the degree to which matter will tend to converge or diverge in time (via the Raychaudhuri equation). It is related to the matter content of the universe by means of the Einstein field equations. If the Ricci tensor satisfies the vacuum Einstein equation, then the manifold is an Einstein manifold, which have been extensively studied. In this connection, the Ricci flow equation governs the evolution of a given metric to an Einstein metric; the precise manner in which this occurs ultimately leads to the solution of the Poincaré conjecture. The Ricci tensor $R_{\mu\nu}$ is the trace of Riemann curvature tensor given by:

$$R^{\mu}_{\nu\sigma\kappa} = \frac{\partial \Gamma^{\mu}_{\nu\kappa}}{\partial x^{\sigma}} + \frac{\partial \Gamma^{\mu}_{\nu\sigma}}{\partial x^{\kappa}} + \Gamma^{\mu}_{\rho\sigma}\Gamma^{\rho}_{\nu\kappa} - \Gamma^{\mu}_{\rho\kappa}\Gamma^{\rho}_{\nu\sigma} \tag{2.3}$$

Therefore, The Ricci tensor and Ricci scalar are formed from the Riemann tensor as follows

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}, \quad R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}.$$

2.4 Energy-Momentum Tensor

The stress energy-momentum tensor (energy-momentum tensor) is a tensor quantity in physics that describes the density and flux of energy and momentum in spacetime, generalizing the stress tensor of Newtonian physics. It is an attribute of matter, radiation, and non-gravitational force fields. The stress-energy tensor is the source of the gravitational field in the Einstein field equations of general relativity, just as mass density is the source of such a field in Newtonian gravity. The stress energy-momentum tensor given by $T^{\mu\nu} = \rho u^\mu u^\nu$, where ρ is the proper density of the fluid, i.e. that measured by an observer moving with the local flow, and u^μ is its 4-velocity. To give a physical interpretation of the components of the energy-momentum tensor, it is convenient to consider a local cartesian inertial frame at P in which the set of components of the 4-velocity of the fluid is $u^\mu = \gamma_u(c, \overrightarrow{u})$, where $\gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$. In this frame, writing out the components in full we have

$$
\begin{aligned}
&T^{00} = \rho u^0 u^0 = \gamma_u^2 \rho c^2, \\
&T^{0i} = T^{i0} = \rho u^0 u^i = \gamma_u^2 \rho c u^i, \\
&T^{ij} = \rho u^i u^j = \gamma_u^2 \rho u^i u^j.
\end{aligned}
$$

Thus the physical meanings of these components in this frame are as follows: T^{00} is the energy density of the particles, T^{0i} is the energy flux$\times c^{-1}$ in the i-direction, T^{i0} is the momentum density$\times c$ in the i-direction and T^{ij} is the rate of flow of the i-component of momentum per unit area in the j-direction. It is because of these identifications that the tensor T is known as the energy-momentum or stress-energy tensor.

2.5 Einstein Field Equations

In this section we bring together metric tensor, Ricci tensor and momentum tensor to get Einstein field equations (2.1). If the space is empty, i.e. free of gravitation, then EFEs take the form $R_{\mu\nu} = 0$, which is similar to Laplace equation of empty space $\nabla^2\phi = 0$, but if we need to derive gravitational field equations affected by the matter or the mass contained within it, this matter could be the mass inside the earth or the dust existing in universe, we consider the general theory of relativity that is equivalent to Poisson equation $\nabla^2\phi = 4\pi\rho G$. Here G is the gravitational constant, ρ is the density and ϕ, the sought function, is the gradient of the gravitation field. Thus to get the required field equations, the Ricci tensor must equal a nonzero tensor. The suitable tensor to be considered is the contravariant tensor of energy $T^{\mu\nu}$. Since $R_{\mu\nu}$ is a covariant tensor then in order to get equality we raise the indices of Ricci tensor using metric tensor. Thus we write $R^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}R_{\rho\sigma}$. Therefore we take $R^{\mu\nu} = kT^{\mu\nu}$ where k is a constant, as a consequence one gets in flat spacetime that $T^{\mu\nu} = 0$. Because the spacetime curved partially due to the existence of matter then k assigned small value. Hence in a flat spacetime we ignore k. In Riemann geometry and depending on Bianchi identity we get, (see Hobson *et.al.* [11]), $(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu})_{;\nu} = 0$, where $(;)$ stands for covariant differentiation of any tensor field. Based on the conservation of energy and momentum we have $T^{\mu\nu}_{;\nu} = 0$. Thus we have immediately the Einstein field equations

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = kT^{\mu\nu} \tag{2.4}$$

We shall also use the equivalent form, [12]

$$\vec{\gamma}^{\star}\mathbf{R} = 8\pi\left(\vec{\gamma}^{\star}\mathbf{T} - \frac{1}{2}T\vec{\gamma}^{\star}\mathbf{g}\right) \tag{2.5}$$

where $T := g^{\mu\nu}T_{\mu\nu}$ stands for the trace (with respect to g) of the stress-energy tensor $\boldsymbol{T}$, $\vec{\gamma}$ is the “extended” induced metric $\boldsymbol{\gamma}$ with the first index raised by the metric g, and we use the $\vec{\gamma}^{\star}$ operation to extend the extrinsic curvature tensor $\boldsymbol{K}$, defined a priori as a bilinear form on Σ.

Multiplying the last equation by $g_{\mu\nu}$ and taking into consideration the constant tensors $T = g_{\mu\nu}T^{\mu\nu}$, $R = g_{\mu\nu}R^{\mu\nu}$, $\delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$ (Kroenker delta) and noting that $g^{\mu\nu}g_{\mu\nu} = \delta_{\rho\rho}$ we get $g_{\mu\nu}R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} = kg_{\mu\nu}T^{\mu\nu}$ which implies that $R - \frac{1}{2}(4)R = kT$ or we write

$$R = -kT. \tag{2.6}$$

Equations (2.4) and (2.6) lead to $R^{\mu\nu} = k(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)$. In a vacuum $T^{\mu\nu}$ vanishes, so from the last equation we see that the Einstein field equations in empty space are just $R^{\mu\nu} = 0$ which is equivalent to $R_{\mu\nu} = 0$. Since $g_{\mu\nu;\nu} = 0$, Einstein introduced a scalar multiple of the metric tenor to his field equations. This term is added on the base of cosmological reasons. Therefore with $k = \frac{8\pi G}{c^4}$ Einstein field equations take the form

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}. \tag{2.7}$$

2.6 Exact Solution of EFEs

Solutions of the Einstein field equations are spacetimes that result from solving the Einstein field equations (EFEs) of general relativity. Solving the field equations actually gives a Lorentz manifold.

The Einstein tensor is built up from the metric tensor and its partial derivatives; thus, the EFEs are a system of ten partial differential equations to be solved for the metric.

Solving of the equations is important to realize that the Einstein field equations alone are not enough to determine the evolution of a gravitational system in many cases. They depend on the stress-energy tensor, which depends on the dynamics of matter and energy (such as trajectories of moving particles), which in turn depends on the gravitational field. If one is only interested in the weak field limit of the theory, the dynamics of matter can be computed using special relativity methods and/or Newtonian laws of gravity and then the resulting stress-energy tensor can be plugged into the Einstein field equations. But if the exact solution is required or a solution describing strong fields, the evolution of the metric and the stress-energy tensor must be solved for together.

To obtain solutions, the relevant equations are the above quoted EFEs plus the continuity equation (to determine evolution of the stress-energy tensor):

$$T^{\mu\nu}_{;\nu} = 0. \tag{2.8}$$

This is clearly not enough, as there are only 14 equations (10 from the field equations and 4 from the continuity equation) for 20 unknowns (10 metric components and 10 stress-energy tensor components). Equations of state are missing. In the most general case, it's easy to see that at least 6 more equations are required, possibly more if there are internal degrees of freedom (such as temperature) which may vary throughout space-time.

In practice, it is usually possible to simplify the problem by replacing the full set of equations of state with a simple approximation. Some common approximations are:

• Vacuum

$$T_{\mu\nu} = 0 \tag{2.9}$$

• Perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \ \ \text{where} \ \ u^\mu u_\mu = -1 \tag{2.10}$$

Here ρ is the mass-energy density measured in a momentary co-moving frame, u_a is the fluid's 4-velocity vector field, and p is the pressure.

• Non-interacting dust (a special case of perfect fluid):

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{2.11}$$

For a perfect fluid, another equation of state relating density ρ and pressure p must be added. This equation will often depend on temperature, so a heat transfer equation is required or the postulate that heat transfer can be neglected.

Next, notice that only 10 of the original 14 equations are independent, because the continuity equation $T^{\mu\nu}_{;\nu} = 0$ is a consequence of Einstein's equations. This reflects the fact that the system is gauge invariant (in general, absent some symmetry, any choice of a curvilinear coordinate net on the same system would correspond to a numerically different solution). A "gauge fixing" is needed, i.e. we need to impose 4 (arbitrary) constraints on the coordinate system in order to obtain unequivocal results. These constraints are known as coordinate conditions.

A popular choice of gauge is the so-called "De Donder gauge", also known as the harmonic condition or harmonic gauge

$$g^{\mu\nu}\Gamma^\sigma_{\mu\nu} = 0. \tag{2.12}$$

In numerical relativity, the preferred gauge is the so-called "3+1 decomposition", based on the ADM formalism. In this decomposition, metric is written in the form

$$ds^2 = (-\alpha + \beta^i\beta^j\gamma_{ij})dt^2 + 2\beta^i\gamma_{ij}dtdx^j + \gamma_{ij}dx^i dx^j, \ \ \ where \ \ i, j = 1...3. \tag{2.13}$$

α and β^i are functions of spacetime coordinates and can be chosen arbitrarily in each point. The remaining physical degrees of freedom are contained in γ_{ij}, which represents the Riemannian metric on 3-hypersurfaces $t = const$. For example, a naive choice of $\alpha = 0, \beta_i = 0$, would correspond to a so-called synchronous coordinate system: one where t-coordinate coincides with proper time for any comoving observer (particle that moves along a fixed x^i trajectory).

Chapter 3

ADM Formalism of Einstein Field Equations

Intoduction

The ADM formalism, named for its authors Richard Arnowitt, Stanley Deser and Charles W. Misner, is a Hamiltonian formulation of general relativity that plays an important role in quantum gravity and numerical relativity. It was first published in 1959 [9]. Many numerical codes to solve Einstein's equations of general relativity in 3+1 dimensional spacetime employ the standard ADM form of the field equations. This form involves evolution equations for the raw spatial metric and extrinsic curvature tensors. Following Shibata and Nakamura [3], these equations were modified by factoring out the conformal factor and introducing three connection functions.

In the numerical relativity literature, the 3+1 Einstein equations are sometimes called the ADM equations or standard ADM equations

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij}, \tag{3.1}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)K_{ij} = & -D_i D_j \alpha \\ & + \alpha\left\{R_{ij} + K K_{ij} - 2K_{ik}K^k_j + 4\pi\left[(S-\rho)\gamma_{ij} - 2S_{ij}\right]\right\},\end{aligned} \tag{3.2}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{3.3}$$

$$D_j K^j_i - D_i K = 8\pi S_i, \tag{3.4}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative along n^a (the normal vector), K_{ij} denotes the extrinsic curvature tensor, K denotes the mean curvature (the trace of the extrinsic curvature),

D_i denotes a spatial, covariant derivative with respect to the coordinate x_i (intrinsic derivative), γ_{ij} denotes the metric tensor, α denotes the lapse function, R_{ij} the Rieman tensor, S denotes the matter stress tensor, E denotes the matter energy density, and p_i denotes the matter momentum density.

Modifying Einstein field equations by factoring out the conformal factor and introducing three connection functions leads to the situation that, the evolution equations can then be reduced to wave equations for the conformal metric components, which are coupled to evolution equations for the connection functions (modified ADM equations or the BSSN equations)

$$\gamma_{ij} = e^{4\phi}\tilde{\gamma}_{ij}, \qquad K_{ij} = e^{4\phi}\tilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{3.5}$$

$$\partial_t\phi = -\frac{1}{6}\alpha K + \beta^i\partial_i\phi + \frac{1}{6}\partial_i\beta^i. \tag{3.6}$$

$$\partial_t K = -\gamma^{ij}D_jD_i\alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(\rho+S) + \beta^i\partial_i K. \tag{3.7}$$

$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k. \tag{3.8}$$

$$\begin{aligned}\partial_t\tilde{A}_{ij} = {} & e^{-4\phi}\left(-(D_iD_j\alpha)^{TF} + \alpha(R_{ij}^{TF} - 8\pi S_{ij}^{TF})\right) \\ & + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j) + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k.\end{aligned} \tag{3.9}$$

$$0 = \mathrm{H} = \tilde{\gamma}^{ij}\tilde{D}_i\tilde{D}_j e^{\phi} - \frac{e^{\phi}}{8}\tilde{R} + \frac{e^{5\phi}}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho. \tag{3.10}$$

$$0 = \mathrm{M}^i = \tilde{D}_j(e^{6\phi}\tilde{A}^{ij}) - \frac{2}{3}e^{6\phi}\tilde{D}^iK - 8\pi e^{6\phi}S^i. \tag{3.11}$$

$$\begin{aligned}\partial_t\tilde{\Gamma}^i = & -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_jK - 8\pi\tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right) \\ & + \beta^j\partial_j\tilde{\Gamma}^i - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\tilde{\gamma}^{li}\partial_l\partial_j\beta^j + \tilde{\gamma}^{lj}\partial_j\partial_l\beta^i.\end{aligned} \tag{3.12}$$

$$\tilde{\Gamma}^i \equiv \tilde{\gamma}^{ik}\tilde{\Gamma}^i_{jk} = -\partial_j\tilde{\gamma}^{ij}. \tag{3.13}$$

Note that obviously not all these variables are independent. In particular, the determinant of $\tilde{\gamma}_{ij}$ has to be unity, and the trace of $\tilde{A}_{ij}$ has to vanish. These conditions can either be used to reduce the number of evolved quantities, or, alternatively, all quantities can be evolved and the conditions can be used as a numerical check. In the following we will derive both ADM and modified ADM methods.

3.1 3 + 1 Spacetime Decomposition

The $3+1$ spacetime decomposition consists of splitting spacetime as a series of spatial hypersurfaces, parameterized by time t. We start by defining the lapse and the shift functions. Consider the two hypersurfaces Σ_t and Σ_{t+dt}, see Figure 3.1. Suppose that their 3-metrics are given by, respectively, $g^{ij}(t,x^k)dx^idx^j$ and $g^{ij}(t+dt,x^k)dx^idx^j$. Let the point P_1, with coordinates (t,x^i), be a point on Σ_t. We define the point P_2 as being the intersection of Σ_{t+dt} with the normal to Σ_t at P_1. The proper time interval $d\tau = \alpha dt$ between P_1 and P_2 then defines the lapse function $\alpha(t,x^k)$. Let us define the P_3 point on Σ_{t+dt} as being a point of this hypersurface having the same space coordinates as the P_1 point. The P_3 point coordinates are thus $(t+dt,x^i)$ whereas the coordinates of P_2 are $(t+dt,x^i-\beta^idt)$. The vector binding P_2 and P_3 then defines the shift functions $\beta^i(t,x^k)$. Let P_4 be the point on Σ_{t+dt} with coordinates $(t+dt,x^i+dx^i)$ and the P_6 point on Σ_t having the same space coordinates as P_4, i.e. (t,x^i+dx^i). We define P_5 as being the intersection of the normal to Σ_{t+dt} in P_4 with Σ_t. The P_5 coordinates are then $(t,x^i+dx^i+\beta^idt)$.

We are now able to express the line element ds^2 between the P_1 and P_4 points with help of the 3-metric g_{ij}, the shift and the lapse functions, [19]. Writing the Pythagoras theorem in the non Euclidean 4-space with signature $(-,+,+,+)$, it comes

$$\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\
&= g_{ij}(t,x^k)(x^i(P_5)-x^i(P_1))(x^j(P_5)-x^j(P_1)) - d\tau^2 \\
&= g_{ij}(t,x^k)(dx^i+\beta^idt)(dx^j+\beta^jdt) - \alpha^2dt^2 \\
&= g_{ij}(t,x^k)dx^idx^j + g_{ij}(t,x^k)\beta^jdx^idt + g_{ij}(t,x^k)\beta^idtdx^j \\
&\quad + g_{ij}(t,x^k)\beta^i\beta^jdt^2 - \alpha^2dt^2
\end{aligned} \tag{3.14}$$

from which we get for the metric

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + g_{ij}\beta^i\beta^j & g_{ij}\beta^i \\ g_{ij}\beta^j & g_{ij} \end{bmatrix} \tag{3.15}$$

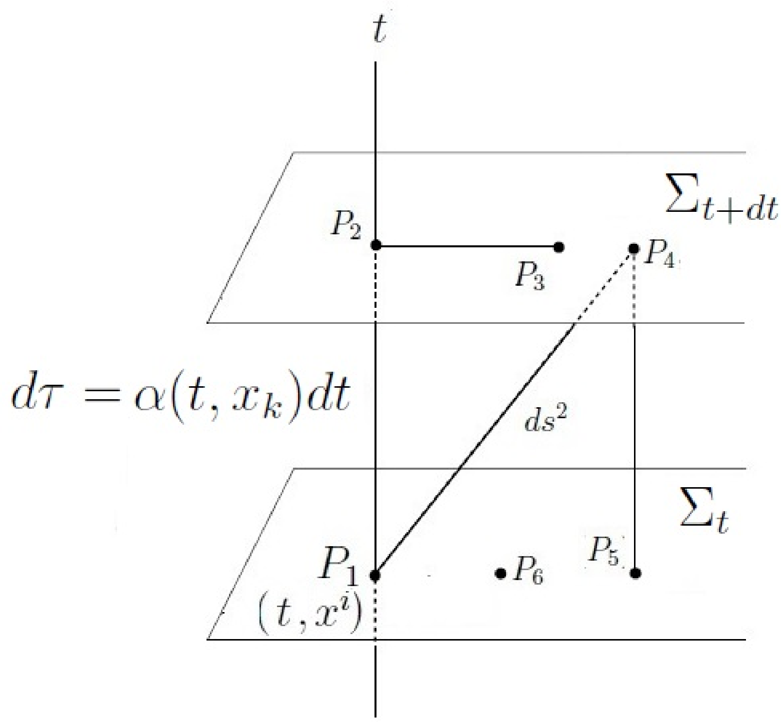

Figure 3.1: **The 3 + 1 spacetime decomposition.**

or

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta_j\beta^j & \beta_j \\ \beta_i & \gamma_{ij} \end{bmatrix} \tag{3.16}$$

where $g_{ij} = \gamma_{ij}$. Equivalently, the line element may be decomposed as

$$\begin{aligned} ds^2 &= \begin{bmatrix} dt & dx^i \end{bmatrix} \begin{bmatrix} -\alpha^2 + \beta_j\beta^j & \beta_j \\ \beta_i & \gamma_{ij} \end{bmatrix} \begin{bmatrix} dt \\ dx^j \end{bmatrix} \\ &= \begin{bmatrix} -\alpha^2 dt + \beta_j\beta^j dt + \beta_i dx^i & \beta_j dt + \gamma_{ij} dx^i \end{bmatrix} \begin{bmatrix} dt \\ dx^j \end{bmatrix} \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
ds^2 &= -\alpha^2 dt^2 + \beta_j \beta^j dt^2 + \beta_i dx^i dt + \beta_j dt dx^j + \gamma_{ij} dx^i dx^j \\
&= -\alpha^2 dt^2 + \underbrace{\gamma_{ij}\beta^i\beta^j}\, dt^2 + \underbrace{\gamma_{ij}\beta^j}\, dx^i dt + \underbrace{\gamma_{ij}\beta^i}\, dt dx^j + \gamma_{ij} dx^i dx^j \\
&= -\alpha^2 dt^2 + \gamma_{ij}(\beta^i \beta^j dt^2 + \beta^i dx^i dt + \beta^i dt dx^j + dx^i dx^j) \\
&= -\alpha^2 dt^2 + \gamma_{ij}(\beta^j dt + dx^j)(\beta^i dt + dx^i),
\end{aligned} \tag{3.18}
$$

then

$$
ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(\beta^j dt + dx^j)(\beta^i dt + dx^i). \tag{3.19}
$$

3.2 Framework and Symbols

3.2.1 Metric Tensor γ_{ij}

It comes from the metric tensor $g_{\mu\nu}$ where μ and ν take the values $0, 1, 2, 3$. From Equation (3.16), we have

$$
g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta_j \beta^j & \beta_j \\ \beta_i & \gamma_{ij} \end{bmatrix} \tag{3.20}
$$

The quantities α and β_i, called the lapse and shift respectively, will be determined independently of the Einstein equations, and Equation (3.20) will be used to determine the time-varying spatial metric γ_{ij}. The projection operator or the intrinsic $3-$metric g_{ij} is defined as

$$
\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \text{ where } n_\mu = (-\alpha, 0, 0, 0), \tag{3.21}
$$

$$
n^\mu = g^{\mu\nu} n_\nu = (\frac{1}{\alpha}, \frac{-\beta^i}{\alpha}) \tag{3.22}
$$

n^μ is the unit normal vector of the spacelike hypersurface Σ.

