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Oscillation and Nonoscillation of First Order Functional Differential Equations with Advanced Arguments

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Dedication

To my parents, my brothers, my sisters, my wife, my sons, and my daughters, I will dedicate this research.

Kamel Khalil Ahmed Noman

Declaration

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: ……………………..

Kamel Khalil Ahmed Noman

29/8/2007

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Abstract

This thesis aimed to study the behavior of solutions and criterion of oscillation for solutions of first order advanced functional differential equations. So we tackle the conditions that limit oscillation for these linear and nonlinear equations, and the unknown function in the general form for this type of equations contains one advanced variable or more about the variable that represents the present state.

Such type of study is studied and classified according to the coefficients even if they are constants, constants and variables or all of them are variables.

This thesis contains in its contents basic concepts of functional differential equations and the definition of oscillation. It also contains several result due to oscillation theorems in addition to a set of examples that explain the main theorems.

The reason why the researcher studied the type of equations is because of anxious, the subject is interesting and important.

This study contains many modern results resulted in oscillation of advanced differential equations in both cases linear and nonlinear, also homogeneous and nonhomogeneous. Nonhomogeneous equations has been transformed by a specific transformation to homogeneous case.

Some theorems of advanced differential equations have been proved by contrasting them with delay differential equations and this is the out put of the study that the researcher accomplished.

الملخص

اهتمت هذه الدراسة بدراسة سلوك حلول ومعايير التذبذب لحلول فئة معينة من المعادلات التفاضلية الاقترانية المتقدمة من الدرجة الأولى، حيث تعرضنا للشروط التي تحدد التذبذب لهذه المعادلات الخطية وغير الخطية، وكذلك تعرضنا للاقتران المجهول في الصورة العامة لهذه الفئة من المعادلات والذي يحتوي على متغير متقدم واحد أو أكثر عن المتغير الذي يمثل الوضع الحالي. تمت دراسة هذه الفئة من المعادلات وتصنيفها بالاعتماد على المعاملات سواء كانت ثابتة أو ثابتة ومتغيرة أو جميعها متغيرة.

تحتوي ثنايا الرسالة على المفاهيم الأساسية للمعادلات التفاضلية الاقترانية وكذلك تعريف التذبذب وتحتوي أيضاً على العديد من النتائج التي تتعلق بنظريات التذبذب لهذه المعادلات بالإضافة إلى مجموعة من الأمثلة التي توضح النظريات الرئيسية.

كانت الرغبة في دراسة هذا النوع من المعادلات لأن الموضوع ممتع وجدير بالاهتمام.

تحتوي الرسالة على العديد من النتائج الحديثة الصادرة في نظرية التذبذب للمعادلات التفاضلية المتقدمة بحالتيها الخطية وغير الخطية وكذلك المعادلات المتجانسة وغير المتجانسة، حيث تم تحويل المعادلة غير المتجانسة إلى معادلة متجانسة باستعمال تحويلاً معيناً.

تم برهنة بعض النظريات للمعادلات التفاضلية المتقدمة (advanced) بمقارنتها مع المعادلات التفاضلية المتأخرة (delay)، وهذه تعتبر من النتائج التي استطعنا التوصل إليها في هذا البحث.

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Introduction

Recently, there has been a lot of activities concerning the oscillatory and nonoscillatory behavior of delay differential equations; for example see [3], [4], [5] and [8] and references therein. But, for the oscillatory and nonoscillatory results of advanced differential equations, compared with those of delay differential equations, less is known up to know.

With the past two decades, the oscillatory behavior of solutions of differential equations with deviating arguments has been studied by many authors. The problem of the oscillations caused by deviating arguments (delays or advanced arguments) has been the subject of intensive investigation. Among numerous works dealing with the study of this problem we choose to refer to L. E. El'sgol'ts [3], Ladde, Lakshmikanthan and Zhang [8], Gyori and Ladas [5], Erbe, Kong and Zhang [4], and Kordonis and Philos [7].

In the special case of an autonomous advanced differential equation a necessary and sufficient condition for the oscillation of all solutions is that its characteristic equation has no real roots, this appears in [5]. Also for advanced differential equations with oscillating coefficients, a necessary and sufficient conditions for the oscillation of all solutions is given by Li, Zhu and Wang [10].

An advanced functional differential equation is one in which the derivatives of the future state or derivatives of functionals of the future state are involved as well as the present state of the system. In fact when the derivatives of the future history are used, most of the literature is devoted to existence, uniqueness, and continuous dependence.

In this research we consider theorems that provide sufficient conditions for the oscillation of solutions of the first order, linear, nonlinear and impulsive advanced differential equations, taking different forms depending on the coefficients and on the advanced argument (which may be constants, variables or constants and variables) and the forcing terms of these equations. Also we consider theorems which give sufficient conditions for the oscillation of mixed type and of an alternating advanced and delay differential equations.

Our research deals with the oscillation of the first order advanced functional differential equations. It consists of four chapters:

Chapter one: contains the main concepts, definitions, lemmas, theorems, and preliminary material that are essential in the following chapters.

Chapter two: devotes the oscillation theory of the linear advanced functional differential equation

$$y'(t) = p(t)y(t) + \sum_{i=1}^{n} p_i(t)y(\tau_i(t)),$$

where

$p(t) \geq 0$, $p_i(t) \geq 0$, and $\tau_i(t) > t$ are continuous $i = 1,2,...,n$, with special cases:

(i) p_i and τ_i are constants $i = 1,2,...,n$,

(ii) p_i are variables, τ_i are constants $i = 1,2,...,n$,

(iii) p_i and τ_i are variables.

Chapter three: deals with oscillatory and nonoscillatory solutions of the nonlinear advanced differential equation of the form

$$y'(t) - \sum_{i=1}^{n} p_i(t) f\left(y(\tau_i(t))\right) = 0,$$

where $p_i(t) \geq 0$, $\tau_i(t) > t$, $i = 1,2,...,n$ are continuous. And as a special case of this nonlinear advanced differential equation: n=1, $p(t) \geq 0$ almost everywhere and $p(t)$ is locally integrable and $\tau(t) > t$.

Chapter four studies oscillation theorems of special kinds of differential equations: impulsive, mixed type and alternately advanced and retarded differential equations.

Symboles

$\Re = (-\infty, \infty)$ the set of real numbers.

$\Re^+ = [0, \infty)$ the set of nonnegative real numbers .

$C[a,b]$: the set of all real valued continuous functions on the closed interval $[a,b]$.

$C^1[a,b]$: the set of all real valued continuously differentiable functions on $[a,b]$.

$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times ... \times A_n$.

The triple (a,b,c) refers to definitions, theorems, examples, lemmas, corollaries, remarks, equations or inequalities where:

a: refers to the chapter's number,

b: refers to the section's number,

c: refers to the number of definitions, theorems, examples, lemmas, corollaries, remarks, equations or inequalities.

The symbol [x] means the reference number.

$\|.\|$: any vector norm.

Chapter one

Preliminaries

1.0 Introduction

The aim of this chapter is to present some preliminary definitions, examples and results which will be used throughout the research.

Section 1.1 introduces definitions of differential equations with deviating arguments and their classification with examples.

Section 1.2 investigates the definition of oscillatory and nonoscillatory solutions of differential equations.

Section 1.3 gives some basic lemmas and theorems of oscillation of differential equations by using the Laplace transform.

Section 1.4 contains a detailed description of possible existence and uniqueness results that are needed in our treatment of the oscillation theory of advanced differential equations.

Finally section 1.5 introduces some theorems which are important tools in oscillation theory, especially, the generalized characteristic equation and the existence of positive solutions of the first order advanced functional differential equation.

1.1 Definitions and examples

Definition 1.1.1: Differential equations with deviating arguments

Differential equations with deviating arguments are differential equations, in which the unknown function appears with various values of the argument, and these, classified in the following three types:

1- differential equations with retarded arguments:

A differential equation with retarded argument is a differential equation with deviating argument, in which the highest order derivative of the unknown function appears for just one value of the argument, and this argument is not less than all arguments of the unknown function, and its derivative appearing in the equation.

2- Differential equations with advanced arguments:

A differential equations with advanced argument is a differential equation with deviating argument, in which the highest order derivative of the unknown function appears of just one value of the argument, and this argument is not larger than the remaining arguments of the unknown function, and its derivative appearing in the equation.

3- Differential equations with neutral arguments:

A differential equation with neutral argument is a differential equation with deviating argument, which is not of retarded argument nor of advanced argument. That is, the highest order derivative of the unknown function in the differential equation with neutral argument, is evaluated both with the present state and at one or more past or future states.

Example 1.1.1: Consider the following differential equations with deviating arguments:

i. $y'(t) = f(t, y(t), y(t-\tau(t)))$

ii. $y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2))$

iii. $y''(t) = f(t, y(t), y'(t), y(t-\tau(t)), y'(t-\tau(t)))$

iv. $y''(t) = f(t, y(\frac{t}{2}), y'(\frac{t}{2}), y(t), y'(t))$

v. $y''(t) = f(t, y(t), y'(t), y(t-\tau(t)), y'(t-\tau(t)), y''(t-\tau(t)))$

Then

(i) and (iii) are with retarded arguments if $\tau(t) \geq 0$, and with advanced argument if $\tau(t) \leq 0$.

(ii) is with retarded argument if $\tau_1 > 0$, $\tau_2 > 0$, and with advanced argument if $\tau_1 < 0$, $\tau_2 < 0$.

(iv) is with retarded argument if $t \geq 0$, and with advanced argument if $t \leq 0$.

(v) is with neutral argument.

It is possible that an equation belongs to one of the above mentioned arguments on one set of values of *t*, and to another type on another set. For example, the differential equation:

$$y'(t) = f(t, y(t), y(t+\tau(t))),$$

is of retarded argument on intervals on which $\tau(t) \leq 0$, and of advanced argument on intervals on which $\tau(t) \geq 0$.

1.2 Definition of oscillation

The most frequently definitions of oscillation, used in the literature are the following two definitions:

Definition 1.2.1: A nontrivial solution *y(t)* of a differential equation is said to be oscillatory solution if and only if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros $\{t_n\}$ $(y(t_n)=0)$ of *y(t)* such that $\lim_{n\to\infty} t_n = +\infty$.

Otherwise, y(t) is called nonoscillatory.

Definition 1.2.2: A nontrivial solution y(t) is said to be oscillatory, if it changes sign on $[T,\infty)$, T is any number.

Remark 1.2.1: Definition 1.2.1 is more general than definition 1.2.2, for example:

y(t) = 1-sin t,

is an oscillatory solution according to definition 1.2.1, and is nonoscillatory solution according to definition 1.2.2.

Example 1.2.1: The equation

$$y'(t) + y(t + \frac{3\pi}{2}) = 0\,,$$

has the oscillatory solutions:

$$y_1(t) = \sin t, y_2(t) = \cos t\,.$$

Example 1.2.2: The equation

$$y'(t) + \frac{3\pi}{2} y(t+1) = 0\,,$$

has the oscillatory solution:

$$y(t) = \sin\frac{3\pi}{2}t + \cos\frac{3\pi}{2}t\,,$$

and also has the bounded nonoscillatory solution $y(t) = Ae^{\lambda t}$ where

A is a constant and λ is a root of the equation $\lambda + \frac{3\pi}{2}e^{\lambda} = 0$

(λ=-1.2931).

Example 1.2.3: The equation

$$y'(t) = y(t)\,,$$

has a nonoscillatory solution

$y(t) = ce^t, c$ is a constant

Lemma 1.2.1: Let p and τ be two positive constants. Let $y(t)$ be an eventually positive solution of the advance differential inequality

$$y'(t) - py(t+\tau) \geq 0 \tag{1.2.1}$$

Then for t sufficiently large,

$$y(t+\tau) \leq By(t), \tag{1.2.2}$$

where $B = (\frac{2}{p\tau})^2$

Proof: Assume that t_0 is such that $y(t) > 0$ for $t < t_0 + \tau$, and $y(t)$ satisfies (1.2.1) for $t \geq t_0$. For given $s \leq t_0 + \tau$, integrate both sides of (1.2.1) from $s - \frac{\tau}{2}$ to s, and by using the fact that y(t) is increasing for $t \geq t_0$, we find that

$$y(s) - y(s - \frac{\tau}{2}) - \frac{p\tau}{2} y(s + \frac{\tau}{2}) \geq 0, \tag{1.2.3}$$

since y(t)>0, then $y(s - \frac{\tau}{2}) \geq 0$, and hence

$$y(s) - \frac{p\tau}{2} y(s + \frac{\tau}{2}) \geq 0, \tag{1.2.4}$$

or

$$\frac{p\tau}{2} y(s + \frac{\tau}{2}) \leq y(s). \tag{1.2.5}$$

Applying (1.2.5) for s=t+$\frac{\tau}{2}$, and for s=t, we have

$$\frac{p\tau}{2} y(t+\tau) \leq y(t + \frac{\tau}{2}), \tag{1.2.6}$$

and

$$\frac{p\tau}{2}y(t+\frac{\tau}{2}) \le y(t), \qquad (1.2.7)$$

respectively. Combining (1.2.6) and (1.2.7) yields

$$(\frac{p\tau}{2})^2 y(t+\tau) \le \frac{p\tau}{2}y(t+\frac{\tau}{2}) \le y(t), \qquad 1.2.8)$$

and hence

$$y(t+\tau) \le (\frac{2}{p\tau})^2 y(t), \qquad (1.2.9)$$

or

$$y(t+\tau) \le By(t), \qquad (1.2.10)$$

where B= $(\frac{2}{p\tau})^2$

Theorem 1.2.1: Consider the advanced differential equation and inequalities:

$$y'(t) - p(t)y(t+\tau) = 0 \qquad (1.2.11)$$

$$y'(t) - p(t)y(t+\tau) \ge 0 \qquad (1.2.12)$$

$$y'(t) - p(t)y(t+\tau) \le 0 \qquad (1.2.13)$$

Assume that $p \in C[(t_0,\infty),\Re^+]$, $\tau > 0$, and $\underline{\lim}_{t\to\infty} \int_t^{t+\tau} p(s)ds > \frac{1}{e}$ (1.2.14)

then

(i) every solution of (1.2.11) oscillates.

(ii) Inequality (1.2.12) has no eventually positive solution.

(iii) Inequality (1.2.13) has no eventually negative solution.

Proof: Assume that (1.2.11) has an eventually positive solution y(t). Then there exists a $t^* \ge t_0 + \tau$, such that for t≥t*, y(t) >0 and y(t+ τ)≥0.

Also $y'(t) \ge 0$ and

$$y(t) \le y(t+\tau), \qquad (1.2.15)$$

since y(t) is increasing. And

$$y'(t) - p(t)y(t) \ge y'(t) - p(t)y(t+\tau) = 0 \qquad (1.2.16)$$

Thus

$$y'(t) - p(t)y(t) \ge 0, \qquad (1.2.17)$$

or

$$\frac{y'(t)}{y(t)} \ge p(t). \qquad (1.2.18)$$

By integrating both sides of (1.2.18) from t to t+τ , we find

$$\ln \frac{y(t+\tau)}{y(t)} \ge \int_t^{t+\tau} p(s)ds. \qquad (1.2.19)$$

Also from (1.2.14) it follows that there exists a constant c>0 and a $t_1 \ge t^*$, such that

$$\int_t^{t+\tau} p(s)ds \ge c > \tfrac{1}{e}, \quad t \ge t_1 \qquad (1.2.20)$$

so

$$\ln \frac{y(t+\tau)}{y(t)} \ge c, \qquad (1.2.21)$$

or

$$e^c y(t) \le y(t+\tau). \qquad (1.2.22)$$

But $e^c \ge ec$, $\forall c \in \Re$, so (1.2.22) becomes

$$ec\, y(t) \le y(t+\tau), \quad t \ge t_1 + \tau. \qquad (1.2.23)$$

Repeating the above procedure, it follows by induction that for any positive integer k

$$(ec)^k y(t) \le y(t+\tau), \quad t \ge t_1 + k\tau. \qquad (1.2.24)$$

Choose k such that

$$\frac{4}{c^2} < (ec)^k , \qquad (1.2.25)$$

which is possible, because ce>1. Now, fix a $t' \geq t_1 + k\tau$. Then because of (1.2.20), there exists a $\xi \in (t' - \tau, t')$ such that

$$\int_{t'-\tau}^{\xi} p(s)ds \geq \frac{c}{2} \quad \text{and} \quad \int_{\xi}^{t'} p(s)ds \geq \frac{c}{2}. \qquad (1.2.26)$$

By integrating (1.2.11) over the intervals $[t' - \tau, \xi], [\xi, t']$, we find

$$y(\xi) - y(t' - \tau) - \int_{t'-\tau}^{\xi} p(s) y(s+\tau) ds = 0 , \qquad (1.2.27)$$

and

$$y(t') - y(\xi) - \int_{\xi}^{t'} p(s) y(s+\tau) ds = 0 . \qquad (1.2.28)$$

By omitting the second terms in (1.2.27) and (1.2.28), and by using the increasing nature of y(t) and (1.2.26), we find

$$y(\xi) \geq \int_{t'-\tau}^{\xi} p(s) y(s+\tau) ds \geq \frac{c}{2} \int_{t'-\tau}^{\xi} y(s+\tau) ds \geq \frac{c}{2} y(t' - \tau + \tau) = \frac{c}{2} y(t') . \qquad (1.2.29)$$

Thus

$$y(\xi) \geq \frac{c}{2} y(t') . \qquad (1.2.30)$$

Also from (1.2.28), we conclude that

$$y(t') \geq \int_{\xi}^{t'} p(s) y(s+\tau) ds \geq \frac{c}{2} y(\xi + \tau) ,$$

or

$$y(t') \geq \frac{c}{2} y(\xi + \tau) . \qquad (1.2.31)$$

Combining (1.2.30) and (1.2.31), gives

$$y(\xi) \geq \frac{c}{2} y(t') > (\frac{c}{2})^2 y(\xi + \tau) \quad , \tag{1.2.32}$$

or

$$\frac{y(\xi + \tau)}{y(\xi)} < (\frac{2}{c})^2 . \tag{1.2.33}$$

But from (1.2.24)

$$(ec)^k \leq \frac{y(\xi + \tau)}{y(\xi)} < (\frac{2}{c})^2 = \frac{4}{c^2} . \tag{1.2.34}$$

This contradicts (1.2.25). So the assumption of y(t) is eventually positive solution is not true. Therefore every solution of equation (1.2.11) is oscillatory.

