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On Continuity of Functions between Vector Metric Spaces

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On Continuity of Functions between Vector Metric Spaces

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Dedication

I dedicate this thesis to my family (father, mother and brothers) and my friends for their support, understanding and giving the wanted facilities.

Declaration

I certify that this thesis submitted for the degree of master, is the result of my own research, except where otherwise acknowledge, and that this study has been not submitted for a higher degree to any other university or institution.

Signature:

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Abstract

In 2014, the researcher “Cuneyt Cevik” studied two types of continuity of functions in vector metric spaces, namely, vectorial and topological continuous functions. Cevik concluded several important relations and theorems in vector metric space, such as Extension Theorem and Uniform Limit theorem.

In this thesis, I studied and developed the Cuneyt Cevik’s work [3], so I concluded and found out many relations. In fact, we proved the Extension Theorem holds for the case of vectorial uniformly continuous instead of a topological uniformly continuous as “Cuneyt Cevik “ proved. Also, we proved the uniform limit theorem for the case of vectorially uniformly continuous and topological continuous.

Introduction

In [4], a vector metric space is defined with a distance map having values in a Riesz space, and some results in metric space theory are generalized to vector metric space theory. In this thesis, we used the Riesz space as a tool for studying the continuity of vector valued functions, for more information about Riesz spaces see [2, 5, 7]. Actually, the study of metric spaces having value on a vector space has started by Zabrejko in [6]. The distance map in the sense of Zabrejko takes values from an ordered vector space. We use the structure of lattice with the vector metrics having values in Riesz spaces; then we have new results.

The outline of the thesis is as follows

In Chapter one a general introduction about Riesz space, vector metric space and two types of continuity on vector metric space is presented. This chapter distinguishes continuities vectorially and topologically. Moreover, vectorial continuity examples are given and the relationship between vectorial continuity of a function and its graph demonstrated.

In Chapter two equivalent vector metrics, vectorial isometry, vectorial homeomorphism definitions, and examples are given.

In Chapter three uniform continuity was discussed, some extension theorems for functions defined on vector metric spaces are given, uniform limit theorem on a vector metric space is given, and the structure of vectorial continuous function spaces is demonstrated.

Chapter one

Vector Metric Spaces

In this chapter we will introduce the concepts of Riesz space, order convergent, vector metric spaces, E-convergent, topological continuous and vectorial continuous spaces and also some related concepts.

1.1 Riesz Space

In order to define the concept of vector metric space we need to define the Riesz space. To do this, we first define an ordered relation and an ordered vector space.

Definition 1.1.1

Let E be a vector space over the real number R, an ordered relation is a partially ordered relation $\preccurlyeq$ which satisfies the following condition if $x, y, z \in E, \lambda \in R, \lambda \geq 0$, then

$x + z \preccurlyeq y + z$ and $\lambda x \preccurlyeq \lambda y$ whenever $x \preccurlyeq y$.

The vector space E over R with an ordered relation $\preccurlyeq$ on E is called an ordered vector space.

The following examples explain what we mean by an ordered vector space

Example 1.1.2

Let R be the set of real numbers and consider R as a vector space over itself (with usual addition and scalar multiplication), then R with usual partially ordering $\leq$, is an ordered vector space.

Example 1.1.3

Consider $M = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R\right\}$ as a vector space over R (with usual addition and scalar multiplications in matrices). Define $\preccurlyeq$ on M by $A \preccurlyeq B$ if and only if $trac(A) \leq trac(B)$, then $(M, \preccurlyeq)$ is an ordered vector space, since

$trac(A + C) = trac(A) + trac(C)$ and $trac(\lambda A) = \lambda\, trac(A)$.

Next, we define a Riesz space.

Definition 1.1.4

Let E be a vector space over R, then E is said to be Riesz Space if it is an ordered vector space and for each pair of elements in E it has a supremum or infimum in E.

The vector spaces defined in Example 1.1.2, 1.1.3 are Riesz spaces.

Now we will introduce several definitions in order to define order convergent and order Cauchy. Not that a set E is bounded if it is bounded from both above and below. Also, we write

$a_n \downarrow a$ if (a_n) is decreasing sequence in E such that $inf\ a_n = a$.

Definition 1.1.5

Let E be a Riesz space. E is called Archimedean if $\frac{a}{n} \downarrow 0, \forall a \in E_+$ where

$E_+ = \{x \in E : x \geq 0\}$. Clearly, for all $a \in R_+$ and $\frac{a}{n} \downarrow 0$, R is Archimedean.

Moreover, we want to define Dedekind complete and Dedekind σ-complete.

Definition 1.1.6

Let E be a Riesz space, E is called Dedekind complete if every nonempty bounded above subset of E has a supremum in E.

Recall that in R, every nonempty bounded above subset has a supremum in R (complete axiom). Therefore, R is Dedekind complete.

Definition 1.1.7

Let E be a riesz space, E is called Dedekind σ- complete if every nonempty countable bounded above subset of E has a supremum in E.

Let E be a Dedekind complete, if A is a nonempty countable bounded above subset of E, then A is bounded above in E, so by Definition1.1.6, A has a supremum, which implies that E is Dedekind σ-complete. In fact we proved the following theorem.

Theorem 1.1.8

Every Dedekind complete is Dedekind σ-complete.

The convers of the last theorem is not true since we can find an example which is Dedekind σ-complete but not Dedekind complete.

Example 1.1.9

Let E be the Riesz space of all real bounded functions on $[0,1]$ such that $f(x) \neq f(0)$ holds for at most countably many x, with pointwise ordering defined as

$(f_1, g_1) \leq (f_2, g_2)$ if and only if $f_1 \leq f_2$ and $g_1 \leq g_2$ for (f_1, g_1) and $(f_2, g_2) \in [0,1]$

If $0 \leq U_n \leq V$ holds for the sequence (U_n) in E, then $\sup U_n$ exists in E, so E is Dedekind σ-complete.

Let $A = \{all\ function\ f \in E\ vanishing\ on\ \left[0, \frac{1}{2}\right]\}$ and

$A^d = \{all\ function\ f \in E\ vanishing\ on\ \left[\frac{1}{2}, 1\right]\}$. So every $f \in A \oplus A^d$ satisfies $f(0) = 0$, so $A \oplus A^d \neq E$, so E does not have the projection property. But we know from theorem12.3 in [1] " Every Dedekind complete has the projection property "

Not that if we let E be a Riesz Space, then the supremum element denoted by $x \vee y$ defined by

$x \vee y = \sup\{x, y\}\ \forall x, y \in E.$

Definition 1.1.10

Let E be a riesz space.

(a) A sequence (b_n) in E is said to be o-convergent (or order convergent) to b if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n.

(denoted by ($b_n \overset{o}{\rightarrow} b$)) where $|a| := a \vee (-a)$ for any a $\in$ E.

(b) A sequence (b_n) in E is said to be o-Cauchy if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $\left|b_n - b_{n+p}\right| \leq a_n\ \ \forall n, p \in N.$

(c) The Riesz space E is said to be o-complete if every o-Cauchy sequence is o- convergent.

Now we will introduce many concepts on operator function T between two Riesz spaces in order to prove that every σ-order continuous operator T is bounded.

Definition 1.1.11

The operator $T: E \to F$ between two Riesz spaces is positive if $T(x) \geq 0$ for all $x \geq 0$.

Definition1.1.12

(a) Let $(E, \preccurlyeq)$ be a Riesz space and let $x, y \in E$, then the order interval $[x, y]$ is the set

$\{z \in E : x \leq z \leq y\}$

(b) The operator $T: E \to F$ between two Riesz spaces is order bounded if it maps bounded subsets of E to bounded subsets of F.

Definition 1.1.13

The operator T is called σ-order continuous if $x_n \xrightarrow{o} 0$ in E implies $T(x_n) \xrightarrow{o} 0$ in F.

Theorem 1.1.14 [3]

Every σ-order continuous operator is order bounded.

Proof:

Let $T: E \to F$ be an σ-order continuous operator and let $x \in E_+$. If we consider the order bounded interval $[0, x] \subseteq E$ and let $\{x_n\}$ be a sequence in $[0, x]$ such that $x_n \xrightarrow{o} 0$, then since T is σ-order continuous operator, $T(x_n) \xrightarrow{o} 0$. So, there is a sequence (y_n) in F such that $|Tx_n| \leq y_n$and $y_n \downarrow 0$. Hence, $T[0, x]$ is an order bounded subsets of F. Thus T is order bounded ■

Definition 1.1.15

Let E and F are Riesz Space. The operator $T: E \to F$ is said to be lattice homomorphism if $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in E$.

Example 1.1.16

Consider the Riesz space R^{+} (with addition and scalar multiplication defined by

$x + y = x \times y$ and $kx = x^{k}$). Define $T: R^{+} \to R$ by $T(x) = x^{2}$ and $x \vee y = sup\{x, y\}$ then T is a lattice homomorphism. To prove it we consider two cases as follows

Case1:

if $x \geq y$ then $T(x \vee y) = T(x) = x^{2}$ and $T(x) \vee T(y) = sup\{x^{2}, y^{2}\} = x^{2}$.

Case2:

if $x \leq y$ then $T(x \vee y) = T(y) = y^{2}$ and $T(x) \vee T(y) = sup\{x^{2}, y^{2}\} = y^{2}$.

Hence, T is lattice homomorphism.

1.2 Convergence in Vector Metric Spaces

In this section we show the type of convergent in vector metric space and present the properties between them.

Definition 1.2.1

Let X be a nonempty set and let E be a Riesz space. The function $d: X \times X \to E$, which satisfies the following condition

(VM1) $d(x, y) = 0$ if and only if $x = y$

(VM2) $d(x, y) \leq d(x, z) + d(y, z), \forall x, y, z \in X$ is said to be vector metric (or E-metric). The triple (X, d, E) is called vector metric space.

Not that, the vector metric function defined in the previous definition have many properties, which is for all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$.

Next we present some examples of vector metric spaces

Example 1.2.2

(a) ARiesz space E is a vector metric space with $d: E \times E \to E$ defined by

$$d(x,y) = |x - y|$$

Since it satisfies the condition of definition 1.2.1 as follow

$d(x,y) = |x - y| = 0$ if and only if $x - y = 0$ if and only if $x = y$ and

$d(x,y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x,z) + d(z,y)$

This vector metric is called the absolute valued metric on E.

(b) The space R^2 is a Riesz space with coordinatwise ordering defined by

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$ for (x_1, y_1) and $(x_2, y_2) \in R^2$. To show that, let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$, then

(1) If $x \leq y$, then $x_1 \leq y_1$ and $x_2 \leq y_2$. For $z_1, z_2 \in R$, we have

$x_1 + z_1 \leq y_1 + z_1$ and $x_2 + z_2 \leq y_2 + z_2$................................($\star$)

but $x + z = (x_1 + z_1, x_2 + z_2)$ and $y + z = (y_1 + z_1, y_2 + z_2)$.