The hypersurface is said to be, [12],

- Spacelike iff the metric γ is positive definite, i.e. has signature $(+, +, +)$;
- Timelike iff the metric γ is Lorentzian, i.e. has signature $(-, +, +)$;
- Null iff the metric γ is degenerate, i.e. has signature $(0, +, +)$.

3.2.2 Intrinsic Derivative D_i

Returning to the formal derivation of the $3+1$ decomposition we will also need a 3-dimensional covariant derivative that maps spatial tensors into spatial tensors. It is uniquely defined by requiring that it be compatible with the 3-dimensional metric γ_{ij}. We can construct this derivative by projecting all indices present in a 4-dimensional covariant derivative into Σ. For a scalar f, for example, we define

$$D_i f \equiv \gamma_i^j \nabla_j f \tag{3.23}$$

and for a rank $\binom{1}{1}$ tensor T_i^k

$$D_i T_k^j \equiv \gamma_i^l \gamma_e^j \gamma_k^f \nabla_l T_f^e. \tag{3.24}$$

The extension to other type tensors is obvious. Note that γ_i^j is the orthogonal projection $\gamma_i^j = \delta_i^j + n^j n_i$.

The 3-dimensional covariant derivative can be expressed in terms of 3-dimensional connection coefficients, which, in a coordinate basis, are given by

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_k \gamma_{jl} + \partial_j \gamma_{kl} - \partial_l \gamma_{jk}). \tag{3.25}$$

The 3-dimensional Riemann tensor associated with γ_{ij} is defined by requiring that

$$2D_{[i}D_{j]}\omega_k = (D_i D_j - D_j D_i)\,\omega_k = R^l_{kji}\omega_l \text{ and } R^l_{kji} n_l = 0, \tag{3.26}$$

for any spatial vector ω_l. In a coordinate basis, the components of the Riemann tensor can be computed from

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^e_{ik}\Gamma^l_{ej} - \Gamma^e_{jk}\Gamma^l_{ei} \tag{3.27}$$

3.2.3 The Lapse Function α

The timelike and future-directed unit vector n normal to the slice Σ is necessarily collinear to the vector $\vec{\nabla} t$ associated with the gradient 1-form dt, where Σ is defined as a level surface of a scalar field t, i.e. t=constant. The vector $\vec{\nabla} t$ defines the unique direction normal to Σ. In other words, any other vector v normal to Σ must be collinear to

$\overrightarrow{\nabla} t, v = \lambda \overrightarrow{\nabla} t$. Notice a characteristic property of null hypersurfaces is a vector normal to them is also tangent to them. This is because null vectors are orthogonal to themselves. Hence we may write

$$n := -\alpha \overrightarrow{\nabla} t \tag{3.28}$$

with

$$\alpha := (-\overrightarrow{\nabla} t \cdot \overrightarrow{\nabla} t)^{-\frac{1}{2}} \tag{3.29}$$

where Σ is not null, we can re-normalize $\overrightarrow{\nabla} t$ to make it a unit vector.

The minus sign in Equation (3.28) is chosen so that the vector n is future-oriented if the scalar field t is increasing towards the future. Notice that the value of α ensures that n is a unit timelike vector

$$n \cdot n = -1$$

.

The scalar field α hence defined is called the **lapse function**, [12].

Notice that by construction Equation (3.29),

$$\alpha > 0. \tag{3.30}$$

In particular, the lapse function never vanishes for a regular foliation. Equation (3.28) also says that $-\alpha$ is the proportionality factor between the gradient 1-form dt and the 1-form $\underline{n}$ associated to the vector n by the metric duality

$$\underline{n} = -\alpha dt. \tag{3.31}$$

3.2.4 The Shift Vector β

The difference between ∂_t (the time vector) and m (the normal evolution vector $m = \alpha n$) is called the shift vector and is denoted by β

$$\partial_t =: m + \beta. \tag{3.32}$$

By combining the equations

$$< dt, \partial_t >= 1, < dt, m >= \nabla_m t = m^\mu \nabla_\mu t = 1, \tag{3.33}$$

we get

$$< dt, \beta >=< dt, \partial_t > - < dt, m >= 1 - 1 = 0, \tag{3.34}$$

or equivalently, since $dt = -\alpha^{-1}\underline{n}$, we have

$$\underline{n} = -\alpha dt, \quad n \cdot \beta = 0. \tag{3.35}$$

Hence the vector β is tangent to the hypersurfaces Σ.

We write Equation (3.32) as

$$\partial_t = \alpha n + \beta. \tag{3.36}$$

Since the vector n is normal to Σ and β tangent to Σ, Equation (3.36) is a decomposition of the time vector ∂_t.

The scalar square of ∂_t is deduced immediately from Equation (3.36), taking into account $n \cdot n = -1$ and Equation (3.35)

$$\partial_t \cdot \partial_t = -\alpha^2 + \beta \cdot \beta. \tag{3.37}$$

Hence we have the following

$$\partial_t \text{ is } \; \text{timelike} \quad \Leftrightarrow \beta \cdot \beta < \alpha^2, \tag{3.38}$$

$$\partial_t \text{ is } \; \text{null} \quad \Leftrightarrow \beta \cdot \beta = \alpha^2, \tag{3.39}$$

$$\partial_t \text{ is } \; \text{spacelike} \quad \Leftrightarrow \beta \cdot \beta > \alpha^2. \tag{3.40}$$

Since β is tangent to Σ, let us introduce the components of β and the metric dual form $\underline{\beta}$ with respect to the spatial coordinates (x^i) according to

$$\beta =: \beta^i \partial_i \quad and \quad \underline{\beta} =: \beta_i dx^i. \tag{3.41}$$

Equation (3.36) then shows that the components of the unit normal vector n with respect to the natural basis (∂_μ) are expressible in terms of α and (β^i) as

$$n^\mu = \left(\frac{1}{\alpha}, \frac{-\beta^1}{\alpha}, \frac{-\beta^2}{\alpha}, \frac{-\beta^3}{\alpha} \right) \tag{3.42}$$

Notice that the covariant components (i.e. the components of $\underline{n}$ with respect to the basis (dx^μ) of $\mathcal{T}_p^*(\mathcal{M})$ are immediately deduced from the relation $\underline{n} = -\alpha dt$, Equation (3.31),

$$n_\mu = (-\alpha, 0, 0, 0). \tag{3.43}$$

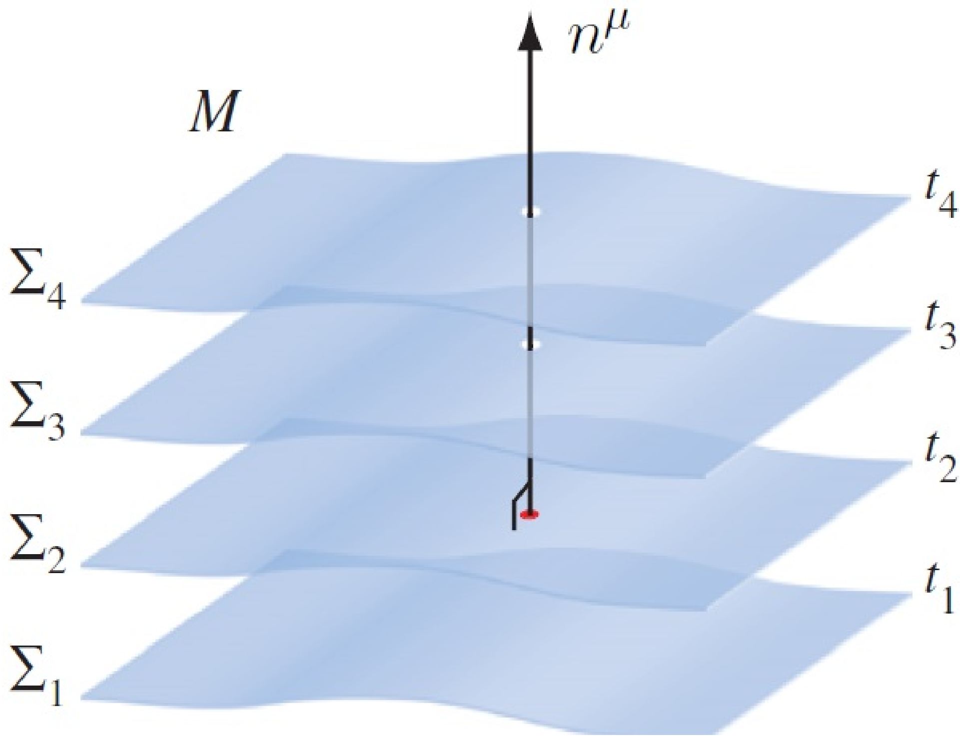

Figure 3.2: **A foliation of the spacetime M. The hypersurfaces Σ are level surfaces of the coordinate time t, $\Omega_\mu = \nabla_\mu t$. The normal vector n^μ is orthogonal to these t = constant spatial hypersurfaces.**

3.2.5 Foliations of spacetime

We assume that the spacetime $(M, g_{\mu\nu})$ can be foliated into a family of nonintersecting spacelike 3-surfaces Σ, which arise, at least locally, as the level surfaces of a scalar function t that can be interpreted as a global time function (see Figure 3.2 for an illustration). From t we can define the 1-form $\Omega_\mu = \nabla_\mu t$, which is closed by construction,

$$\nabla_{[\mu}\Omega_{\nu]} = \nabla_{[\mu}\nabla_{\nu]}t = 0. \tag{3.44}$$

The 4-metric $g_{\mu\nu}$ allows us to compute the norm of Ω, which we call $-\alpha^{-2}$,

$$\|\Omega\|^2 = g^{\mu\nu}\nabla_\mu t \nabla_\nu t \equiv -\frac{1}{\alpha^2}. \tag{3.45}$$

As we will see more clearly later, α measures how much proper time elapses between neighboring time slices along the normal vector Ω^μ to the slice, and is therefore called the lapse function. We assume that $\alpha > 0$, so that Ω^μ is timelike and the hypersurface Σ is spacelike everywhere.

The normalized 1-form $\omega_\mu \equiv \alpha\Omega_\mu$ is rotation-free

$$\omega_{[\mu}\nabla_\nu\omega_{\kappa]} = 0$$

.

We can now define the unit normal to the slices as

$$n^\mu \equiv -g^{\mu\nu}\omega_\nu. \tag{3.46}$$

Here the negative sign has been chosen so that n^μ points in the direction of increasing t,

$$n^\mu\omega_\mu = -g^{\mu\nu}\omega_\mu\omega_\nu = 1. \tag{3.47}$$

By construction, n^μ is normalized and timelike,

$$n^\mu n_\mu = g^{\mu\nu}\omega_\mu\omega_\nu = -1, \tag{3.48}$$

and may therefore be thought of as the 4-velocity of a "normal" observer whose worldline is always normal to the spatial slices Σ.

With the normal vector we can now construct the spatial metric γ_{ij} that is induced by g_{ij} on the 3-dimensional hypersurfaces Σ,

$$\gamma_{ij} = g_{ij} + n_i n_j. \tag{3.49}$$

3.2.6 The extrinsic curvature tensor and mean curvature

Einstein's equations relate contractions of the 4-dimensional Riemann tensor $R^\mu_{\nu\sigma\kappa}$ to the stress-energy tensor. Since we want to rewrite these equations in terms of $3-$dimensional objects, we decompose $R^\mu_{\nu\sigma\kappa}$ into spatial tensors. Not surprisingly, this decomposition involves its $3-$dimensional form $R^\mu_{\nu\sigma\kappa}$, but obviously this cannot contain all the information needed. $R^\kappa_{\mu\nu\sigma}$ is a purely spatial object and can be computed from spatial derivatives of the spatial metric alone, while $R^\kappa_{\mu\nu\sigma}$ is a spacetime creature which also contains time derivatives of the 4−dimensional metric. Stated differently, the 3−dimensional curvature $R^{\mu}_{\nu\sigma\kappa}$ only contains information about the curvature intrinsic to a slice Σ, but it gives no information about what shape this slice takes in the spacetime M in which it is embedded. This information is contained in a tensor called the extrinsic curvature.

The extrinsic curvature K_{ij} can be found by projecting gradients of the normal vector into the slice Σ. We will also see that the extrinsic curvature is related to the first time derivative of the spatial metric γ_{ij}. The metric and the extrinsic curvature (γ_{ij}, K_{ij}) can therefore be considered as the equivalent of positions and velocities in classical mechanics, they measure the "instantaneous" state of the gravitational field, and form the fundamental variables in our initial value formulation.

The projection of the gradient of the normal vector $\gamma_i^k \; \gamma_j^l \; \nabla_k \; n_l$ can be split into a symmetric part, also known as the expansion tensor

$$\theta_{ij} = \gamma_i^k \gamma_j^l \nabla_{(k} n_{l)}, \tag{3.50}$$

and an antisymmetric part, also known as the rotation 2−form or twist,

$$\omega_{ij} = \gamma_i^k \gamma_j^l \nabla_{[k} n_{l]}, \tag{3.51}$$

We now define the extrinsic curvature, K_{ij}, as the negative expansion

$$K_{ij} \equiv -\gamma_i^k \gamma_j^l \nabla_{(k} n_{l)} = -\gamma_i^k \gamma_j^l \nabla_k n_l. \tag{3.52}$$

By definition, the extrinsic curvature is symmetric and purely spatial. It measures the gradient of the normal vectors n^i. Since the latter are normalized, they can only differ in the direction in which they are pointing, and the extrinsic curvature therefore provides information on how much this direction changes from point to point across a spatial hypersurface. As a consequence, the extrinsic curvature measures the rate at which the hypersurface deforms as it is carried forward along a normal.

Alternatively, we can express the extrinsic curvature in terms of the acceleration of the unit normal vector field

$$i_i \equiv n^j \nabla_j n_i. \tag{3.53}$$

Expanding the right hand side of Equation 3.52 and using the identity $n^l \nabla_k n_l = 0$ together with the definition of Equation (3.53) we find

$$
\begin{aligned}
K_{ij} &= -\gamma_i^k \gamma_j^l \nabla_k n_l \\
&= -(\delta_i^k + n_i n^k)(\delta_j^l + n_j n^l)\nabla_k n_l \\
&= -(\delta_i^k + n_i n^k)\delta_j^l \nabla_k n_l = -\nabla_i n_j - n_i i_j \\
&= -\nabla_i n_j - n_i n^j \nabla_j n_i.
\end{aligned}
\tag{3.54}
$$

Finally, we can write the extrinsic curvature as

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}, \tag{3.55}$$

where $\mathcal{L}_n$ denotes the Lie derivative along n^i, see the next subsection. The trace of the extrinsic curvature, often called the mean curvature, $K = g^{ij}K_{ij} = \gamma^{ij}K_{ij}$. Taking the trace of K_{ij}, we find that

$$K = \gamma^{ij}K_{ij} = -\frac{1}{2}\gamma^{ij}\mathcal{L}_n\gamma_{ij} = -\frac{1}{2\gamma}\mathcal{L}_n\gamma = -\frac{1}{\gamma^{\frac{1}{2}}}\mathcal{L}_n\gamma^{\frac{1}{2}} = -\mathcal{L}_n \ln\gamma^{\frac{1}{2}}. \tag{3.56}$$

Since $\gamma^{\frac{1}{2}}d^3x$ is the proper volume element in the spatial slice Σ, the negative of the mean curvature measures the fractional change in the proper $3-$volume along n^i, [7].

3.2.7 The Lie derivative

Consider a (nonzero) vector field X^μ in a manifold M. We can find the integral curves $x^\mu(\lambda)$ (or orbits, or trajectories) of X^μ by integrating the ordinary differential equations

$$\frac{dx^\mu}{d\lambda} = X^\mu(x(\lambda)). \tag{3.57}$$

Here λ is some affine parameter, and we use notation x instead of index notation x^μ for the coordinate location in the argument to make the expressions more transparent.

We would now like to define a derivative of a tensor field, say T^μ_ν, using X^μ. This involves comparing the tensor field at two different points along X^μ, say P and Q, see Figure 3.3, and taking the limit as Q tends to P.

Now, by comparing two tensors at two different locations in the manifold M, we could simply compare components of the tensor field T^μ_ν at P, $T^\mu_\nu(P)$, and at Q, $T^\mu_\nu(Q)$. This

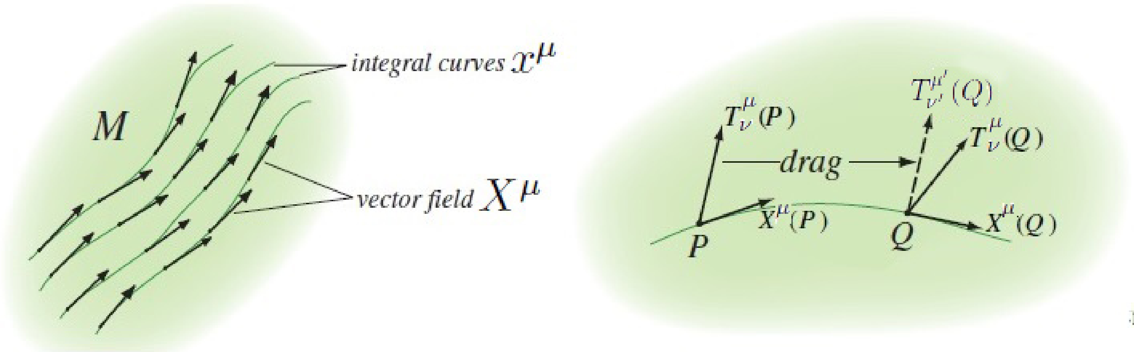

Figure 3.3: **A vector field X^μ generates a congruence of curves x^μ, dragging a tensor T^μ_ν from P to Q.**

leads to the definition of the partial derivative. We have to drag one tensor to the other point before we can compare the two tensors. For example, we can drag $T^\mu_\nu(P)$ along X^μ to the point Q. At Q, we can then compare the dragged tensor, which we will denote with primes, $T^{\mu'}_{\nu'}(Q)$, with the tensor already present at Q, $T^\mu_\nu(Q)$.

Parallel-transporting, leads to the definition of the covariant derivative, is not the only way of dragging T^μ_ν along X^μ. We can view the dragging as a coordinate transformation from P to Q. This, viewing defines the concept of the Lie derivative.

Thus the Lie derivative along a vector field X^μ measures by how much the changes in a tensor field along X^μ differ from a mere infinitesimal coordinate transformation generated by X^μ. Unlike the covariant derivative, the Lie derivative does not require an affine connection and hence requires less structure.

Consider now the infinitesimal coordinate transformation

$$x^{\mu'} = x^\mu + \delta\lambda X^\mu(x), \tag{3.58}$$

which maps the point P, with coordinates x^μ, into the point Q, with coordinates $x^{\mu'}$. We regard this as an active coordinate transformation, which maps points (and tensors) to new locations in the old coordinate system.

Assuming a coordinate basis, we can differentiate Equation (3.58) to find

$$\frac{\partial x^{\mu'}}{\partial x^\nu} = \delta^\mu_\nu + \delta\lambda \partial_\nu X^\mu, \tag{3.59}$$

and, to first order in $\delta\lambda$,

$$\frac{\partial x^{\mu}}{\partial x^{\nu'}} = \delta^{\mu}_{\nu} - \delta\lambda \partial_{\nu} X^{\mu}. \tag{3.60}$$

We now start at point P, where the components of the tensor field T^{μ}_{ν} are $T^{\mu}_{\nu}(x)$. We map this tensor into the primed tensor $T^{\mu'}_{\nu'}(x)$ at Q with the help of the coordinate transformation Equation (3.58)

$$\begin{aligned} T^{\mu'}_{\nu'}(x') &= \frac{\partial x^{\mu'}}{\partial x^{\sigma}} \frac{\partial x^{\kappa}}{\partial x^{\nu'}} T^{\nu}_{\kappa}(x) \\ &= (\delta^{\mu}_{\sigma} + \delta\lambda \partial_{\sigma} X^{\mu})(\delta^{\kappa}_{\nu} - \delta\lambda \partial_{\nu} X^{\kappa}) T^{\sigma}_{\kappa}(x) \\ &= T^{\mu}_{\nu}(x) + \delta\lambda(\partial_{\sigma} X^{\mu} T^{\sigma}_{\nu}(x) - \partial_{\nu} X^{\sigma} T^{\mu}_{\sigma}(x)) + \mathcal{O}(\delta\lambda^2). \end{aligned} \tag{3.61}$$

For the purpose of defining the Lie derivative this is the result of dragging T^{μ}_{ν} along X^{μ} from P to Q. The components of the unprimed tensor already present at Q, $T^{\mu}_{\nu}x'$, can be related to $T^{\mu}_{\nu}x'$ by Taylor expanding

$$\begin{aligned} T^{\mu}_{\nu}x' = T^{\mu}_{\nu}x^{\sigma'} &= T^{\mu}_{\nu}(x^{\sigma} + \delta\lambda X^{\sigma}) \\ &= T^{\mu}_{\nu}(x) + \delta\lambda X^{\sigma} \partial_{\sigma} T^{\mu}_{\nu} + \mathcal{O}(\delta\lambda^2). \end{aligned} \tag{3.62}$$

We now denote the Lie derivative of T^{μ}_{ν} with respect to X^{μ} as $\mathcal{L}_X T^{\mu}_{\nu}$ and define

$$\mathcal{L}_X T^{\mu}_{\nu} \equiv \lim_{\delta\lambda \to 0} \left(\frac{T^{\mu}_{\nu}(x') - T^{\mu'}_{\nu'}(x')}{\delta\lambda} \right). \tag{3.63}$$

An important consequence of this is that the Lie derivative along m of any tensor field T tangent to Σ_t is a tensor field tangent to Σ_t

$$T \text{ tangent } to\ \Sigma_t \ \Rightarrow \ \mathcal{L}_m T \text{ tangent } to\ \Sigma_t \tag{3.64}$$

This definition holds for any tensor of arbitrary rank and type *i.e.*, covariant and contravariant.