By using parallel arguments we can prove (ii) and (iii) of the theorem.

1.3 Some basic definitions, lemmas and theorems

Definition 1.3.1: A function F is analytic at z_0 if and only if there exist r>0, such that $F'(z)$ exists for all $z \in B(z_0, r)$, where $B(z_0, r)$ is the ball centered at z_0 and has radius =r.

Definition 1.3.2: The function F has an isolated singular point at z=a if there exist, $R > 0$, such that F is analytic in $B(a, R) \setminus \{a\}$.

Definition 1.3.3: The Laplace transform

Let $x : [0, \infty) \to \Re$ be a real valued function. The Laplace transform of x(t), denoted by $L[x(t)]$ or X(s), is given by

$$\text{L[x(t)] = X(s) = } \int_0^\infty e^{-st} x(t) dt \tag{1.3.1}$$

X(s) is defined for all values of the complex variable s, for which the integral in (1.3.1) converges in the sense that:

$\lim_{u\to\infty}\int_0^u e^{-st}x(t)dt$ exists and is finite.

Definition 1.3.4: Compact set

A set $K\subseteq\Re$ is said to be compact if whenever it is contained in the union of a collection $T=\{G_\alpha\}$ of open sets in $\Re$, then it is contained in the union of some finite number of sets in T.

Definition 1.3.5: Locally integrable function

A function is said to be locally integrable on an open set S in a finite dimensional Euclidean space if it is defined almost everywhere in S and has a finite integral on compact subset of S.

Definition 1.3.6: Locally summable function

$L^1(\mu)$: All complex measurable functions f on a set Ω such that

$\int_\Omega |f|d\mu<\infty$. The members of $L^1(\mu)$ are called Lebesgue integrable (or summable) functions with respect to μ.

Remark 1.3.1: There exists $\sigma_0\in\Re$ (possibly $\neq\infty$),such that the integral in (1.3.1) converges for all s with Re s> σ_0, and diverges for all s with Re s< σ_0, σ_0 is called the abscissa of convergence of X(s), where Re s is the real part of s.

Lemma 1.3.1: Let $x\in C[[0,\infty),\Re]$, and suppose that there exist positive constants M and α such that

$$\|x(t)\|\leq Me^{\alpha t}\ ,\ \text{for}\ t\geq 0,$$

then the abscissa of convergence σ_0 of the Laplace transform X(s) of x(t) satisfies

$\sigma_0 \le \alpha$.

Furthermore, X(s) exists, and is an analytic function of s for Re s> σ_0 .

Lemma 1.3.2:

(i) Let $x \in C^1[[0,\infty),\Re]$, and let $\sigma_0 < \infty$, be the abscissa of convergence of the Laplace transform X(s) of x(t). Then the Laplace Transform of x'(t) has the same abscissa of convergence, and

$$L[x'(t)] = \int_0^\infty e^{-st} x'(t)dt = sX(s) - x(0) \tag{1.3.2}$$

for all s, with Re s> σ_0

(ii) Let

$x \in C[[0,\infty),\Re]$

and let $\sigma_0 < \infty$, be the abscissa of convergence of the Laplace transform X(s) of x(t). Then the Laplace transform of the shift function x(t+ τ) has the same abscissa of convergence, and

$$L[x(t+\tau)] = \int_0^\infty e^{-st} x(t+\tau)dt = e^{s\tau} X(s) - e^{s\tau} \int_0^\tau e^{-st} x(t)dt\,, \tag{1.3.3}$$

for all s with Re s> σ_0

Remark 1.3.2: It is well known that if $x(t)$ satisfies $\|x(t)\| \le Me^{\alpha t}$, then the Laplace transform $X(s)$ of $x(t)$ which is given by (1.3.1) exists for $\operatorname{Re} s > \alpha$, M and α are positive constants.

Theorem 1.3.1: Let $x \in C[[0,\infty),\Re^{+}]$, and assume that the abscissa of convergence σ_0 of the Laplace transform X(s) of x(t) is finite, then X(s) has a singularity at the point $s=\sigma_0$, more precisely, there exist a sequence

$S_n=\alpha_n+iB_n$, n=1,2,…. Such that

$\alpha_n \geq \sigma_0$, for n≥1, $\lim_{n\to\infty}\alpha_n=\sigma_0, \lim_{n\to\infty} B_n=0$, and $\lim_{n\to\infty}|X(s)|=\infty$.

Proof: see[5].

Chapter two

Oscillation of linear advanced functional differential equations

2.0 Introduction

Our aim is to discuss oscillatory and nonoscillatory behavior of solutions of the first order functional differential equation

$$y'(t) = p(t)y(t) + \sum_{i=1}^{n} p_i(t)y(t+\tau_i(t)), \tag{2-A}$$

where

$$p(t) \geq 0,\ p_i(t) \geq 0, \text{ and } \tau_i(t) > 0 \text{ are continuous and } i = 1,2,...,n.$$

In order to reach what will we hope, special cases for $p(t)$, $p_i(t)$ and $\tau_i(t)$ are taken to obtain oscillation and nonoscillation criteria for all solutions of (2-A).

In this chapter we present some of the oscillation results that recently have been obtained for this form of equations.

In section 2.1 we introduce sufficient conditions for the oscillation of equation (2-A) with constant coefficients, single and several deviating arguments and $p(t) = 0$. That is, we consider the following two equations:

$$y'(t) = py(t+\tau),$$

$$y'(t) = \sum_{i=1}^{n} p_i y(t+\tau_i).$$

In section 2.2 we study some oscillation results of equation (2-A) with variable coefficients, constant deviating arguments and $p(t) = 0$. In section 2.3 we present oscillation criteria for the solutions of (2-A) with variable coefficients, variable deviating arguments (with both several and single deviating arguments) and with $p(t) = 0$.

Finally section 2.4 concerns with the results of oscillation theorem of nonhomogeneous equations (with forcing terms).

2.1. Equations with constant coefficients and constant advanced argument

In this section we will consider equation (2-A) with the following assumptions:

$$p(t) = 0, \; p_i(t) = p > 0, \; \tau_i(t) = \tau > 0 \text{ and } n = 1 \tag{2.1.1}$$

so that equation (2-A) becomes

$$y'(t) = py(t+\tau). \tag{2.1.2}$$

Theorem 2.1.1: Assume that p and τ are positive numbers, and assume that $p\tau e \le 1$, then equation (2.1.2) has a nonoscillatory solution.

Proof: Let $y(t) = e^{\lambda t}, \lambda$ constant, be a solution of equation (2.1.2), then the characteristic equation of equation (2.1.2) will be

$$F(\lambda) = \lambda - pe^{\lambda\tau}. \tag{2.1.3}$$

Observe that

$$F(0) = -p < 0,$$

and

$$F\left(\frac{1}{\tau}\right) = \frac{1}{\tau} - pe = \frac{1 - p\tau e}{\tau} \ge 0.$$

Hence, there exists a positive real number $\lambda \in \left(0, \frac{1}{\tau}\right]$, such that

$e^{\lambda t}$ is a nonoscillatory solutions of equation (2.1.2)

Corollary 2.1.1: If $p(t) = p > 0$, $\tau(t) = t + \tau, \tau > 0$, then the condition $p\tau e > 1$ is necessary and sufficient for all solutions of equation (2.1.2) to oscillate.

Example 2.1.1: The equation:

$$y'(t) = \frac{1}{3}y(t+1), \text{ with } p = \frac{1}{3}, \ \tau = 1$$

has a nonoscillatory solution

$y(t) = Ae^{\lambda t}$, where A is any constant and λ is a constant satisfying the equation

$$\lambda = \frac{1}{3}e^{\lambda}, \qquad \lambda \in (0,1), \quad (\lambda \approx 0.6190615)$$

Remark 2.1.1: The oscillatory theory of differential equations with deviating argument present some new problems which are not present in the theory of corresponding ordinary differential equations. First order differential equations with deviating arguments can have oscillatory solutions while first order ordinary differential equations do not possess oscillatory solution. The following example explains this idea.

Example 2.1.2: The ordinary differential equation

$$y' = y(t),$$

has the non-oscillatory solution

$$y(t) = e^{t}.$$

The delay differential equation

$$y'(t) = y\left(t - \frac{3\pi}{2}\right),$$

has both oscillatory solutions:

$y_1(t) = \sin t$, $y_2(t) = \cos t$ and nonoscillatory solution

$$y(t)=e^{\lambda_0 t},\ \lambda_0 \text{ satisfies } \lambda_0=e^{\frac{-3}{2}\pi\lambda_0},\ (\lambda_0=0.277410633).$$

While all solutions of advanced differential equation

$$y'(t)=y\left(t+\frac{3\pi}{2}\right),$$

are oscillatory by Corollary (2.1.1) (p=1, $\tau=\frac{3\pi}{2}$ and $p\tau e=\frac{3\pi}{2}e>1$).

From remark (2.1.1), the nature of solution changes completely after the appearance of the deviating argument in the equation.

It is important to discuss oscillatory and nonoscillatory behavior of solutions of equation (2-A) with

$p(t)=0$, $p_i(t)=p_i>0$, $\tau_i(t)=\tau_i>0$, $i=1,2,...,n$. So we have the following form

$$y'(t)=\sum_{i=1}^{n}p_i y(t+\tau_i). \tag{2.1.4}$$

The following results concerning oscillatory and nonoscillatory behavior of equation (2.1.4).

Theorem 2.1.2: If $F(\lambda_0)=\lambda_0-\sum_{i=1}^{n}p_i e^{\lambda_0\tau_i}<0$, (2.1.5)

where λ_0 satisfies the equation

$$\sum_{i=1}^{n}p_i\tau_i e^{\lambda_0\tau_i}=1. \tag{2.1.6}$$

Then all solutions of (2.1.4) oscillate.

Proof: Let $y(t)=e^{\lambda t}$ be a solution of equation (2.1.4), then the characteristic equation of (2.1.4) is

$$F(\lambda)=\lambda-\sum_{i=1}^{n}p_i e^{\lambda\tau_i}=0, \tag{2.1.7}$$

and so

$$F'(\lambda) = 1 - \sum_{i=1}^{n} \tau_i p_i e^{\lambda \tau_i} , \tag{2.1.8}$$

and

$$F''(\lambda) = -\sum_{i=1}^{n} \tau_i^2 p_i e^{\lambda \tau_i} . \tag{2.1.9}$$

Thus $F(\lambda)$ is concave down and has a maximum value.

The relation (2.1.6), shows that $F(\lambda_0)$ is a maximum value. But since $F(\lambda_0) < 0$, then the characteristic equation has no real roots.

Hence all solutions of equation (2.1.4) oscillate.

Theorem 2.1.3: If there exist

$$N_i > 0 \quad , \quad \sum_{1 \le i \le n} N_i = 1 \quad \text{such that}$$

$$\sum_{i=1}^{n} \frac{N_i}{\tau_i} \left(1 - \ln \frac{N_i}{p_i \tau_i} \right) > 0 \tag{2.1.10}$$

Then all solutions of (2.1.4) oscillate.

Proof: Let

$y(t) = e^{\lambda t}$, then

$y'(t) = \lambda e^{\lambda t}$, so

$$\lambda - \sum_{i=1}^{n} p_i e^{\lambda \tau_i} = 0 . \tag{2.1.11}$$

write

$$F(\lambda) = \lambda - \sum_{i=1}^{n} p_i e^{\lambda \tau_i} , \tag{2.1.12}$$

or

$$F(\lambda)=\sum_{i=1}^{n}\left(N_i\lambda-p_ie^{\lambda\tau_i}\right). \qquad (2.1.13)$$

let

$$f_i(\lambda)=N_i\lambda-p_ie^{\lambda\tau_i} \quad , \qquad (2.1.14)$$

thus

$$F(\lambda)=\sum_{i=1}^{n}f_i(\lambda), \qquad (2.1.15)$$

$$f_i'(\lambda)=N_i-p_i\tau_ie^{\lambda\tau_i}. \qquad (2.1.16)$$

The extreme value of $f_i(\lambda)$ is at

$$\lambda=\frac{1}{\tau_i}\ln\frac{N_i}{p_i\tau_i} \quad , \qquad (2.1.17)$$

so

$$\max \quad f_i(\lambda)=\frac{N_i}{\tau_i}\ln\frac{N_i}{p_i\tau_i}-p_ie^{\left(\frac{1}{\tau_i}\ln\frac{N_i}{p_i\tau_i}\right)\tau_i} \qquad (2.1.18)$$

$$=\frac{N_i}{\tau_i}\left(\ln\frac{N_i}{p_i\tau_i}-1\right). \qquad (2.1.19)$$

And thus

$$\text{Max } F(\lambda)=\max\sum_{i=1}^{n}f_i(\lambda)\le\sum\frac{N_i}{\tau_i}\left(\ln\frac{N_i}{p_i\tau_i}-1\right)<0, \qquad (2.1.20)$$

so the maximum value of $F(\lambda)$ is negative, which means that the characteristic equation of (2.1.4) has no real roots. Therefore, all solutions of (2.1.4) oscillate.

Theorem (2.1.4): Each of the following conditions is sufficient for all solutions of equation (2.1.4) to be oscillatory.

$$\text{(i)} \ \sum_{i=1}^{n}p_i\tau_i>\frac{1}{e} \qquad (2.1.21)$$

(ii) $$\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}}\left(\sum_{i=1}^{n}\tau_i\right) > \frac{1}{e} \tag{2.1.22}$$

(iii) There exists some j, such that

$$\sum_{i\neq j} p_i\tau_i + p_j\tau_j e > e^{-\left(\frac{\sum_{i\neq j} p_i}{\sum_{i\neq j} p_i + p_j e}\right)} .$$

Proof: The proof of this theorem follows by an application of Theorem (2.1.3), for the following choices of N_i

(i) $N_i = \dfrac{p_i\tau_i}{\sum_{i=1}^{n} p_i\tau_i}$, $\qquad i = 1,2,......,n$

(ii) $N_i = \dfrac{\tau_i}{\sum_{i=1}^{n} \tau_i}$

(iii) $N_i = \dfrac{p_i\tau_i}{\sum_{k\neq j} p_K\tau_K + p_j\tau_j e}$, $\qquad i \neq j$ and $N_j = \dfrac{p_j\tau_j e}{\sum_{k\neq j} p_K\tau_K + p_j\tau_j e}$

Example 2.1.3: The equation

$$y'(t) = \frac{1}{2}y\left(t+\frac{1}{e}\right) + y\left(t+\frac{1}{2e}\right) , \tag{2.1.23}$$

with

$$p_1 = \frac{1}{2} \quad , \quad p_2 = 1 \quad , \quad \tau_1 = \frac{1}{e} \quad \text{and} \quad \tau_2 = \frac{1}{2e},$$

satisfies

$$\sum_{i=1}^{2} p_i\tau_i = \frac{1}{2e} + \frac{1}{2e} = \frac{1}{e}.$$

So (2.1.23) doesn't satisfy condition (i) of Theorem (2.1.4), but

$$\left(\prod_{i=1}^{2} p_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{2}\tau_i\right)=\sqrt{p_1p_2}\left(\tau_1+\tau_2\right)=\frac{3}{2\sqrt{2e}}>\frac{1}{e}.$$

Which satisfies condition (ii) of Theorem (2.1.4). So all solutions of (2.1.23) oscillate.

Theorem 2.1.5: If

$$\tau_{\max}\sum_{i=1}^{n} p_i e^{\left(\frac{\tau_i}{\tau_{\max}}\right)}<1, \tag{2.1.24}$$

where $\tau_{\max}=\max\{\tau_i\}$, $i=1,2,...,n$, then (2.1.4) has a nonoscillatory solution.

Proof: The characteristic equation of (2.1.4) is

$$F(\lambda)=\lambda-\sum_{i=1}^{n} p_i e^{\lambda\tau_i}.$$

Obviously

$$F(0)=-\sum_{i=1}^{n} p_i<0,$$

and

$$F\left(\frac{1}{\tau_{\max}}\right)=\frac{1}{\tau_{\max}}-\sum_{i=1}^{n} p_i e^{\frac{\tau_i}{\tau_{\max}}}.$$

By using (2.1.24), we have

$$F\left(\frac{1}{\tau_{\max}}\right)>0.$$

Hence $F(\lambda)=0$, has a real root $\lambda_0\in\left(0,\frac{1}{\tau_{\max}}\right)$.

This means (2.1.4) has a nonoscillatory solution

$$y(t)=e^{\lambda_0 t}.$$

Example 2.1.4: The equation

$$y'(t)=e^{-\frac{a\pi}{2}}y\left(t+\frac{\pi}{2}\right)+ae^{-2a\pi}y(t+2\pi), \tag{2.1.25}$$

has the oscillatory solution

$y(t)=e^{at}\sin t$, $0\le a\le 0.95$. Equation (2.1.25) satisfies condition (i) of Theorem (2.1.4).