So, from ($\star$) we have $x + z \leq y + z$.

(2) If $x \leq y$ and $\lambda \in R, \lambda \geq 0$, then $x_1 \leq x_2$ and $y_1 \leq y_2$ and so $\lambda x_1 \leq \lambda x_2$ and $\lambda y_1 \leq \lambda y_2$ and so $\lambda x \leq \lambda y$.

Now, let $x, y \in R^2$ where $= (x_1, y_1)$ and (x_2, y_2), then $x_1, x_2, y_1, y_2 \in R$ and we have foure cases:

Case (1): If $x_1 \leq y_1$ and $x_2 \leq y_2$, then $x < y$ and $\sup\{x, y\} = y \in R^2$.

Case (2): If $x_1 \leq y_1$ and $x_2 \geq y_2$, then $\sup\{x, y\} = (y_1, x_2) \in R^2$.

Case (3): If $x_1 \geq y_1$ and $x_2 \leq y_2$, then $\sup\{x, y\} = (x_1, y_2) \in R^2$.

Case (4): If $x_1 \geq y_1$ and $x_2 \geq y_2$, then $x > y$ and $\sup\{x, y\} = x \in R^2$.

So in all cases, the supremum belong to R^2.

Hence, R^2 is Riesz Space.

Also R^2 is a Riesz space with coordinatewise defined by

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 < x_2$ or $x_1 = x_2$,$y_1 \leq y_2$.

Therefore $d: R^2 \times R^2 \rightarrow R^2$ defined by

$$d\big((x_1, y_1), (x_2, y_2)\big) = (\alpha|x_1 - y_1|, \beta|x_2 - y_2|)$$

is a vector metric, where α and β are positive real numbers.

Proof: Want to show that $d\big((x_1, y_1), (x_2, y_2)\big)$ satisfies (VM1) and (VM2)

$d\big((x_1, y_1), (x_2, y_2)\big) = (\alpha|x_1 - y_1|, \beta|x_2 - y_2|) = (0,0)$ if and only if $\alpha|x_1 - y_1| = 0$ and $\beta|x_2 - y_2| = 0$, but α, β are positive so $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$ which implies $x_1 = x_2$ and $y_1 = y_2$ and

$$\begin{aligned} d\big((x_1, y_1), (x_2, y_2)\big) &= (\alpha|x_1 - y_1|, \beta|x_2 - y_2|) \\ &= (\alpha|x_1 - z_1 + z_1 - y_1|, \beta|x_2 - z_2 + z_2 - y_2|) \\ &\leq (\alpha|x_1 - z_1| + \alpha|z_1 - y_1|, \beta|x_2 - z_2| + \beta|z_2 - y_2|) \\ &\leq (\alpha|x_1 - z_1|, \beta|x_2 - z_2|) + (\alpha|z_1 - y_1|, \beta|z_2 - y_2|) \end{aligned}$$

$$= d\big((x_1, z_1), (x_2, z_2)\big) + d\big((z_1, y_1), (z_2, y_2)\big)$$

(c) Let $d: R \times R \to R^2$ defined by

$d(x, y) = (\alpha|x - y|, \beta|x - y|)$, where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Then d is a vector metric with coordinatewise.

Proof : Want to show that $d(x, y)$ satisfies (VM1) and (VM2)

$d(x, y) = (\alpha|x - y|, \beta|x - y|) = (0,0)$ if and only if $\alpha|x - y| = 0$ and $\beta|x - y| = 0$, but α, β are positive not both zero, so $|x - y| = 0$ if and only if $x - y = 0$ if and only if $x = y$.

$$d(x, y) = (\alpha|x - y|, \beta|x - y|) = (\alpha|x - z + z - y|, \beta|x - w + w - y|)$$

$$\leq (\alpha|x - z| + \alpha|z - y|, \beta|x - w| + \beta|w - y|)$$
$$\leq (\alpha|x - z|, \beta|x - w|) + (\alpha|z - y|, \beta|w -|)$$

$$= d\big((x_1, z_1), (x_2, z_2)\big) + d\big((z_1, y_1), (z_2, y_2)\big)$$

In the rest of this section, we introduce E-convergent, E- Cauchy, E-complet, E-bounded and prove some relations between them.

Definition 1.2.3

Let (X, d, E) be a vector metric space.

(a) A sequence (x_n) in X is vectorially convergent (or is E-convergent) to some $x \in X$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ $\forall n$, denoted by $x_n \xrightarrow{d,E} x$.

(b) A sequence (x_n) in X is called E-Cauchy whenever there exists a sequence $(\mathrm{a_n})$ in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$, $\forall n, p$.

(c) The vector metric space X is called E-Complete if each E-Cauchy sequence in X is E-convergent to a limit in X.

(d) The set X is said to be E-bounded if there exists an element $a > 0$ in E such that $d(x, y) \leq a$ for x and y in X.

(e) A subset U of vector metric space (X, d, E) is called E-closed if for any sequence $(x_n) \subseteq U$ such that $x_n \xrightarrow{d,E} x$ then $x \in U$.

(f) A subset Y of X is called E-dense whenever for every $x \in X$ there exists a sequence (x_n) in Y satisfying $x_n \xrightarrow{d,E} x$.

Theorem 1.2.4 [4]

For the vector metric space (X, d, E) the following properties hold:

(a) Every E-convergent sequence is an E-Cauchy sequence.

(b) Every E-Cauchy sequence is E-bounded.

(c) If an E-Cauchy sequence (x_n) has a subsequence $\left(x_{n_k}\right)$ such that $x_{n_k} \xrightarrow{d,E} x$ then $x_n \xrightarrow{d,E} x$.

(d) If (x_n) and (y_n) are E-Cauchy sequence, then $(d(x_n, y_n))$ is an o-Cauchy.

Proof:

(a) Let (x_n) be a sequence in X such that $x_n \xrightarrow{d,E} x$. Want to show that (x_n) is E-Cauchy in X. Since there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq \frac{a_n}{2}$ $\forall n$, then $d\left(x_n, x_{n+p}\right) \leq d(x_n, x) + d\left(x_{n+p}, x\right) \leq \frac{a_n}{2} + \frac{a_n}{2} \leq a_n$ for all n and p, then (x_n) is an E-Cauchy sequence in X.

(b) Let (x_n) be an E-Cauchy sequence in X. Want to show that (x_n) is E-bounded, that is there exist an element $a > 0$ in E such that $d(x_n, x_m) \leq a, \ \forall n, m \in N$. Since (x_n) is E-Cauchy then there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ for all n and p. Now, let $m > n$, then let $p = m - n$, so $m = n + p$ and $d(x_n, x_m) = d(x_n, x_{n+p}) \leq a_n \ \ \forall n, m \in N$. But $a_n \downarrow 0$ so $a_1 > a_n \ \ \forall n \geq 1$.

Therefore, let $a = a_1$, then we have $d(x_n, x_m) < a \quad \forall n, m \in N$

(c) Let (x_n) be an E-Cauchy sequence and let (x_{n_k}) be a subsequence of (x_n) such that $x_{n_k} \xrightarrow{d,E} x$ in X. Want to find $c_n \downarrow 0$ such that $d(x_n, x) \leq c_n$. Since (x_n) is E-Cauchy, then there exist $a_n \downarrow 0$ such that $d(x_n, x_{n+p}) \leq a_n, \ \forall n, p$ and since $x_{n_k} \xrightarrow{d,E} x$, then there exist $b_n \downarrow 0$ such that $d(x_{n_k}, x) \leq b_n$. Now, $(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \ \leq a_n + b_n$. Take $c_n = a_n + b_n$, clearly, $c_n \downarrow 0$ and $d(x_n, x) \leq c_n$,therefore, $x_n \xrightarrow{d,E} x$.

(d) Let (x_n) and (y_n) are E-Cauchy sequence. Want to show that $(d(x_n, y_n))$ is an o-Cauchy. Since (x_n) and (y_n) are E-Cauchy sequences, then there exist $a_n \downarrow 0$ and $b_n \downarrow 0$ in E such that $d(x_n, x_{n+p}) \leq a_n$ and $d(y_n, y_{n+p}) \leq b_n$. Since

$$\mathrm{d}(\mathrm{x}_{\mathrm{n+p}}, \mathrm{y}_{\mathrm{n+p}}) \leq \mathrm{d}(\mathrm{x}_{n+p}, \mathrm{x}_{\mathrm{n}}) + \mathrm{d}(\mathrm{x}_{\mathrm{n}}, \mathrm{y}_{\mathrm{n}}) + \mathrm{d}(\mathrm{y}_{\mathrm{n}}, \mathrm{y}_{\mathrm{n+p}}),$$

then

$$d(x_n, y_n) - d(x_{n+p}, y_{n+p}) \leq d(x_n, x_{n+p}) + d(y_n, y_{n+p})$$

and

$$d(x_{n+p}, y_{n+p}) - d(x_n, y_n) \leq d(x_{n+p}, x_n) + d(y_{n+p}, y_n).$$

So from the last two inequalities we get

$\left|d(x_n, y_n) - d(x_{n+p}, y_{n+p})\right| \leq d\left(x_n, x_{n+p}\right) + d\left(y_n, y_{n+p}\right) \leq a_n + b_n$ for all n and p. Therefore, the sequence $(d(x_n, y_n))$ is an o-Cauchy sequence in E ■

Example 1.2.5

(1) If $E = R$, then the concepts of vectorial convergence and convergence in metric are the same. Since $d(x, y) = |x - y|$, for any $x, y \in R$.

For more details, if $x_n \overset{d,E}{\longrightarrow} x$, then there exist a_n in E such that $d(x_n, x) \leq a_n$ and $a_n \downarrow 0$. Now, if $\epsilon > 0$, then there exist N such that $a_n \leq \in$, $\forall n \geq N$ and so $d(x_n, x) = |x_n - x| < \in, \forall n \geq N$, which implies (x_n) converge to x.

(2) If $E = R$, then the concepts of E-Cauchy sequence and Cauchy sequence are the same. Since $d(x, y) = |x - y|$, for any $x, y \in R$.

For more details, let (x_n) be E-cauchy sequence, then there exist b_n in E such that $d(x_n, x_{n+p}) \leq b_n$ and $b_n \downarrow 0$. Now, if $\epsilon > 0$, then there exist N such that $b_n \leq \in$, $\forall n \geq N$ and so $d\left(x_n, x_{n+p}\right) = \left|x_n - x_{n+p}\right| < \in, \forall n, p \geq N$, which implies (x_n) Cauchy sequence.