Note that we evaluate both tensors at the same point, so that the Lie derivative of a tensor is again a tensor, and moreover a tensor of the same rank. Note also that the expression $\mathcal{L}_X T^{\mu}_{\nu} = (\mathcal{L}_X T)^{\mu}_{\nu}$ implies that the Lie derivative of the tensor T^{μ}_{ν} is again a tensor of rank $\binom{1}{1}$; it does not denote the Lie derivative of the $\mu - \nu$ component of T^{μ}_{ν}. For our tensor T^{μ}_{ν} of rank $\binom{1}{1}$ we can insert Equations (3.61) and (3.62) into Equation

(3.63) to find

$$\mathcal{L}_X T^\mu_\nu = X^\sigma \partial_\sigma T^\mu_\nu - T^\mu_\nu \partial_\sigma X^\mu + T^\mu_\sigma \partial_\nu X^\sigma. \tag{3.65}$$

The Lie derivative of a general tensor field can be found by first taking a partial derivative of the tensor and contracting it with X^μ, and then adding additional terms involving derivatives of X^μ as in Equation (3.65) for each index, with a negative sign for contravariant indices and a positive sign for covariant indices.

3.3 The equations of Gauss, Codazzi and Ricci

The metric γ_{ij} and the extrinsic curvature K_{ij} cannot be chosen arbitrarily. Instead, they have to satisfy certain constraints, so that the spatial slices fit into the spacetime M. In order to find these relations, we have to relate the 3-dimensional Riemann tensor R^i_{jkl} of the hypersurfaces Σ to the 4-dimensional Riemann tensor $R^\mu_{\nu\sigma\kappa}$ of M. To do so, we first take a completely spatial projection of $R^\mu_{\nu\sigma\kappa}$, then a projection with one index projected in the normal direction, and finally a projection with two indices projected in the normal direction. All other projections vanish identically because of the symmetries of the Riemann tensor. A decomposition of $R^\mu_{\nu\sigma\kappa}$ into spatial and normal pieces therefore involves these three different types of projections.

We can write the 4-dimensional Riemann tensor $R_{\mu\nu\sigma\kappa}$ as

$$\begin{aligned} R_{\mu\nu\sigma\kappa} = {} & \gamma^\lambda_\mu \gamma^\upsilon_\nu \gamma^\zeta_\sigma \gamma^\xi_\kappa R_{\lambda\upsilon\zeta\xi} - 2\gamma^\lambda_\mu \gamma^\upsilon_\nu \gamma^\zeta_{[\sigma} n_{\kappa]} n^\xi R_{\lambda\upsilon\zeta\xi} \\ & - 2\gamma^\lambda_\sigma \gamma^\upsilon_\kappa \gamma^\zeta_{[\mu} n_{\nu]} n^\xi R_{\lambda\upsilon\zeta\xi} + 2\gamma^\lambda_\mu \gamma^\zeta_{[\sigma} n_{\kappa]} n_\nu n^\upsilon n^\xi R_{\lambda\upsilon\zeta\xi} \\ & - 2\gamma^\lambda_\nu \gamma^\zeta_{[\sigma} n_{\kappa]} n_\mu n^\upsilon n^\xi R_{\lambda\upsilon\zeta\xi}. \end{aligned} \tag{3.66}$$

The above projections give rise to the equations of Gauss, Codazzi and Ricci, which we will derive below. Given that $R^\mu_{\nu\sigma\kappa}$ involves up to second time derivatives of the metric, while R^i_{jki} only contains space derivatives, we may already anticipate that these relations will involve the extrinsic curvature and its time derivative.

The Riemann tensor is defined in terms of second covariant derivatives of a vector. To relate the 4-dimensional Riemann tensor to its 3-dimensional counterpart, it is therefore natural to start by relating the corresponding covariant derivatives to each other. We first expand the definition of the spatial gradient of a spatial vector V^j as

$$\begin{aligned} D_iV^j &= \gamma_i^p\gamma_q^j\nabla_pV^q = \gamma_i^p(g_q^j + n_qn^j)\nabla_pV^q = \gamma_i^p\nabla_pV^p - \gamma_i^pn^jV^q\nabla_pn_q, \\ &= \gamma_i^p\nabla_pV^j - n^jV^e\gamma_i^p\gamma_e^q\nabla_pn_q = \gamma_i^p\nabla_pV^j + n^jV^eK_{ie}, \end{aligned} \tag{3.67}$$

where we have used $n_qV^q = 0$, and hence $n_q\nabla_pV^q = -V^q\nabla_pn_q$, as well the definition of the extrinsic curvature Equation 3.52.

Also, we can obtain the following equations

$$\nabla_iV^i = \frac{1}{\alpha}D_i(\alpha V^i), \tag{3.68}$$

for any spatial vector V^i and

$$\begin{aligned} D_iD_jV^k &= D_i(D_jV^k) = \gamma_i^p\gamma_j^l\gamma_e^k\nabla_p\left(\gamma_l^q\gamma_r^e\nabla_qV^r\right) \\ &= \gamma_i^p\gamma_j^l\gamma_e^k\left(n^q\nabla_pn_l\gamma_r^e\nabla_qV^r + \gamma_l^q\nabla_pn^e\underbrace{n_r\nabla_qV^r}_{=-V^r\nabla_qn_r} + \gamma_l^q\gamma_r^e\nabla_p\nabla_qV^r\right) \\ &= \gamma_i^p\gamma_j^l\gamma_r^k\nabla_pn_ln^q\nabla_qV^r - \gamma_i^p\gamma_j^q\gamma_e^kV^r\nabla_pn^e\nabla_qn_r + \gamma_i^p\gamma_j^q\gamma_r^k\nabla_p\nabla_qV^r \\ &= -K_{ij}\gamma_r^kn^p\nabla_pV^r - K_i^kK_{jp}V^p + \gamma_i^p\gamma_j^q\gamma_r^k\nabla_p\nabla_qV^r. \end{aligned} \tag{3.69}$$

We can now use Equation (3.69) to relate the 3- and 4-dimensional Riemann tensors to each other. Writing the definition of the 3-dimensional Riemann tensor (3.26) as

$$R_{ji}^{lk}V_l = 2D_{[i}D_{j]}V^k \tag{3.70}$$

we can insert the second derivative Equation (3.69) to find

$$R_{ji}^{lk}V_l = 2\gamma_i^p\gamma_p^q\gamma_r^k\nabla_{[p}\nabla_{q]}V^r - 2K_{[ij]}\gamma_r^kn^p\nabla_pV^r - 2K_{[i}^kK_{j]p}V^p. \tag{3.71}$$

The second term on the right hand side vanishes because K_{ij} is symmetric, and the first term can be rewritten in terms of the 4-dimensional Riemann tensor, which yields

$$R_{lkji}V^l = \gamma_i^p\gamma_j^q\gamma_c^rR_{lrqp}V^l - 2K_{k[i}K_{j]l}V^l \tag{3.72}$$

after relabeling some indices and lowering the index k. Since this relation has to hold for any arbitrary spatial vector V^l, we have

$$R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{kj} = \gamma_i^p\gamma_j^q\gamma_k^r\gamma_l^sR_{pqrs}. \tag{3.73}$$

This equation is called Gauss' equation. It relates the full spatial projection of $R^{\mu}_{\nu\sigma\kappa}$ to the 3-dimensional R^{i}_{jkl} and terms quadratic in the extrinsic curvature.

Next, we want to consider projections of $R^{\mu}_{\nu\sigma\kappa}$ with one index projected in the normal direction. This will involve a spatial derivative of the extrinsic curvature

$$D_i K_{jk} = \gamma_i^p \gamma_j^q \gamma_k^r \nabla_p K_{qr} = -\gamma_i^p \gamma_j^q \gamma_k^r (\nabla_p \nabla_q n_r + \nabla_p (n_q i_r)). \tag{3.74}$$

Since $\gamma_b^q n_q = 0$, only the gradient of n_q will give a nonzero contribution in the second term, namely

$$\gamma_i^p \gamma_j^q \gamma_k^r i_r \nabla_p n_q = -i_k K_{ij}. \tag{3.75}$$

We therefore have

$$D_i K_{jk} = -\gamma_i^p \gamma_j^q \gamma_k^r \nabla_p \nabla_q n^r + i_k K_{ij}. \tag{3.76}$$

Since K_{ij} is symmetric, the last term disappears when antisymmetrizing to give

$$D_{[i} K_{j]k} = -\gamma_i^p \gamma_j^q \gamma_k^r \nabla_{[p} \nabla_{q]} n^r. \tag{3.77}$$

By the definition of the Riemann tensor, this can be rewritten as

$$D_j K_{ik} - D_i K_{jk} = \gamma_i^p \gamma_j^q \gamma_k^r n^s R_{pqrs}. \tag{3.78}$$

This equation is known as the Codazzi equation. Note that Gauss Equation (3.73) and the Codazzi Equation (3.78) depend only on the spatial metric, the extrinsic curvature and their spatial derivatives. They can be thought of as the integrability conditions allowing the embedding of a 3-dimensional slice Σ with data $(\gamma_{ij},\ K_{ij})$ inside a 4-dimensional manifold M with g_{ij}. As we will see in the next section, these two equations give rise to the constraint equations.

However, we first consider the last remaining projection of $R^{\mu}_{\nu\sigma\kappa}$, namely with two indices projected in the normal direction. This will involve a time derivative of K_{ij}, and therefore we first compute

$$\begin{aligned} \mathcal{L}_n K_{ij} =& n^k \nabla_k K_{ij} + 2K_{k(i} \nabla_{j)} n^k \\ =& - n^k \nabla_k \nabla_i n_j - n^k \nabla_k (n_i i_j) - 2K_{k(} K^k_{j)} - 2K_{k(i} n_{j)} i^k. \end{aligned} \tag{3.79}$$

Here we have used Equation (3.54) to expand both terms. We can now insert

$$R_{\kappa\nu\mu\sigma} n^{\kappa} = 2\nabla_{[\sigma}\nabla_{\mu]} n_{\nu}, \tag{3.80}$$

which yields

$$\begin{aligned}\mathcal{L}_n K_{ij} = &- n^l n^k R_{ljik} - n^k \nabla_i \nabla_k n_j - n^k i_j \nabla_k n_i \\ &- n^k n_i \nabla_k i_j - 2K^k_{(i} K_{j)k} - 2K_{k(i} n_{j)} i^k.\end{aligned} \tag{3.81}$$

Using the definition of $i_j = n^k \nabla_k n_j$ and the relation

$$n^k \nabla_i \nabla_k n_j = \nabla_i i_j - (\nabla_i n^k)(\nabla_k n^j) = \nabla_i i_j - K^k_k K_{kl} - n_i i^k K_{kj}, \tag{3.82}$$

several terms cancel and we find

$$\mathcal{L}_n K_{ij} = -n^l n^k R_{ljik} - \nabla_i i_j - n^k n_i \nabla_k i_j - i_i i_j - K^k_j K_{ik} - K_{ki} n_j i^k. \tag{3.83}$$

Since $\mathcal{L}_n K_{ij}$ is purely spatial, projecting the two free indices in Equation (3.83) leaves the left hand side unchanged and results in

$$\mathcal{L}_n K_{ij} = -n^l n^k \gamma^q_i \gamma^r_j R_{lrqk} - \gamma^q_i \gamma^r_j \nabla_q i_r - i_i i_j - K^k_j K_{ik}. \tag{3.84}$$

Finally, we simplify Equation (3.84) with the help of equation

$$D_i i_j = -i_i i_j + \frac{1}{\alpha} D_i D_j \alpha, \tag{3.85}$$

and find

$$\mathcal{L}_n K_{ij} = -n^l n^k \gamma^q_i \gamma^r_j R_{lrqk} - \frac{1}{\alpha} D_i D_j \alpha - K^k_j K_{ik}. \tag{3.86}$$

Equation (3.86) is Ricci equation. It relates the time derivative of K_{ij} to a projection of the 4-dimensional Rieman tensor with two indices projected in the time direction, [7].

3.4 The constraint and evolution equations

To rewrite Einstein's field equations in a $3+1$ form, we need to take the equations of Gauss, Codazzi and Ricci and eliminate the 4-dimensional Rieman tensor using Einsteins equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{3.87}$$

To accomplish this aim, we will first derive the constraint equations from Gauss Equation (3.73) and the Codazzi Equation (3.78), and will then derive the evolution equations from Equation (3.55) and the Ricci Equation (3.86).

Contracting Gauss Equation (3.73) once, we find

$$\gamma^{pr}\gamma^q_j\gamma^s_l R_{pqrs} = R_{jl} + KK_{jl} - K^k_l K_{kj}, \tag{3.88}$$

where K is the trace of the extrinsic curvature, $K = K^i_i$. A further contraction yields

$$\gamma^{pr}\gamma^{qs} R_{pqrs} = R + K^2 - K_{ij}K^{ij}. \tag{3.89}$$

The left hand side can be expanded into

$$\gamma^{pr}\gamma^{qs} R_{pqrs} = (g^{pr} + n^p n^r)(g^{qs} + n^q n^s) R_{pqrs} = R + 2n^p n^r R_{pr}. \tag{3.90}$$

Note that the term $n^p n^r n^q n^s R_{pqrs}$ vanishes identically because of the symmetry properties of the Riemann tensor. We also have

$$\begin{aligned} 2n^p n^r G_{pr} &= 2n^p n^r R_{pr} - n^p n^r g_{pr} R = 2n^p n^r R_{pr} - n^p n^r(\gamma_{pr} - n_p n_r)R \\ &= 2n^p n^r R_{pr} + R = \gamma^{pr}\gamma^{qs} R_{pqrs}, \end{aligned} \tag{3.91}$$

where we have used Equation (3.90) in the last equality. Inserting this into the contracted Gauss Equation (3.89) yields

$$2n^p n^r G_{pr} = R + K^2 - K_{ij}K^{ij}. \tag{3.92}$$

We now define the energy density ρ to be the total energy density as measured by a normal observer n^i,

$$\rho \equiv n_i n_j T^{ij}. \tag{3.93}$$

Using Einstein's Equation (3.87) together with Equations (3.92) and (3.93), we get

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho. \tag{3.94}$$

Equation (3.94) is the Hamiltonian constraint.

Contracting the Codazzi Equation (3.78) once gives

$$D_j K^j_i - D_i K = \gamma^p_i \gamma^{qr} n^s R^{pqrs}. \tag{3.95}$$

The right hand side is

$$\gamma_i^p \gamma^{qr} n^s R^{pqrs} = -\gamma_i^p (g^{qr} + n^q n^r) n^s R_{qprs} = -\gamma_i^p n^s R_{ps} - \gamma_i^p n^q n^r n^s R_{qprs}. \tag{3.96}$$

The last term vanishes again because of the symmetries of R_{efgd}, while the first term on the right hand side can be rewritten using

$$\gamma_i^q n^s G_{qs} = \gamma_i^q n^s R_{qs} - \frac{1}{2}\gamma_i^q n^s g_{qs} R = \gamma_i^q n^s R_{qs}. \tag{3.97}$$

Here the last equality holds because $\gamma_i^q n^s g_{qs} = \gamma_{is} n^s = 0$. Collecting terms and inserting into Equation (3.95) we get

$$D_j K_i^j - D_i K = -\gamma_i^q n^s G_{qs}. \tag{3.98}$$

Now define S_i to be the momentum density as measured by a normal observer n^i,

$$S_i \equiv -\gamma_i^j n^k T_{jk}, \tag{3.99}$$

and find

$$D_j K_i^j - D_i K = 8\pi S_i. \tag{3.100}$$

Equation (3.100) is the momentum constraint.

The evolution equations that evolve the data $(\gamma_{ij},\ K_{ij})$ forward in time can be found from Equation (3.55), which can be considered as the definition of the extrinsic curvature, and the Ricci Equation (3.86). However, the Lie derivative along n^i, $\mathcal{L}_n$, is not a natural time derivative since n^i is not dual to the surface 1-form $\Omega_a = \nabla_a t$, i.e., their dot product is not unity but rather

$$n^i \Omega_i = -\alpha g^{ij} \nabla_i t \nabla_j t = \alpha^{-1}. \tag{3.101}$$

Instead, consider the vector

$$t^i = \alpha n^i + \beta^i, \tag{3.102}$$

which is dual to Ω_i for any spatial shift vector β^i,

$$t^i \Omega_i = \alpha n^i \Omega_i + \beta^i \Omega_i = 1. \tag{3.103}$$

Note that we use $\partial_t = t = \alpha n + \beta$.

Consider now the Lie derivative of K_{ij} along t^i,

$$\mathcal{L}_t K_{ij} = \mathcal{L}_{\alpha n} + \beta K_{ij} = \alpha \mathcal{L}_n K_{ij} + \beta K_{ij}, \tag{3.104}$$

which follows from the definition of the Lie derivative. Here we can insert the Ricci Equation (3.86) to eliminate $\mathcal{L}_n K_{ij}$.

Before we do so, we first rewrite the projection of R_{ijkl} that appears in Equation (3.86) as

$$n^l n^k \gamma_i^q \gamma_j^r R_{lrkq} = \gamma^{kl} \gamma_i^q \gamma_j^r R_{lrkq} - \gamma_i^q \gamma_j^r R_{rq}. \tag{3.105}$$

Next we can replace the first term on the right hand side above by substituting Gauss Equation (3.88) and the second term by substituting Einstein's equations

$$n^l n^k \gamma_i^q \gamma_j^r R_{lrkq} = R_{jl} + K K_{jl} - K_l^k K_{kj} - 8\pi \gamma_i^q \gamma_j^r (T_{rq} - \frac{1}{2} g_{rq} T), \tag{3.106}$$

where $T = T_{ij} g^{ij}$. We now define the spatial stress and its trace according to

$$S_{ij} \equiv \gamma_i^k \gamma_j^l T_{kl} \qquad S \equiv S_i^i. \tag{3.107}$$

We can then evaluate the last term in Equation (3.106) as

$$\gamma_i^q \gamma_j^r g_{rq} g^{ef} T_{ef} = \gamma_{ij} (\gamma^{ef} - n^e n^f) T_{ef} = \gamma_{ij} (S - \rho). \tag{3.108}$$

Inserting these expressions into Equations (3.86) and (3.104), we get

$$\mathcal{L}_t K_{ij} = -D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K_j^k + K K_{ij}) - 8\pi \alpha (S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho)) + \mathcal{L}_\beta K_{ij}. \tag{3.109}$$

This is the evolution equation for the extrinsic curvature. Note that all differential operators and the Ricci tensor R_{ij} are associated with the spatial metric γ_{ij}.

The evolution equation for the spatial metric γ_{ij}, the last missing piece, can be found directly from Equation (3.55), again using Equation (3.102),

$$\mathcal{L}_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}. \tag{3.110}$$

For t equal to normal evolution vector m, we get:

$$\mathcal{L}_m \gamma = -2\alpha K. \tag{3.111}$$

The coupled evolution Equations (3.109) and (3.110) determine the evolution of the gravitational field data (γ_{ij}, K_{ij}). Together with the Constraint Equations (3.94) and (3.100) they are completely equivalent to Einstein's Equations 3.87. Note we have succeeded in recasting Einstein's equations, which are second order in time in their original form, as a coupled set of partial differential equations that are now first order in time. As in electrodynamics, the evolution equations conserve the constraint equations, i.e., if the field data (γ_{ij}, K_{ij}) satisfy the constraints at some time t and are evolved with the evolution equations, then the data will also satisfy the constraint equations at all later times, [7].

3.5 3+1 decomposition of the stress-energy tensor

Let us assume that the spacetime $(M,\ g)$ is globally hyperbolic and let $(\Sigma_t)_{t\in\mathbb{R}}$ be a foliation of M by a family of spacelike hypersurfaces. The foundation of the 3+1 formalism amounts to projecting the Einstein equation onto Σ_t and perpendicularly to Σ_t. To this purpose let us first consider the 3+1 decomposition of the stress-energy tensor.

From the definition of a stress-energy tensor, the matter energy density as measured by the Eulerian observer introduced is

$$E := \mathbf{T}(\mathbf{n}, \mathbf{n}). \tag{3.112}$$

This follows from the fact that the 4-velocity of the Eulerian observer in the unit normal vector n.

Similarly, also from the very definition of a stress-energy tensor, the matter momentum density as measured by the Eulerian observer is the linear form

$$\mathbf{p} := -\mathbf{T}(\mathbf{n}, \overrightarrow{\gamma}(.)). \tag{3.113}$$

i.e. the linear form defined by

$$\forall v \in \mathcal{T}_{\mathbf{p}}(\mathbf{M}), \quad \langle \mathbf{p}, v \rangle = -\mathbf{T}(\mathbf{n}, \overrightarrow{\gamma}(v)). \tag{3.114}$$

In components

$$p_\alpha = -T_{\mu\nu} n^\mu \gamma^\nu_\alpha. \tag{3.115}$$

Notice that, $\mathbf{p}$ is a linear form tangent to Σ_t.