Example 2.1.5: The equation

$$y'(t)=\frac{1}{10e}\left[y(t+1)+y(t+9)\right] . \qquad (2.1.26)$$

This equation does not satisfy conditions (i) and (ii) of Theorem 2.1.4, but does satisfy condition (iii) of the same Theorem. In fact, set $p_1=p_2=\frac{1}{10e}$, $\tau_1=1$, $\tau_2=9$, so

$$\ln\left(p_1\tau_1+p_2\tau_2 e\right)=\ln\left(\frac{1}{10e}+\frac{9}{10}\right),$$

and

$$\frac{-p_1}{p_1+p_2e}=-\frac{1}{1+e} .$$

But

$$\ln\left(\frac{1}{10e}+\frac{9}{10}\right)>-\frac{1}{1+e} .$$

Therefore (2.1.26) satisfies condition (iii) of Theorem (2.1.4), hence all solutions of (2.1.26) oscillate.

We also can connect the phenomena of oscillation of equation (2.1.4) with the roots of its characteristic equation by using the Laplace transform for the functions $y(t)$ and $y(t+\tau)$respectively.

The proof of the following result, will explain this idea.

Theorem 2.1.6: Assume that $p_i\in\Re$, $\tau_i\in\Re^+$, $i=1,2,...,n$, then every solution of the linear advanced functional differential equation (2.1.4) oscillates if and only if the characteristic equation

$$\lambda - \sum_{i=1}^{n} p_i e^{\lambda \tau_i} = 0, \tag{2.1.27}$$

has no real roots

Proof: $\Rightarrow$Assume that equation (2.1.27) has a real root λ_0, then $y(t) = e^{\lambda_0 t} > 0$ is a nonoscillatory solution of equation (2.1.4) (contradiction).

$\Leftarrow$Assume equation (2.1.27) holds, and equation (2.1.4) has an eventually positive solution $y(t)$. By the fact that if $y(t)$ is a solution of

$$y'(t) - \sum_{i=1}^{n} p_i(t) y(t + \tau_i(t)) = 0,$$

then $y(t)$ is exponentially bounded, that is there exist positive constants M and α such that $\|y(t)\| \leq \mathrm{M}e^{\alpha t}$, so by Remark (1.3.2) the Laplace transform

$$Y(s) = \int_0^\infty e^{-st} y(t) dt,$$

exist for $\operatorname{Re} s > \alpha$. Let σ_0 be the abscissa of convergence of $Y(s)$, that is

$$\sigma_0 = \inf\{\sigma \in \Re, Y(\sigma) \ \ exists\}$$

Then for any $i = 1,2,...,n$, the Laplace transform of the shift function $y(t+\tau)$, exists and has abscissa of convergence σ_0.

Also by Lemma 1.3.2

$$\int_0^\infty e^{-st} y'(t) dt = sY(s) - y(0), \ \operatorname{Re} s > \sigma_0,$$

and

$$\int_0^\infty e^{-st} y(t + \tau_i) dt = e^{s\tau_i} Y(s) - e^{s\tau_i} \int_0^{\tau_i} e^{-st} y(t) dt,$$

with $\operatorname{Re} s > \sigma_0$

Therefore by taking the Laplace transform of both sides of (2.1.4), we obtain

$$sY(s)-y(0)-\sum_{i=1}^{n} p_i\left[e^{s\tau_i}Y(s)-e^{s\tau_i}\int_0^{\tau_i} e^{-st}y(t)dt\right]=0, \tag{2.1.28}$$

and so

$$Y(s)\left[s-\sum_{i=1}^{n} p_i e^{s\tau_i}\right]=y(0)-\sum_{i=1}^{n} p_i e^{s\tau_i}\int_0^{\tau_i} e^{-st}y(t)dt. \tag{2.1.29}$$

Set

$$F(s)=s-\sum_{i=1}^{n} p_i e^{s\tau_i}$$

and

$$\phi(s)=y(0)-\sum_{i=1}^{n} p_i e^{s\tau_i}\int_0^{\tau_i} e^{-st}y(t)dt.$$

Equation (2.1.29) becomes

$$Y(s)=\frac{\phi(s)}{F(s)},\ \text{Re s}>\sigma_0 \tag{2.1.30}$$

Clearly, $F(s)$ and $\phi(s)$ are entire functions. $F(s)\neq 0$, for all real s. Since $y(t)>0$ (by hypothesis), then $Y(s)$ is positive. $F(s)$ is negative since $F(-\infty)=-\infty$ and the characteristic equation has no real roots. Claim that

$$\sigma_0=-\infty,$$

otherwise,

$$\sigma_0>-\infty.$$

And by Theorem (1.3.1), the point $s=\sigma_0$ must be a singularity of the quotient $\dfrac{\phi(s)}{F(s)}$.

But this quotient has no singularity on the real axis, since $F(s)$ is an entire function, and has no real roots. Thus $\sigma_0=-\infty$, and so

$$Y(s)=\frac{\phi(s)}{F(s)}\text{ , for all } s\in R.$$

As $s\to-\infty$, through real values, then

$Y(s)=\dfrac{\phi(s)}{F(s)}$, leads to a contradiction because $Y(s)$ is positive and $F(s)$ is negative, while

$$\lim_{t\to\infty}\phi(s)=y(0),$$

which is eventually positive. The proof is complete.

Theorem 2.1.7: Assume that $p_i>0$ and $\tau_i\geq 0$, $i=1,2,...,n$.

The following statements are equivalent:

a) $y'(t)-\sum_{i=0}^{n}p_i y(t+\tau_i)=0$, (2.1.31)

has a positive solution

b) The characteristic equation

$$\lambda-\sum_{i=1}^{n}p_i e^{\lambda\tau_i}=0, \tag{2.1.32}$$

has a real root

c) The advanced differential inequality

$$y'(t)-\sum_{i=1}^{n}p_i y(t+\tau_i)\geq 0, \tag{2.1.33}$$

has a positive solution

Proof: See [5].

2.2 Equations with variable coefficients and constant advanced argument.

In this section, some sufficient conditions are established for the oscillation of all solutions of the advanced differential equation

$$y'(t) - p(t)y(t+\tau) = 0\,,\ t \ge t_0 \tag{2.2.1}$$

Where the coefficient $p(t) \in C\left[(t_0, \infty), \Re^+\right]$, and τ is a positive constant.

The previous works for the studies of the oscillation of (2.2.1) are done by Ladas [5] and Stavroulakis [11]. They proved that all solutions of (2.2.1) oscillate if

$$p(t) \ge 0\,,\ \liminf_{t\to\infty} \int_t^{t+\tau} p(s)ds > \frac{1}{e}\,. \tag{2.2.2}$$

Recently, Li and Zhu [9] improved the above result to the following form.

Theorem 2.2.1 [9]: Suppose that there exist a $t_1 > t_0 + \tau$, and a positive integer K, such that

$$p_K(t) \ge \frac{1}{e^K}\,,\ q_K(t) \ge \frac{1}{e^K}\,,\ t \ge t_1 + K\tau\,, \tag{2.2.3}$$

$$\int_{t_1+kt}^{\infty} p(t)\left[\exp\left(e^{K-1}p_K(t) - \frac{1}{e}\right) - 1\right]dt = \infty\,. \tag{2.2.4}$$

Then every solution of (2.2.1) oscillates. Here $p(t) \in c[(t_0, \infty), [0, \infty)]$ and the sequences $\{p_n(t)\}$, $\{q_n(t)\}$ of functions are defined as follows:

$$p_1(t) = \int_t^{t+\tau} p(s)ds$$

$$p_n(t) = \int_t^{t+\tau} p(s)p_{n-1}(s)ds \quad n \ge 2\,,\ \ t \ge t_0 \tag{2.2.5}$$

$$q_1(t)=\int_{t-\tau}^{t} p(s)ds \quad , \qquad t \ge t_0+\tau$$

$$q_n(t)=\int_{t-\tau}^{t} p(s)q_{n-1}(s)ds\,, \qquad n \ge 2\,, \quad t \ge t_0+n\tau \tag{2.2.6}$$

Proof: see [9].

Remark 2.2.1: If $p(t)\equiv p \in (0,\infty)$, then (2.2.3) reduces to $p\tau \ge \frac{1}{e}$, which together with (2.2.4) indicates $p\tau > \frac{1}{e}$, which is necessary and sufficient condition for (2.2.1) to have only oscillatory solutions.

Corollary 2.2.1: If there exists a positive integer K such that

$$\liminf_{t\to\infty} p_K(t) > \frac{1}{e^K}\,, \quad \liminf_{t\to\infty} q_K(t) > \frac{1}{e^K}\,,$$

where $p_K(t)$, $q_K(t)$ are defined by (2.2.5) and (2.2.6) respectively, then every solution of equation (2.2.1) oscillates.

Corollary 2.2.2: Suppose that there exist a $t_1 > t_0+\tau$ and a positive integer K such that (2.2.3) holds and

$$\int_{t_1+K\tau}^{\infty} p(t)\left(e^{K-1}p_k(t)-\frac{1}{e}\right)dt=\infty\,, \tag{2.2.7}$$

where $p_K(t)$ is defined by (2.2.5). Then every solution of equation (2.2.1) oscillates.

Proof: Since $e^x - 1 \ge x$ for all $x \ge 0$, so (2.2.7) implies (2.2.4). Accordingly, Theorem (2.2.1) indicates the truth of the corollary.

Example 2.2.1 [9]: Consider the following advanced differential equation

$$y'(t)-\frac{1}{2e}(1+\sin t)y(t+\pi)=0\,,\ t \ge 0 \tag{2.2.8}$$

Compared with (2.2.1), one has $p(t)=\frac{1}{2e}(1+\sin t)$, $\tau=\pi$. Clearly,

$$\liminf_{t\to\infty}\int_t^{t+\tau}\frac{1}{2e}(1+\sin s)ds=\frac{1}{2e}(\pi-2)<\frac{1}{e},$$

which implies that condition (2.2.2) does not hold. But

$$p_1(t)=\int_t^{t+\tau}\frac{1}{2e}(1+\sin s)ds=\frac{1}{2e}(\pi+2\cos t)$$

$$p_2(t)=\int_t^{t+\tau}p(s)p_1(s)ds=\int_t^{t+\tau}\frac{1+\sin s}{4e^2}(\pi+2\cos s)ds=\frac{\pi^2+2\pi\cos t-4\sin t}{4e^2}$$

$$p_3(t)=\int_t^{t+\tau}p(s)p_2(s)ds=\int_t^{t+\tau}\frac{1+\sin s}{8e^3}(1+\sin s)(\pi^2+2\cos s-4\sin s)ds$$

$$=\frac{1}{8e^3}\left(\pi^3-2\pi+(2\pi^2-8)\cos t-4\pi\sin t\right)$$

$$p_4(t)=\int_t^{t+\tau}p(s)p_3(s)ds=\int_t^{t+\tau}\frac{1+\sin s}{16e^4}\left(\pi^3-2\pi+(2\pi^2-8)\cos s-4\pi\sin s\right)ds$$

$$=\frac{1}{16e^4}\left[\pi^4-4\pi^2+2(\pi^3-6\pi)\cos t-4(\pi^2-4)\sin t\right]$$

$$\liminf_{t\to\infty}p_4(t)=\frac{1}{16e^4}\left[\pi^4-4\pi^2-2\sqrt{(\pi^3-6\pi)^2+4(\pi^2-4)^2}\right]>\frac{22}{16e^4},$$

and

$$q_1(t)=\int_{t-\tau}^{t}\frac{1}{2e}(1+\sin s)ds=\frac{1}{2e}(\pi-2\cos t)$$

$$q_2(t)=\int_{t-\tau}^{t}p(s)q_1(s)ds=\int_{t-\tau}^{t}\frac{1+\sin s}{4e^2}(\pi-2\cos s)ds=\frac{1}{4e^2}\left(\pi^2-2\pi\cos t-4\sin t\right)$$

$$q_3(t)=\int_{t-\tau}^{t}p(s)q_2(s)ds=\int_{t-\tau}^{t}\frac{1+\sin s}{8e^3}\left(\pi^2-2\cos s-4\sin s\right)ds$$

$$= \frac{1}{8e^3}\left(\pi^3 - 2\pi + \left(2\pi^2 - 8\right)\cos t - 4\pi \sin t\right)$$

$$q_4(t) = \int_{t-\tau}^{t} p(s)q_3(s)ds = \int_{t-\tau}^{t} \frac{1+\sin s}{16e^4}\left(\pi^3 - 2\pi - \left(2\pi^2 - 8\right)\cos s - 4\sin s\right)ds$$

$$= \frac{1}{16e^4}\left[\pi^4 - 4\pi^2 + 2\left(\pi^3 - 6\pi\right)\cos t - 4\left(\pi^2 - 4\right)\sin t\right]$$

$$\liminf_{t\to\infty} q_4(t) = \frac{1}{16e^4}\left[\pi^4 - 4\pi^2 - 2\sqrt{\left(\pi^3 - 6\pi\right)^2 + 4\left(\pi^2 - 4\right)^2}\right] > \frac{22}{16e^4}.$$

Hence by corollary (2.2.1) every solution of (2.2.8) oscillates.

Now let us generalize the result above to the differential equation with several advanced arguments.

$$y'(t) - \sum_{i=1}^{n} p_i(t)y(t+\tau_i) = 0, \quad t \ge t_0 \tag{2.2.9}$$

where $p(t)$, $p_i(t) \in C[[t_0,\infty),[0,\infty)]$, τ_i are positive constants, $i = 1,2,...,n$.

First, define the sequence $\{p_i^m(t)\}$ and $\{q_i^m(t)\}$ of functions for some $i = 1,2,...,n$ as follows

$$p_i^{(1)}(t) = \int_{t}^{t+\tau_i} p_i(s)ds, \quad t \ge t_0$$

$$p_i^{(2)}(t) = \int_{t}^{t+\tau_i} p_i(s)p_i^{(1)}(s)ds, \; t \ge t_0$$

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$$p_i^{(m)}(t) = \int_{t}^{t+\tau_i} p_i(s)p_i^{(m-1)}(s)ds, \; m \ge 2, \; t \ge t_0 \tag{2.2.10}$$

and

$$q_i^{(1)}(t) = \int_{t-\tau_i}^{t} p_i(s)ds, \quad t \ge t_0 + \tau_i$$

$$q_i^{(2)}(t) = \int_{t-\tau_i}^{t} p_i(s)q_i^{(1)}(s)ds, \quad t \ge t_0 + 2\tau_i$$

.

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$$q_i^{(m)}(t) = \int_{t-\tau_i}^{t} p_i(s)q_i^{(m-1)}(s)ds, \quad m \ge 2, \quad t \ge t_0 + m\tau_i \tag{2.2.11}$$

X. Li and Deming Zhu [9] used the above sequences to introduce oscillation criteria for equation (2.2.9), which appears in the following result.

Theorem 2.2.2 [9]: Suppose that for some $i \in \{1,2,...,n\}$ there exist a $t_1 > t_0 + \tau_i$ and a positive integer m such that

$$p_i^{(m)}(t) \ge \frac{1}{e^m}, \quad q_i^{(m)}(t) \ge \frac{1}{e^m}, \quad t \ge t_1 + m\tau_i \tag{2.2.12}$$

and

$$\int_{t_1+m\tau_i}^{\infty} p_i(t)\left[\exp\left(e^{m-1}p_i^{(m)}(t) - \frac{1}{e}\right) - 1\right]dt = \infty \tag{2.2.13}$$

Where $p_i^{(m)}(t)$ and $q_i^{(m)}(t)$ are defined by (2.2.10) and (2.2.11) respectively. Then every solution of equation (2.2.9) oscillates.

Proof: see [9].

Corollary 2.2.3: If for some $i \in \{1,2,...,n\}$ there exist a positive integer m such that

$$\liminf_{t\to\infty} p_i^{(m)}(t) > \frac{1}{e^m}, \quad \liminf_{t\to\infty} q_i^{(m)}(t) > \frac{1}{e^m} \tag{2.2.14}$$

Where $p_i^{(m)}(t)$ and $q_i^{(m)}(t)$ are defined by (2.2.10) and (2.2.11), respectively, then every solution of (2.2.9) is oscillatory.

Proof: Condition (2.2.14) holding implies that so do conditions (2.2.12) and (2.2.13). Thus, by Theorem (2.2.2), the conclusion is true and the proof is finished.

Corollary 2.2.4: If for some $i \in \{1,2,...,n\}$ there exist a $t_1 > t_0 + \tau_i$ and a positive integer K such that (2.2.12) holds and

$$\int_{t_1+K\tau_i}^{\infty} p_i(t)\left(e^{k-1}p_i^{(k)}(t) - \frac{1}{e}\right)dt = \infty \ , \tag{2.2.15}$$

where $p_i^{(K)}$ is defined by (2.2.10), then every solution of equation (2.2.9) oscillates.

Proof: According to $e^x - 1 \geq x$ for all $x \geq 0$, and by the condition (2.2.15) implies that (2.2.13) will be satisfied. Therefore, Theorem (2.2.2) shows that the claim is true.