1.3 Topological and Vectorial Continuity

In this section, we study two types of continuity in a vector metric space and give many relations between them. Also present δ-double vector metric and $E \times F$-valued product vector metric.

Definition 1.3.1

Let (X, d_1, E) *and* (Y, d_2, F) be vector metric spaces, and let $x \in X$.

(a) A function $f: X \to Y$ is said to be topologically continuous at x if for every $b > 0$ in F there exists some a in E such that $d_2\big(f(x), f(y)\big) < b$ whenever $x, y \in X$ and $d_1(x, y) < a$.

(b) A function $f: X \to Y$ is said to be vectorially continuous at x if $x_n \xrightarrow{d_1,E} x$ in X implies $f(x_n) \xrightarrow{d_2,F} f(x)$ in Y.

Theorem 1.3.2 [3]

Let (X, d_1, E) *and* (Y, d_2, F) be vector metric spaces where F is Archimedean. If a function $f: X \to Y$ is topologically continuous, then f is vectorially continuous.

Proof:

Let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x$, then there is a sequence a_n in E such that $a_n \downarrow 0$ and $d_1(x_n, x) \leq a_n$. Let $b > 0$ in F, since f is a topological continuous at x, then

for n=1, there exists $b_1 > 0$ in E such that $d_1(x, x_1) < b_1$ implies $d_2(f(x), f(x_1)) < b$

for n=2, there exists $b_2 > 0$ in E such that $d_1(x, x_2) < b_2$ implies $d_2(f(x), f(x_2)) < \frac{b}{2}$

if we continue in this manner we get

for $n = k$, there exists $b_k > 0$, in E such that $d_1(x, x_k) < b_k$ implies $d_2(f(x), f(x_k)) < \frac{b}{k}$

Take $c_k = a_k \wedge b_k$ in E such that if $d_1(x_k, x) \leq c_k \leq b_k$, then $d_2\big(f(x_k), f(x)\big) < \frac{b}{k}$. But since F is Archimedean, then $\frac{b}{k} \downarrow 0$, so if $d_k = \frac{b}{k}$, then $d_2(\mathrm{f(x_k), f(x)}) < d_k$ and $d_k \downarrow 0$. That is $f(x_k) \xrightarrow{d_2,F} f(x)$, so f is vectorial continuous at x ■

The following corollary summarize some of the nice characterization of vectorially continuous functions.

Corollary 1.3.3

For a function $f: X \to Y$ between two vector metric spaces (X, d_1, E) and (Y, d_2, F) the following statements hold for a sequence (x_n) in E and $x \in E$.
(a) If F is Dedekind σ-complete, f is vectorially continuous and $d_1(x_n, x) \downarrow 0$, then $d_2(f(x_n), f(x)) \downarrow 0$.

(b) If E is Dedekind σ-complete and $d_1(x_n, x) \downarrow 0$ implies $d_2(f(x_n), f(x)) \downarrow 0$, then the function f is vectorially continuous.

(c) Suppose that E and F are Dedekind σ-complete. Then the function f is vectorially continuous if and only if $\mathrm{d}_1(\mathrm{x}_n, \mathrm{x}) \downarrow 0$ implies $d_2(f(x_n), f(x)) \downarrow 0$

Proof:

(a) Let (x_n) be a sequence in E such that $d_1(x_n, x) \downarrow 0$, from definition of E- convergent, then $x_n \xrightarrow{d_1,E} x$. Want to prove that $d_2(f(x_n), f(x)) \downarrow 0$.

Since f is a vectorially continuous, then $f(x_n) \xrightarrow{d_2,F} f(x)$, and so there exist $b_n \downarrow 0$ such that $d_2\big(f(x_n), f(x)\big) \le b_n \; \forall n$. Since $\{d_2\big(f(x_n), f(x)\big): n \in N\}$ is a non-empty countable bounded subset of F and F is Dedekind σ- complete then this set has a supermum in F, but $0 \le d_2(f(x_n), f(x)) \le b_n \;\; \forall n$, by sandwich theorem $d_2(f(x_n), f(x)) \downarrow 0$.

(b) Let x_n be a sequence in X such that $x_n \xrightarrow{d_1,E} x$, then there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \le a_n$. Want to show that $f(x_n) \xrightarrow{d_2,F} f(x)$. Since E is Dedekind σ-complete, then $d(x_n, x) \downarrow 0$ hold, and $d_2(f(x_n), f(x)) \downarrow 0$ hold (by hypothesis), and so $f(x_n) \xrightarrow{d_2,F} f(x)$.

(c) There is two sided to prove this part, one side get from the proof of part (a) and another side get from the proof of part (b) ■

Now, we will give an example

Example 1.3.4

Let (X, d, E) be a vector metric space and suppose $d: X^2 \to E$ be a vector metric function.

If $x_n \xrightarrow{d,E} x \;\; and \; y_n \xrightarrow{d,E} y$, then $d(x_n, y_n) \xrightarrow{o} d(x, y)$ and d is vectorially continuous.

Proof: Since $x_n \xrightarrow{d,E} x \;\; and \; y_n \xrightarrow{d,E} y$, then there exist $a_n \downarrow 0$ and $b_n \downarrow 0$ such that $d(x_n, x) \leq a_n$ and $d(y_n, y) \leq b_n$ implies $|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \leq a_n + b_n$.

Let $c_n = a_n + b_n$, then $c_n \downarrow 0$ and $|d(x_n, y_n) - d(x, y)| \leq c_n$. So $d(x_n, y_n) \xrightarrow{o} d(x, y)$

Therefore d is vectorially continuous.

In this example X^2 is equipped with the E-valued vector metric $\tilde{d}$ defined by

$\tilde{d}(z, w) = d(x_1, x_2) + d(y_1, y_2)$ for all $z = (x_1, y_1)$, $w = (x_2, y_2) \in X^2$ and E is equipped with the absolute valued vector metric $|.|$.

Theorem 1.3.5 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces. If a function $f: X \to Y$ is vectorially continuous, then for every F-closed subset B of Y the set $f^{-1}(B)$ is E-closed in X.

Proof:

Let (x_n) be a sequence in $f^{-1}(B)$ such that $x_n \xrightarrow{d_1,E} x$. Want to show that $x \in f^{-1}(B)$. Since the function f is vectorially continuous, $f(x_n) \xrightarrow{d_2,F} f(x)$ but the set B is F-closed, so $f(x) \in B$, that is $x \in f^{-1}(B)$. Therefore, the set $f^{-1}(B)$ is E-closed ■

Next, we present some results related to Riesz space. To this end note that if E and F are two Riesz spaces, then $E \times F$ is also Riesz space with coordinatewise ordering defined by

$(e_1, f_1) \leq (e_2, f_2) \leftrightarrow e_1 \leq e_2, f_1 \leq f_2$ for all $(e_1, f_1), (e_2, f_2) \in E \times F$.

Further, the Riesz space $\mathrm{E} \times \mathrm{F}$ is a vector metric space equipped with the biabsolute valued vector metric $|.|$ defined as $|a - b| = (|e_1 - e_2|, |f_1 - f_2|)$ for all $a = (e_1, f_1)$, $b = (e_2, f_2) \in E \times F$. To achieve our goal, first define δ-double vector metric and give some examples about vectorially continuous function.

Remark 1.3.6

Let d_1 and d_2 be two vector metrics on X which are E-valued and F-valued respectively. The map δ defined by $\delta(x, y) = (d_1(x, y), d_2(x, y))$ for all $x, y \in X$ is an $E \times F$-valued vector metric on X.

Definition 1.3.7

The $E \times F$-Falued vector metric given in the remark is called δ double vector metric.

Example 1.3.8

Let (X, d_1, E) and (X, d_2, F) are vector metric spaces and $f : X \to E$, $g : X \to F$ are vectorially continuous functions, then the function $h: X \to E \times F$ defined by $h(x) = (f(x), g(x))$ for all $x \in X$ is vectorially continuous with the double vector metric δ and the biabsolute valued vector metric $|.|$. To show this, let (x_n) be a sequence in X such that $x_n \xrightarrow{\delta, E\times F} x$.

Want to show that $h(x_n) \xrightarrow{bi, E\times F} h(x)$.

Now for some (a_n, b_n) in $E \times F$ with $(a_n, b_n) \downarrow 0$ we have

$\delta(x_n, x) = (d_1(x_n, x), d_2(x_n, x)) \leq (a_n, b_n)$ which implies

$d_1(x_n, x) \leq a_n$ and $d_2(x_n, x) \leq b_n$ with $a_n \downarrow 0$ and $b_n \downarrow 0$.

Since f and g are vectorially continuous, then there exist $c_n \downarrow 0$ and $d_n \downarrow 0$ such that $|f(x_n) - f(x)| \leq c_n$ and $|g(x_n) - g(x)| \leq d_n$. Therefore

$(|f(x_n) - f(x)|, |g(x_n) - g(x)|) \leq (c_n, d_n)$. Let $w_n = (c_n, d_n)$, then $w_n \downarrow 0$ and

$((f(x_n), g(x_n)), (f(x), g(x))) \leq w_n$

Hence, $h(x_n) = (f(x_n), g(x_n)) \xrightarrow{bi, E\times F} (f(x), g(x)) = h(x)$

Therefore, h is vectorially continuous function.

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces. Then $X \times Y$ is a vector metric space equipped with the $E \times F$-valued product vector metric π defined by

$\pi(z, w) = (d_1(x_1, x_2), d_2(y_1, y_2))$ for all $z = (x_1, y_1)$, $w = (x_2, y_2) \in X \times Y$

Corollary 1.3.9 [3]

(a) If $f:(X,d,E)\to(Y,\eta,G)$ and $g:(X,\zeta,F)\to(Z,\xi,H)$ are vectorially contiuous functions, then the function $h: X\to Y\times Z$ defined by $h(x)=(f(x),g(y))$ for all $x\in X$ is vectorially continuous with $E\times F$-valued double vector metric δ on X and the $G\times H$-valued product vector metric π on $Y\times Z$ and the absolute valued vector metric $|.|$.

(b) Let G be a Riesz space. If $f:(X,d_1,E)\to G$ and $g:(Y,d_2,F)\to G$ are vectorially continuous functions, then the function $h: X\times Y\to G$ defined by $h(x,y)=|f(x)-g(y)|$ for all $x\in X, y\in Y$ is vectorially continuous with $E\times F$-valued product vector metric π on $X\times Y$ and the absolute valued vector metric $|.|$ on G.

(c) If $f:(X,d,E)\to(Z,\eta,G)$ and $g:(Y,\zeta,F)\to(W,\xi,H)$ are vectorially continuous function, then the function $h: X\times Y\to Z\times W$ defined by $h(x,y)=(f(x),g(y))$ for all $x\in X, y\in Y$ is vectorially continuous with the $E\times F$-valued and $G\times H$-valued product vector metrics on $X\times Y$ and $Z\times W$ respectively.