Finally, still from the very definition of a stress-energy tensor, the matter stress tensor as measured by the Eulerian observer is the bilinear form

$$\mathbf{S} := \vec{\gamma}^{\star}\mathbf{T}, \tag{3.116}$$

or, in components,

$$S_{\alpha\beta} = T_{\mu\nu}\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}. \tag{3.117}$$

As for $\mathbf{p}$, $\mathbf{S}$ is a tensor field tangent to Σ_t. Let us denote by S the trace of $\mathbf{S}$ with respect to the metric γ (or equivalently with respect to the metric g)

$$S := \gamma^{ij}S_{ij} = g^{\mu\nu}S_{\mu\nu}. \tag{3.118}$$

3.6 Projection of the Einstein equations and derivation of ADM equations

There are only three possibilities of projection of the Einstein equation:

Full projection onto Σ_t

This amounts to applying the operator $\vec{\gamma}^{\star}$ to the Einstein equation. Doing so we get

$$\vec{\gamma}^{\star}\mathbf{R} = 8\pi\left(\vec{\gamma}^{\star}\mathbf{T} - \frac{1}{2}T\vec{\gamma}^{\star}\mathbf{g}\right), \tag{3.119}$$

where $\vec{\gamma}^{\star}\mathbf{R}$ is given by the $3+1$ decomposition of the Riemann tensor,

$$\vec{\gamma}^{\star}\mathbf{R} = \frac{-1}{\alpha}\mathcal{L}_m\mathbf{K} - \frac{1}{\alpha}\mathbf{DD}\alpha + \mathbf{R} + K\mathbf{K} - 2\mathbf{K}\cdot\vec{\mathbf{K}}, \tag{3.120}$$

and in components

$$\gamma^{\mu}_{\sigma}\gamma^{\nu}_{\kappa}R_{\mu\nu} = -\frac{1}{\alpha}\mathcal{L}_m K_{\sigma\kappa} - \frac{1}{\alpha}D_\sigma D_\kappa\alpha + R_{\sigma\kappa} + KK_{\sigma\kappa} - 2K_{\sigma\mu}K^{\mu}_{\kappa}. \tag{3.121}$$

Inserting this expression into Equation (3.119), we get

$$\frac{-1}{\alpha}\mathcal{L}_m\mathbf{K} - \frac{1}{\alpha}\mathbf{DD}\alpha + \mathbf{R} + K\mathbf{K} - 2\mathbf{K}\cdot\vec{\mathbf{K}} = 8\pi\left(\mathbf{S} - \frac{1}{2}(S-E)\gamma\right), \tag{3.122}$$

or equivalently

$$\mathcal{L}_m\mathbf{K} = -\mathbf{DD}\alpha + \alpha\left\{\mathbf{R} + K\mathbf{K} - 2\mathbf{K}\cdot\overrightarrow{\mathbf{K}} + 4\pi\left[(S-E)\gamma - 2\mathbf{S}\right]\right\}, \tag{3.123}$$

where $\overrightarrow{\gamma}^{\star}T = S$, $T = S - E$, $\overrightarrow{\gamma}^{\star}g$ is simply γ.

In components

$$\begin{aligned}\mathcal{L}_m K_{\sigma\kappa} = &-D_\sigma D_\kappa\alpha \\ &+ \alpha\left\{R_{\sigma\kappa} + KK_{\sigma\kappa} - 2K_{\sigma\mu}K^{\mu}_{\kappa} + 4\pi[(S-E)\gamma_{\alpha\beta} - 2S_{\sigma\kappa}]\right\}.\end{aligned} \tag{3.124}$$

Notice that each term in the above equation is a tensor field tangent to Σ_t. For $\mathcal{L}_m\mathbf{K}$, this results from the fundamental property Equation (3.64) of $\mathcal{L}_m$. Consequently, we may restrict to spatial indices without any loss of generality and write Equation (3.109) as

$$\mathcal{L}_m K_{ij} = -D_iD_j\alpha + \left\{R_{ij} + KK_{ij} - 2K_{ik}K^{k}_{j} + 4\pi[(S-E)\gamma_{ij} - 2S_{ij}]\right\}. \tag{3.125}$$

Therefore, we get the first equation of the ADM equations

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)K_{ij} = &- D_iD_j\alpha \\ &+ \alpha\left\{R_{ij} + KK_{ij} - 2K_{ij}K^{k}_{j} + 4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\right\}.\end{aligned} \tag{3.126}$$

To summarize, the Einstein equation is equivalent to the system of the three Equations (3.126),(3.94),(3.110) and (3.100).

3.7 Einstein equations in 3+1 form

3.7.1 Lie derivatives along m as partial derivatives

Let us consider the term $\mathcal{L}K$ which occurs in the $3+1$ Einstein Equation (3.126). With the aid of $\partial_t \;=:\; m \;+\; \beta$, we can write

$$\mathcal{L}_m\mathbf{K} = \mathcal{L}_{\partial_t}\mathbf{K} - \mathcal{L}_\beta\mathbf{K}. \tag{3.127}$$

This implies that $\mathcal{L}_{\partial_t}\mathbf{K}$ is a tensor field tangent to Σ_t, since both $\mathcal{L}_m\mathbf{K}$ and $\mathcal{L}_\beta\mathbf{K}$ are tangent to Σ_t, the former by the property Equation (3.64) and the latter because β and K are tangent to Σ_t. Moreover, if one uses tensor components with respect to a coordinate system $(x^\alpha) = (t,\ x^i)$ adapted to the foliation, the Lie derivative along ∂_t reduces simply to the partial derivative with respect to t [cf. Eq. (A.3), [7]]

$$\mathcal{L}_{\partial_t} K_{ij} = \frac{\partial K_{ij}}{\partial t}. \tag{3.128}$$

By means of formula (A.6, [7]), one can also express $\mathcal{L}_\beta \mathbf{K}$ in terms of partial derivatives

$$\mathcal{L}_{\partial_t} K_{ij} = \beta^k \frac{\partial K_{ij}}{\partial x^k} + K_{ij}\frac{\partial \beta_k}{\partial x^i} + K_{ik}\frac{\partial \beta^k}{\partial x^j}. \tag{3.129}$$

Similarly, the relation Equation (3.111) between $\mathcal{L}_\beta \gamma$ and $\mathbf{K} = K_{ij}$ becomes

$$\mathcal{L}_{\partial_t}\gamma - \mathcal{L}_\beta \gamma = -2\alpha \mathbf{K}, \tag{3.130}$$

with

$$\mathcal{L}_{\partial_t}\gamma_{ij} = \frac{\partial \gamma_{ij}}{\partial t} \tag{3.131}$$

and, evaluating the Lie derivative with the connection covariant derivatives $\mathbf{D}$ instead of partial derivatives [cf.Eq. (A.8), [7]]

$$\mathcal{L}_\beta \gamma_{ij} = \beta^k \underbrace{D_k \gamma_{ij}}_{=0} + \gamma_{kj} D_i \beta^k, \tag{3.132}$$

i.e.

$$\mathcal{L}_\beta \gamma_{ij} = D_i \beta_j + D_j \beta_i. \tag{3.133}$$

Using Equations (3.127) and (3.128), as well as Equations (3.130) and (3.131), we rewrite the 3+1 Einstein system Equations (3.109), (3.94) and (3.100) as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij}, \tag{3.134}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right) K_{ij} = & -D_i D_j \alpha \\ & + \alpha\left\{R_{ij} + K K_{ij} - 2K_{ik}K^k_j + 4\pi\left[(S-\rho)\gamma_{ij} - 2S_{ij}\right]\right\},\end{aligned} \tag{3.135}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{3.136}$$

$$D_j K^j_i - D_i K = 8\pi S_i, \tag{3.137}$$

In this system, the covariant derivatives D_i can be expressed in terms of partial derivatives with respect to the spatial coordinates (x^i) by means of the Christoffel symbols Γ^i_{jk} of $\mathbf{D}$ associated with (x^i)

$$D_i D_j \alpha = \frac{\partial^2 \alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \alpha}{\partial x^k} \tag{3.138}$$

$$D_j K^j_i = \frac{\partial K^j_i}{\partial x^j} + \Gamma^j_{jk} K^k_i - \Gamma^k_{ji} K^j_k, \tag{3.139}$$

$$D_i K = \frac{\partial K}{\partial x^i}. \tag{3.140}$$

The Lie derivatives along β can be expressed in terms of partial derivatives with respect to the spatial coordinates (x^i), via Equations (3.129) and (3.133)

$$\mathcal{L}_\beta \gamma_{ij} = \frac{\partial \beta_i}{\partial x^j} + \frac{\partial \beta_j}{\partial x^i} - 2\Gamma^k_{ij} \beta_k \tag{3.141}$$

$$\mathcal{L}_\beta K_{ij} = \beta^k \frac{\partial K_{ij}}{\partial x^k} + K_{kj} \frac{\partial \beta^k}{\partial x^i} + K_{ik} \frac{\partial \beta^k}{\partial x^j}. \tag{3.142}$$

Finally, the Ricci tensor and scalar curvature of γ are expressible according to the standard expressions

$$R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \Gamma^k_{ij} \Gamma^l_{kl} - \Gamma^l_{ik} \Gamma^k_{lj} \tag{3.143}$$

$$R = \gamma^{ij} R_{ij}. \tag{3.144}$$

For completeness, let us recall the expression of the Christoffel symbols in terms of partial derivatives of the metric

$$\Gamma^k_{ij} = \frac{1}{2} \gamma^{kl} \left(\frac{\partial \gamma_{li}}{\partial x^i} + \frac{\partial \gamma_{il}}{\partial x^j} - \frac{\partial \gamma_{ij}}{\partial x^l} \right). \tag{3.145}$$

The determinant $\gamma = det(\gamma_{ij})$ of the spatial metric and the trace $K = K^i_i$ of the extrinsic curvature satisfy the equations

$$\partial_t \ln \gamma^{1/2} = \alpha K + D_i \beta^i \tag{3.146}$$

and

$$\partial_t K = -D^2 \alpha + \alpha (K_{ij} K^{ij} + 4\pi(\rho + S)) + \beta^i D_i K, \tag{3.147}$$

where $D^2 = \gamma^{ij} D_i D_j$ is the Laplace operator associated with γ_{ij}. The matter source terms appearing in the above equations are defined by

$$\rho = n_i n_j T^{ij}, \quad S^i = -\gamma^{ij} n^k T_{kj}, \quad S_{ij} = \gamma_{ik} \gamma_{jl} T^{kl}, \quad S = \gamma^{ij} S_{ij}. \tag{3.148}$$

Assuming that matter "source terms" ($E,\ p_i,\ S_{ij}$) are given, the system Equations (3.134) - (3.137), with all the terms explicated according to Equations (3.138) - (3.145) constitutes a second-order non-linear PDE system for the unknowns

$$(\gamma_{ij},\ K_{ij},\ \alpha,\ \beta^i).$$

Remark: In the numerical relativity literature, the 3+1 Einstein Equations (3.134) - (3.137) are sometimes called the "ADM equations".

3.8 Recasting the evolution equation

Recasting Maxwell's equations

Maxwell's equations naturally split into two groups. The first group can be written as

$$D_i E^i - 4\pi\rho = 0 \tag{3.149}$$

$$D_i B^i = 0, \tag{3.150}$$

where E^i and B^i are the electric and the magnetic fields and ρ is the charge density. Here D_i denotes a spatial, covariant derivative with respect to the coordinate x^i. In flat space and Cartesian coordinates, it reduces to an ordinary partial derivative.

The above equations involve only spatial derivatives of the electric and magnetic fields and hold at each instant of time. They therefore constrain any possible configurations of the fields, and are correspondingly called the **constraint equations**.

We can bring Maxwell's evolution equations in a Minkowski spacetime into the form

$$\partial_t A_i = -E_i - D_i\Phi \tag{3.151}$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi j_i. \tag{3.152}$$

Where A_i is the three-vector potential, the magnetic field satisfies $B_i = \epsilon_{ijk} D^j A^k$ and therefore is automatically divergence-free, ϕ is a gauge potential, and the electric field E_i

has to satisfy the constraint equation 3.149,

$$D_i E^i = 4\pi\rho. \tag{3.153}$$

In the above equations ρ is the electric charge density and j_i the current density. In "ADM equations" we have discussed some of the similarities of Maxwell's equations in the above form with the ADM Evolution Equations 3.154 and 3.155, namely

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \tag{3.154}$$

and

$$\begin{aligned}\partial_t K_{ij} =& \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - D_iD_j\alpha - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\ &+ \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k.\end{aligned} \tag{3.155}$$

If we identify the vector potential A_i with the spatial metric γ_{ij} and the electric field E_i with the extrinsic curvature K_{ij}, we see that the right-hand sides of both Equations (3.151) and (3.154) contain a field variable and a spatial derivative of a gauge variable, while the right-hand sides of both equations involve (3.152) and (3.154) matter sources as well as second spatial derivatives of the second field variable. In Equation (3.155) these second derivatives are hidden in the Ricci tensor R_{ij}, which we can write, for example, as

$$R_{ij} = \frac{1}{2}\gamma^{kl}\left(\partial_l\partial_i\gamma_{kj} + \partial_j\partial_k\gamma_{ij} - \partial_j\partial_i\gamma_{kl} - \partial_l\partial_k\gamma_{ij}\right) + \gamma^{kl}\left(\Gamma^m_{il}\Gamma_{mkj} - \Gamma^m_{ij}\Gamma_{mkl}\right). \tag{3.156}$$

We can now exploit these similarities by focusing on the simpler Maxwell system of equations to identify some of the computational shortcomings of these forms of the evolution equations.

First note that Equations (3.151) and (3.152) almost can be combined to yield a wave equation, which would make the system symmetric hyperbolic. To see this, take a time derivative of Equation (3.151) and insert Equation (3.152) to form a single equation for the vector potential A_i

$$-\partial_t^2 A_i + D^jD_jA_i - D_iD^jA_j = D_i\partial_t\Phi - 4\pi j_j. \tag{3.157}$$

On the left-hand side, the second time derivative combines with the Laplace operator $D^j D_j A_i$ to form a wave operator (d'Alembertian). Equations 3.151 and 3.152 would then constitute a wave equation for the components A_i if it weren't for the mixed derivative term $D_i D^j A_j$. In general relativity the situation is very similar. The Ricci tensor R_{ij} on the right hand side of Equation (3.155) contains three mixed derivative terms in addition to the term with a Laplace-like operator acting on γ_{ij}, i.e., $\gamma^{kl}\partial_l\partial_k\gamma_{ij}$. Without these mixed derivative terms the standard ADM equations could be written as a set of wave equations for the components of the spatial metric, which would make them symmetric hyperbolic.

These considerations suggest that it would be desirable to eliminate the mixed derivative terms. In electrodynamics, three different approaches can be taken to eliminate the $D_i D^j A_j$ term: one can make a special gauge choice; one can bring Maxwell's equations into a first order symmetric hyperbolic form; or, one can introduce an auxiliary variable. Regarding the third strategy, we introduce the auxiliary variable Γ defined by $\Gamma = D^i A_i$. We treat this variable as a new, independent field that we evolve. We can derive an evolution equation for Γ from Equation 3.151,

$$\partial_t \Gamma = \partial_t D^i A_i = D^i \partial_t A_i = -D^i E_i - D_i D^i \Phi = -D_i D^i \Phi - 4\pi\rho. \tag{3.158}$$

In terms of Γ, the Evolution Equation 3.152 for E_i becomes

$$\partial_t E_i = -D_j D^j A_i + D_i \Gamma - 4\pi j_i. \tag{3.159}$$

In this formulation the mixed derivative term $D_i D^j A_j$ has been eliminated without using up any gauge freedom, which is still imposed via the choice of Φ. A similar situation could be applied to Einstein field equations.

3.9 Conformal transformation of the spatial metric

Consider the equation for the electric field E_i

$$D_i E^i = 4\pi\rho. \tag{3.160}$$

Given an electrical charge density ρ, we can solve this equation for one of the components of E^i, but not all three of them. For example, we could make certain choices for E^x and E^y, and then solve Equation (3.160) for E^z, even though we might be troubled by the asymmetry in singling out one particular component in this approach. Alternatively, we may prefer to write E^i as some "background" field $\bar{E}^i$ times some overall scaling factor, say ψ

$$E^i = \psi \bar{E}^i \tag{3.161}$$

We could now insert Equation 3.161 into Equation 3.160, make certain choices for all three components of the background field $\bar{E}^i$, and then solve Equation (3.160) for the scaling factor ψ. Though it might not be so useful for treating Maxwell's equations, such an approach leads to a very convenient and tractable system for Einstein's equations.

By analogy with our electromagnetic example Equation (3.161) we begin by writing the spatial metric γ_{ij} as a product of some power of a positive scaling factor ψ and a background metric $\bar{\gamma}_{ij}$

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}. \tag{3.162}$$

This identification is a conformal transformation of the spatial metric. We call ψ the conformal factor, and $\bar{\gamma}_{ij}$ the conformally related metric. We take conventionally ψ to the fourth power. In three dimensions it is natural to use

$$\bar{\gamma}_{ij} = \gamma^{-1/3} \gamma_{ij}, \tag{3.163}$$

where γ is the determinant of γ_{ij} and $\gamma = \psi^{12}$. This particular choice results in $\bar{\gamma} = 1$. Loosely speaking, the conformal factor absorbs the overall scale of the metric, which leaves five degrees of freedom in the conformally related metric.

Superficially, the conformal transformation (3.162) is just a mathematical trick, namely, rewriting one unknown as a product of two unknowns in order to make solving some equations easier. At a deeper level, however, the conformal transformation serves to define an equivalence class of manifolds and metrics.

Inserting the transformation law Equation (3.162) into Equation 3.145 we find that, in three dimensions, the connection coefficients must transform according to

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + 2(\delta^i_j \bar{D}_k \ln\psi + \delta^i_k \bar{D}_j \ln\psi - \bar{\gamma}_{ik}\bar{\gamma}^{il}\bar{D}_l \ln\psi). \tag{3.164}$$

Here we have used

$$\gamma^{ij} = \psi^{-4}\bar{\gamma}^{ij}, \tag{3.165}$$

where $\bar{\gamma}^{ij}$ is the inverse of $\bar{\gamma}_{ij}$. Note that bar mean objects associated with the conformal metric $\bar{\gamma}_{ij}$. In equations ψ must be treated as a scalar function in covariant derivatives (as opposed to a scalar density; cf. Appendix A.3, [7]).

The covariant derivative associated with the connection 3.164 is compatible with the conformally related metric,

$$\bar{D}_i\bar{\gamma}_{jk} = 0. \tag{3.166}$$

For the Ricci tensor we similarly find

$$\begin{aligned} R_{ij} =& \bar{R}_{ij} - 2(\bar{D}_i\bar{D}_j \ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m \ln\psi \\ &+ 4((\bar{D}_i \ln\psi)(\bar{D}_j\ ln\psi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l \ln\psi)(\bar{D}_m \ln\psi)), \end{aligned} \tag{3.167}$$

and for the scalar curvature

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi. \tag{3.168}$$

Here $\bar{D}^2 = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j$ is the covariant Laplace operator associated with $\bar{\gamma}_{ij}$.

3.10 The BSSN formulation of ADM

The BSSN formalism adopts a similar strategy to simplify the three-dimensional, spatial Ricci tensor. In addition, the conformal factor and the trace of the extrinsic curvature are evolved separately in the BSSN formalism, which follows the philosophy of separating transverse from longitudinal, or, equivalently, radiative from nonradiative, degrees of freedom.

To derive this formulation, we begin by writing the conformal factor ψ as $\psi = e^{4\phi}$, so that we have

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}. \tag{3.169}$$

We then require that the determinant of the conformally related metric $\tilde{\gamma}_{ij}$ be equal to that of the flat metric η_{ij} in whatever coordinate system we are using, i.e.,

$$\phi = \frac{1}{12} \ln \left(\frac{\gamma}{\eta} \right). \tag{3.170}$$

and choose

$$e^{4\phi} = \gamma^{1/3} \equiv \ det(\gamma_{ij})^{1/3}, \tag{3.171}$$

so that the determinant of $\tilde{\gamma}_{ij}$ is unity. We also write the trace-free part of the extrinsic curvature K_{ij} as

$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K, \tag{3.172}$$

where $K = \gamma^{ij}K_{ij}$. It turns out to be convenient to introduce

$$\tilde{A}_{ij} = e^{-4\phi}A_{ij}. \tag{3.173}$$

Indices of $\tilde{A}_{ij}$ will be raised and lowered with the conformal metric $\tilde{\gamma}_{ij}$, so that $\tilde{A}^{ij} = e^{-4\phi}A^{ij}$.