Example 2.2.2 [9]: Consider the advanced differential equation

$$y'(t) - \frac{1}{2e}(1+\cos t)y(t+\pi) - \frac{1}{2e}(1+\sin t)y\left(t+\frac{\pi}{2}\right) = 0 \tag{2.2.16}$$

Rewriting this equation in form of equation (2.2.9), then

$$p_1(t) = \frac{1}{2e}(1+\cos t), \ \ p_2(t) = \frac{1}{2e}(1+\sin t)$$

$$\tau_1 = \pi, \ \ \tau_2 = \frac{\pi}{2}$$

For this equation the conclusion in Laddas and Stavroulakis are not suitable since the condition (2.2.2) does not satisfied:

$$\liminf_{t\to\infty} \int_t^{t+\tau_1} p_1(s)ds = \liminf_{t\to\infty} \int_t^{t+\pi} \frac{1}{2e}(1+\cos s)ds = \frac{1}{2e}(\pi - 2) < \frac{1}{e},$$

and

$$\liminf_{t\to\infty} \int_t^{t+\tau_2} p_2(s)ds = \liminf_{t\to\infty} \int_t^{t+\frac{\pi}{2}} \frac{1}{2e}(1+\sin s)ds = \frac{\frac{\pi}{2} - \sqrt{2}}{2e} < \frac{1}{e}.$$

While

$$p_1^{(1)}(t)=\int_t^{t+\tau_1} p_1(s)ds=\int_t^{t+\pi}\frac{1}{2e}(1+\cos s)ds=\frac{1}{2e}(\pi-2\sin t)$$

$$p_1^{(2)}(t)=\int_t^{t+\tau_1} p_1(s)p_1^{(1)}(s)ds=\int_t^{t+\pi}\frac{1+\cos s}{4e^2}(\pi-2\sin s)ds=\frac{\pi^2-2\pi\sin t-4\cos t}{4e^2}$$

$$p_1^{(3)}(t)=\int_t^{t+\tau_1} p_1(s)p_1^{(2)}(s)ds=\int_t^{t+\pi}\frac{1+\cos s}{8e^3}\left(\pi^2-2\pi\sin s-4\cos s\right)ds$$

$$=\frac{1}{8e^3}\left(\pi^3-2\pi-\left(2\pi^2-8\right)\sin t-4\pi\cos t\right)$$

$$p_1^{(4)}(t)=\int_t^{t+\tau_1} p_1(s)p_1^{(3)}(s)ds=\int_t^{t+\pi}\frac{1+\cos s}{16e^4}\left(\pi^3-2\pi-\left(2\pi^2-8\right)\sin s-4\pi\cos s\right)ds$$

$$=\frac{1}{16e^4}\left[\pi^4-4\pi^2-2\left(\pi^3-6\pi\right)\sin t-4\left(\pi^2-4\right)\cos t\right]$$

$$\liminf_{t\to\infty} p_1^{(4)}(t)=\frac{1}{16e^4}\left[\pi^4-4\pi^2-2\sqrt{\left(\pi^3-6\pi\right)^2+4\left(\pi^2-4\right)^2}\right]>\frac{22}{16e^4},$$

and

$$q_1^{(1)}(t)=\int_{t-\tau_1}^{t} p_1(s)ds=\int_{t-\pi}^{t}\frac{1}{2e}(1+\cos s)ds=\frac{1}{2e}(\pi+2\sin t)$$

$$q_1^{2}(t)=\int_{t-\tau_1}^{t} p_1(s)q_1^{(1)}(s)ds=\int_{t-\pi}^{t}\frac{1+\cos s}{4e^2}(\pi+2\sin s)ds=\frac{\pi^2+2\pi\sin t-4\cos t}{4e^2}$$

$$q_1^{(3)}(t)=\int_{t-\tau_1}^{t} p_1(s)q_1^{(2)}(s)ds=\int_{t-\pi}^{t}\frac{1+\cos s}{8e^3}\left(\pi^2+2\pi\sin s-4\cos s\right)ds$$

$$=\frac{1}{8e^3}\left(\pi^3-2\pi+\left(2\pi^2-8\right)\sin t-4\pi\cos t\right)$$

$$q_1^{(4)}(t) = \int_{t-\tau_1}^{t} p_1(s) q_1^{(3)}(s) ds = \int_{t-\pi}^{t} \frac{1+\cos s}{16e^4} \left(\pi^3 - 2\pi + \left(2\pi^2 - 8\right)\sin s - 4\cos s\right) ds$$

$$= \frac{1}{16e^4}\left[\pi^4 - 4\pi^2 + 2\left(\pi^3 - 6\pi\right)\sin t - 4\left(\pi^2 - 4\right)\cos t\right]$$

$$\liminf_{t\to\infty} q_1^{(4)}(t) = \frac{1}{16e^4}\left[\pi^4 - 4\pi^2 - 2\sqrt{(\pi^3 - 6\pi)^2 + 4(\pi^2 - 4)^2}\right] > \frac{22}{16e^4}.$$

It follows from corollary (2.2.3) that every solution of equation (2.2.16) is oscillatory.

Since equation (2.2.1) is a linear differential equation, if it has eventually positive solution, then it also has eventually negative solution, that is, it has nonoscillatory solutions. Thus, in order to study the nonoscillation of (2.2.1), it suffices to consider the existence of eventually positive solution of (2.2.1).

All previous work of Ladas, Stavroulakis [11] and Li and Zhu [9], are under the assumption that the coefficient $p(t)$ has constant sign, that is, $p(t) \in C[t_0, \infty), \Re^+]$. These investigations, in general make use of the observation that if $y(t)$ is an eventually positive solution of (2.2.1), then

$$y'(t) - p(t)y(t+\tau) \ge 0,$$

for all large t, so that $y(t)$ is eventually nondecreasing. However, when the coefficient $p(t)$ is oscillatory, that is, $p(t)$ takes positive and negative values, the monotonicity does not hold any longer. All known results cannot be applied to the case where $p(t)$ is oscillatory. The following result gives necessary conditions for oscillation of equation (2.2.1) when $p(t)$ is an oscillatory function.

Theorem 2.2.3 [10]: Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequence in $[t_0, \infty)$, satisfying

$$a_n + 2\tau \le b_n \le a_{n+1} - 2\tau \tag{2.2.17}$$

Assume that

$$p(t) \ge 0\text{, for } t \in \cup_{n=1}^{\infty}[a_n, b_n] \tag{2.2.18}$$

Define function $P(t)$ as follows

$$P(t) = \begin{cases} p(t), t \in \cup_{n=1}^{\infty}[a_n, b_n - \tau] \\ 0, otherwise \end{cases} \tag{2.2.19}$$

If

$$\int_{t_0}^{\infty} P(t) \ln\left[e \int_{t-\tau}^{t} P(s) ds + 1 - sign\left(\int_{t-\tau}^{t} P(s) ds \right) \right] dt = \infty \tag{2.2.20}$$

then every solution of (2.2.1) is oscillatory.

Proof: see [10].

Remark 2.2.1: The function sign (.) is the signum function, that is:

$$sign(r) = \begin{cases} -1, & r < 0 \\ 0, & r = 0 \\ 1, & r > 0 \end{cases}$$

Example 2.2.1 [10]: As an application of Theorem (2.2.3), we consider the oscillation of the following equation

$$y'(t) - p(t) y(t+1) = 0, \qquad t \ge 0, \tag{2.2.21}$$

where $\tau = 1$ and the function $p(t)$ is 6-periodic one with

$$p(t)=\begin{cases}-t \quad , \quad 0\le t\le 1 \\ t-2, \quad 1\le t\le 4 \\ 6-t, \quad 4\le t\le 6\end{cases} \tag{2.2.22}$$

Obviously

$$\liminf_{t\to\infty}\int_t^{t+\tau} p(s)ds=\frac{-1}{2}<0$$

Therefore, the result of Ladas and Stavroulakis (equation (2.2.2)) cannot be applied to (2.2.21). But if we denote.

$$a_n=2+6(n-1) \quad , \; b_n=6n \quad , \; n\ge 1$$

Then clearly $a_n, b_n \in [0,\infty)$

$$a_n+2\tau\le b_n\le a_{n+1}-2\tau \quad , \quad n=1,2,... \tag{2.2.23}$$

and $p(t)\ge 0$ for $t\in\cup_{n=1}^{\infty}[a_n,b_n]$. Furthermore, if we set

$$P(t)=\begin{cases}p(t), t\in\cup_{n=1}^{\infty}[a_n,b_n-\tau] \\ 0, otherwise\end{cases} \tag{2.2.24}$$

Then we have.

$$\int_{a_n}^{b_n-\tau} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds+1-sign\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt$$

$$=\int_2^5 P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds+1-sign\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt$$

$$=\int_2^4 P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds+1-sign\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt$$
$$+\int_4^5 P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds+1-sign\left(\int_{t-\tau}^{t} P(s)ds\right)\right]dt$$

$$= \int_{2}^{4} (t-2)\ln\left[e\int_{t-1}^{2} P(s)ds + e\int_{2}^{t} P(s)ds + 1 - sign\left(\int_{t-\tau}^{t} P(s)ds \right) \right] dt$$

$$+ \int_{4}^{5} (6-t)\ln\left[e\left(\int_{t-1}^{2} P(s)ds + \int_{2}^{4} P(s)ds + \int_{4}^{t} P(s)ds \right) + 1 - sign\left(\int_{t-\tau}^{t} P(s)ds \right) \right] dt$$

$$= \left[\int_{2}^{t} (s-2)ds.\ln e\int_{2}^{t} (s-2)ds - \int_{2}^{t} (s-2)ds \right]_{2}^{4}$$

$$+ \left[\left(2 + \int_{4}^{t} (6-s)ds \right).\ln\left(e\left(2 + \int_{4}^{t} (6-s)ds \right) \right) - \left(2 + \int_{4}^{t} (6-s)ds \right) \right]_{4}^{5}$$

$$= \int_{2}^{4} (t-2)\ln\left(e\int_{2}^{t} (s-2)ds \right) dt + \int_{4}^{5} (6-t)\ln\left(e\left(2 + \int_{4}^{t} (6-s)ds \right) \right) dt$$

$$= 2\ln 2 + \left(\frac{7}{2}\ln\frac{7}{2} - 2\ln 2 \right) = \frac{7}{2}\ln\frac{7}{2} > 0$$

which means that,

$$\int_{a_1}^{\infty} P(t)\ln\left[e\int_{t-\tau}^{t} P(s)ds + 1 - sign\left(\int_{t-\tau}^{t} P(s)ds \right) \right] dt = \infty$$

So by Theorem (2.2.2), every solution of (2.2.21) is oscillatory.

2.3 Equations with variable coefficients and variable advanced argument

In this section we will study the behavior of oscillatory solutions of the advanced differential equation (2-A)

$$y'(t) = p(t)y(t) + \sum_{i=1}^{n} p_i(t)y(t + \tau_i(t)), \tag{2.3.1}$$

where

$p(t) \geq 0$, $p_i(t) \geq 0$, and $\tau_i(t) > 0$, are continuous, $i = 1,2,...,n$.

Before studying the general form (2.3.1), let us take special cases:

Let $p(t) = 0$, $n = 1$, then (2.3.1) becomes.

$$y'(t) - p(t)y(t+\tau(t)) = 0 \quad , \qquad \tau(t) > 0 \tag{2.3.2}$$

First, we will introduce the following result for the advanced inequality

$$y'(t)\operatorname{sgn} y(t) - p(t)|y(\tau(t))| \geq 0 , \tag{2.3.3}$$

where

$$p(t), \tau(t) \in C[\Re^+, \Re^+], \text{ and } \tau(t) > t \tag{2.3.4}$$

Theorem 2.3.1: If (2.3.4) holds and

$$\varliminf_{t\to\infty} \int_t^{\tau(t)} p(s)ds > \frac{1}{e} , \tag{2.3.5}$$

then all solutions of (2.3.3) are oscillatory.

Proof: Assume that there exists an eventually positive solution $y(t)$ of (2.3.3). From (2.3.5), there exists a $t_2 \geq t_1$ such that

$$\int_t^{\tau(t)} p(s)ds \geq c > e^{-1} , \quad t \geq t_2$$

and $y(t) > 0$, $y'(t) > 0$ for $t \geq t_2$. Hence,

$$y'(t) \geq p(t)y(\tau(t)) \geq p(t)y(t), \qquad t \geq t_2 .$$

Dividing by $y(t)$ and integrating from t to $\tau(t)$ we obtain:

$$\ln \frac{y(\tau(t))}{y(t)} \geq \int_t^{\tau(t)} p(s)ds , \qquad t \geq t_2$$

which is equivalent to

$$\frac{y(\tau(t))}{y(t)} \geq \exp\left(\int_t^{\tau(t)} p(s)ds \right) \geq e^c \geq ec ,$$

for $t \geq t_2$. Repeating the above procedure, there exists a sequence t_k such that.

$$\frac{y(\tau(t))}{y(t)} \geq (ec)^k , \qquad t \geq t_k$$

this implies that

$$\lim_{t\to\infty}\frac{y(\tau(t))}{y(t)}=+\infty\,.$$

On the other hand, using the argument in the proof of Theorem (1.2.1), we can get

$$\frac{y(\tau(t))}{y(t)}\le\left(\frac{2}{c}\right)^2,$$

for large t, this leads to a contradiction. Thus all solutions of (2.3.3) are oscillatory.

The following examples illustrate the sharpness of conditions of Theorem 2.3.1.

Example 2.3.1: Consider the equation.

$$y'(t)-\frac{2}{e(\ln 2)\ \ t}y(2t)=0\ ,\quad t\ge t_0>0\,. \tag{2.3.6}$$

Here

$$p(t)=\frac{2}{e(\ln 2)\ \ t}>0\quad,\quad \tau(t)=2t\text{ , and therefore}$$

$$\varliminf_{t\to\infty}\int_t^{\tau(t)}p(s)ds=\varliminf_{t\to\infty}\int_t^{2t}\frac{2}{e(\ln 2)}\frac{ds}{s}=\frac{2}{e}>\frac{1}{e}\,.$$

So all solutions of (2.3.6) are oscillatory.

Example 2.3.2: Consider the equation

$$y'(t)-\frac{1}{e(\ln 2)\ \ t}y(2t)=0, \tag{2.3.7}$$

where

$$p(t)=\frac{1}{e(\ln 2)\ \ t}>0,\quad \tau(t)=2t\,.$$

Then

$$\lim_{t\to\infty}\int_t^{\tau(t)} p(s)ds = \lim_{t\to\infty}\int_t^{2t}\frac{1}{e(\ln 2)}\frac{ds}{s} = \frac{1}{e}$$

Consequently, (2.3.7) does not satisfy the conditions of Theorem (2.3.1), and therefore (2.3.7) has the non-oscillatory solution

$$y(t) = t^{\alpha} \quad , \quad \alpha = \frac{1}{\ln 2} \ .$$

In the following result, we establish the asymptotic behavior of solutions of (2.3.2).

Theorem 2.3.2: Assume that $p(t) > 0$, and

$$\overline{\lim_{t\to\infty}}\int_t^{t+\tau} p(s)ds < 1 . \tag{2.3.8}$$

Then the amplitude of every oscillatory solution of (2.3.2) tends to ∞ as $t \to \infty$.

Proof: Let $y(t)$ be an oscillatory solution of (2.3.2).

Then there exists a sequence t_n, $n = 1,2,...$ of zeros of $y(t)$ with the property that

$t_{n+1} - t_n \geq \tau$ and $y(t) \neq 0$ on (t_n, t_{n+1}) for $n = 1,2,...$

Setting $S_n = \max_{t_n \leq t \leq t_{n+1}} |y(t)|$, $n = 1,2,...$, we see that

$$S_n = |y(\zeta_n)|, \text{ for some } \quad \zeta_n \in (t_n, t_{n+1}) \quad \text{and} \quad y'(\zeta_n) = 0$$

Hence

$$y(\zeta_n + \tau) = 0 .$$

Let

$$\tau_n = \min\{t_{n+1}, \zeta_n + \tau\}, \qquad n = 1,2,... \ .$$

Integrating (2.3.2) from ζ_n to τ_n we get,

$$-y(\zeta_n) = \int_{\zeta_n}^{\tau_n} p(s)y(s+\tau)ds .$$

Hence

$$|y(\zeta_n)| \le \int_{\zeta_n}^{\tau_n} p(s)|y(s+\tau)|ds \le \left(\max_{[t_n,t_{n+1}]}|y(t)|\right)\int_{\zeta_n}^{\zeta_n+\tau} p(s)ds\,.$$

Which yields,

$$S_n \le \max\{s_n, s_{n+1}\}\int_{\zeta_n}^{\zeta_n+\tau} p(s)ds\,. \tag{2.3.9}$$

From (2.3.8), we have

$$\int_{\zeta_n}^{\zeta_n+\tau} p(s)ds \le \mu < 1\,,$$

for sufficiently large n, say $n \ge N$. From (2.3.9), $s_n > s_{n+1}$ is impossible. Therefore $s_n \le s_{n+1}\mu$.

This implies that.

$$S_{n+1} \ge \frac{1}{\mu}S_n \ge \left(\frac{1}{\mu}\right)^2 S_{n-1} \ge \ldots.. \ge \left(\frac{1}{\mu}\right)^{n-N+1} S_N\ , \quad n \ge N\,.$$

Letting $n \to \infty$, we get

$\lim_{s\to\infty} S_n = \infty$, and the proof is complete.

Remark 2.3.1: Condition (2.3.8) guarantees that the amplitude of every oscillatory solution tends to infinity. But it is possible that the equation has a bounded non-oscillatory solution even though condition (2.3.8) holds.

The following example explains Remark 2.3.1.

Example 2.3.3: The equation

$$y'(t) = \frac{N}{e^{Nt} - e^{-N}}\, y(t+1), \tag{2.3.10}$$

satisfies condition (2.3.8), but it has the bounded non-oscillatory solution

$$y(t)=A\left(1-e^{-Nt}\right),$$

where N is a positive integer and A is any constant.

