

Deanship of Graduate Studies

Al-Quds University

SEMI-CONNECTEDNESS IN
BITOPOLOGICAL
SPACES

By

Hani Mahmoud Sallah El-Din

M. Sc. Thesis



2002

SEMI-CONNECTEDNESS IN BITOPOLOGICAL
SPACES

By

Hani M. Ahmad Sallah El-Din

B.Sc. : Al-Quds University , Palestine

A thesis Submitted in Partial fulfillment of requirements for the degree
of Master of Science , Department of Mathematics /Graduate Studies

Al-Quds University

February ,2002

Graduate Studies \ Mathematics

Deanship of Graduate Studies

SEMI-CONNECTEDNESS IN BITOPOLOGICAL
SPACES

By

Student Name : Hani Mahmoud Ahmad Sallah El-Din

Registration No :9720089

Supervisor : Dr. Yousef Bdeir

Master thesis submitted and accepted ,Date :4\2\2002

The names and signatures of the examining committee members are as follows :

1- Dr. Yousef Bdeir

Head of Committee

Signature 

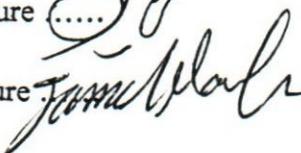
2- Dr. Muhammad Khaleel

Internal Examiner

Signature 

3- Dr. Jamal Nuoh

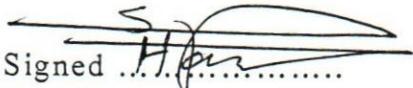
External Examiner

Signature 

Al-Quds University

Declaration :

I Certify that this thesis submitted for the degree of Master is the result of my own research except where otherwise acknowledged , and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution .


Signed

Hani Mahmoud Ahmad Sallah El-Din

Date : 4\2\2002

Acknowledgment

First , I am very grateful to my supervisor Dr. Yousef for his many valuable ideas , helpful , suggestions and excellent guidance . I am also grateful to my external referee Dr. Jamal Nuoh for his useful comments in regard to this thesis . I am also grateful to my internal referee Dr.Muhmmad Kaleel for his fruitful comments and advices .

My thanks to the members of the department of mathematics at Al-Quds University .

Special thanks to my Parents , my wife and my brother and sisters .

ABSTRACT

In this thesis , we study some properties of pairwise connected and pairwise semi-connectedness in bitopological spaces , proffering some properties of semi-open sets in topological space .

Also , we study the totally disconnectedness and the semi-components in bitopological spaces .

Finally , we study connectedness and semi-connectedness between sets in bitopological spaces .

In most of these , we make generalization of similar concepts in topological spaces , and have generalization for results in topological spaces .

DEDICATION

To my mother

To my wife

To my brothers and sisters

To my children

To free people in Palestine and Afghanistan

Table of Contents

Introduction	1
Chapter .1	Pairwise Connected Spaces
1.1	Pairwise separation . 3
1.2	Continuous functions in bitopological spaces . 12
Chapter .2	Pairwise Semi-Connected Space
2.1	Semi-open and semi-closed sets . 21
2.2	Pairwise Semi-separation . 24
2.3	T_1T_2 pairwise semi-connected sets . 34
2.4	Pairwise semi-connected sets .
Chapter 3	Totally disconnectedness in bitopological spaces
3.1	Pairwise separation axioms . 50
3.2	Pairwise semi-separation axioms. 53
3.3	The Components in bitopological Spaces . 56
3.4	Semi-Components in bitopological Spaces. 60
Chapter 4	Pairwise set semi-connected spaces
4.1	Pairwise connected between sets . 65
4.2	Pairwise semi-connectedness between sets . 68

4.3 Subspaces of pairwise semi-connected spaces	
between sets.	79
References	83

Introduction

The triple (X, T_1, T_2) , where X is a set and $T_i, i = 1, 2$ are arbitrary topologies on X , is called a bitopological space. In 1963, Kelly [5] introduced the notion of bitopological spaces. Several authors have studied this notion and other related concepts .e .g Weston, and Kim , Pervin [13] .

Pervin [13] defined connectedness and studied its properties for bitopological spaces. J.Swart [4] defined total disconnectedness in bitopological spaces. And Fora and Hdeib defined pairwise connectedness and semi pairwise connectedness for bitopological spaces.

This thesis consists of 4 chapters. The first chapter deals with pairwise connectedness and some types of continuous functions between bitopological spaces and their effects on some types of connected spaces.

The properties of semi-open sets, the definition of pairwise semi-connected and semi-continuous function and some related results in bitopological spaces are studied in the second chapter .

The third chapter concerns with the study of the totally disconnected Bitopological spaces and the components in bitopological spaces.

The study of pairwise connectedness between sets and pairwise semi-connectedness between sets and some relations between

them in bitopological spaces are covered in the fourth chapter.

We use p- and s- to denote pairwise and semi, respectively. e.g. p-connected and s-connected stand for pairwise connected and semi connected, respectively.