Evolution equations for ϕ and K can now be found by taking the trace of the evolution equations (3.146) and (3.147), to give

$$\partial_t\phi = -\frac{1}{6}\alpha K + \beta^i\partial_i\phi + \frac{1}{6}\partial_i\beta^i \tag{3.174}$$

and

$$\partial_t K = -\gamma^{ij}D_jD_i\alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + 4\pi\alpha(\rho + S) + \beta^i\partial_i K. \tag{3.175}$$

Subtracting these equations from the evolution equations (3.154, 3.155) leaves the trace-less parts of the evolution equations for $\tilde{\gamma}_{ij}$ and $\tilde{A}_{ij}$ according to

$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k, \tag{3.176}$$

and

$$\begin{aligned}\partial_t\tilde{A}_{ij} =& e^{-4\phi}\left(-(D_iD_j\alpha)^{TF} + \alpha(R_{ij}^{TF} - 8\pi S_{ij}^{TF})\right) + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j \\ &+ \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k.\end{aligned} \tag{3.177}$$

Note that, the superscript TF denotes the trace-free part of a tensor, e.g., $R_{ij}^{TF} = R_{ij} - \gamma_{ij}R/3$ also in Equation (3.174) through Equation (3.177) the shift terms arise from Lie derivatives $\mathcal{L}_\beta$ of the respective evolution variable appearing on the left-hand side. The divergence of the shift, $\partial_i \beta^i$, appears in the Lie derivative because the choice $\tilde{\gamma} = 1$ makes ϕ a tensor density of weight $1/6$, and $\tilde{\gamma}_{ij}$ and $\tilde{A}_{ij}$ tensor densities of weight $-2/3$.

According to Equation (3.167) we can split the Ricci tensor into two terms

$$R_{ij} = \tilde{R}_{ij} + R^{\phi}_{ij}, \tag{3.178}$$

where only R^{ϕ}_{ij} depends on the conformal function ϕ. We can identify the form of R^{ϕ}_{ij} by inserting $\phi = \ln \psi$ into equation 3.167. We could compute the conformally related Ricci tensor $\tilde{R}_{ij}$ by inserting $\tilde{\gamma}_{ij}$ into Equation (3.156), but that would again introduce the mixed second derivatives that we are trying to avoid. Analogously to the way we introduced a new variable Γ to eliminate the mixed derivatives in Maxwell's evolution equations, we can now define "conformal connection functions"

$$\tilde{\Gamma}^i \equiv \tilde{\gamma}^{ik} \tilde{\Gamma}^i_{jk} = -\partial_j \tilde{\gamma}^{ij}, \tag{3.179}$$

to accomplish the same task in the above evolution equations for the gravitational field. Here the $\tilde{\Gamma}^i_{jk}$ are the connection coefficients associated with $\tilde{\gamma}_{ij}$, and the last equality holds in Cartesian coordinates when $\tilde{\gamma} = 1$. In terms of these conformal connection functions we can now write the Ricci tensor as

$$\tilde{R}_{ij} = -\frac{1}{2} \tilde{\gamma}^{lm} \partial_m \partial_l \tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i} \partial_{j)} \tilde{\Gamma}^k + \tilde{\Gamma}^k \tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm} \left(2 \tilde{\Gamma}^k_{l(i} \tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im} \tilde{\Gamma}_{klj} \right) \tag{3.180}$$

The only explicit second-derivative operator acting on $\tilde{\gamma}_{ij}$ in this expression involves a Laplacian, $\tilde{\gamma}^{lm} \partial_m \partial_l$ all other second derivatives are absorbed in first derivatives of $\tilde{\Gamma}^i$. Adopting this approach requires us to derive separate evolution equations for the $\tilde{\Gamma}^i$. By analogy with the derivation of Equation (3.158) we interchange a partial time and space derivative in the definition Equation (3.179) to obtain

$$\partial_t \tilde{\Gamma}^i = -\partial_j \left(2\alpha \tilde{A}^{ij} - 2\tilde{\gamma}^{m(j} \partial_m \beta^{i)} + \frac{2}{3} \tilde{\gamma}^{ij} \partial_l \beta^l + \beta^l \partial_l \tilde{\gamma}^{ij} \right). \tag{3.181}$$

We can now eliminate the divergence of the extrinsic curvature with the help of the momentum constraint (3.100), which then yields the desired evolution equation

$$\begin{aligned} \partial_t \tilde{\Gamma}^i = & - 2\tilde{A}^{ij} \partial_j \alpha + 2\alpha \left(\tilde{\Gamma}^i_{jk} \tilde{A}^{kj} - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K - 8\pi \tilde{\gamma}^{ij} S_j + 6 \tilde{A}^{ij} \partial_j \phi \right) \\ & + \beta^j \partial_j \tilde{\Gamma}^i - \tilde{\Gamma}^j \partial_j \beta^i + \frac{2}{3} \tilde{\Gamma}^i \partial_j \beta^j + \frac{1}{3} \tilde{\gamma}^{li} \partial_l \partial_j \beta^j + \tilde{\gamma}^{lj} \partial_j \partial_l \beta^i. \end{aligned} \tag{3.182}$$

Equations (3.174) through (3.177), together with Equation (3.182), form a new system of evolution equations that is equivalent to Equations (3.154) and (3.155). Since the $\tilde{\Gamma}^i$ are evolved as independent functions, the defining relation Equation (3.179) serves as a new constraint equation, in addition to Equations (3.94) and (3.100).

Summary

In the BSSN formulation of the 3 + 1 equations the spatial metric γ_{ij} is decomposed into a conformally related metric $\tilde{\gamma}_{ij}$ with determinant $\tilde{\gamma} = 1$ (assuming Cartesian coordinates) and a conformal factor e^{ϕ},

$$\tilde{A}_{ij} = e^{-4\phi} A_{ij}. \tag{3.183}$$

We also decompose the extrinsic curvature into its trace and traceless parts and conformally transform the traceless part as we do the metric,

$$K_{ij} = e^{4\phi} \tilde{A}_{ij} + \frac{1}{3}\gamma_{ij} K. \tag{3.184}$$

In terms of these variables the Hamiltonian constraint (3.94) becomes

$$0 = \mathcal{H} = \tilde{\gamma}^{ij} \tilde{D}_i \tilde{D}_j e^{\phi} - \frac{e^{\phi}}{8}\tilde{R} + \frac{e^{5\phi}}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho, \tag{3.185}$$

while the momentum constraint (3.100) becomes

$$0 = \mathcal{M}^i = \tilde{D}_j(e^{6\phi}\tilde{A}^{ij}) - \frac{2}{3}e^{6\phi}\tilde{D}^i K - 8\pi e^{6\phi} S^i. \tag{3.186}$$

The Evolution Equation (3.146) for γ_{ij} splits into two equations,

$$\partial_t \phi = -\frac{1}{6}\alpha K + \beta^i \partial_i \phi + \frac{1}{6}\partial_i \beta^i \tag{3.187}$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{kj}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k, \tag{3.188}$$

while the Evolution Equation (3.155) for K_{ij} splits into the two equations

$$\partial_t K = -\gamma^{ij} D_j D_i \alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + 4\pi\alpha(\rho + S) + \beta^i \partial_i K. \tag{3.189}$$

$$\begin{aligned} \partial_t \tilde{A}_{ij} =& e^{-4\phi}\left(-(D_i D_j \alpha)^{TF} + \alpha(R_{ij}^{TF} - 8\pi S_{ij}^{TF})\right) + \alpha(K\tilde{A}_{ij} - 2\tilde{\gamma}_{il}\tilde{A}^l_j \\ &+ \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \end{aligned} \tag{3.190}$$

In the last equation the superscript TF denotes the trace-free part of a tensor, e.g., $R_{ij}^{TF} = R_{ij} - \gamma_{ij} R/3$. We also split the Ricci tensor into $R_{ij} = \tilde{R}_{ij} + R_{ij}^{\phi}$, where R_{ij}^{ϕ} can be found by inserting $\phi = \ln(\psi)$ into Equation (3.167). We express $\tilde{R}_{ij}$ in terms of the conformal connection functions $\tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\partial_j \tilde{\gamma}^{ij}$, which yields

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{lm}\partial_m\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\left(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}\right). \tag{3.191}$$

The $\tilde{\Gamma}^i$ are now treated as independent functions that satisfy their own evolution equations,

$$\begin{aligned}\partial_t\tilde{\Gamma}^i =& -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - 8\pi\tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right)\\ &+ \beta^j\partial_j\tilde{\Gamma}^i - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\tilde{\gamma}^{li}\partial_l\partial_j\beta^j + \tilde{\gamma}^{lj}\partial_j\partial_l\beta^i.\end{aligned} \tag{3.192}$$

Both the standard 3 + 1 ADM and BSSN systems give very similar results early on, but the standard system crashes very soon, while the BSSN system remains stable. Similar improvements have been found for many other applications that employ a BSSN scheme, including the propagation of nonlinear gravitational waves and the evolution of spacetimes containing black holes and neutron stars. The BSSN system, or a variation closely related to it, is currently the form of the Einsteins equations most commonly used in numerical relativity.

3.11 Crank Nicolson method

The Crank Nicolson method is a finite difference method used to solve the partial differential equations numerically. It is a second-order implicit method in time. It is numerically stable, in the sense that small error due either to arithmetic inaccuracies or to the truncation approximation of the involved derivatives leads to small error in the output.

For the differential equation

$$u_t = F, \tag{3.193}$$

where F depends on u, the first and the second partial derivatives of u, letting $u(N\Delta t, I\Delta x, J\Delta y, K\Delta z) = u^N_{IJK}$, the equation for Crank Nicolson method is a combination of the forward Euler method at N and the backward Euler method at $N+1$ as follows:

$$\frac{u_{IJK}^{N+1} - u_{IJK}^{N}}{\Delta t} = F_{IJK}^{N} \quad \text{(explicit forward Euler)} \tag{3.194}$$

$$\frac{u_{IJK}^{N+1} - u_{IJK}^{N}}{\Delta t} = F_{IJK}^{N+1} \quad \text{(implicit backward Euler).} \tag{3.195}$$

Then Crank Nicolson reads

$$\frac{u_{IJK}^{N+1} - u_{IJK}^{N}}{\Delta t} = \frac{1}{2}[F_{IJK}^{N+1} + F_{IJK}^{N}]. \tag{3.196}$$

Note that the first and second partial derivatives appear in the right hand side of Crank Nicolson are approximated by central schemes.

3.12 Numerical Implementation

In order to compare the properties of standard ADM system and BSSN system, we implemented them numerically in an identical environment. We integrate the evolution equations with a two-level, iterative Crank-Nicholson method. The iteration is truncated after a certain accuracy has been achieved. However, we iterate at least twice, so that the scheme is second order accurate. The gridpoints on the outer boundaries are updated with a Sommerfeld condition,

$$\lim_{r\to\infty}(e^{2\phi}\partial_t + \alpha\partial_r)rQ = 0. \tag{3.197}$$

Hence we assume that on the outer boundaries, the fundamental variables behave like outgoing radial waves

$$Q(t,r) = \frac{G(\alpha t - e^{2\phi}r)}{r}. \tag{3.198}$$

where Q is any of the fundamental variables (except for the diagonal components of $\tilde{\gamma}_{ij}$, for which the radiative part is $Q = \bar{\gamma}_{ii} - 1$), and G can be found by considering backward characteristic to the previous timestep and interpolating linearly the corresponding variable to that point. we assume actant symmetry in order to minimize the number of gridpoints, and impose corresponding symmetry boundary conditions on the symmetry plains. The calculations presented in this thesis were taken from [1]. The calculations

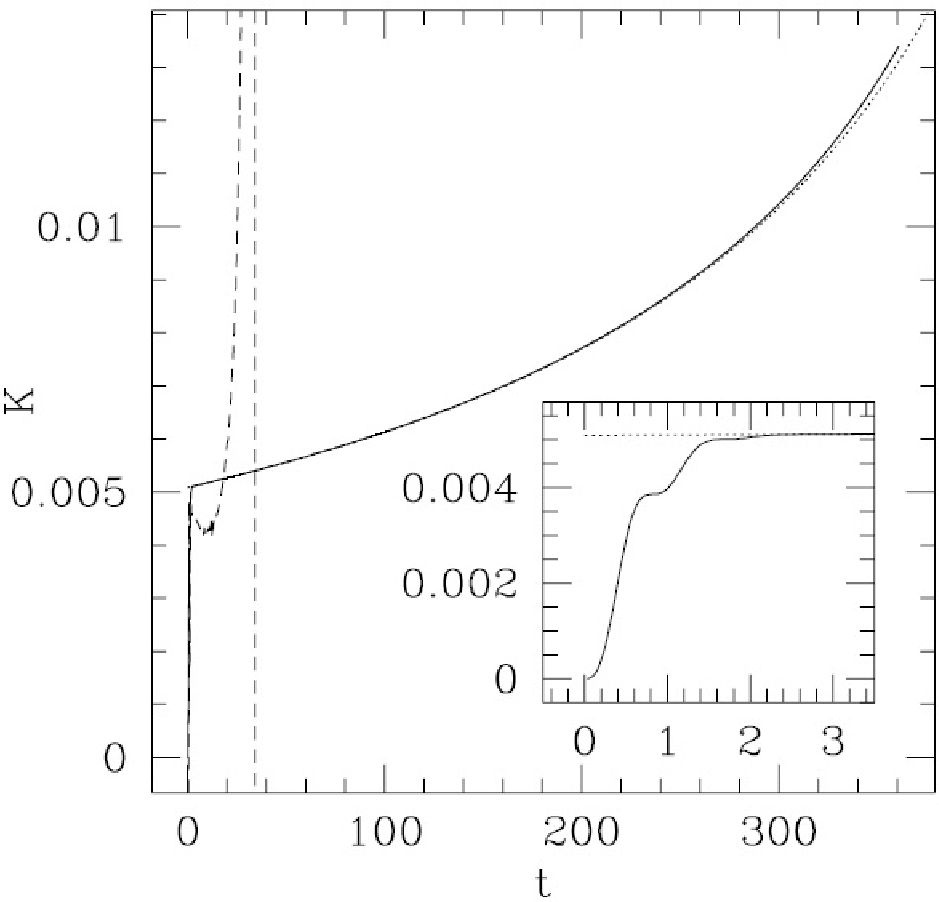

Figure 3.4: **Evolution of the trace of the extrinsic curvature K for a small amplitude wave in geodesic slicing at the origin. The solid line is the result for System II, and the dashed line for System I. The dotted line is the approximate solution $K \sim \frac{3K_0}{3-K_0(t-t_0)}$.**

were performed on grids of $(32)^2$ gridpoints, and used a Courant factor of 1/4. The spatial domain considered is $[-4,4]^3$ and a uniform grid with step size $h=\frac{1}{8}$ is used. Hence the timestep is taken to be $\Delta t=\frac{h}{8}$.

3.13 Results

3.13.1 Initial data

For the initial data we choose a linearized wave solution that evolved with the full non-linear systems ADM and BSSN.

We follow Teukolsky, [4], and construct the solution (called by Teukolsky even-parity $L=2,\ M=0$ solution).

$$\begin{aligned} ds^2 = -dt^2 &+ (1+Af_{rr})dr^2 + 2Bf_{r\theta}r\,dr\,d\theta + 2Bf_{r\phi}r\sin\theta\,dr\,d\phi \\ &+ (1+Cf^{(1)}_{\theta\theta}+Af^{(2)}_{\theta\theta})r^2d\theta^2 + [2(A-2C)f_{\theta\phi}]r^2\sin\theta\,d\theta\,d\phi \\ &+ (1+Cf^{(1)}_{\phi\phi}+Af^{(2)}_{\phi\phi})r^2\sin^2\theta\,d\phi^2, \end{aligned} \tag{3.199}$$

where

$$A = 3\left[\frac{F^{(2)}}{r^3}+\frac{3F^{(1)}}{r^4}+\frac{3F}{r^5}\right], \tag{3.200}$$

$$B = -\left[\frac{F^{(3)}}{r^2}+\frac{3F^{(2)}}{r^3}+\frac{6F^{(1)}}{r^4}+\frac{6F}{r^5}\right], \tag{3.201}$$

$$C = \frac{1}{4}\left[\frac{F^{(4)}}{r}+\frac{2F^{(3)}}{r^2}+\frac{9F^{(2)}}{r^3}+\frac{21F^{(1)}}{r^4}+\frac{21F}{r^5}\right], \tag{3.202}$$

$$F = F(t-r) = 10^{-3}\left(e^{(t-r)^2}+e^{(t+r)^2}\right), \quad F^{(n)} \equiv \left[\frac{d^nF(x)}{dx^n}\right]_{x=t-r}, \tag{3.203}$$

and

$$\begin{aligned} &f_{rr}=2-3\sin^2\theta, \quad f_{r\theta}=-3\sin\theta\cos\theta, \quad f_{r\phi}=0, \quad f^{(1)}_{\theta\theta}=3\sin^2\theta, \\ &f^{(2)}_{\theta\theta}=-1, \quad f_{\theta\phi}=0, \quad f^{(1)}_{\phi\phi}=-f^{(1)}_{\theta\theta}, \quad f^{(2)}_{\phi\phi}=3\sin^2\theta-1. \end{aligned} \tag{3.204}$$

Further, the outer boundary conditions are imposed at $x,\ y,\ z=4$. The initial data evolved for zero shift $\beta^i=0$, and compare the performance of systems ADM and BSSN for both geodesic and harmonic slicing.

3.13.2 Geodesic Slicing

In geodesic slicing, the lapse is unity

$$\alpha = 1. \tag{3.205}$$

Since the acceleration of normal observers satisfies $a_a = D_a \ln \alpha = 0$, these observers follow geodesics. The energy content of even a small, linear wave packet will therefore focus these observers, and even after the wave has dispersed, the observers will continue to coast towards each other. Since $\beta^i = 0$, normal observers are identical to coordinate observers, hence geodesic slicing will ultimately lead to the formation of a coordinate singularity even for arbitrarily small waves.

The timescale for the formation of this singularity can be estimated from equation

$$\frac{d}{dt}K = -\gamma^{ij} D_j D_i \alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + \frac{1}{2}\alpha(\rho + S),$$

with $\alpha = 1$ and $\beta^i = 0$. The $\tilde{A}_{ij}$, which can be associated with the gravitational waves, will cause K to increase to some finite value, say K_0 at time t_0, even if K was zero initially. After roughly a light crossing time, the waves will have dispersed, and the further evolution of K is described by $\partial_t K \sim K^3/3$, or

$$K \sim \frac{3K_0}{3 - K_0(t - t_0)} \tag{3.206}$$

Obviously, the coordinate singularity forms at $t \sim 3/K_0 + t_0$ as a result of the nonlinear evolution.

We can now evolve the wave initial data with Systems I and II and compare how well they reproduce the formation of the coordinate singularity.

In Figure (3.4), we show K at the origin ($x = y = z = 0$) as a function of time both for System I (dashed line) and System II (solid line). We also plot the approximate analytic solution Equation (3.206) as a dotted line, which we have matched to the System I solution with values $K_0 = 0.00518$ and $t_0 = 10$. For these values, Equation (3.206) predicts that the coordinate singularity appears at $t \sim 590$. In the insert, we show a blow-up of System II for early times. It can be seen very clearly how the initial wave content lets K grow from zero to the "seed" value K_0. Once the waves have dispersed,

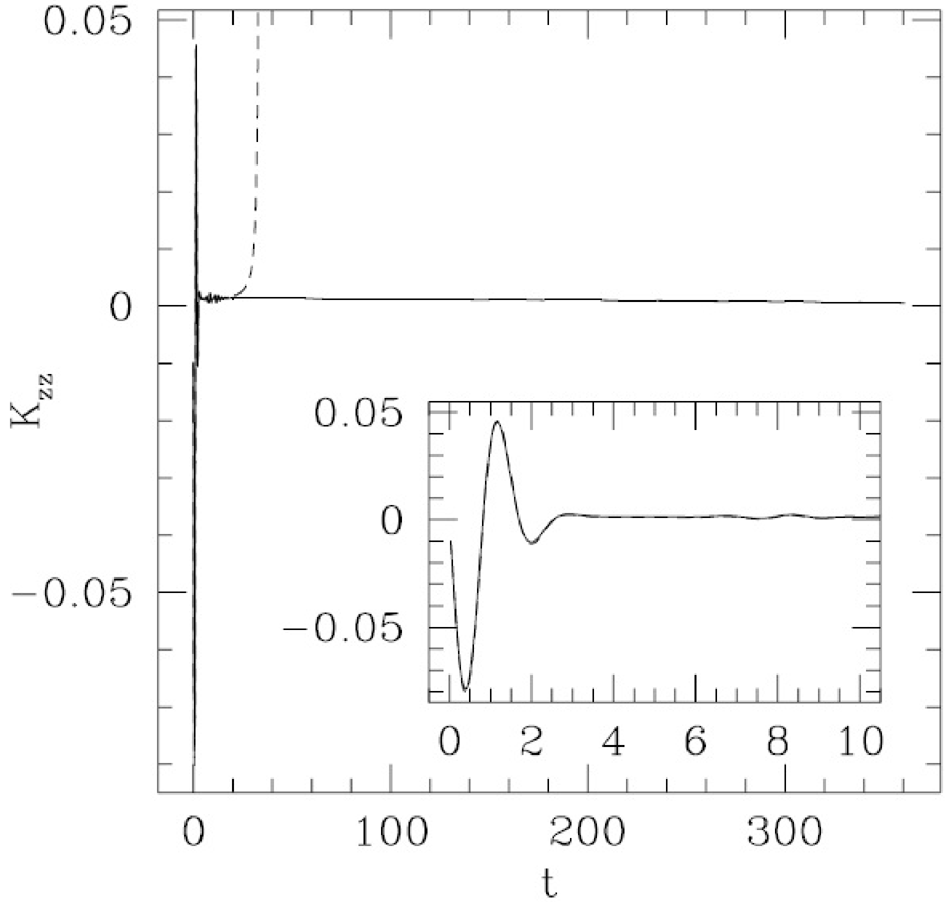

Figure 3.5: **Evolution of the extrinsic curvature component K_{zz} at the origin in geodesic slicing. The solid line is the result for System II, and the dashed line for System I. For System II, we constructed K_{zz} from $\tilde{A}_{zz}$, ϕ, K and $\tilde{\gamma}_{zz}$**

.