Now we introduce the following result for the advanced equation

$$y'(t)=p(t)y(\tau(t)), \qquad (2.3.11)$$

where

$$p(t)\geq 0 \ , \ \tau(t)>t \quad \text{are continuous.}$$

Theorem 2.3.3: If $\overline{\lim_{t\to\infty}}\int_t^{\tau(t)} p(s)ds>1$, (2.3.12)

and $\tau(t)$ is nondecreasing with $\lim_{t\to\infty}\tau(t)=\infty$,then every solution of (2.3.11) is oscillatory.

Proof: Without loss of generality, let $y(t)>0$ be a nonoscillatory solution of (2.3.11) such that

$y(\tau(t))>0$, $t>t_1$. Integrating (2.3.11) from t to $\tau(t)$, we have

$$y(\tau(t))-y(t)-\int_t^{\tau(t)} p(s)y(\tau(s))ds=0,$$

or equivalently

$$-y(t)\geq y(\tau(t))\left[\int_t^{\tau(t)} p(s)ds-1\right]. \qquad (2.3.13)$$

From (2.3.13) and $\int_t^{\tau(t)} p(s)ds\geq 1$, when t is sufficiently large, therefore (2.3.13) is a contradiction. The proof is complete.

We can obtain the following results by utilizing the ideas of section 1.2. We shall merely state the following results and omit the proof.

Theorem 2.3.4: If $$\overline{\lim_{t\to\infty}}\int_t^{\tau(t)} p(s)ds < \frac{1}{e},$$

then (2.3.11) has a non-oscillatory solution.

We shall now try to extend the above results to the case of a more complicated advanced argument. Consider

$$y'(t) = p(t)y(\Delta(t, y(t))) \ , \tag{2.3.14}$$

where

$p \in C[\Re^+, \Re^+]$, $\Delta \in C[\Re^+ \times \Re, \Re]$, Δ is nondecreasing in t for fixed v and $\Delta(t,v) > t$ and $\Delta(t,v_1) \le \Delta(t,v_2)$ for $|v_2| \ge |v_1|$, $v_1 v_2 \ge 0$.

Corollary 2.3.1: In addition to the above conditions if

$$\underline{\lim_{t\to\infty}}\int_t^{\Delta(t,\eta)} p(s)ds > e^{-1} \quad \text{, for any } \eta \tag{2.3.15}$$

then all solutions of (2.3.14) oscillate.

Proof: Without loss of generality, assume that there exists a positive solution $y(t) > 0$ for $t \ge t_1 \ge t_0$, then $y'(t) \ge 0$ and hence

$$y(t) \ge y(t_1) = \eta\ , \quad \Delta(t, y(t)) \ge \Delta(t,\eta).$$

Thus

$$y'(t) \ge p(t)y(\Delta(t,\eta)),$$

which contradicts Theorem 2.3.1

Example 2.3.4: Consider the equation

$$y'(t) = \sqrt{t}\, y(t + y^2(t)) \ , \tag{2.3.16}$$

where $\Delta(t,v) = t + v^2$, $p(t) = \sqrt{t}$, (2.3.16) satisfies the conditions of corollary (2.3.1). Therefore all solutions of (2.3.16) oscillate.

Let us present another form of advanced differential equation.

Consider the advanced differential equation

$$y'(t) + p(t)y(t + \tau(t)) = 0, \tag{2.3.17}$$

where $p(t) > 0$ and $\tau(t) > 0$ are continuous.

Theorem 2.3.5: Assume that

$$\lim_{t\to\infty} \int_t^{t+\tau(t)} p(s)ds, \tag{2.3.18}$$

exists, then (2.3.17) has a bounded nonoscillatory solution.

Proof: see [8]

Example 2.3.5 [8]: The equation

$$y'(t) + \frac{3\pi}{2} y(t+1) = 0 , \tag{2.3.19}$$

satisfies the conditions of Theorem (2.3.5), so (2.3.19) has a bounded solution, which is

$$y(t) = Ae^{st},$$

where A is any constant, and s is a root of the equation $s + \frac{3\pi}{2}e^{s} = 0$ $(s = -1.2931)$.

Also (2.3.19) has the oscillatory solution

$$y(t) = \cos\frac{3\pi}{2}t + \sin\frac{3\pi}{2}t.$$

Back to equation (2.3.1) with $p(t) = 0$, then we have the advanced equation with several deviating arguments

$$y'(t) = \sum_{i=1}^{n} p_i(t)y(t + \tau_i(t)), \tag{2.3.20}$$

where $p_i(t) \geq 0$ and $\tau_i(t) \geq 0$ are continues, $i = 1,2,...,n$.

Theorem 2.3.6: If for some $i = 1,2,...,n$, either

$$\varliminf_{t\to\infty} \int_t^{t+\tau_i(t)} p_i(s)ds > \frac{1}{e},$$

or

$$\varliminf_{t\to\infty} \int_t^{t+\tau_{\min}(t)} \sum_{i=1}^{n} p_i(s)ds > \frac{1}{e},$$

then all solutions of (2.3.20) oscillate, where

$$\tau_{\min}(t) = \min\{\tau_1(t), \tau_2(t), ..., \tau_n(t)\}$$

Proof: Without loss of generality, assume that there exists a positive nonoscillatory solution $y(t) > 0$, for $t \ge t_0$. This implies that there exists a t_1 such that $y(t + \tau_i(t)) > 0$ for $t \ge t_1$, $i \in I_n$. From (2.3.20) we have

$$y'(t) - \sum_{i=1}^{n} p_i(t) y(t + \tau_i(t)) \ge 0, \tag{2.3.21}$$

and

$$y'(t) - y(t + \tau_{\min}(t)) \sum_{i=1}^{n} p_i(t) \ge 0 \ . \tag{2.3.22}$$

Comparing (2.3.21) and (2.3.22), we obtain a contradiction to Theorem (1.2.1) and the proof is complete.

Also Kordonis and Philos [7] gave a nice result for the advanced differential equation

$$y'(t) - \sum_{j\in J} p_j(t) y(t + \tau_j(t)) = 0, \tag{2.3.23}$$

where J is an (nonempty) initial segment of natural numbers and for $j \in J$, p_j and τ_j are nonnegative continuous real-valued functions on the interval $[0, \infty)$. The set J may finite or infinite.

The result of Kordonis and Philos is the following Theorem.

Now we are able to discuss oscillatory and non-oscillatory behavior of solutions of equation (2-A) which is:

$$y'(t) = p(t)y(t) + \sum_{i=1}^{n} p_i(t)y(t+\tau_i(t)), \qquad (2.3.24)$$

where

$p(t) \geq 0$, $p_i(t) \geq 0$, and $\tau_i(t) > 0$ are continuous, $i = 1,2,...,n$

The discussion will be done by transforming (2.3.24) to the form of that of equation (2.3.20) with satisfaction of the conditions of Theorem (2.3.6), on the resulting equation after transformation. To do that, let

$$y(t) = \exp\left[\int_{t_1}^{t} p(u)du\right].z(t) \quad , \quad t \geq t_1 . \qquad (2.3.25)$$

So

$$y'(t) = \exp\left[\int_{t_1}^{t} p(u)du\right].z'(t) + p(t)z(t).\exp\left[\int_{t_1}^{t} p(u)du\right],$$

or

$$y'(t) = p(t)y(t) + \exp\left[\int_{t_1}^{t} p(u)du\right].z'(t),$$

thus

$$\exp\left[\int_{t_1}^{t} p(u)du\right].z'(t) = \sum_{i=1}^{n} p_i(t)y(t+\tau_i(t)),$$

or

$$z'(t) = \exp\left[-\int_{t_1}^{t} p(u)du\right]\sum_{i=1}^{n} p_i(t)y(t+\tau_i(t))$$

$$=\sum_{i=1}^{n}\exp\left[-\int_{t_1}^{t}p(u)du\right].p_i(t)y(t+\tau_i(t))$$

$$=\sum_{i=1}^{n}\exp\left[-\int_{t_1}^{t+\tau_i(t)}p(u)du+\int_{t}^{t+\tau_i(t)}p(u)du\right].p_i(t)y(t+\tau_i(t))$$

$$=\sum_{i=1}^{n}\exp\left[\int_{t}^{t+\tau_i(t)}p(u)du\right].p_i(t)\exp\left[-\int_{t_1}^{t+\tau_i(t)}p(u)du\right]y(t+\tau_i(t))\ . \tag{2.3.26}$$

But from (2.3.25)

$$z(t)=\exp\left[-\int_{t_1}^{t}p(u)du\right]y(t).$$

Therefore (2.3.26) will be of the form

$$z'(t)=\sum_{i=1}^{n}q_i(t)z(t+\tau_i(t)), \tag{2.3.27}$$

where

$$q_i(t)=\exp\left[\int_{t}^{t+\tau_i(t)}p(u)du\right].p_i(t) \tag{2.3.28}$$

Equation (2.3.27) is of the form of (2.3.20). We see that the transformation (2.3.25) preserves oscillation. Therefore we can apply the above results with respect to (2.3.20) to equation (2.3.24). For example we have the following Theorem.

Theorem 2.3.7: If any one of the following conditions holds

1. $\underline{\lim}_{t\to\infty}\int_{t}^{t+\tau_i(t)}q_i(s)ds>\frac{1}{e}$, for some $i=1,2,...,n$.

2. $\underline{\lim}_{t\to\infty}\int_{t}^{t+\tau_{\min}(t)}\sum_{i=1}^{n}q_i(s)ds>\frac{1}{e}$.

3. $\left[\prod_{i=1}^{n}\sum_{j=1}^{n}\varliminf_{t\to\infty}\int_{t}^{t+\tau_j} q_i(s)ds\right]^{\frac{1}{n}} > \frac{1}{e}$,

and $q_i(t)$ satisfies the condition

$$\varliminf_{t\to\infty}\int_{t_1}^{t+\tau_{\min}(t)} q_i(s)ds > 0 .$$

4. $\varlimsup_{t\to\infty}\int_{t}^{t+\tau_{\max}(t)}\sum_{i=1}^{n} q_i(s)ds > 1$, where $\tau_{\max}(t) = \max\{\tau_1(t),\ldots,\tau_n(t)\}$.

Then all solutions of (2.3.24) oscillate, where $q_i(t)$ is defined by (2.3.28)

Example 2.3.6: Consider the advanced differential equation

$$y'(t) = y(t) + 2y\left(t+\frac{\pi}{2}\right) + y(t+\pi) + y\left(t+\frac{3\pi}{2}\right) . \qquad (2.3.29)$$

Here $p(t)=1$, $p_1(t)=2$, $p_2(t)=p_3(t)=1$, and $\tau_1(t)=\frac{\pi}{2}$, $\tau_2(t)=\pi$, $\tau_3(t)=\frac{3\pi}{2}$.

And $q_1(t)=2e^{\frac{\pi}{2}}$, $q_2(t)=e^{\pi}$, $q_3(t)=e^{\frac{3\pi}{2}}$.

Equation (2.3.29) satisfies any one of the conditions of Theorem (2.3.7) for example, for condition (1): $\varliminf_{t\to\infty}\int_{t}^{t+\frac{\pi}{2}} 2e^{\frac{\pi}{2}}dt = \pi e^{\frac{\pi}{2}} > \frac{1}{e}$. Similarly we can make sure for the rest of the conditions. So by Theorem (2.3.7) all solutions of equation (2.3.29) oscillate. In fact $y(t)=\sin t$ is a solution of equation (2.3.29).

2.4. Equations with forcing terms

In this section we want to discuss oscillation of solution of the non-homogeneous advanced differential equation

$$y'(t)-\sum_{i=1}^{n} p_i(t)y(t+\tau_i(t))=q(t), \qquad (2.4.1)$$

where $q(t), p_i(t)\geq 0$ and $\tau_i(t)>0$ are continuous, $i=1,2,...,n$. (2.4.2)

The following Theorem gives the main result of oscillation of equation (2.4.1).

Theorem 2.4.1: Assume that

(i) (2.4.2) holds.

(ii) There exists a function $Q(t)$ and two constants q_1, q_2 and sequences $\{t'_m\}$, $\{t''_m\}$ such that

$Q'(t)=q(t)$, $Q(t'_m)=q_1$, $Q(t''_m)=q_2$, $\lim\limits_{m\to\infty} t'_m=\infty$, $\lim\limits_{m\to\infty} t''_m=\infty$ and $q_1\leq Q(t)\leq q_2$ for $t\geq 0$.

(iii) $p_i(t)$, $i=1,2,...,n$ satisfy any one of the conditions

$$P^*_{ij}=\varliminf_{t\to\infty}\int_t^{t+\tau_j} p_i(s)ds>e^{-1}, \text{ for some } i=1,2,...,n \text{ and } j=1,2,...,n, \qquad (2.4.3)$$

$$\left[\prod_{i=1}^{n}\sum_{j=1}^{n} P^*_{ij}\right]^{\frac{1}{n}}>e^{-1}, \qquad (2.4.4)$$

and

$$\varliminf_{t\to\infty}\int_t^{t+\tau_{\min}(t)}\sum_{i=1}^{n} p_i(s)ds>\frac{1}{e}, \qquad (2.4.5)$$

where

$$\tau_{\min}(t)=\min\{\tau_1(t),\tau_2(t),.....,\tau_n(t)\}$$

Then every solution of equation (2.4.1) oscillates.

Proof: Let $y(t)$ be a non-oscillatory solution of (2.4.1) such that

$$y(t) > 0\,,\; y(t+\tau_i(t)) > 0 \;\text{, for } t > t_1 \text{ and let}$$

$$x(t) \equiv y(t) - Q(t)\,,$$

then

$$x'(t) = y'(t) - Q'(t)\,,$$

$$= \sum_{i=1}^{n} p_i(t) y(t+\tau_i(t)) > 0\,, \quad \text{for } t > t_1\,.$$

Suppose

$$x(t) + q_1 \le 0\,, \quad \text{for } t \ge t_2 \ge t_1\,,$$

since

$$x(t) + Q(t) \equiv y(t) > 0\,,$$

especially

$$x(t'_m) + Q(t'_m) = y(t'_m) \quad , \quad t'_m > t_2$$

this is a contradiction. So

$$x(t) + q_1 > 0 \quad , \quad \text{for all } t \ge t_2\,.$$

Let

$$z(t) \equiv x(t) + q_1\,,$$

then

$$z'(t) = x'(t) = y'(t) - Q'(t)$$

$$= \sum_{i=1}^{n} p_i(t) y(t+\tau_i(t))$$

$$= \sum_{i=1}^{n} p_i(t)\left[x(t+\tau_i(t)) + Q(t+\tau_i(t))\right] \geq \sum_{i=1}^{n} p_i(t)\left[x(t+\tau_i(t)) + q_1\right]$$

$$= \sum_{i=1}^{n} p_i(t) z(t+\tau_i(t)).$$

That is

$$z'(t) - \sum_{i=1}^{n} p_i(t) z(t+\tau_i(t)) \geq 0,$$

has an eventually positive solution. But it is impossible according to condition (iii). The proof is complete.

Example 2.4.1: Consider the diffrerential equation

$$y'(t) = \frac{1}{2} y\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos t, \tag{2.4.6}$$

$p(t) = \frac{1}{2}$, $\tau(t) = \frac{\pi}{2}$, $q(t) = \frac{1}{2}\cos t$, $Q(t) = \frac{1}{2}\sin t$.

Since $|Q(t)| \leq \frac{1}{2}$, then $q_1 = -\frac{1}{2}$, $q_2 = \frac{1}{2}$, and $t'_m = \frac{\pi}{2}(4m+3) \Rightarrow Q(t'_m) = -\frac{1}{2} = q_1$ and

$t''_m = \frac{\pi}{2}(4m+1) \Rightarrow Q(t''_m) = \frac{1}{2} = q_2$.

$$p^*_{ij} = \varliminf_{t\to\infty} \int_t^{t+\frac{\pi}{2}} \frac{1}{2} ds = \frac{\pi}{4} > \frac{1}{e}.$$

So by Theorem (2.4.1) all solutions of equation (2.4.6) oscillate. In fact

$y(t) = \sin t$ is a solution of (2.4.6).

Example 2.4.2: Consider the equation

$$y'(t) = y\left(t + \frac{\pi}{2}\right) + y\left(t + \frac{3\pi}{2}\right) + \cos t\,. \tag{2.4.7}$$

By applying Theorem (2.4.1) on equation (2.4.7) all conditions of the theorem are satisfied, so all solutions of equation (2.4.7) are oscillatory. In fact

$y(t) = \sin t$ is a solution of (2.4.7).

Chapter Three

Oscillatory and nonoscillatory solutions of first order nonlinear advanced differential equations

3.0 Introduction:

In this chapter we will discuss oscillatory and nonoscillatory behavior of solutions of the first order nonlinear advanced differential equation

$$y'(t) - \sum_{i=1}^{n} p_i(t) f_i\big(y(\tau_i(t))\big) = 0\,, \tag{3-A}$$

where

$p_i(t) \in C[\Re^+, \Re^+]$, with $p_i(t) \geq 0$; $i \in I_n$, $\tau_i(t) \in [\Re^+, \Re^+]$, $\tau_i(t) > t$

$f \in C[\Re, \Re]$.

This chapter contains two sections. In section 3.1 we introduce sufficient conditions for the oscillation of equation (3-A) when $n = 1$.

In section 3.2 we study some oscillatory results for equation (3-A) with several deviating arguments.

3.1 Oscillation of first order nonlinear homogeneous advanced differential equations

Consider the equation

$$y'(t) = p(t) f\big(y(\tau(t))\big). \tag{3.1.1}$$

We have the following result.