By saying that the bitopological space (X, T_1, T_2) has a certain topological property, we mean that both (X, T_1) and (X, T_2) have this property. By saying that a function $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ satisfies a property that is used for functions between topological spaces, we mean that both $f : (X, T_1) \rightarrow (Y, T^*_1)$ and $f : (X, T_2) \rightarrow (Y, T^*_2)$ satisfy this property. T_S , T_L , T_R , and T_D will denote the standard topology, the left ray topology, the right ray topology, and the discrete topology, respectively.

Chapter .1

Pairwise Connected Spaces

1.1 Pairwise Separation .

1.1.1 Definition : [13]

A bitopological space (X, T_1, T_2) is called p -connected if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap \text{cl}_1(B)) \cup (\text{cl}_2(A) \cap B) = \emptyset$, where $\text{cl}_1(B)$ and $\text{cl}_2(A)$ denote the closures of B and A with respect to T_1 and T_2 , respectively.

When X can be expressed as the union of such A and B , we write $X = A|B$ and call it p -disconnection of X , or pairwise separation, and denoted by p -separation. (This is called a separation of X according to William [13]).

1.1.2 Definition :

A bitopological space (X, T_1, T_2) which is not p -connected is called p -disconnected .

1.1.3 Theorem : [13]

For any bitopological space (X, T_1, T_2) , the following conditions are equivalent :

- 1- (X, T_1, T_2) is p -connected .
- 2- X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is T_1 open and B is T_2 open .
- 3- X contains no nonempty proper subset which is both T_1 open and T_2 closed .

Proof:

1 \Rightarrow 2) Suppose $X = A \cup B$, where A and B are nonempty T_1 open and T_2 open, respectively , such that $A \cap B = \emptyset$. A is T_1 open, then B is T_1 closed and this implies that $cl_1(B) = B$. Then $A \cap cl_1(B) = \emptyset$. And since B is T_2 open , then A is T_2 closed and so $cl_2(A) = A$, then $cl_2(A) \cap B = \emptyset$. So $[A \cap cl_1(B)] \cup [cl_2(A) \cap B] = \emptyset$. Then X is not p -connected which contradicts the assumption .

2 \Rightarrow 1) Suppose that X is not p -connected . Then $X = A \cup B$ is a p -disconnection of X , for some nonempty subsets A and B of X . Then $[A \cap cl_1(B)] = \emptyset$ and $[cl_2(A) \cap B] = \emptyset$. Now since $[A \cap cl_1(B)] = \emptyset$ and $B = X \setminus A$, then $[A \cap cl_1(X \setminus A)] = \emptyset$.

But $cl_1(X \setminus A) = X \setminus int_1(A)$, so $[A \cap (X \setminus int_1(A))] = \emptyset$. So $int_1(A) \subset A$, and so $A = int_1(A)$.i.e. A is T_1 open set. Similarly,

we show that B is a T_2 open set. Therefore $X = A \cup B$, and A is T_1 open and B is T_2 open and both of them are nonempty sets. This contradicts our assumption. Hence X is p -connected.

$2 \Rightarrow 3$) Suppose that there exists a nonempty proper subset A of X which is both T_1 open and T_2 -closed. Then $X = A \cup (X \setminus A)$, where A is T_1 open and $X \setminus A$ is T_2 open, which contradicts our assumption of X , as A and $X \setminus A$ are nonempty disjoint subsets of X .

$3 \Rightarrow 2$) Suppose $X = A \cup B$, A is T_1 open and B is T_2 open and A, B are nonempty disjoint sets. Since $B = X \setminus A$ then A is T_2 closed. This contradicts the assumption. This completes the proof of the theorem.

The following corollary results from the equivalence of 1 and 3 in the previous theorem.

1.1.4 Corollary : [1]

The space (X, T_1, T_2) is p -connected if and only if X and \emptyset are the only subsets of the space X which are both open in T_i and closed in T_j ; ($i, j = 1, 2$, and $i \neq j$).

1.1.5 Remark : [13]

For any topology T , (X, T, T) is p -connected if and only if (X, T) is connected.

Proof:

\Leftarrow) Suppose that (X, T, T) is p -disconnected. Then there exist two nonempty disjoint open sets in T , A and B such that $X = A \cup B$. So (X, T) is disconnected which contradicts the assumption.

\Rightarrow) Suppose (X, T) is disconnected. Then there exist two nonempty disjoint open sets A and B such that $X = A \cup B$. Then A and B are closed sets, so $A = \text{cl}(A)$ and $B = \text{cl}(B)$.

Hence $A \cap \text{cl}(B) = \emptyset = \text{cl}(A) \cap B$.

1.1.6 Note : [13]

If the topological spaces (X, T_1) , (X, T_2) are both connected, then (X, T_1, T_2) may fail to be p -connected. Also, E. Berven [13] showed that if (X, T_1, T_2) is p -connected, then (X, T_1) and (X, T_2) need not be connected.

The following examples show these and all other possibilities.

1.1.7 Example :

Let $X = \{1, 2, 3, 4, 5\}$, $T_1 = \{X, \emptyset, \{2\}, \{3\}, \{2, 3\}\}$ and

$T_2 = \{X, \emptyset, \{1, 3, 4, 5\}\}$.