System II approximately follows the solution (3.206) up to fairly late times. System I, on the other hand, crashes long before the coordinate singularity appears.

In Figure (3.5), we compare the extrinsic curvature component K_{zz} evaluated at the origin. The noise around $t \sim 8$, which is present in the evolutions of both systems, is caused by reflections of the initial wave off the outer boundaries. It is obvious from these plots that System II evolves the equations stably to a fairly late time, at which the integration eventually becomes inaccurate as the coordinate singularity approaches. We stopped this calculation when the iterative Crank-Nicholson scheme no longer converged after a certain maximum number of iterations. It is also obvious that System I performs extremely poorly, and crashes at a very early time, well before the coordinate singularity. It is important to realize that the poor performance of System I is not an artifact of our numerical implementation. For example, the ADM code currently being used by the Black Hole Grand Challenge Alliance, is based on the equations of System I, and also crashes after a very similar time. This shows that the codes crashing is intrinsic to the equations and slicing, and not to our numerical implementation.

3.13.3 Harmonic Slicing

Since geodesic slicing is known to develop coordinate singularities for generic, nontrivial initial data, it is obviously not a very good slicing condition. We therefore also compare the two Systems using harmonic slicing. In harmonic slicing, the coordinate time t is a harmonic function of the coordinates $\nabla^\alpha \nabla_\alpha t = 0$, which is equivalent to the condition

$$\Gamma^0 \equiv g^{\alpha\beta}\Gamma^0_{\alpha\beta} = 0, \tag{3.207}$$

where the $\Gamma^\alpha_{\beta\gamma}$ are the connection coefficients associated with the four-dimensional metric $g_{\alpha\beta}$. For $\beta^i = 0$, the above condition reduces to

$$\partial_t \alpha = -\alpha^2 K. \tag{3.208}$$

Inserting $\frac{d}{dt}\phi = -\frac{1}{6}\alpha K$, this can be written as

$$\partial_t(\alpha e^{-6\phi}) = 0 \quad or \quad \alpha = C(x^i)e^{6\phi}, \tag{3.209}$$

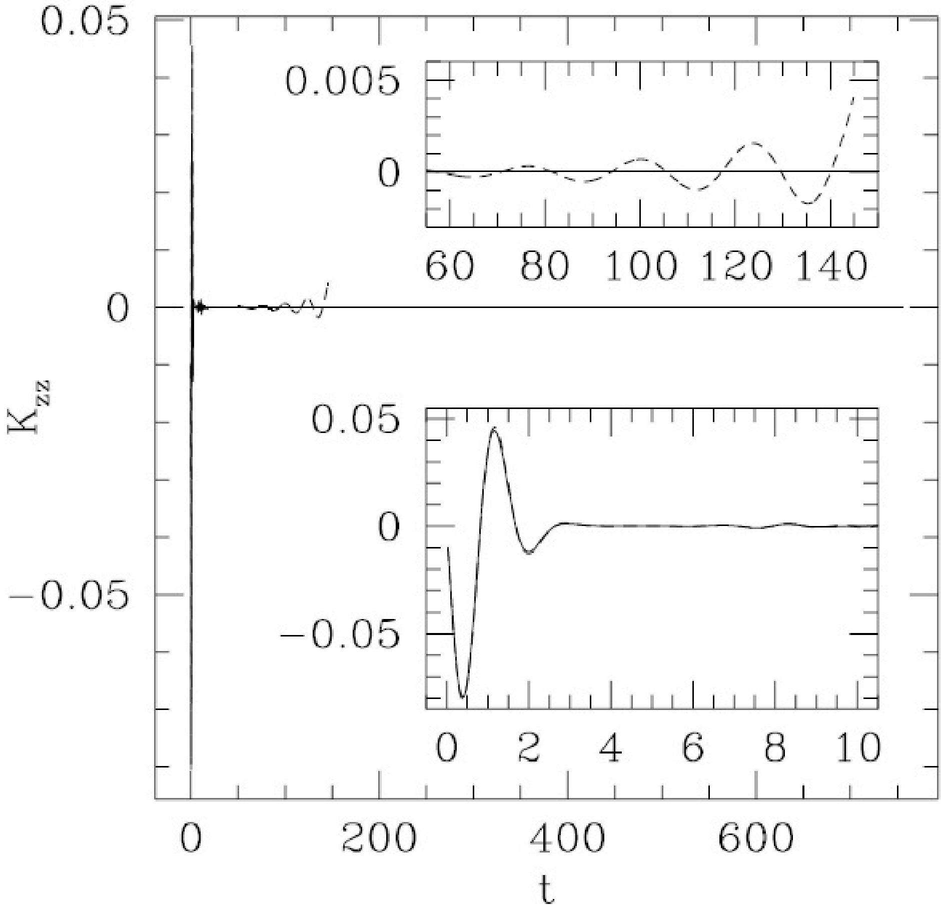

Figure 3.6: **Evolution of the extrinsic curvature component K_{zz} at the origin in harmonic slicing. The solid line is the result for System II, and the dashed line for System I. For System II, we constructed K_{zz} from $\tilde{A}_{zz}$, ϕ, K and $\tilde{\gamma}_{zz}$**

.

where $C(x^i)$ is a constant of integration, which depends on the spatial coordinates only. In practice, we choose $C(x^i) = 1$. In Figure (3.6), we show results for the same initial data as in the last section. Obviously, both Systems do much better for this slicing condition. System I crashes much later than in geodesic slicing (after about 40 light crossing times, as opposed to about 10 for geodesic slicing), but it still crashes. System II, on the other hand, did not crash after even over 100 light crossing times.

Chapter 4

The Initial Data Problem

Intoduction

The goal of numerical relativity is to study spacetime that cannot be studied by analytic means. The focus is therefore primarily on dynamical systems. Numerical relativity has been applied in many areas such that: cosmological models, critical phenomena, perturbed black holes and neutron stars, and the coalescence of black holes and neutron stars. In any of these cases, Einstein field equations can be formulated in several ways that allow us to evolve the dynamics. While Cauchy methods (standard ADM) have received a majority of the attention, characteristic and Reggi calculus based methods have also been used. All of these methods begin with a snapshot of the gravitational fields on some hypersurface, the initial data, and evolve these data to neighboring hypersurfaces. The material of this chapter covered in [7], [8], [12], [18].

4.1 Initial Data

Initial data are the starting point for any numerical simulation. In the case of numerical relativity, Einstein field equations constrain the choices of these initial data. We will examine several of the formalisms used for specifying Cauchy initial data in the 3+1 decomposition of Einstein field equations. We will then explore how these formalisms have been used in constructing initial data for spacetime containing black holes and neutron stars.

An important problem is to find initial data which represents two black holes which, when evolved, orbit about each other and eventually collide and merge into a single black hole, spewing forth gravity waves along the way. There is a great deal of freedom in developing initial data compatible with the constraints, but it is not so clear how to find data which is physically relevant to black hole collisions.

In the Cauchy formulation of Einstein field equations, we begin by foliating the 4-dimensional manifold as a set of spacelike, 3-dimensional hypersurfaces (or slices) Σ. These slices are labeled by a parameter t or, more simply, each slice of the 4-dimensional manifold is a $t =$ constant hypersurface. Following the standard 3+1 decomposition, we let n^μ be the future-pointing timelike unit normal to the slice, with

$$n^\mu \equiv -\alpha \nabla^\mu t. \tag{4.1}$$

Here, α is the lapse function. The scalar lapse function sets the proper interval measured by observers as they move between slices on a path that is normal to the hypersurface (so-called normal observers)

$$ds|_{along\ n^\mu} = \alpha dt. \tag{4.2}$$

To formulate the initial data problem for general relativity, we start by foliating spacetime with a family of spacelike hypersurfaces Σ_t parameterized by t. The normal vector to these surfaces n^μ and the generator of time translations (the time vector) t^μ satisfy

$$t^\mu = \alpha n^\mu + \beta^\mu, \tag{4.3}$$

where

$$\beta^\mu n_\mu \equiv 0. \tag{4.4}$$

Here, β^μ is the shift vector. Because of Equation (4.4), β^μ has only three independent components and is a spatial vector, tangent to the hypersurface on which it resides. At this point, it is convenient to introduce a coordinate system adapted to the foliation Σ. Let x^i be the spatial coordinates in the slice. The fourth coordinate, t, is the parameter labeling each slice. With this adapted coordinate system, we find that 3-dimensional coordinate values remain constant as we move between slices along the t^μ direction Equation (4.3). The four parameters, α and β^i, are a manifestation of the 4-dimensional coordinate invariance, or gauge freedom, in Einstein's theory. If we let γ_{ij} represent the metric of the spacelike hypersurfaces, then we can rewrite the interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ as Equation (3.19). In the language of the $3+1$ decomposition, initial data for the Einstein field equations (and any matter evolution equations) are a set of 24 functions representing the components of α, β^i, γ_{ij}, K_{ij}, ρ, S and S_{ij} on the initial slice Σ_t that together satisfy the constraint equations [18].

In the Cauchy formulation of Einstein field equations, γ_{ij} is regarded as the fundamental variable and values for its components must be given as part of a well-posed initial-value problem. Since Einstein's equations are second order, we must also specify something like a time derivative of the metric. For this, we use the second fundamental form, or extrinsic curvature, of the slice, K_{ij}, defined by

$$K_{ij} \equiv -\frac{1}{2}\mathcal{L}_n \gamma_{ij}, \tag{4.5}$$

where $\mathcal{L}_n$ denotes the Lie derivative along the n^μ direction.

Together, γ_{ij} and K_{ij} are the minimal set of initial data that must be specified for a Cauchy evolution of Einstein's equations. The metric γ_{ij} on a hypersurface is induced on that surface by the 4-metric $g_{\mu\nu}$. This means that the values γ_{ij} receives depend on how Σ is embedded in the full spacetime. In order for the foliation of slices Σ to fit into the higher-dimensional space, they must satisfy the Gauss-Codazzi-Ricci conditions. Combining these conditions with Einstein's equations, and using Equation (3.19), the six evolution equations become

$$\begin{aligned} \partial_t K_{ij} =& \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - D_i D_j \alpha - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) \\ & \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k. \end{aligned} \tag{4.6}$$

Here, D_i is the spatial covariant derivative compatible with γ_{ij}, R_{ij} is the Ricci tensor associated with γ_{ij}, $K \equiv K^i_i$, ρ is the matter energy density, S_{ij} is the matter stress tensor, and $S \equiv S^i_i$. The set of second-order evolution equations is completed by rewriting the definition of the extrinsic curvature (4.5) as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i. \tag{4.7}$$

Equations (4.6) and (4.7) are a first-order representation of a complete set of evolution equations for given initial data γ_{ij} and K_{ij}. However, the data cannot be freely specified in their entirety. The four constraint equations are

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{4.8}$$

and

$$D_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^i. \tag{4.9}$$

Here, S^i is the matter momentum density, and ρ is the matter energy density. Equation (4.9) is referred to as the momentum or vector constraints. Valid initial data for the evolution equations (4.6) and (4.7) must satisfy this set of constraints.

The Hamiltonian constraint (4.8) most naturally constrains the 3-metric γ_{ij}, while the momentum constraints (4.9) naturally constrain the extrinsic curvature K_{ij}.

The four constraint equations, Equations (4.8) and (4.9), represent conditions which the 3-metric and extrinsic curvature must satisfy. But, they do not specify which components (or combination of components) are constrained and which are freely specifiable.

The goal is to transform the equations into standard elliptic forms which can be solved given appropriate boundary conditions. Each different decomposition yields a unique set of elliptic equations to be solved and a unique set of freely specifiable parameters which must be fixed somehow.

4.1.1 York-Lichnerowicz Conformal Decompositions

For general initial-data configurations, the most common procedure of constraint decompositions are the York-Lichnerowicz conformal decompositions [8]. At their center are a conformal decomposition of the metric and certain components of the extrinsic curvature, together with a transverse-traceless decomposition of the extrinsic curvature.

First, the metric is decomposed into a conformal factor ψ (a positive scaling factor) multiplying an auxiliary 3-metric $\tilde{\gamma}_{ij}$ (a background metric)

$$\gamma_{ij} \equiv \psi^4 \tilde{\gamma}_{ij}. \tag{4.10}$$

$\tilde{\gamma}_{ij}$ carries five degrees of freedom. Its natural definition is given by

$$\tilde{\gamma}_{ij} = \gamma^{-1/3}\gamma_{ij}, \quad (\gamma = \det(\gamma_{ij})) \tag{4.11}$$

leaving the determinant of $\tilde{\gamma} = 1$. Using Equation (4.10), we can rewrite the Hamiltonian constraint as

$$\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{8}\psi^5 K^2 + \frac{1}{8}\psi^5 K_{ij}K^{ij} = -2\pi\psi^5\rho, \tag{4.12}$$

where $\tilde{\nabla}^2 \equiv \tilde{\nabla}^i\tilde{\nabla}_i$ is the scalar Laplace operator, and $\tilde{\nabla}_i$ and $\tilde{R}$ are the covariant derivative and Ricci scalar associated with $\tilde{\gamma}_{ij}$. Equation (4.12) is a quasilinear elliptic equation for the conformal factor ψ, and we see that the Hamiltonian constraint naturally constrains the 3-metric.

The conformal decomposition of the Hamiltonian constraint was proposed by Lichnerowicz. But, the key to the full decomposition is the treatment of the extrinsic curvature introduced by York. This begins by splitting the extrinsic curvature into its trace (K) and tracefree (traceless A_{ij}) parts

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{4.13}$$

The decomposition proceeds by using the fact that we can covariantly split any symmetric tracefree tensor as follows

$$S^{ij} \equiv (\mathbb{L}X)^{ij} + T^{ij}. \tag{4.14}$$

Here, T^{ij} is a symmetric, transverse-traceless tensor (i. e., $\nabla_j T^{ij} = 0$ and $T^i_i = 0$) and

$$(\mathbb{L}X)^{ij} \equiv \nabla^i X^j + \nabla^j X^i - \frac{2}{3}\gamma^{ij}\nabla_\ell X^\ell. \tag{4.15}$$

After separating out the transverse-traceless portion of S^{ij}, what remains, $(\mathbb{L}X)^{ij}$, is referred to as its "longitudinal" part. We now want to apply this transverse traceless decomposition to the tracefree part of the extrinsic curvature A_{ij}.

The goal of the decomposition is to produce a coupled set of elliptic equations to be solved with some prescribed boundary conditions. We have already reduced the Hamiltonian constraint to an elliptic equation being solved on a background space in terms of differential operators that are compatible with the conformal 3-metric. In the end, we want to reduce the momentum constraints to a set of elliptic equations based on differential operators that are compatible with the same conformal 3-metric.

Conformal Transverse-Traceless Decomposition

Let us first consider decomposing A^{ij} with respect to the conformal 3-metric. As we will see, when certain assumptions are made, this decomposition has the advantage of producing a simpler set of elliptic equations that must be solved. The first step is to define the conformal tracefree extrinsic curvature $\tilde{A}^{ij}$ by

$$A^{ij} \equiv \psi^{-10}\tilde{A}^{ij} \quad or \quad A_{ij} \equiv \psi^{-2}\tilde{A}_{ij}. \tag{4.16}$$

Next, the transverse-traceless decomposition is applied to the conformal extrinsic curvature,

$$\tilde{A}^{ij} \equiv (\tilde{\mathbb{L}}X)^{ij} + \tilde{Q}^{ij}. \tag{4.17}$$

Note that the longitudinal operator $\tilde{\mathbb{L}}$ and the symmetric, transverse-tracefree tensor $\tilde{Q}^{ij}$ are both defined with respect to covariant derivatives compatible with $\tilde{\gamma}_{ij}$.
Applying equations (4.10), (4.13), (4.15), (4.16), and (4.17) to the momentum constraints, we find that they simplify to

$$\tilde{\Delta}_{\mathbb{L}}X^i = \frac{2}{3}\psi^6\tilde{\nabla}^i K + 8\pi\psi^{10}S^i, \tag{4.18}$$

where

$$\tilde{\Delta}_{\mathbb{L}}X^i \equiv \tilde{\nabla}_j(\tilde{\mathbb{L}}X)^{ij} = \tilde{\nabla}^2 X^i + \frac{1}{3}\tilde{\nabla}^i(\tilde{\nabla}_j X^j) + \tilde{R}^i_j X^j, \tag{4.19}$$

and we have used the fact that

$$\bar{\nabla}_j S^{ij} = \psi^{-10}\tilde{\nabla}_j(\psi^{10}S^{ij}), \tag{4.20}$$

for any symmetric tracefree tensor S^{ij}.
In deriving equation (4.18), we have also used the fact that $\tilde{Q}^{ij}$ is transverse (i. e. $\tilde{\nabla}_j\tilde{Q}^{ij}$).

However, in general, we will not know if a given symmetric tracefree tensor, say $\tilde{M}^{ij}$, is transverse. By using (4.14) we can obtain its transverse-traceless part $\tilde{Q}^{ij}$ via

$$\tilde{Q}^{ij} \equiv \tilde{M}^{ij} - (\tilde{\mathbb{L}}Y)^{ij}, \tag{4.21}$$

and using the fact that if $\tilde{Q}^{ij}$ is transverse, we find

$$\tilde{\nabla}_j \tilde{Q}^{ij} \equiv 0 = \tilde{\nabla}_j \tilde{M}^{ij} - \tilde{\Delta}_{\mathbb{L}} Y^i. \tag{4.22}$$

Thus, Equations (4.21) and (4.22) give us a general way of constructing the required symmetric transverse-traceless tensor from a general symmetric traceless tensor. Using the linearity of $\mathbb{L}$, we can rewrite (4.17) as

$$\tilde{A}^{ij} = (\tilde{\mathbb{L}}V)^{ij} + \tilde{M}^{ij}, \tag{4.23}$$

where

$$V^i \equiv X^i - Y^i. \tag{4.24}$$

Similarly, using the linearity of $\tilde{\Delta}_{\mathbb{L}}$, we can rewrite (4.18) as

$$\tilde{\Delta}_{\mathbb{L}} V^i = \frac{2}{3}\psi^6 \tilde{\nabla}^i K - \tilde{\nabla}_j \tilde{M}^{ij} + 8\pi\psi^{10} S^i. \tag{4.25}$$

By solving directly for V^i, we can combine the steps of decomposing $\tilde{M}^{ij}$ with that of solving the momentum constraints.

After applying (4.13) and (4.16) to the Hamiltonian constraint (4.12), we obtain the following full decomposition

$$\begin{aligned}
\gamma_{ij} =& \psi^4 \tilde{\gamma}_{ij}, \\
K^{ij} =& \psi^{-4}\left(\tilde{A}^{ij} + \frac{1}{3}\tilde{\gamma}^{ij} K\right), \\
\tilde{A}^{ij} =& (\tilde{\mathbb{L}}V)^{ij} + \psi^{-6}\tilde{M}^{ij}, \\
\tilde{\Delta}_{\mathbb{L}} + 6(\tilde{\mathbb{L}}V)^{ij}\tilde{\nabla}_j \ln\psi =& \frac{2}{3}\tilde{\nabla}^i K - \psi^{-6}\tilde{\nabla}_j \tilde{M}^{ij} + 8\pi\psi^4 S^i, \\
\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^5 \tilde{A}_{ij}\tilde{A}^{ij} =& -2\pi\psi^5\rho.
\end{aligned} \tag{4.26}$$

In the decomposition given by Equation (4.26), we are free to specify a symmetric tensor $\tilde{\gamma}_{ij}$ as the conformal 3-metric, a symmetric tracefree tensor $\tilde{M}^{ij}$, and a scalar function

K. Then, with given matter energy and momentum densities, ρ and S^i, and appropriate boundary conditions, the coupled set of constraint equations for ψ and V^i are solved. Finally, given the solutions, we can construct the physical initial data, γ_{ij} and K^{ij}.

The decomposition outlined above has the interesting property that if we choose K to be constant and if the momentum density vanishes, then the momentum constraint equations fully decouple from the Hamiltonian constraint.

4.1.2 Conformal Thin-Sandwich Decomposition

The resulting constraint equations are independent of the kinematical variables α and β^i that govern how the coordinates move through spacetime, and thus there is no connection to dynamics. York's conformal thin-sandwich decomposition takes a different approach by considering the evolution of the metric between two neighboring hypersurfaces (the thin sandwich).