Theorem 3.1.1: If

(i) $\tau(t) \in C[\Re^+, \Re]$, $\tau(t) > t$ for $t \in \Re^+$, $\tau(t)$ is strictly increasing on $\Re^+$.

(ii) $p(t)$ is locally integrable and $p(t) \geq 0$, almost everywhere.

(iii) $uf(u) > 0$ for $u \neq 0$, $f \in [\Re, \Re]$, $f(u)$ is nondecreasing in u,

$$\lim_{u \to \infty} \frac{u}{f(u)} = \mathrm{M} < \infty, \tag{3.1.2}$$

and if

$$\overline{\lim_{t \to \infty}} \int_t^{\tau(t)} p(s)ds > \mathrm{M}. \tag{3.1.3}$$

Then every solution of (3.1.1) oscillates.

Proof: Let $y(t)$ be a nonoscillatory solution of (3.1.1), without loss generality, assume that $y(t) > 0$ for $t_0 < t < \tau(t^*)$. Then

$y'(t) = p(t)f(y(\tau(t))) \geq 0$, for $t \geq t_0$. Thus $y(t)$ is nondecreasing.

From (3.1.1), it follows that

$$y(\tau(t)) - y(t) = \int_t^{\tau(t)} p(s)f(y(\tau(s)))ds,$$

or

$$y(t) - y(\tau(t)) + \int_t^{\tau(t)} p(s)f(y(\tau(s)))ds = 0.$$

This implies

$$y(t) - y(\tau(t))\left[1 - \frac{f(y(\tau(t)))}{y(\tau(t))}\int_t^{\tau(t)} p(s)ds\right] \leq 0,$$

and hence

$$\int_t^{\tau(t)} p(s)ds < \frac{y(\tau(t))}{f(y(\tau(t)))},$$

for sufficiently large t. Therefore

$$\overline{\lim_{t\to\infty}} \int_{t}^{\tau(t)} p(s)ds < \mathrm{M}.$$

This is a contradiction to condition (3.1.3). Therefore $y(t)$ is oscillatory .

Now we present a result concerning the asymptotic behavior of the equation

$$y'(t) = p(t) f\left(y(t+\tau(t))\right), \tag{3.1.4}$$

Theorem 3.1.2: Assume that equation (3.1.4) satisfies the following conditions:

p, $\tau \in C[\Re^+, \Re^+]$, $p(t) > 0$, $f \in C[\Re, \Re]$

$0 \le \tau(t) \le q$, and $yf(y) > 0$ for $y \ne 0$.

If

$$\int^{\infty} p(t)dt = \infty.$$

Then all nonoscillatory solutions of (3.1.4) tend to ∞ as $t \to \infty$.

Proof: Let $y(t) > 0$ be a nonoscillatory solution of (3.1.4) for sufficiently large t. Then

$y'(t) > 0$, and so $y(t)$ is nondecreasing.

Claim that $\lim\limits_{t\to\infty} y(t) = c = \infty$, (3.1.5)

otherwise $c < \infty$, and then there exists a $t^* \ge t > t_0$ such that

$$f\left(y(t+\tau(t))\right) \ge k > 0 \text{ for } t^* \ge t \text{ and } f(c) \ge k > 0.$$

Thus

$$y'(t) = p(t) f\left(y(t+\tau(t))\right) \ge p(t)k > 0.$$

That is,

$$y'(t) \ge kp(t) > 0. \tag{3.1.6}$$

Integrating (3.1.6) from t to t^* yields

$$y(t^*)-y(t)\geq k\int_t^{t^*}p(s)ds>0,$$

or

$$y(t)\leq y(t^*)-k\int_t^{t^*}p(s)ds.$$

Hence $y(t)$ will become negative for sufficiently large t. This is a contradiction to the fact that $y(t)>0$. Therefore $c=\infty$, which completes the proof.

Theorem 3.1.3: Assume that the hypothesis of Theorem (3.1.1) hold except that the relation (3.1.3) is replaced by

$$\varliminf_{t\to\infty}\int_t^{\tau(t)}p(s)ds>\frac{\mathrm{M}}{e}. \qquad (3.1.7)$$

Then every solution of (3.1.1) oscillates.

Proof: Assume that there is a nonoscillatory solution $y(t)>0$, $y(\tau(t))>0$ for $t\geq t_0\geq 0$. So $y'(t)\geq 0$ and hence $y(t)\to\infty$ as $t\to\infty$ (by Theorem (3.1.2)). There exists a $t^*\in(t,\tau(t))$ such that

$$\int_t^{t^*}p(s)ds>\frac{M}{2e} \text{ and } \int_{t^*}^{\tau(t)}p(s)ds>\frac{M}{2e}. \qquad (3.1.8)$$

Now integrating (3.1.1) from t to t^* yields

$$y(t^*)-y(t)=\int_t^{t^*}p(s)f(y(\tau(s)))ds\geq f(y(\tau(t)))\int_t^{t^*}p(s)ds\geq f(y(\tau(t)))\frac{M}{2e},$$

and from t^* to $\tau(t)$, gives

$$y(\tau(t))-y(t^*)=\int_{t^*}^{\tau(t)}p(s)f(y(\tau(s)))ds\geq f(y(\tau(t^*)))\frac{M}{2e}, \qquad (3.1.9)$$

which implies

$$y(t^*) > f(y(\tau(t)))\frac{M}{2e}$$

$$\geq \frac{f(y(\tau(t)))}{y(\tau(t))} f(y(\tau(t^*)))\left(\frac{M}{2e}\right)^2,$$

and hence

$$\frac{y(\tau(t^*))}{y(t^*)} \leq \frac{y(\tau(t))}{f(y(\tau(t)))} \cdot \frac{y(\tau(t^*))}{f(y(\tau(t^*)))}\left(\frac{2e}{M}\right)^2. \tag{3.1.10}$$

Setting

$$w(t) = \frac{y(\tau(t))}{y(t)} \geq 1, \ \lim_{t\to\infty} w(t) = l \geq 1$$

l is finite because of (3.1.10). From (3.1.1) we have

$$\ln w(t) = \int_t^{\tau(t)} p(s)\frac{f(y(\tau(s)))}{y(\tau(s))} w(s)ds,$$

$$= w(\xi)\frac{f(y(\tau(\xi)))}{y(\tau(\xi))}\int_t^{\tau(t)} p(s)ds, \tag{3.1.11}$$

where $t < \xi < \tau(t)$. Taking the limit inferior in equation (3.1.11), we obtain

$$\ln l \geq \frac{l}{M}\varliminf_{t\to\infty}\int_t^{\tau(t)} p(s)ds.$$

But $\max_{l\geq 1}\frac{\ln l}{l} = \frac{1}{e}$, and therefore

$$\frac{M}{e} \geq \varliminf_{t\to\infty}\int_t^{\tau(t)} p(s)ds.$$

This is a contradiction because (3.1.8) hold, which completes the proof.

Example 3.1.1: Consider the nonlinear advanced differential equation

$$y'(t)=\frac{2}{(e\ln\lambda)t}\,y(\lambda t),\ \lambda>1\ . \tag{3.1.12}$$

Note that

$$\int_t^{\lambda t}\frac{2}{(e\ln\lambda)s}\,ds=\frac{2}{e}\,,$$

and

$$M=\lim_{y\to\infty}\frac{y}{f(y)}=1.$$

Therefore (3.1.12) satisfies the conditions of Theorem (3.1.3), so all solutions of (3.1.12) oscillate.

While the equation

$$y'(t)=\frac{1}{(e\ln\lambda)t}\,y(\lambda t)\ ,\ \lambda>1, \tag{3.1.13}$$

does not satisfy the conditions of Theorem (3.1.3). In fact (3.1.13) has the nonoscillatory solution

$$y(t)=t^{m},\quad m=\frac{1}{\ln\lambda}\,.$$

3.2 Nonlinear advanced differential equations with several deviating arguments

Consider the advanced nonlinear differential equation

$$y'(t)=\sum_{i=1}^{n} p_i(t)f_i(y(\tau_i(t))), \tag{3.2.1}$$

where

$p_i(t)\geq 0$, $\tau_i(t)>t$, $i=1,2,...,n$, are continuous. For oscillatory solutions of (3.2.1) we have the following result.

Theorem 3.2.1: If $uf_i(u)>0$ for $u\neq 0$, $f_i(u)$ in nondecreasing in u,

$$\lim_{|u|\to\infty}\frac{u}{f_i(u)}=M_i>0,\ \ i=1,2,...,n, \tag{3.2.2}$$

And if

$$\overline{\lim_{t\to\infty}}\int_t^{\tau_*(t)}\left(\sum_{i=1}^{n}p_i(s)\right)ds>M^*, \tag{3.2.3}$$

where $M^*=\max\{M_1,M_2,....,M_n\}$, and $\tau_*(t)=\min\{\tau_1(t),.....,\tau_n(t)\}$.

Then every solution of (3.2.1) oscillates.

Proof: Let $y(t)$ be a nonoscillatory solution of (3.2.1). Without loss of generally assume that $y(t)>0$. So $y'(t)>0$ and thus $y(t)$ is nondecreasing and $y(t)\to\infty$ as $t\to\infty$ (as in the proof of Theorem (3.1.2)). From (3.2.1), we have

$$y(\tau_*(t))-y(t)=\int_t^{\tau_*(t)}\sum_{i=1}^{n}p_i(s)f_i(y(\tau_i(s)))ds$$

$$\geq\sum_{i=1}^{n}f_i(y(\tau_i(t)))\int_t^{\tau_*(t)}p_i(s)ds$$

$$\geq \sum_{i=1}^{n} f_i(y(\tau_*(t))) \int_t^{\tau_*(t)} p_i(s)ds$$

and so

$$y(\tau_*(t))\left[1-\sum_{i=1}^{n}\frac{f_i(y(\tau_*(t)))}{y(\tau_*(t))}\int_t^{\tau_*(t)} p_i(s)ds\right] \geq y(t) > 0 .$$

Therefore

$$1 > \sum_{i=1}^{n}\frac{f_i(y(\tau_*(t)))}{y(\tau_*(t))}\int_t^{\tau_*(t)} p_i(s)ds$$

$$1 \geq \frac{1}{M^*}\overline{\lim_{t\to\infty}}\int_t^{\tau_*(t)}\left(\sum_{i=1}^{n} p_i(s)\right)ds .$$

This is a contradiction to condition (3.2.3). Therefore $y(t) > 0$ is an oscillatory solution of (3.2.1).

Now let us introduce the oscillation criteria of the first order nonlinear advanced differential inequalities

$$y'(t) + a(t)y(t) - p(t)f\left(y(t+\tau_1(t)),\dots,y(t+\tau_m(t))\right) \geq 0, \tag{3.2.4}$$

$$y'(t) + a(t)y(t) - p(t)f\left(y(t+\tau_1(t)),\dots,y(t+\tau_m(t))\right) \leq 0, \tag{3.2.5}$$

and equation

$$y'(t) + a(t)y(t) - p(t)f\left(y(t+\tau_1(t)),\dots,y(t+\tau_m(t))\right) = 0 . \tag{3.2.6}$$

For these we have the following result.

Theorem 3.2.2: Assume that p, $\tau_i \in C[\Re^+, \Re^+]$, $p(t) > 0$, $\tau_i(t) > 0$, $i = 1,2,\dots,m$, $a \in C[\Re^+, \Re]$, and f satisfies:

$f \in C[\Re^m, \Re]$, $y_1 f(y_1, y_2,\dots, y_m) > 0$. Furthermore , assume that:

$$\liminf_{t\to\infty}\int_t^{t+\tau_i(t)}(-a(s))ds = k_i > -\infty,\ i = 1,2,\dots,m, \tag{3.2.7}$$

where $k_i \in \Re$, and there exist nonnegative numbers k and α_j, $j = 1,2,...,m$ such that

$$\sum_{i=1}^{m} \alpha_i = 1, \quad k > 0$$

$$\left| f(s_1, s_2, ..., s_n) \right| \geq k |s_1|^{\alpha_1} |s_2|^{\alpha_2} ... |s_m|^{\alpha_m}, \tag{3.2.8}$$

for all $s \in \Re^m$, and

$$\liminf_{t \to \infty} \int_{t}^{t+\tau_*(t)} p(s) ds > \frac{1}{ekc}, \tag{3.2.9}$$

where

$$c = \min_{1 \leq i \leq m} e^{k_i} \text{ and } \tau_*(t) = \min\{\tau_1(t), ..., \tau_m(t)\}.$$

Then (3.2.4) has no eventually positive solution, (3.2.5) has no eventually negative solution, and every solution of (3.2.6) is oscillatory.

Proof: See [8].

Example 3.2.1: The equation

$$y'(t) - 3\left[y\left(t + \frac{\pi}{2} \right) \right]^{\frac{1}{3}} \left[y(t + 2\pi) \right]^{\frac{2}{3}} = 0, \tag{3.2.10}$$

note that $a(t) = 0$, $p(t) = 3$, $\tau_1 = \frac{\pi}{2}$, $\tau_2 = 2\pi$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$, and

$$k_i = \liminf_{t \to \infty} \int_{t}^{t+\tau_i} (-a(s)) ds = 0 > -\infty, \; i = 1,2$$

$\tau_*(t) = \frac{\pi}{2}$, so $\liminf_{t \to \infty} \int_{t}^{t+\frac{\pi}{2}} 3 ds = \frac{3\pi}{2}$, and $c = \min e^{k_i} = 1$.

So equation (3.2.10) satisfies the conditions of Theorem (3.2.2), so every solution of (3.2.10) is oscillatory. In fact, the functions $y_1(t) = \cos^3(t)$, $y_2(t) = \sin^3(t)$ are oscillatory solutions of (3.2.10).

Theorem 3.2.3 [8]: If $a(t) \le 0$ in Theorem (3.2.2), then (3.2.7), (3.2.8) and (3.2.9) can be replaced by the condition

$$\varliminf_{t\to\infty} \int_t^{t+\tau_*(t)} p(s)ds > \frac{\mathrm{M}}{e} \exp\left(-\varliminf_{t\to\infty} \int_t^{t+\tau_*(t)} (-a(s))ds \right), \tag{3.2.11}$$

where

$$\mathrm{M} = \varlimsup_{\substack{|y_i|\to\infty \\ 1\le i\le m}} \frac{|y_1|^{\alpha_1} |y_2|^{\alpha_2} \ldots |y_m|^{\alpha_m}}{|f(y_1,\ldots,y_m)|}, \tag{3.2.12}$$

and the conclusion of theorem (3.2.2) remains valid.

Example 3.2.2: Consider the advanced type differential inequality

$$y'(t) - e^{-t} y(t) - e^{-2t} [y(t+1)]^{\frac{1}{3}} \left[y\left(t + \frac{1}{2} \right) \right]^{\frac{2}{3}} \ge 0, \tag{3.2.13}$$

It does not satisfy conditions of Theorem (3.2.3), since

$$\liminf_{t\to\infty} \int_t^{t+\tau_*} e^{-s} ds = 0, \ \tau_*(t) = \frac{1}{2}$$

$$\liminf_{t\to\infty} \int_t^{t+\frac{1}{2}} e^{-2s} ds = 0, \ M = 1.$$

In fact (3.2.13) has the positive solution $y(t) = e^{2t}$.

Another kind of advanced nonlinear differential equations, consider the equation:

$$y'(t) = f\left(t, y(\tau_1(t)), \ldots, y(\tau_m(t))\right), \tag{3.2.14}$$

where $f \in C[\Re^+ \times \Re^m, \Re]$, $\tau_i(t) > t$ on $t \in \Re^+$ and $\tau_i(t) \in C[\Re^+, \Re^+]$, $i \in I_m$.

Theorem 3.2.4: Assume that there exists a function $a \in C[\Re^+, \Re^+]$ such that

$$f(t, y_1, \ldots, y_m)\operatorname{sgn} y_0 \geq a(t)|y_0|, \tag{3.2.15}$$

for $t \geq 0$, $|y_i| \geq |y_0|$, $y_i y_0 \geq 0$, $i = 1, 2 \ldots, m$, and

$$\varliminf_{t \to \infty} \int_t^{\tau_*(t)} a(s)ds > \frac{1}{e}, \tag{3.2.16}$$

where $\tau_*(t) = \min\{\tau_1(t), \ldots, \tau_m(t)\}$. Then every solution of (3.2.14) is oscillatory.

Proof: Assume that $y(t)$ is a nonoscillatory solution of (3.2.14). Without loss of generality, assume that $y(t) > 0$, then from (3.2.14) and (3.2.15), we obtain a first order advanced differential inequality

$$y'(t) - a(t)y(\tau_*(t)) \geq 0, \tag{3.2.17}$$

this implies that (3.2.17) has a positive solution $y(t)$. On the other hand, from Theorem (2.3.1), equation (3.2.17) has no eventually positive solution under condition (3.2.16). this contradiction completes the Proof.

Example 3.2.3: Consider the advanced nonlinear differential equation

$$y'(t) = \frac{2}{(e \ln 2)\ t} y^{1/3}(2t)y(3t)y^{1/3}(4t), \tag{3.2.18}$$

which satisfies condition (3.2.15), and

$$\int_t^{2t} a(s)ds = \frac{2}{e} > \frac{1}{e}.$$

Then all solution of (3.2.18) oscillate.

Theorem 3.2.5: Assume that there exists a function $a(t)$ such that $a \in C[\Re^+, \Re^+]$ and

$$0 \le f(t, y_1, ..., y_m) \operatorname{sgn} y_0 \le a(t)|y_0|, \tag{3.2.19}$$

on $t \in \Re^+$, $|y_i| \le |y_0|$, $y_i y_0 \ge 0$, $i = 1,2,...,m$ and

$$\limsup_{t\to\infty} \int_t^{\tau^*(t)} a(s)ds < \frac{1}{e}, \tag{3.2.20}$$

where $\tau^*(t) = \max\{\tau_1(t), ..., \tau_m(t)\}$. Then equation (3.2.14) has a nonoscillatory solution.