Proof:

(a) Let (x_n) be a sequence in X such that $x_n \xrightarrow{\delta,E\times F} x$.

Want to show that $h(x_n)\xrightarrow{\pi,G\times H} h(x)$.

Since f and g are vectorially continuous, then $f(x_n)\xrightarrow{\eta,G} f(x)$ and $g(x_n)\xrightarrow{\xi,H} g(x)$ implies $\big(f(x_n),g(x_n)\big)\xrightarrow{\pi,G\times H}\big(f(x),g(x)\big)$ because

Since $f(x_n)\xrightarrow{\eta,G} f(x)$ and $g(x_n)\xrightarrow{\xi,H} g(x)$, then there exist $a_n\downarrow 0$ and $b_n\downarrow 0$ such that

$\eta(f(x_n), f(x)) \leq a_n$ and $\xi(g(y_n), g(y)) \leq b_n$. Take $c_{n=}(a_n, b_n)$, clearly $c_n \downarrow 0$ and

$\pi\Big(\big(f(x_n), f(x)\big), (g(x_n), g(x))\Big) = \big(\eta(f(x_n), f(x)), \xi(g(y_n), g(y))\big) \leq (a_n, b_n) = c_n$.

Therefore $h(x_n) = \big(f(x_n), g(x_n)\big) \xrightarrow{\pi, G\times H} \big(f(x), g(x)\big) = h(x)$.

Therefore, h is vectorially continuous function

(b) Let (x_n) and (y_n) be a sequence in X and Y respectively such that $x_n \xrightarrow{d_1, E} x$ and $y_n \xrightarrow{d_2, F} y$.

Want to show that $h(x_n, y_n) \xrightarrow{\pi, E\times F} h(x, y)$.

since f and g are vectorially continuous function ($f(x_n) \xrightarrow{|.|, G} f(x)$ and $g(y_n) \xrightarrow{|.|, G} g(y)$), then

$h(x_n, y_n) = |f(x_n) - g(y_n)| \xrightarrow{\pi, E\times F} |f(x) - g(y)| = h(x, y)$

Therefore, h is vectorially continuous function.

(c) Let (x_n) and (y_n) be a sequence in X and Y respectively such that $x_n \xrightarrow{d, E} x$ and $y_n \xrightarrow{\zeta, F} y$.

Want to show that $h(x_n, y_n) \xrightarrow{\pi, G\times H} h(x, y)$

Since f and g are vectorially continuous, then $f(x_n) \xrightarrow{d, G} f(x)$ and $g(y_n) \xrightarrow{\zeta, H} g(x)$.

Hence $h(x_n, y_n) = \big(f(x_n), g(x_n)\big) \xrightarrow{\pi, G\times H} \big(f(x), g(x)\big) = h(x, y)$.

Therefore, h is vectorially continuous function.

Proposition 1.3.10 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces and $(z_n) = (x_n, y_n)$ be a sequence in $(X \times Y, \pi, E \times F)$ and let $z = (x, y) \in X \times Y$

then, $z_n \xrightarrow{\pi, E\times F} z$ if and only if $x_n \xrightarrow{d_1, E} x$ and $y_n \xrightarrow{d_2, F} y$.

Proof:

First, suppose $z_n \xrightarrow{\pi, E\times F} z$ where $z_n = (x_n, y_n)$, $z = (x, y)$ in $X \times Y$, then there exist $a_n \downarrow 0$ such that $\pi(z_n, z) \leq a_n, \forall n$. But $a_n \in E \times F$, then $a_n = (b_n, c_n)$. Since $a_n \downarrow 0$, then clearly $b_n \downarrow 0$ and $c_n \downarrow 0$ and

$\pi(z_n, z) = \big(d_1(x_n, x), d_2(y_n, y)\big) \leq (b_n, c_n)$, $\forall n$. Hence $d_1(x_n, x) \leq b_n$, $d_2(y_n, y) \leq c_n$ $\forall n$.

Thus, $x_n \xrightarrow{d_1, E} x$ and $y_n \xrightarrow{d_2, F} y$.

Conversely, suppose $x_n \xrightarrow{d_1, E} x$ and $y_n \xrightarrow{d_2, F} y$. Want to show that $z_n \xrightarrow{\pi, E\times F} z$. Since $x_n \xrightarrow{d_1, E} x$ and $y_n \xrightarrow{d_2, F} y$, then there exist $a_n \downarrow 0$ and $b_n \downarrow 0$ such that $d_1(x_n, x) \leq a_n$ and $d_1(y_n, y) \leq b_n$. Take $c_{n=}(a_n, b_n)$, clearly $c_n \downarrow 0$ and $\pi(z_n, z) = \big(d_1(x_n, x), d_2(y_n, y)\big) \leq (a_n, b_n) = c_n$. Therefore, $z_n \xrightarrow{\pi, E\times F} z$ ■

In the next corollary, we show the relation between vectorially continuous function and its graph

Corollary 1.3.11 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces and let $f\colon X \to Y$ be a function. Then for the graph G_f of f the following statements are hold.

(a) The graph G_f is $E \times F$-closed in $(X \times Y, \pi, E \times F)$ if and only if for every sequence (x_n) with $x_n \xrightarrow{d_1, E} x$ and $f(x_n) \xrightarrow{d_2, F} y$ we have $y = f(x)$.

(b) If the function f is vectorially continuous then the graph G_f is $E \times F$-closed.

(c) If the function f is vectorially continuous at $x_0 \in X$ then the induced function $h: X \to G_f$ defined by $h(x) = (x, f(x))$ is vectorially continuous at $x_0 \in X$.

Proof:

(a) The proof of this part contains two sides.

$\Rightarrow$) Suppose the graph G_f is $E \times F$-closed. If $x_n \xrightarrow{d_1,E} x$ and $f(x_n) \xrightarrow{d_2,F} y$ then we have $\big(x_n, f(x_n)\big) \xrightarrow{\pi,E\times F} (x, y)$ by proposition 1.3.10, but G_f is closed,

so $(x, y) \in G_f$ and so $y = f(x)$.

$\Leftarrow$) Let $y = f(x)$ and $(z_n) = (x_n . f(x_n))$ be a sequence in G_f such that

$z_n \xrightarrow{\pi,E\times F} z = (x, y) \in X \times Y$. Want to show that $z \in G_f$. By proposition 1.3.10, $x_n \xrightarrow{d_1,E} x$ and $f(x_n) \xrightarrow{d_2,F} y$, so $y = f(x)$ (that's mean $z = (x, f(x)) \in G_f$).

(b) Let $z_n = (x_n, y_n)$ be a sequence in G_f such that $z_n = (x_n, y_n) \xrightarrow{\pi,E\times F} z = (x, y)$. Want to show that $z \in G_f$. By proposition 1.3.10, $x_n \xrightarrow{d_1,E} x$ and $y_n \xrightarrow{d_2,F} y$ but f is vectorially continuous, so $f(x_n) \to f(x)$ and $f(y_n) \to f(y)$ and thus $(f(x_n), f(y_n)) \to (f(x), f(y))$

so, $((x_n, y_n), (f(x_n), f(y_n)) \to ((x, y), \big(f(x), f(y)\big))$ and so $z = (x, y) \in G_f$.

Therefore G_f is $E \times F$-closed.

(c) Let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x_0$. Want to show that $h(x_n) \xrightarrow{\pi,E\times F} h(x_0)$. $h(x_n) = \big(x_n, f(x_n)\big) \xrightarrow{\pi,E\times F} \big(x_0, f(x_0)\big) = h(x_0)$, since f is vectorially continuous. Therefore $h(x_n) \xrightarrow{\pi,E\times F} h(x_0)$ and so h is vectorially continuous at x_0.

Chapter Two

Fundamental vector valued function classes

In this chapter, we will present the concept of (E, F)-equivalent, vector isometry and vector homeomorphism. Also, we prove many theorems that give the relation between these concepts.

2.1 Equivalent vector metrics

Definition 2.1.1

Let d_1 and d_2 be E-valued vector metric and F-valued vector metric respectively on X, then d_1 and d_2 are called (E, F)-equivalent if for any $x \in X$ and any sequence (x_n) in X, $x_n \xrightarrow{d_1,E} x$ iff $x_n \xrightarrow{d_2,F} x$.

Lemma 2.1.2 [3]

For any two E-valued vector metric d_1 and F-valued vector metric d_2 on X, the following statements are equivalent

(a) There exist some $\alpha, \beta > 0$ in R such that $\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y)$ for all $x, y \in X$.

(b) There exist two positive and σ-order continuous operators $T: E \rightarrow E$ and $S: E \rightarrow E$ such that $d_2(x, y) \leq T(d_1(x, y))$ and $d_1(x, y) \leq S(d_2(x, y))$ for all $x, y \in X$.

Proof:

First we will prove that if (a) holds then (b) holds. From (a) there is $\alpha, \beta > 0$. Define $T: E \rightarrow E$ and $S: E \rightarrow E$ by $T(a) = \beta a$ and $S(a) = \alpha^{-1} a$, for all $a \in E$. Since $\alpha, \beta > 0$, then

$T(a) = \beta a \geq 0$ and $S(a) = \alpha^{-1}a \geq 0, \forall a \geq 0$, so T and S are positive operators. Let (x_n) be a sequence in E such that $x_n \overset{o}{\to} 0$. Since $\beta x_n \overset{o}{\to} 0$ and $\alpha^{-1}x \overset{o}{\to} 0$, then $T(x_n) \overset{o}{\to} 0$ and

$S(x_n) \overset{o}{\to} 0$ and so T and S are σ- order continuous. From (a),

$d_2(x,y) \leq \beta d_1(x,y) = T(d_1(x,y))$ and $\alpha d_1(x,y) \leq d_2(x,y)$ implies

$d_1(x,y) \leq \alpha^{-1}d_2(x,y) = S(d_2(x,y))$ for all $x, y \in X$ and therefore we proved (b). Conversely, suppose (b) holds. Let $T: E \to E$ and $S: E \to E$ be positive and σ-order continuous operators that satisfied part (b). By Theorem 1.1.14, T and S are order bounded operators and so there exists $\alpha, \beta > 0$ such that

$$T(d_1(x,y)) \leq \beta d_1(x,y) \text{ and } S(d_2(x,y)) \leq (1/\alpha)d_2(x,y).$$

But

$$d_2(x,y) \leq T(d_1(x,y)) \leq \beta d_1(x,y) \text{ and } d_1(x,y) \leq S(d_2(x,y)) \leq (1/\alpha)d_2(x,y),$$

then

$$d_2(x,y) \leq \beta d_1(x,y) \text{ and } \alpha d_1(x,y) \leq d_2(x,y).$$

That is $\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$ ■

Theorem 2.1.3 [3]

Let d_1 and d_2 be E-valued vector metric and F-valued vector metric respectively on X, then d_1 and d_2 are (E,F)-equivalent if there exist two positive and σ-order continuous operators $T: E \to F$ and $S: F \to E$ such that $d_2(x,y) \leq T(d_1(x,y))$ and $d_1(x,y) \leq S(d_2(x,y))$ for all $x, y \in X$.