The decomposition begins with the standard conformal decomposition of the 3-metric (4.10). However, we next make use of the evolution equation for the metric in order to connect the 3-metrics on the two neighboring hypersurfaces. Label the two slices by t and t', with $t' = t + \delta t$, then ${\gamma'}_{ij} = \gamma_{ij} + (\partial_t \gamma_{ij})\delta t$. We would like to specify how the 3-metric evolves, but we do not have full freedom to do this. We know we can freely specify only the conformal 3-metric, and similarly, we are free to specify only the evolution of the conformal 3-metric. We make the following definitions

$$u_{ij} \equiv \gamma^{1/3}\partial_t(\gamma^{-1/3}\gamma_{ij}), \tag{4.27}$$

$$\tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij}, \tag{4.28}$$

and

$$\tilde{\gamma}^{ij}\tilde{u}_{ij} \equiv 0. \tag{4.29}$$

The latter definition is made for convenience, so that we can treat ψ, $\tilde{\gamma}_{ij}$, and $\tilde{u}_{ij}$ as regular scalars and tensors instead of as scalar- and tensor-densities within this thin-sandwich formalism.

Now, by previous way, we find group together all the equations that constitute the conformal thin-sandwich decomposition

$$\begin{aligned}
\gamma_{ij} =& \psi^4 \tilde{\gamma}_{ij}, \\
K^{ij} =& \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K, \\
\tilde{A}^{ij} =& \frac{1}{2\tilde{\alpha}} \left((\mathbb{L}\beta)^{ij} - \tilde{u}^{ij} \right), \\
\tilde{\Delta}_{\mathbb{L}} \beta^i - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \tilde{\alpha} - \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K =& \tilde{\alpha} \tilde{\nabla}_j (\frac{1}{\tilde{\alpha}} \tilde{u}^{ij}) + 16\pi \tilde{\alpha} \psi^{10} S^i, \\
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} =& -2\pi \psi^5 \rho.
\end{aligned} \tag{4.30}$$

In this decomposition (4.30), we are free to specify a symmetric tensor γ_{ij} as the conformal 3-metric, a symmetric tracefree tensor $\tilde{u}^{ij}$, a scalar function K, and the scalar function $\tilde{\alpha}$. Solving this set of equations with appropriate boundary conditions yields initial data γ_{ij} and K_{ij} on a *single* hypersurface.

4.1.3 Stationary Solutions

When there is sufficient symmetry present, it is possible to construct initial data that are in true equilibrium. These solutions possess at least two Killing vectors, one that is timelike at large distances and one that is spatial, representing an azimuthal symmetry. When these symmetries are present, solving for the initial data produces a global solution of Einstein's equations and the solution is said to be stationary. The familiar Kerr-Neumann solution for rotating black holes is an example of a stationary solution in vacuum. Stationary configurations supported by matter are also possible, but the matter sources must also satisfy the Killing symmetries, in which case the matter is said to be in hydrostatic equilibrium.

The basic approach for finding stationary solutions begins by simplifying the metric to take into account the symmetries. Many different forms have been used for the metric. I will use a decomposition that makes comparison with the previous decompositions straightforward. First, define the interval as

$$ds^2 = -\psi^{-4} dt^2 + \psi^4 \left[A^2 (dr^2 + r^2 d\theta^2) + B^2 r^2 \sin^2\theta (d\phi + \beta^\phi dt)^2 \right]. \tag{4.31}$$

This form of the metric can describe any stationary spacetime. Notice that the lapse is related to the conformal factor by

$$\alpha = \psi^{-2}, \tag{4.32}$$

and that the shift vector has only one component

$$\beta^i = (0, 0, \beta^\phi) \tag{4.33}$$

I have used the usual conformal decomposition of the 3-metric (4.10) and have written the conformal 3-metric with two parameters as

$$\begin{bmatrix} A^2 & 0 & 0 \\ 0 & A^2 r^2 & 0 \\ 0 & 0 & B^2 r^2 \sin^2\theta \end{bmatrix} \tag{4.34}$$

The four functions ψ, β^ϕ, A, and B are functions of r and θ only.

The equations necessary to solve for these four functions are derived from the constraint equations, and the evolution equations. For the evolution equations, we use the fact that $\partial_t \gamma_{ij} = 0$ and $\partial_t K_{ij} = 0$. The metric evolution equation defines the extrinsic curvature in terms of derivatives of the shift

$$K_{ij} \equiv \frac{1}{2\alpha}(\bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i). \tag{4.35}$$

With the given metric and shift, we find that $K = 0$ and the divergence of the shift also vanishes. This means we can write the tracefree part of the extrinsic curvature as

$$A^{ij} = \psi^{-10} \tilde{A}^{ij} = \frac{1}{2}\psi^{-2}(\tilde{\mathbb{L}}\beta)^{ij}. \tag{4.36}$$

We find that the Hamiltonian and momentum constraints take on the forms given by the conformal thin-sandwich decomposition (4.30) with $\tilde{u}^{ij} = K \equiv 0$ and $\tilde{\alpha} \equiv \psi^{-8}$. Only one of the momentum constraint equations is non-trivial, and we find that the constraints yield elliptic equations for ψ and β^ϕ. What remains unspecified as yet are A and B (i.e., the conformal 3-metric).

The conformal 3-metric is determined by the evolution equations for the traceless part of the extrinsic curvature. Of these five equations, one can be written as an elliptic equation for B, and two yield complementary equations that can each be solved by quadrature for A. The remaining equations are redundant as a result of the Bianchi identities.

4.2 Linearized waves-vacuum solutions

Gravitational waves are ripples in the curvature of spacetime that propagate at the speed of light. Once the waves move away from their source in the near zone, their wavelengths are generally much smaller than the radius of curvature of the background spacetime through which they propagate. The waves usually can be described by linearized theory in this far zone region. Introducing Minkowski coordinates, one has

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \tag{4.37}$$

where we assume Cartesian coordinates and, ignoring any quasistatic contributions to the perturbations $h_{\mu\nu}$ from weak-field sources, consider only the wave contributions. Defining the trace-reversed wave perturbation $\bar{h}_{\mu\nu}$ according to

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^c_c \eta_{\mu\nu}, \tag{4.38}$$

the key equation governing the propagation of a linear wave in vacuum is

$$\Box \bar{h}_{\mu\nu} \equiv \nabla^c \nabla_c \bar{h}_{\mu\nu} = 0 \quad (vacuum). \tag{4.39}$$

Assuming it satisfies the Lorentz gauge condition

$$\nabla_\nu \bar{h}^{\mu\nu} = 0. \tag{4.40}$$

Return now to Einstein's linearized equations (4.39) in vacuum,

$$\Box \bar{h}_{\mu\nu} = 0. \tag{4.41}$$

Just as in electrodynamics, this type of equation admits simple plane wave solutions of the form

$$\bar{h}_{\mu\nu} = Re(A_{\mu\nu} e^{ik_c x^c}). \tag{4.42}$$

Here $k_\mu = (\omega, k^i)$ is a 4-dimensional wave vector, $x^\mu = (t, x^i)$ denote the inertial coordinates of a point in spacetime and $A_{\mu\nu}$ is a constant tensor representing the wave amplitude. Einstein's equations (4.41) then demand that k^μ be a null vector,

$$k_\mu K^\mu = 0, \tag{4.43}$$

whereby $\omega = |k^i|$. This dispersion relation implies that gravitational waves propagate at the speed of light. The Lorentz condition (4.40) requires

$$k^\mu A_{\mu\nu} = 0, \tag{4.44}$$

implying that gravitational waves are transverse. The above results are quite general, since we can always decompose an arbitrary, linear gravitational wave propagating in a nearly flat, vacuum spacetime into a superposition of the plane wave solutions (4.42).

From a numerical point of view the plane wave solutions found above are not the most useful. Most numerical simulations treat spacetimes with finite, bounded sources, for which the waves propagate radially outward at large distance. Moreover such spacetimes approach asymptotic flatness at least as fast as r^{-1}. Clearly spacetimes containing with plane waves Equation (4.42) do not share these properties. More useful for simulation purposes are multiple expansions of linear, vacuum solutions to Equation (4.42) expressed in terms of tensor spherical harmonics. When working in spherical coordinates, it is more convenient to express the two polarization states of gravitational radiation in terms of polar and axial modes.

For polar quadruple modes, the metric takes the form

$$\begin{aligned} ds^2 = &- dt^2 + (1 + Af_{rr})dr^2 + (2Bf_{r\theta})rdrd\theta + (2Bf_{r\phi})r\ \sin\theta drd\phi \\ &+ (1 + Cf^{(1)}_{\theta\theta} + Af^{(2)}_{\theta\theta})r^2 d\theta^2 + [2(A - 2C)f_{\theta\phi}]r^2 \sin\theta d\theta d\phi \\ &+ (1 + Cf^{(1)}_{\phi\phi} + Af^{(2)}_{\phi\phi})r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{4.45}$$

Here the coefficients A, B and C can be constructed from an arbitrary function $F(x)$, where we have $x = t - r$ for an outgoing solution or $x = t + r$ for an ingoing solution. In the general case, we may take

$$F = F_1(t - r) + F_2(t + r), \tag{4.46}$$

and define

$$F^{(n)} \equiv \left[\frac{d^n F_1(x)}{dx^n}\right]_{x=t-r} + (-1)^n \left[\frac{d^n F_2(x)}{dx^n}\right]_{x=t+r}. \tag{4.47}$$

In terms of $F(x)$ and its derivatives we then have

$$A = 3\left[\frac{F^{(2)}}{r^3} + \frac{3F^{(1)}}{r^4} + \frac{3F}{r^5}\right] \tag{4.48}$$

$$B = -\left[\frac{F^{(3)}}{r^2} + \frac{3F^{(2)}}{r^3} + \frac{6F^{(1)}}{r^4} + \frac{6F}{r^5}\right] \tag{4.49}$$

$$C = \frac{1}{4}\left[\frac{F^{(4)}}{r} + \frac{2F^{(3)}}{r^2} + \frac{9F^{(2)}}{r^3} + \frac{21F^{(1)}}{r^4} + \frac{21F}{r^5}\right]. \tag{4.50}$$

The angular functions f_{ij} in the metric Equation (4.45) depend on the axial parameter, denoted by M. We list these functions in the order $M = \pm 2, \pm 1, 0$, with the functions corresponding to the upper sign displayed on top of those corresponding to the lower sign:

$$\begin{aligned}
f_{rr} &= \sin^2\theta \begin{pmatrix}\cos 2\phi \\ \sin 2\phi\end{pmatrix},\ 2\sin\theta\cos\theta \begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix},\ 2 - 3\sin^2\theta, \\
f_{r\theta} &= \sin\theta\cos\theta \begin{pmatrix}\cos 2\phi \\ \sin 2\phi\end{pmatrix},\ (\cos^2\theta - \sin^2\theta)\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix},\ \ -3\sin\theta\cos\theta, \\
f_{r\phi} &= \sin\theta \begin{pmatrix}-\sin 2\phi \\ \cos 2\phi\end{pmatrix},\ \cos\theta \begin{pmatrix}-\sin\phi \\ \cos\phi\end{pmatrix},\ 0, \\
f^{(1)}_{\theta\theta} &= (1 + \cos^2\theta)\begin{pmatrix}\cos 2\phi \\ \sin 2\phi\end{pmatrix},\ 2\sin\theta\cos\theta \begin{pmatrix}-\cos\phi \\ -\sin\phi\end{pmatrix}, 3\sin^2\theta, \\
f^{(2)}_{\theta\theta} &= \begin{pmatrix}-\cos 2\phi \\ -\sin 2\phi\end{pmatrix},\ 0,\ \ -1, \\
f_{\theta\phi} &= \cos\theta \begin{pmatrix}\sin 2\phi \\ -\cos 2\phi\end{pmatrix},\ \sin\theta \begin{pmatrix}-\sin\phi \\ \cos\phi\end{pmatrix},\ 0, \\
f^{(1)} &= -f^{(1)}_{\theta\theta} \\
f^{(2)}_{\phi\phi} &= \cos^2\theta \begin{pmatrix}\cos 2\phi \\ \sin 2\phi\end{pmatrix},\ 2\sin\theta\cos\theta \begin{pmatrix}-\cos\phi \\ -\sin\phi\end{pmatrix},\ 3\sin^2\theta - 1.
\end{aligned} \tag{4.51}$$

Similarly, the axial metric takes the form

$$\begin{aligned}
ds^2 = &-dt^2 + dr^2 + (2Zd_\theta)rdrd\theta + (2Zd_{r\phi})r\sin\theta drd\phi + (1 + Vd_{\theta\theta})r^2 d\theta^2 \\
&+ (2Vd_{\theta\phi})r^2\sin\theta d\theta d\phi + (1 + Vd_{\phi\phi})r^2\sin^2\theta d\phi^2.
\end{aligned} \tag{4.52}$$

Here we construct the coefficients Z and V from a function

$$G = G_1(t - r) + G_2(t + r), \tag{4.53}$$

and its derivatives

$$G^{(n)} \equiv \left[\frac{d^n G_1(x)}{dx^n}\right]_{x=t-r} + (-1)^n \left[\frac{d^n G_2(x)}{dx^n}\right]_{x=t+r}, \tag{4.54}$$

according to

$$Z = \frac{G^{(2)}}{r^2} + \frac{3G^{(1)}}{r^3} + \frac{3G}{r^4}, \tag{4.55}$$

$$V = \frac{G^{(3)}}{r} + \frac{2G^{(2)}}{r^2} + \frac{3G^{(1)}}{r^3} + \frac{3G}{r^4}. \tag{4.56}$$

The angular functions d_{ij} are again listed in the order $M = \pm 2, \pm 1, 0$, yielding

$$\begin{aligned}
d_{r\theta} =& 4\sin\theta \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix}, \; -2\cos\theta \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \; 0 \\
d_{r\phi} =& -4\sin\theta\cos\theta \begin{pmatrix} \sin 2\phi \\ -\cos 2\phi \end{pmatrix}, \; -2(\cos^2\theta - \sin^2\theta) \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}, \\
& 4\cos\theta\sin\theta, \\
d_{\theta\theta} =& -2\cos\theta \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix}, \; -\sin\theta \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \; 0, \\
d_{\theta\phi} =& (2 - \sin^2\theta) \begin{pmatrix} \sin 2\phi \\ -\cos 2\phi \end{pmatrix}, \; \cos\theta\sin\theta \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}, \; -\sin^2\theta, \\
d_{\phi\phi} =& 2\cos\theta \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix}, \; \sin\theta \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \; 0.
\end{aligned} \tag{4.57}$$

4.3 Black Hole Initial Data

In this section, we will look at Cauchy initial data that represent one black hole in an asymptotically at spacetime (Schwarzschild geometry).

Next, we will explore one of the existing black-hole solution and the scheme for generating it.

4.3.1 Classic Solution (Schwarzschild)

The simplest black-hole solution is the Schwarzschild solution. It represents a static spacetime containing a single black hole that connects two causally disconnected, asymptotically at universes. There are actually many different coordinate representations of the Schwarzschild solutions. The simplest representations are time-symmetric ($K_{ij} = 0$), and so exist on a "maximally embedded" spacelike hypersurface ($K = 0$). These choices fix the foliation Σ. Spherical symmetry fixes two of the three spatial gauge choices. If we choose an "areal-radial coordinate", then the interval is written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.58}$$

If we choose an isotropic radial coordinate, then the interval is written as

$$ds^2 = -\left(\frac{1-\frac{M}{2\tilde{r}}}{1+\frac{M}{2\tilde{r}}}\right)^2 dt^2 + \left(1+\frac{M}{2\tilde{r}}\right)^4 (d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta d\phi^2). \tag{4.59}$$

In both Equations (4.58) and (4.59), M represents the mass of the black hole as measured at spacelike infinity. Both of these solutions exist on the same foliation of $t = const.$ slices. But, notice that the 3-geometry of the slice associated with Equation (4.59) is conformally flat, while the 3-geometry associated with Equation (4.58) is not.

The solution given in Equation (4.59) is easily generated by any of the methods in Section (**York-Lichnerowicz Conformal Decompositions**) or Section (**Conformal Thin-Sandwich Decomposition**). By choosing a time-symmetric initial-data hypersurface, we immediately get $K_{ij} = 0$, which eliminates the need to solve the momentum constraints. If we choose the conformal 3-geometry to be given by a flat metric (in spherical coordinates in this case), then the vacuum Hamiltonian constraint (3.94) becomes

$$\tilde{\nabla}^2\psi = 0, \tag{4.60}$$

where $\tilde{\nabla}^2$ is the flat-space Laplace operator. For the solution ψ to yield an asymptotically flat physical 3-metric, we have the boundary condition that $\psi(\tilde{r} \to \infty) = 1$. The simplest solution of this equation is

$$\psi = 1 + \frac{M}{2\tilde{r}}, \tag{4.61}$$

where we have chosen the remaining integration constant to give a mass at infinity of M. We now have full Cauchy initial data representing a single black hole. If we want to generate a full solution of Einstein's equations, we must choose a lapse and a shift vector and integrate the evolution equations. In this case, a reasonable approach for specifying the lapse is to demand that the time derivative of K vanish. For the case of $K = 0$, this yields the so-called maximal slicing equation which, for the current situation, takes the form

$$\tilde{\nabla}^2(\alpha\psi) = 0. \tag{4.62}$$

If we choose boundary conditions so that the lapse is frozen on the event horizon ($\alpha(\tilde{r} = M/2) = 0$) and goes to one at infinity, we find that the solution is

$$\alpha = \frac{1 - \frac{M}{2\tilde{r}}}{1 + \frac{M}{2\tilde{r}}}. \tag{4.63}$$

If we now choose $\beta^i = 0$, we find that the left-hand sides of the evolution equations vanish identically, and we have found the static solution of Einstein's equations given in (4.59). We can, of course, recover the usual Schwarzschild coordinate solution (4.58) by using the purely spatial coordinate transformation $r = \tilde{r}(1 + \frac{M}{2\tilde{r}})^2$.

It is interesting to examine the differences in these two representations of the Schwarzschild solution. The isotropic radial coordinate representation is well behaved everywhere except, it seems, at $\tilde{r} = 0$. However, even here, the solution is well behaved. The 3-geometry is invariant under the coordinate transformation

$$\tilde{r} \to \left(\frac{M}{2}\right)^2 \frac{1}{r'}. \tag{4.64}$$

The event horizon at $\tilde{r} = \frac{M}{2}$ is a fixed-point set of the isometry condition (4.64) which identifies points in two causally disconnected, asymptotically flat universes. We see that $\tilde{r} = 0$ is simply an image of infinity in the other universe.

Given our choice for the lapse (4.63), which is frozen on the event horizon, we find that the solution can cover only the exterior of the black hole. To cover any of the interior with the lapse pinned to zero at the horizon would require we use a slice that is not spacelike everywhere. This is exactly what happens when the usual Schwarzschild areal-radial coordinate is used. At the event horizon, $r = 2M$, there is a coordinate singularity, and inside this radius the $t = const.$ hypersurface is no longer spacelike. It is impossible to perform a Cauchy evolution interior to the event horizon using the areal-radial coordinate and the given time slicing.

We find that a Cauchy evolution, using the usual Schwarzschild time slicing that is frozen at the horizon, is capable of evolving only the region exterior to the black hole's event horizon. Portions of the interior of the black hole can be covered by an evolution that begins with data on a standard Schwarzschild time slice, but the result is not a time-independent solution. As we will see later, there are other slicings of the Schwarzschild spacetime that cover the interior of the black hole and yield time-independent solutions. Here we will construct maximal slicing of the same spacetime, one that gives a time-dependent or "dynamical slicing" of a Schwarzschild black hole. This spacetime is also analytic, if by "analytic" we allow 1- dimensional quadratures. The metric for this solution has some generic features that characterize the metric that develops at late times during stellar collapse to a black hole when maximal slicing is employed. Not only will our examination of this solution be useful to illustrate how maximal slicing works, but our reconstruction of the solution will provide a convenient opportunity to review the typical steps required to build a spacetime in the 3 + 1 formalism, at least in spherical symmetry.

We start with the line element in the form

$$ds^2 = -(\alpha^2 - \beta^2/A)d\bar{t}^2 + 2\beta d\bar{t}dr + Adr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.65}$$

where r is the Schwarzschild radial coordinate (we drop the subscript "s" in this section), $\beta = \beta_r = A\beta^r$, $\bar{t}$ is the maximal time coordinate, and the functions α, β and A depend only on $\bar{t}$ and r. Given this form of the metric we compute all the 3-dimensional Christoffel symbols, which we will need in the evaluation of the standard 3 + 1 or ADM equations. We find

$$\begin{aligned}\Gamma^r_{rr} =& \partial_r A/(2A), \quad \Gamma^r_{\theta\theta} = -r/A, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta/A,\\ \Gamma^\theta_{r\theta} =& \Gamma^\theta_{\theta r} = 1/r, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\\ \Gamma^\phi_{\theta\phi} =& \Gamma^\phi_{\phi\theta} = \cot\theta, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r,\end{aligned} \tag{4.66}$$

with the remaining coefficients equal to zero.