Proof: see [8].

Now we shall present sufficient conditions for the existence of nonoscillatory solutions of the nonlinear advanced differential equation:

$$y'(t) = \sum_i^n q_i(t) f_i(y(\tau_1(t)), ..., y(\tau_m(t))), \tag{3.2.21}$$

where

(i) $q_i, \tau_j \in C[[a,\infty), \Re]$, $q_i(t) \ge 0$ and $\lim_{t\to\infty} \tau_j(t) = \infty$, $i = 1,2,...,n$ and $j = 1,2,...,m$ and there is at least one q_i which is different from zero.

(ii) $f_i \in C[\Re^m, \Re]$, f_i is nondecreasing with respect to every element, and

$$u_1 f_i(u_1, ..., u_m) > 0 \text{ as } u_1 u_j > 0,\ j = 1,2,...,m.$$

Theorem 3.2.6 : Let conditions (i) and (ii) hold. If

$$\sum_{i=1}^n \int^\infty q_i(t)dt < \infty. \tag{3.2.22}$$

Then equation (3.2.21) has a nonoscillatory solution.

Proof: see [8].

Example 3.2.4: Consider the equation

$$y'(t) = \frac{1}{2e} e^{-t} y(t+1) y^2\left(t + \frac{1}{2}\right), \tag{3.2.23}$$

so by Theorem (3.2.6), equation (3.2.23) should have a nonoscillatory solution.We see that $q(t) = \frac{1}{2e} e^{-t}$, $\tau_1(t) = t+1$, $\tau_2(t) = t + \frac{1}{2}$ and

$$\int^{\infty} \frac{1}{2e} e^{-t} dt = 0 < \infty .$$

In fact $y(t) = e^{\frac{t}{2}}$ is such a solution of (3.2.23).

Chapter Four

Oscillation of solutions of Special Kinds of differential equations

4.0 Introduction

In this chapter we will study oscillation criteria for three Kinds of differential equations, impulsive differential equations with advanced argument, mixed type differential equations and an equation of alternately advanced and retarded argument.

Section 4.1 introduces sufficient conditions for the oscillation of the first order impulsive differential equation with advanced argument:

$$\left.\begin{aligned} &y'(t) = p(t)y(t+\tau), && t \neq t_k \\ &y(t_k^+) - y(t_k) = b_k y(t_k), && k \in N \end{aligned}\right\} \qquad \text{4-A}$$

where

$0 \le t_0 < t_1 < \ldots < t_k < \ldots$ are fixed points with $\lim_{k\to\infty} t_k = \infty$

$$b_k \in \Re - \{-1\},\ k \in N = \{1,2,\ldots\}$$

$p \in ([t_{0,}\infty),\Re)$ is locally summable function and $\tau > 0$ is constant.

Section 4.2 deals with oscillation of the mixed type equation

$$y'(t) + a_1(t)y(\tau(t)) + a_2(t)y(\sigma(t)) = 0,\ t \ge t_0 \qquad \text{4-B}$$

with nonnegative coefficients $a_i(t), i = 1,2$, one delayed argument $(\tau(t) \le t)$ and one advanced argument $(\sigma(t) \ge t)$.

Section 4.3 concerns with oscillations in one equation of alternately advanced and retarded argument.

4.1 Impulsive differential equations with advanced argument

Some times it is necessary to deal with phenomena of an impulsive nature, for example, voltage or forces of large magnitude that act over very short time intervals.

The purpose of this section is to study oscillation and nonoscillation of the solutions of impulsive differential equations with advanced argument. Let $N=\{1,2,3...,\}$. Consider the impulsive differential equation with an advanced argument

$$\left.\begin{aligned} &y'(t)=p(t)y(t+\tau) \quad , \; t\neq t_k \\ &y(t_k^+)-y(t_k)=b_k y(t_k), \; k\in N \end{aligned}\right\} \tag{4.1.1}$$

under the following hypothesis:

(A1) $0\le t_0<t_1<...<t_k<...$ are fixed points with $\lim_{k\to\infty} t_k=\infty$;

(A2) $p\in([t_0,\infty),\Re)$ is locally summable function, $\tau>0$ is constant;

(A3) $b_k\in(-\infty,-1)\cup(-1,\infty)$ are constants for $k\in N$.

Definition 4.1.1: A function $y\in([t_0,\infty),\Re)$ is said to be a solution of equation (4.1.1) on $[t_0,\infty)$ if the following conditions are satisfied:

(i) $y(t)$ is absolutely continuous on each interval (t_k,t_{k+1}), $k\in N$, and (t_0,t_1);

(ii) for any $t_k\in[t_0,\infty)$, $y(t_k^+)$ and $y(t_k^-)$ exists and $y(t_k^-)=y(t_k)$, $k\in N$;

(iii) for $t\neq t_k$, $k\in N$, $y(t)$ satisfies $y'(t)=p(t)y(t+\tau)$ almost everywhere and for each $t=t_k$, $y(t_k^+)-y(t_k)=b_k y(t_k)$, $k\in N$.

Definition 4.1.2: A solution of (4.1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

Bainov and Dimitrova [1] established the following results for oscillation of solutions of (4.1.1), under the assumption that $p \in C([t_0, \infty), [0, \infty))$, $\tau > 0$, and $\{t_k\}$ satisfies (A_1). They introduced the following conditions:

(H_1) $0 < \tau < t_1$

(H2) There exists a positive constant $T > \tau$ such that $t_{k+1} - t_k \geq T$, $k \in N$.

(H3) There exists a constant $M > 0$ such that for any $k \in N$ the inequality $0 \leq M \leq b_k$ is valid

Theorem 4.1.1 [1]: Suppose that

(a) Conditions (H_1) and (H_2) hold.

(b) $\limsup_{k \to \infty} [(1 + b_k) \int_{t_k - \tau}^{t_k} p(s) ds] > 1$.

Then all solutions of (4.1.1) are oscillatory.

Proof: let $y(t)$ be a nonoscillatory solution of (4.1.1). Without loss of generality assume that $y(t) > 0$ for $t \geq t_0 > 0$. Then $y(t + \tau) > 0$ for $t \geq t_0$. From (4.1.1), it follows that $y(t)$ is nondecreasing in $(t_0, t_k) \cup [\bigcup_{i=k}^{\infty} (t_i, t_{i+1})]$, where $t_k > t_0 > t_{k-1}$.

Integrate (4.1.1) from $t_i - \tau$ to $t_i (i \geq k + 1)$ we obtain

$$y(t_i) - y(t_i - \tau) = \int_{t_i - \tau}^{t_i} p(s) y(s + \tau) ds$$

$$y(t_i) - y(t_i - \tau) \geq y(t_i + 0) \int_{t_i - \tau}^{t_i} p(s) y(s + \tau) ds \qquad (4.1.2)$$

Since

$$y(t_i + 0) = (1 + b_i) y(t_i - 0) = (1 + b_i) y(t_i), \qquad (4.1.3)$$

then (4.1.2) and (4.1.3) yield the inequality

$$y(t_i - \tau) + y(t_i) \left[(1 + b_i) \int_{t_i - \tau}^{t_i} p(s) ds - 1 \right] \leq 0. \qquad (4.1.4)$$

Inequality (4.1.4) is valid only if

$\lim_{i\to\infty}\sup(1+b_i)\int_{t_i-\tau}^{t_i} p(s)ds \le 1$, which contradicts condition (b) of the theorem. So the proof is complete.

Together with (4.1.1), consider the differential equation with an advanced argument

$$\left.\begin{aligned} &x'(t) = P(t)x(t+\tau) \\ \text{where}\quad &P(t) = \prod_{t\le t_k<t+\tau}(1+b_k)p(t), \quad t\ge t_0 \end{aligned}\right\} \tag{4.1.5}$$

Assume that a product equals to unit if the numbers of factors is equal to zero.

Theorem 4.1.2 [12]: Assume that (A_1)-(A_3) hold. Then all solutions of (4.1.1) are oscillatory if and only if all solutions of (4.1.5) are oscillatory.

Proof: see [12].

Jurang Yan [12] also established the following results for equation (4.1.1). He also used the following condition:

(A_4) $p \in ([t_0,\infty),(0,\infty))$ is locally summable function and $\tau > 0$ is constant.

Theorem 4.1.3 [12]: Assume that (A_1)-(A_3) hold and there exists a sequence of intervals $\{(\xi_n,\eta_n)\}$, such that $\lim_{n\to\infty}\xi_n = \infty$ and $\eta_n - \xi_n > \tau$ for all $n \ge N \ge 1$. If $p(t)\ge 0$ for all $t \in \bigcup_{n=N}^{\infty}(\xi_n,\eta_n)$ and

$$\lim_{t\to\infty}\sup\int_t^{t+\tau}\prod_{s\le t_k<s+\tau}(1+b_k)p(s)ds > 1, \text{ for } t \in \bigcup_{n=N}^{\infty}(\xi_n,\eta_n-\tau), \tag{4.1.6}$$

then all solutions of (4.1.1) are oscillatory.

Proof: let $y(t)$ be a nonoscillatory solution of (4.1.1) and suppose that $y(t) > 0$ for $t \ge T \ge t_0$.

From Theorem (4.1.2), equation (4.1.5) has also a positive solution $x(t)$ on $[T,\infty)$. Thus, for $t \in \bigcup_{n=N}^{\infty}(\zeta_n,\eta_n-\tau)$,

$P(t)=\prod_{t\le t_k<t+\tau}(1+b_k)p(t)\ge 0,$ and hence,

$x'(t)>0$ almost everywhere for $t\in\bigcup_{n=N}^{\infty}(\zeta_n,\eta_n-\tau)$, which implies $x(t)$ is nondecreasing in $\bigcup_{n=N}^{\infty}(\zeta_n,\eta_n-\tau)$. Integrating (4.1.5) from t to $t+\tau$, we obtain that for $t\in\bigcup_{n=N}^{\infty}(\zeta_n,\eta_n-\tau)$,

$$x(t)-x(t+\tau)+\int_t^{t+\tau}P(s)x(s+\tau)ds=0.$$

By using the nondecreasing character of $x(t)$, we derive that

$$x(t)+x(t+\tau)\left[\int_t^{t+\tau}P(s)ds-1\right]\le 0 \text{ for } t\in\bigcup_{n=N}^{\infty}(\zeta_n,\eta_n-\tau),$$

which contradicts (4.1.6).

Theorem 4.1.4 [12]: Assume that (A_1), (A_3), (A_4) hold and

$$\limsup_{t\to\infty}\int_t^{t+\tau}\prod_{s\le t_k<s+\tau}(1+b_k)p(s)ds>1,$$

then all solutions of (4.1.1) are oscillatory.

Proof: The proof of this theorem can be obtained by applying Theorem (4.1.3) immediately.

Theorem 4.1.5 [12]: Assume (A_1), (A_3), (A_4) hold and

$$\liminf_{t\to\infty}\int_t^{t+\tau}\prod_{s\le t_k<s+\tau}(1+b_k)p(s)ds>\frac{1}{e},$$

then all solutions of (4.1.1) are oscillatory.

For existence of a nonoscillatory solution of (4.1.1), we have the following result.

Theorem 4.1.6 [12]: Assume **(A_1)**, **(A_3)**, **(A_4)** with $b_k > -1$ hold and there exists a $T \geq t_0$ such that for all $t \geq T$

$$\int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1+b_k) p(s) ds] \leq \frac{1}{e}$$

Then equation (4.1.1) has a nonoscillatory solution.

Proof: see [12].

Example 4.1.1: Let $t_k = \sigma + km\tau$, m is a positive integer, $p(t) \geq 0$ is a locally summable function and $\tau > 0$, $b_k \in (-1, \infty)$, $k \in N$, are constants.

Consider the impulsive differential equation (4.1.1). Since $t_{k+1} - t_k = m\tau$, there is at most one point of impulsive effect on each $[t, t+\tau]$, $t \geq \sigma$. So,

$$\int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1+b_k) p(s) ds = (1+b_k) \int_t^{t+\tau} p(s) ds, \text{ if } t_k \in [t, t+\tau)$$

or

$$\int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1+b_k) p(s) ds = \int_t^{t+\tau} p(s) ds, \text{ if some } t_k \notin [t, t+\tau),\ k \in N$$

Then we have the following cases

(i) Let

$$d_1 = \limsup_{t \to \infty} \{(1+b_k) \int_t^{t+\tau} p(s) ds,\ t \leq t_k < t+\tau\}$$

and

$$d_2 = \limsup_{t \to \infty} \int_t^{t+\tau} p(s) ds$$

If $d = \max\{d_1, d_2\} > 1$, then by Theorem (4.1.4) all solutions of equation (4.1.1) are oscillatory.

(ii) Let $c_1 = \liminf_{t\to\infty}\{(1+b_k)\int_t^{t+\tau} p(s)ds\}$

and

$$c_2 = \liminf_{t\to\infty} \int_t^{t+\tau} p(s)ds$$

If $c = \min\{c_1, c_2\} > \frac{1}{e}$, then by Theorem (4.1.5) all solution of (4.1.1) are oscillatory.

(iii) If there is $T \geq t_0$ such that

$$\phi(t) \leq \frac{1}{e}, \text{ for all } t \geq T,$$

where

$$\phi(t) = \max\{(1+b_k)\int_t^{t+\tau} p(s)ds, t \leq t_k < t+\tau, \int_t^{t+\tau} p(s)ds\},\ t \geq T,$$

then by Theorem (4.1.6), equation (4.1.1) has a nonoscillatory solution on $[T, \infty)$.

Bainov and Dimitrava [1] established a sharp result for oscillation of the nonhomogeneous impulsive differential equation with deviating argument:

$$\left.\begin{array}{l} y'(t) - p(t)y(t+\tau) = q(t),\quad t \neq t_k \\ y(t_k^+) - y(t_k) = b_k y(t_k) \end{array}\right\}, \qquad (4.1.7)$$

under the following assumptions:

(H4) $q \in C([0,\infty), \Re)$

(H5) there exists a function $v \in C^1(\Re^+, \Re)$ such that $v'(t) = q(t), t \geq 0$

(H6) there exist constants q_1, and q_2 and two sequences $\{t_i'\} \subset \Re^+$ and $\{t_i''\} \subset \Re^+$ with $\lim_{t\to\infty} t_i' = \lim_{t\to\infty} t_i'' = \infty$ and $v(t_i') = q_1$, $v(t_i'') = q_2$, $q_1 \le v(t) \le q_2$.

Theorem 4.1.7 [1]: Suppose that

(i) conditions (H_1), (H_2), (H_4)- (H_6) hold.

(ii) $\limsup_{k\to\infty} \int_{t_k}^{t_k+\tau} p(s)ds > 1$.

(iii) $b_k \ge 0, k \in N$.

Then all solutions of equation (4.1.7) oscillate.

Proof: : Let $y(t)$ be a solution of (4.1.7) for $t \ge t_0 > 0$.

Set

$$z(t) = y(t) - v(t) + q_1.$$

Then from (4.1.7) we obtain

$$\left.\begin{aligned} & z'(t) \ge p(t)z(t+\tau) \\ & z(t_k^+) - z(t_k^-) = b_k z(t_k) + A_k \end{aligned}\right\} \tag{4.1.8}$$

where

$$A_k = b_k v(t_k) - b_k q_1 \ge 0.$$

Let the inequality (4.1.8) has a positive solution $z(t)$ for $t \ge t_1 \ge t_0$. Integrating (4.1.8) from t_k to $t_k + \tau$, $t_k \ge t_1$, we get

$$z(t_k + \tau) - z(t_k + 0) \ge z(t_k + \tau) \int_{t_k}^{t_k+\tau} p(s)ds,$$

$$z(t_k + \tau)\left[\int_{t_k}^{t_k+\tau} p(s)ds - 1\right] \le 0.$$

The last inequality contradicts condition (ii) of the theorem.

If $z(t) < 0$, for $t \ge t_1$ be a solution of the inequality (4.1.8), then

$$z(t_i') = x(t_i') - v(t_i') + q_1 = x(t_i') > 0, \text{ for}$$

$t_i' \ge t_1$. Also a contradiction.

Jankowski [6] studied the existence of solutions for first order impulsive ordinary differential equations, with advanced argument with boundary conditions.

For $J=[0,T]$, $T>0$, let $0=t_0<t_1<...<t_m<t_{m+1}=T$.

Put $J'=J\setminus\{t_1,t_2,...,t_m\}$. Consider the advanced impulsive differential equation

$$\begin{Bmatrix} y'(t)=f(t,y(t),y(\alpha(t)))\equiv Fy(t), t\in J' \\ \Delta y(t_k)=I_k(y(t_k)) \qquad k=1,2,...,m \\ 0=g(y(0),y(T)) \end{Bmatrix} \tag{4.1.9}$$

where $\Delta y(t_k)=y(t_k^+)-y(t_k^-)$, and the hypothesis

(H₇) $f\in C(J\times\Re\times\Re,\Re)$, $\alpha\in C(J,J)$, $t\le\alpha(t)\le T$, $t\in J$, $I_k\in C(\Re,\Re)$ for $k=1,2,...,m$, $g\in C(\Re\times\Re,\Re)$ and if there exists a point $\tilde{t}\in J$ such that $\alpha(\tilde{t})\in\{t_1,t_2,...,t_m\}$, then $\tilde{t}\in\{t_1,t_2,...,t_m\}$.