Proof:

Let (x_n) be a sequence such that $x_n \xrightarrow{d_1,E} x$, then there exist a sequence(a_n) in E such that $(a_n) \downarrow 0$ and $d_1(x_n, x) \le a_n$. Want to show that $x_n \xrightarrow{d_2,F} x$. Since there exist two positive and σ-order continuous operators $T: E \to F$ and $S: F \to E$ such that $d_2(x,y) \le T(d_1(x,y))$ and $d_1(x,y) \le S(d_2(x,y))$ for all $x, y \in X$, then part (a) of lemma 2.1.2 hold. So, there is $\beta > 0$ such that $d_2(x_n, x) \le \beta d_1(x_n, x) \le \beta a_n$ for all n. Let $b_n = \beta a_n$ then $b_n \downarrow 0$ and $\mathrm{d}_2(\mathrm{x_n}, \mathrm{x}) \le \mathrm{b_n}$. So $x_n \xrightarrow{d_2,F} x$. Conversely, let (x_n) be a sequence such that $x_n \xrightarrow{d_2,F} x$, then there exist a sequence b_n such that $(b_n) \downarrow 0$ and $d_2(x_n, x) \le b_n$. Want to show that $x_n \xrightarrow{d_1,E} x$. Since there is $\alpha > 0$ such that $\alpha d_1(x_n, x) \le d_2(x_n, x) \le b_n$, let $a_n = \alpha^{-1} b_n$, then $(a_n) \downarrow 0$ and $d_1(x_n, x) \le a_n$, so $x_n \xrightarrow{d_1,E} x$. Therefore, E-valued vector metric d_1 and F-valued vector metric d_2 on X are (E,F)-equivalent ■

Now, we will give an example.

Example 2.1.4

Suppose that the ordered of R^2 is coordinatewise

(a) Let d_1 and d_2 be R-valued and R^2-valued vector metrics on R, respectively defined by $d_1(x,y) = a|x-y|$, $d_2(x,y) = (b|x-y|, c|x-y|)$ where $a, b, c > 0$. Consider the two operators $T: R \to R^2$ and $S: R^2 \to R$ where's defined by $T(x) = a^{-1}(bx, cx)$ and $S(x,y) = ab^{-1}x$ for all $x, y \in R$. Then the metrics $d_1$and d_2 are (R, R^2)-equivalent on R since the operators T and S are positive operators (since for all $x, y \ge 0$, $T(x), S(x,y) \ge 0$), also let x_n and y_n be sequences such that $x_n \xrightarrow{o} x$ and $y_n \xrightarrow{o} y$, then

$T(x_n) = a^{-1}(bx_n, cx_n) \xrightarrow{o} a^{-1}(bx, cx) = T(x)$ and $S(x_n, y_n) = ab^{-1}x_n \xrightarrow{o} ab^{-1}x = S(x, y)$, so T and S are σ-order continuous.

In addition

$d_2(x, y) = a^{-1}(bd_1(x, y), cd_1(x, y)) = T(d_1(x, y))$ and

$S\big(d_2(x, y)\big) = S(b|x - y|, c|x - y|) = ab^{-1}b|x - y| = d_1(x, y)$.

So by Theorem 2.1.3, d_1 and d_2 are (R, R^2)-equivalent.

(b) Let d_1 and d_2 be R-valued and R^2-valued vector metrics on R^2, respectively, defined by

$$d_1(x, y) = a|x_1 - y_1| + b|x_2 - y_2|, \; d_2(x, y) = (c|x_1 - y_1|, e|x_2 - y_2|)$$

where $= (x_1, x_2)$, $y = (y_1, y_2)$ and $a, b, c, e > 0$. Let $T: R \to R^2$ and $S: R^2 \to R$ be two operators defined as $T(x) = (ca^{-1}x, eb^{-1}x)$ and $S(x, y) = ac^{-1}x + be^{-1}y$. Then the vector metrics $d_1$and d_2 are (R, R^2)-equivalent on R^2 since the operators T and S are positive operators (since $T(x)$,$S(x, y) \geq 0$ for all $x, y \geq 0$), also let x_n and y_n be a sequences such that $x_n \xrightarrow{o} x$ and $y_n \xrightarrow{o} y$, then $T(x_n) = (ca^{-1}x_n, eb^{-1}x_n) \xrightarrow{o} (ca^{-1}x, eb^{-1}x) = T(x)$ and $S(x_n, y_n) = ac^{-1}x_n + be^{-1}y_n \xrightarrow{o} ac^{-1}x + be^{-1}y = S(x, y)$, so T and S are σ-order continuous. In addition,

$$T\big(d_1(x, y)\big) = \big(ca^{-1}d_1(x, y), eb^{-1}d_1(x, y)\big)$$

$$= (ca^{-1}a|x_1 - y_1|, eb^{-1}b|x_2 - y_2|) = d_2(x, y)$$

and $\quad S\big(d_1(x, y), d_2(x, y)\big) = ac^{-1}d_1(x, y) + be^{-1}d_2(x, y)$

$$= ac^{-1}a|x_1 - y_1| + ac^{-1}b|x_2 - y_2| + be^{-1}c|x_1 - y_1|$$

$$+be^{-1}e|x_2 - y_2| \geq a|x_1 - y_1| + b|x_2 - y_2| = d_1(x, y)).$$

So by theorem 2.1.3 d_1 and d_2 are (R, R^2)-equivalent.

(c) Let d_1, d_2 and d_3 be R-valued, R^2-valued and R-valued vector metrics on R^2 respectively defined by $d_1(x, y) = a|x_1 - y_1| + b|x_2 - y_2|$, $d_2(x, y) = (c|x_1 - y_1|, e|x_2 - y_2|)$ and $d_3(x, y) = max\{a|x_1 - y_1|, b|x_2 - y_2|\}$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $a, b, c, e > 0$. Let $T: R \to R^2$ and $S: R^2 \to R$ be two operators defined as $T(x) = (ca^{-1}x, eb^{-1}x)$ and $S(x, y) = max\{ac^{-1}x, be^{-1}y\}$, then the vector metrics d_3 and d_2 are (R, R^2)-equivalent on R^2 since T and S are positive ($T(x)$,$S(x, y) \geq 0$ for all $x, y \geq 0$), also let x_n and y_n be sequences such that $x_n \xrightarrow{o} x$ and $y_n \xrightarrow{o} y$, then

$$T(x_n) = (ca^{-1}x_n, eb^{-1}x_n) \xrightarrow{o} (ca^{-1}x, eb^{-1}x) = T(x)$$

and

$$S(x_n, y_n) = max\{ac^{-1}x_n, be^{-1}y_n\} \xrightarrow{o} max\{ac^{-1}x, be^{-1}y\} = S(x, y),$$

so T and S are σ-order continuous. In addition there exist two cases

Case (1):

If we take $max\{a|x_1 - y_1|, b|x_2 - y_2|\} = a|x_1 - y_1|$, then

$$\begin{aligned} T\big(d_3(x, y)\big) &= \big(ca^{-1}d_3(x, y),\ eb^{-1}d_3(x, y)\big) \\ &= (ca^{-1}(max\{a|x_1 - y_1|, b|x_2 - y_2|\}), eb^{-1}(max\{a|x_1 - y_1|, b|x_2 - y_2|\})) \\ &= (c|x_1 - y_1|, eb^{-1}a|x_1 - y_1|) \geq (c|x_1 - y_1|, e|x_2 - y_2|) = d_2(x, y). \end{aligned}$$

Case (2):

If we take $max\{a|x_1 - y_1|, b|x_2 - y_2|\} = b|x_2 - y_2|$, then

$$T\big(d_3(x,y)\big) = \big(ca^{-1}d_3(x,y),\ eb^{-1}d_3(x,y)\big)$$

$$= (ca^{-1}(max\{a|x_1 - y_1|, b|x_2 - y_2|\})), eb^{-1}(max\{a|x_1 - y_1|, b|x_2 - y_2|\}))$$

$$= ((ca^{-1}b|x_2 - y_2|, e|x_2 - y_2|) \geq (c|x_1 - y_1|, e|x_2 - y_2|) = d_2(x,y) \text{ and}$$

$$S\big(d_2(x,y)\big) = max\{ac^{-1}c|x_1 - y_1|, be^{-1}e|x_2 - y_2|\}$$

$$= max\{a|x_1 - y_1|, b|x_2 - y_2|\} = d_3(x,y)$$

So by theorem 2.1.3 d_3 and d_2 are (R, R^2)-equivalent.

Lemma 2.1.5

Let $d_1: X \times X \to E$, $d_2: X \times X \to F$ and $d_3: Y \times Y \to M$ be vector metrics, where d_1 and d_2 are (E,F)-equivalent on X, then $f : (X, d_1, E) \to (Y, d_3, M)$ vectorially continuous if and only if $f: (X, d_2, F) \to (Y, d_3, M)$ is vectorially continuous.

Proof:

There exist two folds to prove the lemma.

$\Longrightarrow$) Let x_n be a sequence such that $x_n \xrightarrow{d_2,F} x$ then $x_n \xrightarrow{d_1,E} x$ since d_1 and d_2 are (E,F)-equivalent on X. But $f: (X, d_1, E) \to (Y, d_3, M)$ is vectorially continuous so $f(x_n) \xrightarrow{d_3,M} f(x)$ and so f is vectorial continuous from (X, d_2, F) to (Y, d_3, M).

$\Leftarrow$) Let x_n be a sequence such that $x_n \xrightarrow{d_1,E} x$ then $x_n \xrightarrow{d_2,F} x$, since d_1 and d_2 are (E,F)-equivalent on X. But $f:(X,d_2,F) \to (Y,d_3,M)$ is vectorially continuous so $f(x_n) \xrightarrow{d_3,M} f(x)$ and so f is vectorial continuous from (X,d_1,E) to (Y,d_3,M) ■

2.2 Vector Isometry and vector homeomorphism

In this section, we will define an isometry between two vector metric spaces

Definition 2.2.1

Let (X,d_1,E) and (Y,d_2,F) be vector metric spaces. A function $f: X \to Y$ is said to be a vector isometry if there exists a linear operator $T_f: E \to F$ satisfying the following conditions:

(I) $T_f\big(d_1(x,y)\big) = d_2(f(x),f(y))$ for all $x,y \in X$

(II) $T_f(a) = 0$ implies $a = 0$ for all $a \in E$

If the function f is onto, and the operator T_f is a lattice homomorphism then the vector metric spaces (X,d_1,E) and $(Y,d_2,T_f(E))$ are called vector isometric.