Next we insert the Christoffel symbols in Equation (2.3) to calculate the nonvanishing components of the 3-dimensional Riemann tensor, R_{ij}, obtaining

$$R_{rr} = \partial_r A/(rA), \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2\theta = 1 - 1/A + r\partial_r A/(2A^2). \tag{4.67}$$

We now can get $R = R^i_i$, required to solve the Hamiltonian constraint equation (3.94)

$$R = 2\partial_r A/(rA^2) + 2(1 - 1/A)/r^2. \tag{4.68}$$

The nonvanishing components of the extrinsic curvature may be calculated from Equation (3.154), yielding

$$K_{rr} = -\left(\partial_{\bar{r}} A + \beta \partial_r A/A - 2\partial_r \beta\right)/(2\alpha), \quad K_{\theta\theta} = K_{\phi\phi}/\sin^2\theta = r\beta/(\alpha A). \tag{4.69}$$

Maximal slicing requires $K = K^i K_i = 0$, in which case equation (4.69) implies

$$K_{rr} = -2\beta/(\alpha r) \qquad K_{ij}K^{ij} = 6(\beta/\alpha A r)^2 \tag{4.70}$$

and

$$\partial_{\bar{t}} \ln A + (\beta + A)\partial_r \ln(\beta^2 r^3/A) = 0. \tag{4.71}$$

The Hamiltonian constraint (3.94) reduces to

$$R = K_{ij}K^{ij}, \tag{4.72}$$

which, inserting Equations (4.68) and (4.70), yields

$$3\beta^2/(\alpha^2 A) = A - 1 + r\partial_r A/A. \tag{4.73}$$

When combined with Equation (4.70) for K_{rr}, the radial component of the momentum constraint (3.100) may be evaluated to give

$$\partial_r \ln(\beta r^2/A\alpha) = 0. \tag{4.74}$$

Maximal slicing also requires $\partial_{\bar{t}} K = 0$, in which case Equation (3.147), combined with equation (4.72), gives $D^2\alpha = \alpha R$. Substituting Equation (4.68) in the right-hand side and expanding the derivative on the left-hand side yields an equation for the lapse,

$$\partial_r\partial_r\alpha + 2\partial_r\alpha/r - (\partial_r \ln A)\partial_r\alpha/2 = 2\alpha(A - 1 + r\partial_r \ln A)/r^2. \tag{4.75}$$

Finally, the evolution equation (4.6) for K_{rr} gives

$$\begin{aligned}\partial_{\bar{t}} \ln(\beta/\alpha) =& (3\beta/A + \alpha^2 A/\beta - \alpha^2/\beta)/r \\ &+ 3(\partial_r\beta)/A + (\alpha^2/\beta - 4\beta/A)(\partial_r \ln A)/2 \\ &- (\beta/A + \alpha^2/\beta)\partial_r \ln\alpha,\end{aligned} \tag{4.76}$$

where we have used Equation (4.70) to replace K_{rr}, Equation (4.67) for R_{rr}, and Equation (4.75) for $\partial_r\partial_r\alpha$.

Chapter 5

Einstein Field Equations and OHAM method

Introduction

In this chapter we apply OHAM, Optimal Homotopy Asymptotic Method, to find the static spherically symmetric solution of Einstein equations, which is called Schwarzschild solution. In the next section we present the Schwarzschild geometry and solution. In Section 5.2 we present OHAM, and finally in Section 5.3 we apply OHAM and derive Schwarzschild solution. The material of Sections 5.1 and 5.2 are covered in [20,21].

5.1 Einstein Field Equations and Schwarzschild Solution

As mentioned before Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{5.1}$$

determine the geometry of spacetime by providing the definition of distance theorem of Pythagora, based on the matter content in that spacetime. On the other hand, motion of matter is determined by this geometry.

The fact that the motion of matter is determined by properties of geometry is called the equivalence principle, and is built in the Einstein equations.

Following we will demonstrate all this in the simplest nontrivial case of the static spherically symmetric solution of Einstein equations, called Schwarzschild geometry. We begin from flat Minkowski spacetime with the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{5.2}$$

and introduce spherical coordinates via the change of variables $z = r\cos\theta$, $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$ and use

$$\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&= -dt^2 + \left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi\right)^2 + \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \varphi}d\varphi\right)^2 \\
&\quad + \left(\frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \varphi}d\varphi\right)^2 \\
&= -dt^2 + \left(\cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\theta\right)dr^2 \\
&\quad + \left(r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta\right)d\theta^2 \\
&\quad + \left(r^2\sin^2\varphi\sin^2\theta + r\cos^2\varphi\sin^2\theta\right)d\varphi^2 \\
&= -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2.
\end{aligned} \tag{5.3}$$

This is still a flat-spacetime line element, just expressed in curvilinear coordinates. We generalize this line-element in such a way to allow for curvature, while preserving the requirements of geometry being static and spherically symmetric. Static means that the metric should not depend on time, while spherically symmetric means that it should not depend on angles θ and φ. Thus it is enough to consider the following generalization for the line element

$$ds^2 = -e^{2F(r)}dt^2 + e^{2H(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{5.4}$$

where $F(r)$ and $H(r)$ are two functions to be determined by Einstein equations. Note that these terms are written in the exponent because of computational convenience. So we construct first the left-hand side of Einstein equations. We get the metric and inverse metric tensors using the line element (5.2)

$$[g_{\mu\nu}] = \begin{bmatrix} -e^{2F(r)} & & & \\ & e^{2H(r)} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix} \tag{5.5}$$

$$[g^{\mu\nu}] = \begin{bmatrix} -e^{-2F(r)} & & & \\ & e^{-2H(r)} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2\theta} \end{bmatrix} \tag{5.6}$$

Next we construct the Ricci tensor. The nonzero components are

$$\begin{aligned} R_{tt} &= e^{2F-2H}\left(F'' + (F')^2 - F'H' + \frac{2}{r}F'\right) \\ R_{rr} &= -\left(F'' + (F')^2 - F'H' - \frac{2}{r}H'\right), \\ R_{\theta\theta} &= 1 - e^{-2H}(1 + rF' - rH'), \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta. \end{aligned} \tag{5.7}$$

Contract the Ricci tensor with the metric to obtain the Ricci scalar

$$R = -2e^{-2H}\left[F'' + \left(F' + \frac{2}{r}\right)(F' - H') + \frac{1}{r^2}(1 - e^{2H})\right]. \tag{5.8}$$

Finally, we put all this together to form the Einstein tensor

$$\begin{aligned} G_{tt} &= -\frac{1}{r^2}e^{2F-2H}(1 - 2rH' - e^{2H}), \\ G_{rr} &= \frac{1}{r^2}(1 + 2rF' - e^{2H}), \\ G_{\theta\theta} &= r^2e^{-2H}\left[F'' + \left(F' + \frac{1}{r}\right)(F' - H')\right], \\ G_{\phi\phi} &= G_{\theta\theta}\sin^2\theta. \end{aligned} \tag{5.9}$$

Note that the G_{tt} component of the Einstein tensor can be rewritten in the form

$$G_{tt} = \frac{1}{r^2}e^{2F}\frac{d}{dr}[r(1 - e^{-2H})]. \tag{5.10}$$

Now we consider the right-hand side of the Einstein equation. We are interesting of the simplest possible stress-energy tensor, namely one that represents a static ball of radius R and density $\rho(r)$ with the center in $r = 0$. The general formula for the stress-energy tensor of a fluid element with density ρ, pressure P, and 4-velocity u^μ is

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}. \tag{5.11}$$

We wish to describe the static fluid ($u_r = u_\theta = u_\varphi = 0$). So the stress-energy takes the form

$$T_{tt} = \rho u_t u_t + P(u_t u_t + g_{tt}),\ T_{rr} = Pg_{rr},\ T_{\theta\theta} = Pg_{\theta\theta},\ T_{\varphi\varphi} = Pg_{\varphi\varphi}, \tag{5.12}$$

while other components vanish. Next, the 4-velocity vector must be normalized, $u_\mu u_\nu g^{\mu\nu} = 1$, which means that $u_t u_t = -g_{tt} = e^{2F}$, so we have

$$T_{tt} = \rho e^{2F}, \quad T_{rr} = Pe^{2H}, \quad T_{\theta\theta} = Pr^2, \quad T_{\varphi\varphi} = Pr^2 \sin^2\theta. \tag{5.13}$$

The density and pressure of the fluid can depend only on r due to the spherical symmetry, and must be zero for $r > R$, that is, outside the ball.

Finally, after substituting all these results into Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we get $t-t$, $r-r$, $\theta-\theta$ equations. Straightforward integration gives

$$H(r) = -\frac{1}{2}\ln\left(1 - \frac{2m(r)}{r}\right), \quad where \;\; m(r) \equiv 4\pi \int dr\, r^2 \rho^2(r). \tag{5.14}$$

Choosing the initial condition $m(0) = 0$, we can interpret $m(r)$ as the total mass inside radius r, since it is defined as an integral of mass density ρ over the volume of a ball of radius r.

Next, we discuss the $r-r$ equation. Solve it for F to obtain

$$F(r) = \int dr \frac{m(r) + 4\pi r^3 P(r)}{r[r - 2m(r)]}. \tag{5.15}$$

For $r > R$ we have $m(r) = M$ (total mass of the ball) and $P(r) = 0$ (zero pressure in vacuum), so $F(r)$ can be easily integrated by partial fraction, we get

$$F(r) = \int -\frac{1}{2r} + \frac{1}{2(r-2M)} dr. \tag{5.16}$$

The result is

$$F(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right), \tag{5.17}$$

where the constant of integration has been chosen so that in the limit $r \to \infty$ the line element recovers its Minkowski form (far away from the ball spacetime should be flat).

Finally, we discuss the $\theta-\theta$ equation. Substitute all previous results and (after a tedious calculation) obtain the following result

$$P'(r) + F'(r)(\rho(r) + P(r)) = 0. \tag{5.18}$$

This is a differential equation that determines the radial pressure distribution of matter within the ball. This distribution is such that the repulsive pressure balances attractive gravity everywhere, thereby maintaining static configuration of matter inside the ball. Therefore for the geometry outside the ball (for $r > R$). The line element has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \tag{5.19}$$

This is the famous Schwarzschild solution of Einstein equations, and defines the so-called Schwarzschild geometry.

5.2 Basic Formulation of Optimal Homotopy Asymptotic Method (OHAM)

Consider the operator equation of the form, [2],

$$A(y(x)) + f(x) = 0, \tag{5.20}$$

where A is an operator, $y(x)$ is unknown function, and $f(x)$ a known analytic function. Assume that A can be decomposed into two operators L (simple) and N (the rest) such that

$$A = L + N. \tag{5.21}$$

According to $OHAM$, one can construct an optimal homotopy map

$$y(x,p) : \Omega \times [0,1] \to R, \tag{5.22}$$

that satisfies the homotopy equation

$$\begin{aligned} \hat{H}(y(x,p),p) =& (1-p)L(y(x,p)) + f(x) \\ &- C(p)A(y(x,p)) + f(x) = 0, \end{aligned} \tag{5.23}$$

where the auxiliary $C(p)$ function is nonzero for $p \neq 0$; $C(0) = 0$ and $p \in [0,1]$ is an embedding parameter. Equation (5.23) is called optimal homotopy equation or zero-order homotopy equation. Note that if $p = 0$, we get $y(x,0) = y_0(x)$, and when $p = 1$, we obtain $y(x,1) = y(x)$; the exact solution. Thus, as p varies from 0 to 1, the solution $y(x,p)$ arrives from $y_0(x)$ at $y(x)$, where $y_0(x)$ is the solution of Equation (5.23) when we substitute $p = 0$, i.e $y_0(x)$ satisfies

$$L(y_0(x)) + f(x) = 0. \tag{5.24}$$

Next, we choose the auxiliary function $C(p)$ to be the power series in p; $C(p) = pc_1 + p^2c_2 + ...$, where c_i are constants for all $i, i = 1, 2, ...$ To get an approximate solution, we expand $y(x, p, c_1, c_2, ...)$ by Taylor's series, about p in the following manner

$$y(x, p, c_1, c_2, ...) = y_0(x) + \sum_{k=1}^{\infty} y_k(x, c_1, ..., c_k)p^k, \tag{5.25}$$

Substituting from Equation (5.25) into Equation (5.23) and equating the coefficients of like powers of p, we obtain the following *zeroth* to the *kth* order problems governing equations of

$$\begin{aligned}
&y_0(x), y_1(x, c_1), ..., y_k(x, c_1, ..., c_k) : \\
&L(y_0(x)) + f(x) = 0, \quad L(y_1(x, c_1)) - L(y_0(x)) = c_1N_0(y_0(x)), \\
&L(y_2(x, c_1, c_2)) - L(y_1(x, c_1)) = c_2N_0(y_0(x)) + c_1L(y_1(x, c_1))+ \\
&+ N_1(y_0(x), y_1(x, c_1), \\
&L(y_k(x, c_1, ..., c_k)) - L(y_{(k-1)}(x, c_1, ..., c_{(k-1)})) = \\
&= c_kN_0(y_0(x)) + \sum_{i=1}^{k-1} c_i[L(y_{(k-i)}x, c_1, ..., c_{(k-i)}))+ \\
&+ N_{(k-i)}(y_0(x), y_1(x, c_1), ..., y_{(k-i)}(x, c_1, , c_{(k-i)}))
\end{aligned} \tag{5.26}$$

for $k = 2, 3, ...$ where $N_{(k-i)}$ are the coefficient of $p^{(k-i)}$ in the expansion of for $N(y(x, p, c_1, c_2, ...))$ about the embedding parameter p;

$$\begin{aligned}
N(y(x, p, c_1, c_2, ...)) =& N_0(y_0(x))+ \\
&+ \sum_{k=1}^{\infty} N_k(y_0(x), y_1(x, c_1), ..., y_k(x, c_1, ..., c_k)p^k).
\end{aligned} \tag{5.27}$$

Note that the governing equations are linear and can be easily solved for $y_k, k \geq 0$.

It has been observed that the convergence of the series in Equation (5.25) depends upon the auxiliary constants $c_1, c_2, c_3, ...$

If it is convergent at $p = 1$, one get

$$y(x, 1, c_1, c_2, ...) = y_0(x) + \sum_{k=1}^{\infty} y_k(x, c_1, ..., c_k). \tag{5.28}$$

This equation is the source of the required approximate solutions. Substituting from Equation (5.25) into

$$L(y(x)) + f(x) + N(y(x)) = 0, \tag{5.29}$$

leads to the following residual formula

$$R(x, c_1, c_2, ...) = L(y(x, c_1, c_2, ...)) + f(x) + N(y(x, c_1, c_2, ...)). \tag{5.30}$$

If $R(x, c_1, c_2, ...) = 0$ then $y(x, c_1, c_2, ...)$ is the exact solution of the problem. For the determination of auxiliary constants c_i; $i = 1, 2, ..., m$, there are different methods. One method is the Least Squares;

$$J(c_1, c_2, ..., c_m) = \int_a^b R^2(x, c_1, c_2, ..., c_m)dx, \tag{5.31}$$

where $[a, b]$ is an interval depending on the given problem. The unknown constants c_i can be identified from the conditions

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = ... = \frac{\partial J}{\partial c_m} = 0. \tag{5.32}$$

With these constants known, the approximate solution is well-determined as

$$y^{(m)}(x) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2) + ... + y_m(x, c_1, ..., c_m). \tag{5.33}$$

5.3 Finding Schwarzschild Solution Using OHAM

We consider the Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu},$$

and construct the homotopy equation

$$(1 - p)(L(g_{\mu\nu}(r, p))) = c_1 p(G_{\mu\nu}(r, p) - 8\pi T_{\mu\nu}(r, p)),$$

where the operator L is defined to be

$$L(g_{\mu\nu}(r, p)) = -8\pi T_{\mu\nu}(r, p).$$

Note that $C(p)$ is taken to be $c_1 p$. Then

$$(1 - p)(-8\pi T_{\mu\nu}(r, p)) = c_1 p(G_{\mu\nu}(r, p) - 8\pi T_{\mu\nu}(r, p)),$$

clearly setting $p = 1$, leads to

$$G_{\mu\nu}(r, p) - 8\pi T_{\mu\nu}(r, p) = 0,$$

while taking $p = 0$, gives

$$T_{\mu\nu}(r, p) = 0.$$

In the following we follow schwarzschild and apply OHAM to (t, t) and (r, r) equations.

First case $(\mu,\ \nu) = (t,\ t)$.

For this case we have

$$L(g_{tt}(r, p)) = -8\pi T_{tt}(r, p) = -8\pi\rho(r)e^{2F(r,p)} = 0,$$

where $\rho(r) = 0$ (zero density in Vacuum).

Hence, we get

$$(1 - p)(-8\pi\rho(r)e^{2F(r,p)}) = c_1 p(G_{tt}(r, p) - 8\pi\rho(r)e^{2F(r,p)}).$$

Therefore

$$c_1 p G_{tt}(r, p) = 0.$$

But from Equation (5.10)

$$G_{tt}(r, p) = \frac{1}{r^2}e^{2F(r,p)}\frac{d}{dr}[r(1 - e^{-2H(r,p)})].$$

Thus

$$\frac{c_1 p}{r^2}e^{2F(r,p)}\frac{d}{dr}[r(1 - e^{-2H(r,p)})] = 0,$$

or

$$\frac{d}{dr}[r(1 - e^{-2H(r,p)})] = 0.$$

Integrating with respect to r gives

$$r(1 - e^{-2H(r,p)}) = c,$$

where c is arbitrary constant.

Hence

$$1 - e^{-2H(r,p)} = \frac{c}{r},$$

or

$$-e^{-2H(r,p)} = \frac{c}{r} - 1.$$

Therefore

$$e^{-2H(r,p)} = 1 - \frac{c}{r}.$$

Apply

$$H(r,p) = H^{(0)}(r) + pH^{(1)}(r, c_1), \tag{5.34}$$

we get

$$e^{-2H^{(0)}(r) - 2pH^{(1)}(r,c_1)} = 1 - \frac{c}{r},$$

or

$$e^{-2H^{(0)}(r)} e^{-2pH^{(1)}(r,c_1)} = 1 - \frac{c}{r},$$

with the aid of the approximation

$$e^{-2pH^{(1)}(r,c_1)} \simeq 1 - 2pH^{(1)}(r, c_1),$$

we obtain in an approximate sense

$$e^{-2H^{(0)}(r)} \left(1 - 2pH^{(1)}(r, c_1)\right) = 1 - \frac{c}{r},$$

or we write

$$e^{-2H^{(0)}(r)} - 2pH^{(1)}(r, c_1) e^{-2H^{(0)}(r)} = 1 - \frac{c}{r}.$$

Equating the coefficient of powers of p, we obtain

$$-2H^{(1)}(r, c_1) e^{-2H^{(0)}(r)} = 0, \quad \text{and} \quad H^{(1)}(r, c_1) = 0.$$

Thus

$$e^{-2H^{(0)}(r)} = 1 - \frac{c}{r}.$$

Taking the logarithm of both sides implies

$$-2H^{(0)}(r) = \ln\left(1 - \frac{c}{r}\right),$$

or

$$H^{(0)}(r) = -\frac{1}{2} \ln\left(1 - \frac{c}{r}\right).$$

Hence Equation (5.34) gives

$$H(r,p) = -\frac{1}{2} \ln\left(1 - \frac{c}{r}\right).$$

Second case $(\mu,\nu)=(r,r)$.

For this case

$$L(g_{rr}(r,p))=-8\pi T_{rr}(r,p)=-8\pi P(r)e^{2H(r,p)}=0,$$

where $P(r)=0$ (zero pressure in Vacuum).

Thus

$$(1-p)(-8\pi Pe^{2H(r,p)})=c_1p(G_{rr}(r,p)-8\pi Pe^{2H(r,p)}).$$

Therefore

$$c_1pG_{rr}(r,p)=0,$$

$$G_{rr}(r,p)=0.$$

Again $G_{rr}(r,p)$ is found to be, Equation (5.9),

$$G_{rr}(r,p)=\frac{1}{r^2}(1+2rF^{'}(r,p)-e^{2H(r,p)}).$$

Hence

$$1+2rF^{'}(r,p)-e^{2H(r,p)}=0,$$

or

$$1+2rF^{'}(r,p)-\frac{1}{1-\frac{c}{r}}=0,$$

or

$$2rF^{'}(r,p)=\frac{1}{1-\frac{c}{r}}-1.$$

Thus

$$F^{'}(r,p)=\frac{1}{2}\left(\frac{1}{r-c}-\frac{1}{r}\right).$$

Integrating with respect to r, we obtain

$$F(r,p)=\frac{1}{2}\left(\ln(r-c)-\ln(r)\right),$$

or

$$F(r,p)=\frac{1}{2}\ln\frac{r-c}{r},$$

or

$$F(r,p)=\frac{1}{2}\ln\left(1-\frac{c}{r}\right),$$

where the constant of integration has been chosen so that is limit $r \to \infty$ the line element recover its Minkowski form.

Since

$$F(r,p) = F^{(0)}(r) + pF^{(1)}(r,c_1), \tag{5.35}$$

we can write

$$F^{(0)}(r) + pF^{(1)}(r,c_1) = \frac{1}{2}\ln\left(1 - \frac{c}{r}\right).$$

Similar to first case we have

$$F^{(1)}(r,c_1) = 0,$$

and

$$F^{(0)}(r) = \frac{1}{2}\ln\left(1 - \frac{c}{r}\right).$$

Therefore Equation (5.35) implies

$$F(r,p) = \frac{1}{2}\ln(1 - \frac{c}{r}).$$

Since c was arbitrary, we choose it to be $2M$.

Hence we get the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

which is exactly the Schwarzschild solution, Equation (5.19).

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