Lemma 2.2.2 [3]

Let (X,d_1,E) and (Y,d_2,F) be vector metric spaces. A vector isometry f is one-to-one mapping.

Proof:

Let $f(x) = f(y)$, to show f is one-to-one we must show $x = y$. Since f is a vector isometry, then there exist T_f such that $T_f\big(d_1(x,y)\big) = d_2\big(f(x),f(y)\big) = 0$ so by Definition 2.2.1 $d_1(x,y) = 0$ and so $x = y$ ■

Now, we will give an example

Example 2.2.3

Let d_1 be R-valued vector metric and let d_2 be R^2-valued vector metric on R defined by $d_1(x,y) = a|x-y|$, $d_2(x,y) = (b|x-y|, c|x-y|)$ where $b, c \geq 0$ and $a, b+c > 0$. Consider the identity mapping $I: R \to R$ defined by $(x) = x$, $\forall x \in R$ and the linear operator $T_I: R \to R^2$ defined by $T_I(x) = a^{-1}(bx, cx)$ for all $x \in R$. Then the identity mapping I is a vector isometry since

$$\begin{aligned}
T_I\big(d_1(x,y)\big) &= a^{-1}\big(bd_1(x,y), cd_1(x,y)\big) \\
&= a^{-1}(ba|x-y|, ca|x-y|) \\
&= (b|x-y|, c|x-y|) \\
&= d_2(x,y) = d_2(I(x), I(y))
\end{aligned}$$

and $T_I(x) = a^{-1}(bx, cx) = (0,0)$ implies $(bx, cx) = (0,0)$ for all $x \in R$, which implies that $bx = 0$ and $cx = 0$. Since b and c are not both zeros, then $x = 0$.

Since I is onto and T_I is lattice homomorphism, then the vector metric spaces (R, d_1, R) and $(R, d_2, \{(x,y): cx = by; x, y \in R\}$ are vector isometric.

Definition 2.2.4

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces. A function $f: X \to Y$ is said to be a vector homeomorphism if f is one-to-one, vectorially continuous and has a vectorially continuous inverse on $f(x)$. If the function f is onto, then the vector metric spaces X and Y are called vector homeomorphic.

Lemma 2.2.5 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces. A vector homeomorphism $f: X \to Y$ is one-to-one function that preserves vectorial convergence of sequences.

Proof:

Let $f: X \to Y$ be a vector homeomorphism, then by definition 2.2.4 f is one-to-one. let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x$. Want to show that $f(x_n) \xrightarrow{d_2,F} f(x)$. It is clear that is satisfied, since f is vectorial continuous (from definition of homeomorphism) ■

The following theorem describe that an onto vector homeomorphism keeping the closed property

Theorem 2.2.6 [3]

An onto vector homeomorphism is one-to-one function that preserve vector closed sets.

Proof:

Let $f: X \to Y$ be an vector homeomorphism. Since f is a one-to-one function and its inverse f^{-1} is vectorially continuous then by Theorem1.3.5 for every E-closed set A in X, $f(A) = (f^{-1})^{-1}(A)$ is F-closed in Y.

Now we will give an example, which shows the relationship between vectorial equivalence and vector homeomorphism ■

Example 2.2.7

Let $d_1$and d_2 be two (E, F)-equivalent vector metrics on X. Then the vector metric spaces (X, d_1, E) and (X, d_2, F) are vector homeomorphic under the identity mapping since let

$f: X \longrightarrow X$ be the identity function defined by $f(x) = x$, then its clearly that f is one-to-one vectorially continuous (since if (x_n) is a sequence in X such that $x_n \xrightarrow{d_1,E} x$ then

$f(x_n) = x_n \xrightarrow{d_2,F} x = f(x)$) and has a vectorially continuous inverse in $f(x)$ (because $f^{-1}(x) = x$ and is the same as f).

Lemma 2.2.8 [3]

Let (X, d_1, E) and (Y, d_2, F) be any two vector metric spaces where are vector homeomorphic under a function $f: X \longrightarrow Y$, and let $d_3(x, y) = d_2(f(x), f(y))$ for all $x, y \in X$, then the vector metrics d_1 and d_3 are (E, F)-equivalent vector metrics on X.

Proof:

There exist two sides to prove this

$\Longrightarrow$) Let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x$, want to show that $x_n \xrightarrow{d_3,F} x$, since f is vector homeomorphism, then f is vectorial continuous so there exist a sequence (b_n) in F such that $b_n \downarrow 0$ and $d_2(f(x_n), f(x)) \leq b_n$, so $d_3(x_n, x) = d_2(f(x_n), f(x)) \leq b_n$. Therefore $x_n \xrightarrow{d_3,F} x$.

$\Longleftarrow$) Let (x_n) be a sequence in X such that $x_n \xrightarrow{d_3,F} x$, then there exist a sequence (b_n) in F such that $b_n \downarrow 0$ and $d_3(x_n, x) \leq b_n$. Want to show that $x_n \xrightarrow{d_1,F} x$. But $d_3(x_n, x) = d_2(f(x_n), f(x))$, so $d_2(f(x_n), f(x)) \leq b_n$ and so $f(x_n) \xrightarrow{d_2,F} f(x)$.

But f^{-1} is vectorial continuous, so $f^{-1}(f(x_n)) \xrightarrow{d_1,E} f^{-1}(f(x))$ implies $x_n \xrightarrow{d_1,F} x$. Therefore, d_1 and d_3 are (E, F)-equivalent vector metrics on X ■

Chapter Three

Extension theorems on continuity

This chapter focuses on two types of uniformly continuous functions on vector metric space, which are topological uniformly continuous function and vectorial uniformly continuous function and give the relation between them. In addition, it will focus on extension theorem.

3.1 Uniformly continuous functions on vector metric spaces

Theorem 3.1.1 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces, and let $f: X \to Y$ and $g: X \to Y$ be vectorially continuous functions. Then the set $\{x \in X: f(x) = g(x)\}$ is an E-closed subset of X.

Proof:

Let $B = \{x \in X: f(x) = g(x)\}$ and let (x_n) be a sequence in B such that $x_n \xrightarrow{d_1,E} x$. Want to show that $x \in B$. Since f and g are vectorially continuous, there exist sequences(a_n) and (b_n) such that $a_n \downarrow 0$ and $b_n \downarrow 0$ and $d_2\big(f(x_n), f(x)\big) \leq a_n$, $d_2\big(g(x_n), g(x)\big) \leq b_n$ for all n. Since $x_n \in B, \forall n$, so $f(x_n) = g(x_n)$ and therefore $d_2\big(f(x_n), g(x_n)\big) = 0$.

Thus, $$d_2\big(f(x), g(x)\big) \leq d_2\big(f(x), f(x_n)\big) + d_2\big(f(x_n), g(x_n)\big) + d_2\big(g(x_n), g(x)\big)$$

$$\leq a_n + b_n .$$

But $a_n \downarrow 0$ and $b_n \downarrow 0$, so $(a_n + b_n) \downarrow 0$ and so $d_2(f(x), g(x)) = 0$, which means that $f(x) = g(x)$ and this implies $x \in B$. Hence, B is an E-closed subset of X ■

Corollary 3.1.2 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces, and let $f: X \to Y$ and $g: X \to Y$ be vectorially continuous functions. If the set $\{x \in X: f(x) = g(x)\}$ is E-dense in X, then $f = g$.

Proof:

Let $B = \{x \in X: f(x) = g(x)\}$, then by Theorem 3.1.1 B is E-closed and since B is dense in X, then $B = \bar{B} = X$. So for all $x \in X$, $f(x) = g(x)$, that is $f = g$ ■

Definition 3.1.3

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces.

(a) A function $f: X \to Y$ is said to be topological uniformly continuous on X if for every $b > 0$ in F there exist some a in E such that for all $\mathrm{x}, \mathrm{y} \in X$,

$$d_2\big(f(x), f(y)\big) < b \text{ whenever } d_1(x, y) < a.$$

(b) A function $f: X \to Y$ is said to be vectorial uniformly continuous on X if for every E-Cauchy sequence (x_n) the sequence$(\, f(x_n))$ is F-Cauchy.

Theorem 3.1.4 [3]

Let (X, d_1, E) and (Y, d_2, F) be vector metric spaces where F is Archimedean. If a function $f: X \to Y$ is topological uniformly continuous, then f is vectorial uniformly continuous.

Proof:

Suppose that (x_n) is an E-Cauchy sequence. Then there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d_1(x_n, x_{n+p}) \leq a_n$ for all n and p. Since f is topological uniformly continuous on X, then for any $b > 0$ in F, there exist $b_n > 0$ in E such that $d_1(x, y) < b_n$ implies

$d_2(f(x), f(y)) < \frac{b}{n}$. Take $c_n = \min\{a_n, b_n\}$, so $d_1(x, y) \leq c_n$ implies $d_2(f(x), f(y)) < \frac{b}{n}$. But $c_n < a_n, \forall n$, so $d_1(x_n, x_{n+p}) \leq c_n < a_n$ implies $d_2(f(x_n), f(x_{n+p})) < (1/n)b$. However, since F is Archimedean, $(1/n)b \downarrow 0$, So $(f(x_n))$ is F-Cauchy■

Now, we will give an example a bout vectorial uniformly continuous function

Example 3.1.5

(a) Let (X, d_1, E) and (Y, d_2, F) be two vector metric spaces and the function $f: X \to Y$ be a vector isometry, then the function f is vectorial uniformly continuous if T_f is positive and σ-order continuous.

Proof:

Suppose T_f is positive and σ-order continuous. Let (x_n) be E-Cauchy sequence, then there exist a sequence $(a_n) \downarrow 0$ in E such that $d_1(x_n, x_{n+p}) \leq a_n$ for all n and p. We want to prove that $(f(x_n))$ is F-Cauchy. Since f is a vector isometry, then $d_1(x_n, x_{n+p}) \leq a_n$ implies $d_2\left(f(x_n), f(x_{n+p})\right) = T_f(d_1(x_n, x_{n+p})) \leq T_f(a_n)$, say $b_n = T_f(a_n)$.

Clearly, $b_n \downarrow 0$ since $(a_n) \downarrow 0$ and T_f is positive and σ-order continuous. So, $d_2\left(f(x_n), f(x_{n+p})\right) \leq b_n$ for all n and p, which implies that $(f(x_n))$ is F-Cauchy.

(b) Let (X, d, E) be vector metric space. Fix $y \in X$, then the function $f_y: X \to E$ defined by $f_y(x) = d(x, y)$ for all $x \in X$ is vectorial uniformly continuous since let (x_n) be E-Cauchy sequence in X. Want to show that $(f_y(x_n))$ is E-Cauchy. Let $y_n = y$, then (y_n) is E-Cauchy. So by Theorem 1.2.4 $(d(x_n, y_n))$ is E-Cauchy, but $(d(x_n, y_n)) = (f_y(x_n))$. So f_y is vectorial uniformly continuous.

Now we will arise to the main theorem of my thesis

Theorem 3.1.6 (Extension Theorem), [3]

Let A be E-dense subset of a vector metric space (X, d_1, E) and (Y, d_2, F) be an F-complete vector metric space where F is Archimedean. If $f: A \to Y$ is topological uniformly continuous function then f has a unique vectorially continuous extension to X which is also topological uniformly continuous.

Proof:

Let $x \in X$. Then there exist a sequence (x_n) in A such that $x_n \xrightarrow{d_1,E} x$, since A is E-dense subset of X. By theorem 1.2.4, (x_n) is E-Cauchy in X, but f is vectorially uniformly continuous function. So $(f(x_n))$ is F-Cauchy sequence in Y. Y is F-complete vector metric space, so $(f(x_n))$ is F- convergent, that is $f(x_n) \xrightarrow{d_2,F} y$. Define an extension function $g: X \to Y$ by

$$g(x) = y \text{ if } x \in X \backslash A, \text{ and } g(x) = f(x) \text{ if } x \in A.$$

Claim: g is well define.

Proof of claim:

Let (y_n) be another sequence in A such that $y_n \xrightarrow{d_1,E} x$. As we said, $(f(y_n))$ is F-Cauchy in Y, then $f(y_n) \xrightarrow{d_2,F} y_1$. Since $(f(x_n))$ and $(f(y_n))$ are F-convergent sequences then $(f(x_n))$ and $(f(y_n))$ are F- Cauchy. By Theorem 1.2.4, we have $(d_2(f(x_n), f(y_n))$ F-Cauchy, so there exist $c_n \downarrow 0$ such that $d_2(f(x_n), f(y_n)) \leq c_n$. Also, since $f(x_n) \xrightarrow{d_2,F} y$ and $f(y_n) \xrightarrow{d_2,F} y_1$, then there exist $a_n \downarrow 0$ and $b_n \downarrow 0$ such that $d_2(f(x_n), y) \leq a_n$ and $d_2(f(y_n), y_1) \leq b_n$ $\forall n$. Therefore

$$d_2(y, y_1) \le d_2\big(y, f(x_n)\big) + d_2\big(f(x_n), f(y_n)\big) + d_2(f(y_n), y_1)$$

$$\le a_n + c_n + b_n. \text{ but } (a_n + c_n + b_n) \downarrow 0,$$

Thus, $d_2(y, y_1) = 0$. Therefore $y = y_1$.

Now, want to show that g is topological uniformly continuous function.

Let $b > 0$ in F, since f is topological uniformly continuous on A, there exist $a > 0$ in E such that $d_1(x, y) < a$ implies $d_2\big(f(x), f(y)\big) < b, \forall x, y \in A$. Now, let $x, y \in X$ with $d_1(x, y) < a$. We want to show $d_2(g(x), g(y)) < b$.

Case I: If $x, y \in A$, then we done.

Case II: Suppose $x, y \in X \backslash A$, since A is dense, there exist two sequences (x_n) and (y_n) in A such that $x_n \xrightarrow{d_1,E} x$ and $y_n \xrightarrow{d_1,E} y$. So $d_1(x_n, y_n) \overset{o}{\to} d_1(x, y)$ in E. Fix n_0 such that $n > n_0$ implies $d_1(x_n, y_n) < a$, so $d_2(f(x_n), f(y_n)) < b$, $\forall n > n_0$. Since f is a vectorial uniformly continuous in A, then $(f(x_n))$ and $(f(y_n))$ are F-Cauchy. But Y is F-complete. So there exist u and $v \in Y$ such that $f(x_n) \xrightarrow{d_2,F} u$ and $f(y_n) \xrightarrow{d_2,F} v$. By definition of g, $g(x) = u$ and $g(y) = v$. Then, $d_2\big(g(x_n), g(y_n)\big) \overset{o}{\to} d_2\big(g(x), g(y)\big) = d_2(u, v) < b$.

Now, want to prove that extension function is unique.

Let g and h are two extension functions of *f*. Want $g(x) = h(x)$, $\forall x \in X$.

Case I: Let $x \in A$, then $g(x) = f(x) = h(x)$.

Case II: Let $x \in X \backslash A$, then there exist a sequence (x_n) in A such that $x_n \xrightarrow{d_1,E} x$. Since g and h are vectorial continuous function, then $g(x_n) \xrightarrow{d_2,F} g(x)$ and $h(x_n) \xrightarrow{d_2,F} h(x)$, but $g(x_n) = f(x_n) = h(x_n)$. So $g(x) = h(x)$ ■

Theorem 3.1.7

Let A be E-dense subset of a vector metric space (X, d_1, E) and (Y, d_2, F) be an F-complete vector metric space where F is Archimedean. If $f: A \to Y$ is vectorially uniformly continuous function then f has a unique vectorially continuous extension to X.

Proof:

The definition of the extension function, the well-defined and the uniqueness of this function as in Theorem 3.1.6 . To show that f is vectorially continuous function, let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x$, since A is E-dense subset of X, there exist sequences (x_{nm}) and (x_{0m}) in A such that $x_{nm} \xrightarrow{d_1,E} x_n$ and $x_{0m} \xrightarrow{d_1,E} x$. Hence, there exist $t_m \downarrow 0, w_n \downarrow 0$ and $z_m \downarrow 0$ such that $d_1(x_{nm}, x_n) \leq t_m$, $d_1(x_n, x) \leq w_n$ and $d_1(x, x_{0m}) \leq z_m$. Now, let $m \geq n$, then $t_m \leq t_n$ and $z_m \leq z_n$ so

$$d_1(x_{nm}, x_{0m}) \leq d_1(x_{nm}, x_n) + d_1(x_n, x) + d_1(x, x_{0m})$$

$\leq t_m + w_n + z_m \leq t_n + w_n + z_n$, let $d_n = t_n + w_n + z_n$, then $d_n \downarrow 0$ and $x_{nm} \xrightarrow{d_1,E} x_{0m}$.

Therefore $f(x_{nm}) \xrightarrow{d_2,F} f(x_{0m})$, $f(x_{nm}) \xrightarrow{d_2,F} g(x_n)$ and $f(x_{0m}) \xrightarrow{d_2,F} g(x)$, so there exist $h_m \downarrow 0$, $j_n \downarrow 0$ and $k_m \downarrow 0$ such that $d_2\big(f(x_{nm}), f(x_{0m})\big) \leq j_n$, $d_2\big(f(x_{0m}), g(x)\big) \leq k_m$ and $d_2(g(x_n), f(x_{nm})) \leq h_m$.

Now, let $m \geq n$, then $k_m \leq k_n$ and $h_m \leq h_n$, so

$$d_2\big(g(x_n), g(x)\big) \leq d_2\big(g(x_n), f(x_{nm})\big) + d_2\big(f(x_{nm}), f(x_{0m})\big) + d_2\big(f(x_{0m}), g(x)\big)$$

$$\leq h_m + j_n + k_m \leq h_n + j_n + k_n, \text{ let } l_n = h_n + j_n + k_n, \text{ then } l_n \downarrow 0.$$

Therefore, $d_2\big(g(x_n), g(x)\big) \leq l_n$ and so g is vectorial continuous extension ■

3.2 Uniformly Convergent in Vector metric spaces

This section focuses on uniformly F-convergent, vectorial bounded function, the uniform limit theorem and many concepts and relations in vector metric space

Definition 3.2.1

Let X be any nonempty set and let (Y, d_2, F) be a vector metric space. Then a sequence (f_n) of functions from X to Y is said to be uniformly F-convergent to a function $f: X \to Y$, if there exists a sequence (a_n) in F such that $a_n \downarrow 0$ and $d_2(f_n(x), f(x)) \leq a_n$ holds for all $x \in X$ and $n \in N$.

Now, we will give an example

Example 3.2.2

(a) Let $X = [0,1]$, $Y = R^2$and $d_2(x, y) = (|x_1 - y_1|, |x_2 - y_2|)$. Define

$$f_n(x) = \left(x + \frac{1}{n^2}, \frac{x}{n^3}\right),$$

then f_n is uniformly F-convergent to $f(x) = (x, 0)$

since

$$d_2(f_n(x), f(x)) = \left(\left|x + \frac{1}{n^2} - x\right|, \left|\frac{x}{n^3} - 0\right|\right)$$

$$= (\left|\frac{1}{n^2}\right|, \left|\frac{x}{n^3}\right|)$$

take $a_n = (\frac{1}{n^2}, \frac{1}{n^3})$, then $a_n \downarrow 0$ and

$$d_2(f_n(x), f(x)) = (\left|\frac{1}{n^2}\right|, \left|\frac{x}{n^3}\right|) \leq (\frac{1}{n^2}, \frac{1}{n^3}).$$

(b) Let $X = R$, $Y = R^2$and $d_2(x, y) = (|x_1 - y_1|, |x_2 - y_2|)$. Define $f_n(x) = (x + \frac{1}{n}, x - \frac{1}{n^2})$, then f_n is uniformly F-convergent to $f(x) = (x, x)$ since, $d_2(f_n(x), f(x)) = (\left|x + \frac{1}{n} - x\right|, \left|x - \frac{1}{n^2} - x\right|)$

$$= (\frac{1}{n}, \frac{1}{n^2})$$

take $a_n = (\frac{1}{n}, \frac{1}{n^2})$, then $a_n \downarrow 0$ and $d_2(f_n(x), f(x)) = (\frac{1}{n}, \frac{1}{n^2})$.

In addition, f_n in example 3.2.2 is F-convergent to f since every uniformly F-convergent is F-convergent.

Now, we will give the main result of this chapter

Theorem 3.2.3 (Uniform Limit Theorem), [3]

Let (f_n) be a sequence of vectorially continuous functions between two vector metric spaces (X, d_1, E) and (Y, d_2, F). If (f_n) is uniformly F-convergent to f, then the function f is vectorially continuous function.

Proof:

Let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x$. Want to show that $f(x_n) \xrightarrow{d_2,F} f(x)$. Since (f_n) is uniformly F-convergent to f, there is a sequence (a_n) in F such that $a_n \downarrow 0$ and $d_2\big(f_n(x), f(x)\big) \leq a_n$ for all $n \in \mathbb{N}$ and $\forall x \in X$. For each $k \in \mathbb{N}$ there is a sequence (b_n) in F such that $b_n \downarrow 0$ and $d_2\big(f_k(x_n), f_k(x)\big) \leq b_n$ for all $n \in \mathbb{N}$ by the vectorial continuity of f_k. For $k = n$ we get,

$$d_2\big(f(x_n), f(x)\big) \leq d_2\big(f(x_n), f_n(x_n)\big) + d_2\big(f(x), f_n(x)\big) + d_2\big(f_n(x_n), f_n(x)\big)$$

$$\leq 2a_n + b_n,$$

let $c_n = 2a_n + b_n$, then $c_n \downarrow 0$ and $d_2\big(f(x_n), f(x)\big) \leq c_n$. So $f(x_n) \xrightarrow{d_2,F} f(x)$.

Therefore f is vectorially continuous function ■

Let A be a nonempty subset of a vector metric space (X, d, E). E-diameter of A denoted by $d(A)$, is defined by $sup\{d(x,y): x, y \in A\}$ if $sup\{d(x,y): x, y \in A\}$ exist in E.

Theorem 3.2.4

Let (f_n) be a sequence of vectorially uniformly continuous functions between two vector metric spaces (X, d_1, E) and (Y, d_2, F). If (f_n) is uniformly F-convergent to f, then the function f is vectorially uniformly continuous function

Proof:

Let (x_n) be an E-Cauchy sequence in X. Want to show that $(f(x_n))$ is F-Cauchy. For all $k \in N$, f_k is vectorial uniformly continuous function, then $(f_k(x_n))$ is F-Cauchy. So there exist $a_{kn} \downarrow 0$ such that $d_2(f_k(x_n), f_k(x_{n+p})) \leq a_{kn}$ $\forall n, p$. Since (f_k) is uniformly F-convergent to

f, there exist $b_k \downarrow 0$ such that $d_2(f_k(x), f(x)) \leq b_k$. Then $d_2(f(x_n), f(x_{n+p})) \leq d_2(f(x_n), f_k(x_n)) + d_2(f_k(x_n), f_k(x_{n+p})) + d_2(f_k(x_{n+p}), f(x_{n+p)}) < b_k + a_{kn} + b_k = a_{kn} + 2b_k$ and $(a_{kn} + 2b_k) \downarrow 0$.

So, $(f(x_n))$ is F-Cauchy sequence and so f is vectorially uniformly continuous function ■

Theorem 3.2.5

Let (f_n) be a sequence of topological continuous functions between two vector metric spaces (X, d_1, E) and (Y, d_2, F). If (f_n) is uniformly F-convergent to f, then the function f is topological continuous function.

Proof:

Let $b > 0$, since f_k is topological continuous, there exist $a_k > 0$ such that $d_1(x, y) < a_k$ implies $d_2\big(f_k(x), f_k(y)\big) < b$. Take $a = \inf\{a_k : k \in N\}$, then $d_1(x, y) < a$ implies

$$d_2\big(f(x), f(y)\big) \leq d_2\big(f(x), f_k(x)\big) + d_2\big(f_k(x), f_k(y)\big) + d_2\big(f_k(y), f(y)\big)$$

$$< a_n + b + a_n = 2a_n + b.$$

As $n \to \infty$, $d_2\big(f(x), f(y)\big) \leq b$. Therefore f is topological continuous function ■

Theorem 3.2.6 [3]

Let A be a nonempty subset of a vector metric space (X, d, E). If E is Dedekind complete, then every E-bounded subset of X has an E-diameter.

Proof:

Let A be E-bounded subset of X, then there exist an element $a > 0$ such that

$d(x,y) \leq a, \forall x,y \in A$. If we take a supremum for both sides we get

$sup\{d(x,y): x,y \in X\} \leq sup\{a\} = a$. Therefore, it has a supremum and $d(A) \leq a$ ∎

Definition 3.2.7

A function $f: X \to Y$ between two vector metric spaces (X, d_1, E) and (Y, d_2, F) is called vectorial bounded if f maps E-bounded subsets of X to F-bounded subsets of Y.

Theorem 3.2.8 [3]

A function $f: X \to Y$ between two vector metric spaces (X, d_1, E) and (Y, d_2, F) is vectorial bounded if there exists a positive operator $T: E \to F$ such that $d_2(f(x), f(y)) \leq T(d_1(x,y))$ for all $x, y \in X$.

Proof:

Suppose there exists a positive operator $T: E \to F$ such that $d_2(f(x), f(y)) \leq T(d_1(x,y))$ for all $x, y \in X$ hold. Let A be E-bounded subset of X, then there exist an element a such that $d_1(x,y) \leq a$, $\forall x, y \in X$, want to show that $f(A)$ is F-bounded. Let $y_1, y_2 \in f(A)$,

take $b = T(a)$, then $\exists\, x_1, x_2 \in A$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ and $d_2(y_1, y_2) = d_2\big(f(x_1), f(x_2)\big) \leq T\big(d_1(x_1, x_2)\big) \leq T(a) = b$ ∎

Therefore $f(A)$ is F- bounded.

Let $C_v(X,F)$ and $C_t(X,F)$ be the collection of all vectorially continuous and topologically continuous functions between a vector metric space (X,d,E) and a Riesz space F, respectively. By theorem 1.3.2 $C_t(X,F) \subseteq C_v(X,F)$ whenever F is Archimedean.

Lemma 3.2.9

Let (X, d_1, E) and (Y, d_2, F) be two vector metric spaces, then $C_v(X, F)$ and $C_t(X, F)$ are closed under addition and scalar multiplication.

Proof:

Let f and g be a function in $C_v(X, F)$ and λ be a scalar. Want to show that $f + g$ and λf are vectorial continuous functions. Let (x_n) be a sequence in X such that $x_n \xrightarrow{d_1,E} x$.

Since $f, g \in C_v(X, F)$, then $f(x_n) \xrightarrow{d_2,F} f(x)$ and $g(x_n) \xrightarrow{d_2,F} g(x)$.

So

$$(f + g)(x_n) = f(x_n) + g(x_n) \xrightarrow{d_2,F} f(x) + g(x) = (f + g)(x)$$

and

$$(\lambda f)(x_n) = \lambda f(x_n) \xrightarrow{d_2,F} \lambda f(x) = (\lambda f)(x).$$

So $f + g$ and λf are vectorial continuous functions (that means $f + g, \lambda f \in C_v(X, F)$). Therefore $C_v(X, F)$ is closed under addition and scalar multiplication

Let f and g be a function in $C_t(X, F)$ and λ be a scalar. Want to show that $f + g$ and λf are topological continuous functions. Let $a, b, c \in E$ such that $b, c > 0$ and $d_1(x, y) < a$, then since $f, g \in C_t(X, F)$, then $d_2(f(x), f(y)) < b$ and $d_2\big(g(x), g(y)\big) < c$ for all $x, y \in X$. So $d_2\big((f + g)(x), (f + g)(y)\big) = d_2\big(f(x) + g(x), f(y) + g(y)\big)$

$$\leq d_2\big(f(x), f(y)\big) + d_2\big(g(x), g(y)\big)$$

$$< b + c.$$

Let $b + c = d$, then $d_2\big((f + g)(x), (f + g)(y)\big) < d$ and

$$d_2\big((\lambda f)(x), (\lambda f)(y)\big) = d_2\big(\lambda f(x), \lambda f(y)\big) \le d_2(f(x), f(y)) < b.$$

So $f + g$ and λf are topological continuous functions (that means $f + g$,$\lambda f \in C_t(X, F)$). Therefore $C_t(X, F)$ is closed under addition and scalar multiplication ■

Theorem 3.2.10 [3]

The spaces $C_v(X, F)$ and $C_t(X, F)$ are Riesz spaces with the natural partial ordering defined by $f \le g$ whenever $f(x) \le g(x)$ for all $x \in X$.

Proof :

Let $f, g, h \in C_v(X, F)$ and $f \le g$, then

(1) $(f + h)(x) = f(x) + h(x) \le g(x) + h(x) = (g + h)(x)$

$\therefore f + h \le g + h$

(2) Let $\lambda \ge 0$, $\lambda f(x) = \lambda\big(f(x)\big) \le \lambda\big(g(x)\big) = \lambda g(x)$

$\therefore\ \lambda f \le \lambda g.$

From (1) and (2), $C_v(X, F)$ is an ordered vector space and the supremum exist inside this space. So, $C_v(X, F)$ is Riesz space. Similarly, $C_t(X, F)$ is Riesz space ■

Conclusion

In this thesis, I studied the relationships between topological continuity and vectorial continuity." Cüneyt Çevik" has concluded that: every topological continuous function is a victorial continuous function. In addition, he proved that every topological uniformly continuous function is vectorial uniformly continuous function. He also studied extension theorem and the uniform limit theorem.

After deep study for the above, I managed to prove the following:

Let A be E-dense subset of a vector metric space (X, d_1, E) and (Y, d_2, F) be an F-complete vector metric space where F is Archimedean. If $f: A \rightarrow Y$ is vectorially uniformly continuous function then f has a unique vectorially continuous extension to X.

Let (f_n) be a sequence of vectorially uniformly continuous functions between two vector metric spaces (X, d_1, E) and (Y, d_2, F). If (f_n) is uniformly F-convergent to f, then the function f is vectorially uniformly continuous function

Let (f_n) be a sequence of topological continuous functions between two vector metric spaces (X, d_1, E) and (Y, d_2, F). If (f_n) is uniformly F-convergent to f, then the function f is topological continuous function.

Finally, I conjecture the vice versa of these results are not true but that will need further study from other researchers.

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الاتصال ما بين اقترانات متجهات الفضاءات المترية

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ملخص

عام 2014، درس الباحث كيونيت سيفيك نوعان من الاتصال (الفيكتوريال والتيبولوجيكال) وتوصل بناءً على ذلك إلى العديد من العلاقات والنظريات المهمة في الفضاءات المترية ومن هذه العلاقات "نظرية التمدد" و " نظرية النهاية المنتظمة".

خلال كتابتي لهذه الرسالة، قمت بدراسة وبتطوير على ما توصل إليه الباحث كيونيت سيفيك حتى توصلت إلى العديد من العلاقات. في الحقيقة انني اثبتت ان نظرية التوسع صحيحة في حالة كون الاقتران اقترانا فضائيا متصلا ومنتظم بدلا من كونة اقترانا طوبولوجي ومتصلا ومنتظم كما اثبت العالم كيونت سيفيك في بحثه، أيضا استطعت ان اثبت نظرية النهاية المنتظمة في حالة الاقتران الفضائي المتصل، وكذلك اثبتنا هذه النظرية في حالة الاقتران الطوبولوجي المتصل.