Put $J_0=[0,t_1]$, $J_k=(t_k,t_{k+1}]$, $k=1,2,...,m$. Introduce the spaces:

$$PC(J)=PC(J,\Re)=\begin{Bmatrix} y:J\to\Re, y\,|\,J_k\in C(J_k,\Re), k=0,1,...m \\ \text{and there exist } y(t_k^+) \text{ for } \quad k=1,2,...,m \end{Bmatrix}$$

and

$$PC^1(J)=PC^1(J,\Re)=\begin{Bmatrix} y\in PC(J), y\,|\,J_k\in C^1(J_k,\Re), k=0,1,...m \\ \text{and there exist } y'(t_k^+) \text{ for } \quad k=1,2,...,m \end{Bmatrix}$$

Note that $PC(J)$ and $PC^1(J)$ are Banach spaces with respective norms:

$$\|y\|_{PC}=\sup_{t\in J}\|y(t)\|,\ \|y\|_{PC^1}=\|y\|_{PC}+\|y'\|_{PC}.$$

By a solution of (4.1.9) we mean a function $y\in PC^1(J)$ which satisfies:

(i) The differential equation in (4.1.9) for every $t\in J'$.

(ii) The boundary condition in (4.1.9).

(iii) At every $t_k, k=1,2,...,m$, the function y satisfies the second condition in (4.1.9).

Definition 4.1.3: Lower and upper solution of problem (4.1.9)

We say that $u \in PC^1(J)$ is a lower solution of (4.1.9) if

$$\begin{cases} u'(t) \le Fu(t), t \in J' \\ \Delta u(t_k) \le I_k(u(t_k)), k = 1,2,...,m \\ g(u(0), u(T)) \le 0 \end{cases},$$

and u is an upper solution of (4.1.9) if the above inequalities are reversed.

Theorem 4.1.8 [6]: Let assumption **(H$_7$)** hold. Moreover, assume that

(H$_8$) $y_0, z_0 \in PC^1(J)$ are lower and upper solutions of problem (4.1.9), respectively, and $z_0(t) \le y_0(t)$ on J,

(H$_9$) there exist functions $K, M \in C(J, \Re)$, M is nonnegative and such that

$$f(t,u,v) - f(t,\bar{u},\bar{v}) \ge -K(t)(\bar{u} - u) - M(t)(\bar{v} - v)$$

for $z_0(t) \le u \le \bar{u} \le y_0(t)$, $z_0(\alpha(t)) \le v \le \bar{v} \le y_0(\alpha(t))$, $t \in J$,

(H$_{10}$) there exist constants $L_k \in [0,1), k = 1,2,...,m$, such that

$$I_k(w(t_k)) - I_k(\bar{w}(t_k)) \ge -L_k[\bar{w}(t_k) - w(t_k)],\ k = 1,2,...,m,$$

for any $w, \bar{w}$ with $z_0(t_k) \le w(t_k) \le \bar{w}(t_k) \le y_0(t_k,)$, $k = 1,2,...,m$,

(H$_{11}$) conditions:

$$\int_0^T M^*(t)dt \left(\prod_{i=1}^{m} (1 + L_i) \right) \le 1 \text{ with } M^*(t) = M(t) e^{\int_t^{\alpha(1)} K(s)ds} .$$

And

$$\delta \equiv \int_0^T M^*(s)ds + \sum_{i=1}^{n} L_i < 1$$

(H$_{12}$) there exists $\gamma > 0$ such that for any $u, \bar{u} \in [z_0(0), y_0(0)]$ with $u \le \bar{u}$ and $v, \bar{v} \in [z_0(T), y_0(T)]$ with $v \le \bar{v}$ we have

$$g(u,v) \geq g(\bar{u},v),$$

$$g(u,v) - g(u,\bar{v}) \leq \gamma(\bar{v} - v).$$

Then there exist solutions $v, w \in [z_0, y_0]$ of problem (4.1.9).

Proof: see [6]

Example 4.1.2 [6]: For $J = [0,T]$, we consider the problem

$$\begin{cases} y'(t) = \lambda_1(t)e^{y(t)} + \lambda_2(t)\sin(y(\alpha(t))) - \lambda_1(t), t \in J \setminus \{t_1\}, \\ \Delta y(t_1) = Ly(t_1), \\ 0 = 2y^2(0) + y(T) - k, \end{cases} \tag{4.1.10}$$

where

$$\lambda_1, \lambda_2 \in C(J, \Re^+), \Re^+ = [0,\infty), \alpha \in C(J,J), t \leq \alpha(t) \leq T, t \in J, 0 < t_1 < T, L \geq 0, 0 \leq k \leq 1.$$

Take $y_0(t) = 0, z_0(t) = -1, t \in J$. Indeed, $z_0(t) < y_0(t)$ on J, and

$$Fy_0(t) = \lambda_1(t) - \lambda_1(t) = 0 = y_0'(t),$$

$$Fz_0(t) = \lambda_1(t)[e^{-1} - 1] - \lambda_2(t)\sin 1 \leq 0 = z_0'(t),$$

$$\Delta y_0(t_1) = L.0 = I_1(y_0(t_1)),$$

$$\Delta z_0(t_1) = 0 \geq L(-1) = I_1(z_0(t_1)),$$

$$g(y_0(0), y_0(T)) = g(0,0) = -k \leq 0,$$

$$g(z_0(0), z_0(T)) = g(-1,-1) = 1 - k \geq 0.$$

It proves that y_0, z_0 are lower and upper solutions of problem (4.1.10), respectively. Moreover $K(t) = \lambda_1(t)$, $M(t) = \lambda_2(t)$, $L_1 = L$, so assumption **(H_9)**, **(H_{10})**, **(H12)** are satisfied. If we extra assume that:

$$\int_0^T \lambda_2(t) e^{\int_t^{\alpha(t)} \lambda_1(s)ds} dt + L < 1, \tag{4.1.11}$$

then problem (4.1.10) has solutions in the segment $[-1,0]$, by Theorem (4.1.8).

For example, if we take $L=\frac{1}{2}$, $T=\pi$, $\lambda_1(t)=\lambda>0$, $\lambda_2(t)=\beta e^{\lambda(t-T)}\sin t$ for $t\in J$, $\alpha(t)=\pi$ then condition (4.1.11) holds if $0<\beta<\frac{1}{4}$.

4.2 Mixed type differential equations

In this section we will introduce the oscillation of the mixed differential equation:

$$y'(t)+a_1(t)y(\tau(t))+a_2(t)y(\sigma(t))=0\,,\ t\geq t_0, \tag{4.2.1}$$

with nonnegative coefficients $a_i(t)$, one delayed argument $(\tau(t)\leq t)$ and one advanced argument $(\sigma(t)\geq t)$.

L. Berezansky and Y. Domshlak [2] studied equation (4.2.1) with both constant and variable coefficients which appears in Corollary (4.2.1) and in Theorem (4.2.1) respectively.

A special case of equation (4.2.1) is the following differential equation

$$y'(t)+a_1y(t-\tau)+a_2y(t+\sigma)=0\,, \tag{4.2.2}$$

where $\tau>0,\sigma>0,a_k>0,k=1,2$.

Corollary 4.2.1 [2]: Suppose for the characteristic polynomial of (4.2.2)

$$F(\lambda)=\lambda+a_1e^{-\lambda\tau}+a_2e^{\lambda\sigma}\,,$$

the following condition holds

$$F(\lambda)>0\text{, for all }\lambda\in(-\infty,\infty).$$

Then all solution of (4.2.2) are oscillatory.

Proof: see [2]

Theorem 4.2.1 [2]: Let $\tau(t)\to\infty$ and assume that there exist functions $b_j(t), j=1,2,$ such that

$$a_j(t)\geq b_j(t)\geq 0,\ j=1,2, t\geq t_0; \tag{4.2.3}$$

the following limits exist and finite:

$$B_{1j}:=\lim_{t\to\infty}\int_{\tau(t)}^{t} b_j(s)ds,\ B_{2j}:=\lim_{t\to\infty}\int_{t}^{\sigma(t)} b_j(s)ds,\ \ j=1,2\ \ , \tag{4.2.4}$$

with

$$B_{11}+B_{22}>0; \tag{4.2.5}$$

and the following system has a positive solution $\{y_1,y_2\}$:

$$\left.\begin{array}{l} -(B_{11}B_{22}-B_{12}B_{21})y_1y_2-B_{11}y_1+B_{22}y_2+1=0 \\ \ln y_1-B_{11}y_1-B_{12}y_2<0 \\ \ln y_2+B_{21}y_1+B_{22}y_2<0. \end{array}\right\} \tag{4.2.6}$$

Then all solution of (4.2.1) are oscillatory.

Proof: see [2]

Example 4.2.1: Consider the equation

$$y'(t)+\frac{a_1}{t}y\left(\frac{t}{\mu}\right)+\frac{a_2}{t}y(t+\sigma)=0,\ \ t\geq t_0>0, \tag{4.2.7}$$

where $\mu>1$, $\sigma>0$, $a_1,a_2>0$. Put $b_1(t):=a_1(t)=\frac{a_1}{t}$ and $b_2(t):=a_2(t)=\frac{a_2}{t}$ in Theorem (4.2.1). Then $B_{11}=a_1\ln\mu$, $B_{12}=a_2\ln\mu$, $B_{21}=B_{22}=0$.

System (4.2.6) turns into the system

$$-a_1y_1\ln\mu+1=0$$

$$\ln y_1-a_1y_1\ln\mu-a_2y_2\ln\mu<0$$

$$\ln y_2<0$$

which is equivalent to the system

$$y_1 = \frac{1}{a_1 \ln \mu}$$

$$-\ln[a_1 \ln \mu] - 1 < y_2 a_2 \ln \mu$$

$$\ln y_2 < 0$$

and this in turn is equivalent to the system

$$y_1 = \frac{1}{a_1 \ln \mu}$$

$$\frac{-\ln[a_1 \ln \mu] - 1}{a_2 \ln \mu} < y_2 < 1$$

The last system has a solution if and only if

$$\frac{-\ln[a_1 \ln \mu] - 1}{a_2 \ln \mu} < 1 \Leftrightarrow a_1 \mu^{a_2} > \frac{1}{e \ln \mu}. \tag{4.2.8}$$

Thus, (4.2.8) is sufficient for oscillation of all solution of (4.2.7). Note that (4.2.8) does not depend on σ.

4.3 Oscillation in equation of alternately retarded and advanced type

In this section we want to study the oscillation of all solutions of the following differential equation

$$y'(t) + py\left(2\left[\frac{t+1}{2}\right]\right) = 0,\ t \geq 0, \tag{4.3.1}$$

where p is a real number and $[.]$ denotes the greatest integer function.

We can look on equation (4.3.1) as equation of the form

$$y'(t) + py(t - \tau(t)) = 0,\ t \geq 0, \tag{4.3.2}$$

where the argument of deviation is given by

$$\tau(t) = t - 2\left[\frac{t+1}{2}\right]. \tag{4.3.3}$$

The argument $\tau(t)$ is a periodic function of period two. Furthermore, for every integer n, $\tau(t)$ is negative for $2n-1 \le t < 2n$ and is positive for $2n < t < 2n+1$. Therefore, in each interval $[2n-1, 2n+1)$, equation (4.3.1) is of alternately advanced and retarded type. More precisely, for every integer n,

$$\tau(t) = t - 2n \text{ for } 2n-1 \le t < 2n+1$$

And

$$-1 \le \tau(t) \le 1 \text{ for } 2n-1 \le t < 2n+1.$$

We can write $\tau(t)$ in the form

$$\tau(t) = \begin{cases} t, & 0 \le t < 1 \\ t-2, & 1 \le t < 3 \\ t-4, & 3 \le t < 5 \\ t-6, & 5 \le t < 7 \\ & \vdots \end{cases}$$

Also the curve of $\tau(t)$ can bee seen in the following figure:

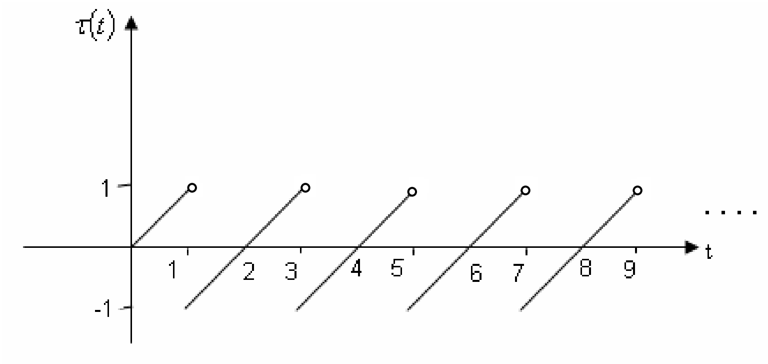

Figure (1): the graph of $\tau(t) = t - 2\left[\frac{t+1}{2}\right]$

Therefore equation (4.3.1) is of advanced type in $[2n-1,2n]$, and of retarded type in $(2n,2n+1)$.

Definition 4.3.1: Solution of equation (4.3.1)

By a solution of equation (4.3.1) we mean a function $y(t)$ which satisfies the following properties:

(i) $y(t)$ is continuous on $[0,\infty)$.

(ii) $y'(t)$ exists at each point $t \in [0,\infty)$, with the possible exception of the points $t = 2n+1$, $n \in \mathrm{N}$ where one-sided derivatives exist.

(iii) Equation (4.3.1) is satisfied on each interval of the form $[2n-1,2n+1) \cap \Re^{+}$ for $n \in \mathrm{N}$.

With equation (4.3.1) we associate an initial condition of the form

$$y(0) = a_0 , \tag{4.3.4}$$

where a_0 is a given real number.

The following lemma deals with existence and uniqueness of solution of equation (4.3.1).

Lemma 4.3.1 [5]: Assume that $p, a_0 \in \Re$ and $p \neq 1$.

Then the initial value problem (4.3.1) and (4.3.4) has a unique solution $y(t)$.

Furthermore, $y(t)$ is given by

$$y(t) = [1 - p(t-2n)]a_{2n} , \text{ for } t \in [2n-1,2n+1) \cap \Re^{+} , \ n \in \mathrm{N} , \tag{4.3.5}$$

where the sequence $\{a_n\}$ satisfies the equations

$$\left.\begin{array}{l} a_{2n+1} = (1-p)a_{2n} \ \text{ for } \ n = 0,1,2,.. \\ a_{2n-1} = (1+p)a_{2n} \ \text{ for } \ n = 1,2,... \end{array}\right\}. \tag{4.3.6}$$

Proof: Let $y(t)$ be a solution of (4.3.1) and (4.3.4). then in the interval $[2n-1,2n+1)\cap\Re^+$, and for any $n\in N$, (4.3.1) becomes

$$y'(t)+pa_{2n}=0, \tag{4.3.7}$$

where we have used the notation $a_n=y(n)$ for $n\in N$. Then the solution of (4.3.7) with initial condition $y(n)=a_{2n}$ is given by (4.3.5). By the continuity of the solution as $t\to 2n+1$ and for $t=2n-1$, (4.3.5) yields (4.3.6) and (4.3.7). So we have proved that if $y(t)$ is a solution of (4.3.1) and (4.3.4) then $y(t)$ is given by (4.3.5) where the sequence $\{a_n\}$ satisfies (4.3.6).

Conversely, given $a_0\in\Re$ and because $p\neq -1$, the equation (4.3.6) has a unique solution $\{a_n\}$. Now by direct substitution into (4.3.1) we can see that $y(t)$ as defined by (4.3.5) is a solution. The proof is complete.

The following Theorem provides necessary and sufficient conditions for the oscillation of solutions of equation (4.3.1).

Theorem 4.3.1 [5]: Assume that $p\in\Re$ and $p\neq -1$. Then every solution of equation (3.4.1) oscillates if and only if

$$p\in(-\infty,-1)\cup[1,\infty). \tag{4.3.8}$$

Proof: Assume that (4.3.8) holds. Then either $p<-1$ or $p\geq 1$ and in either case it follows from (4.3.6) that the sequence $\{a_n\}$ oscillates. As $y(n)=a_n$ for $n\in N$, $y(t)$ also oscillates. Conversely, assume that every solution $y(t)$ of (4.3.1) oscillates, and for the sake of contradiction, assume that

$$|p|<1. \tag{4.3.9}$$

Let $y(t)$ be the solution of (4.3.1) with $y(0)=a_0=1$. Then from (4.3.6) and because of (4.3.9),

$$a_n>0 \qquad \text{for } n=0,1,2,\ldots\ .$$

Hence for $t\in[2n-1,2n+1)$ and $n\in N$, $|2n-t|\leq 1$, so (4.3.5) yields

$$y(t) = [1 - p(t-2n)]a_{2n} \geq [1 - |p| \; |t-2n|]a_{2n} \geq (1-|p|)a_{2n} > 0.$$

This contradicts the assumption that $y(t)$ oscillates and the proof is complete.

Another example of alternately retarded and advanced equations is the differential equation

$$y'(t) + py\left(\left[t + \frac{1}{2}\right]\right) = 0, \; t \geq 0 \tag{4.3.10}$$

where p is a real number and $[.]$ denotes the greatest integer function.

Equation (4.3.10) can be written in the form

$$y'(t) + py(t - \sigma(t)) = 0, \; t \geq 0 \tag{4.3.11}$$

where the argument deviation is given by

$$\sigma(t) = t - \left[t + \frac{1}{2}\right],$$

is linear periodic function with period1. More precisely, for every integer n,

$$\sigma(t) = t - n, \text{ for } n - \frac{1}{2} \leq t < n + \frac{1}{2}.$$

Also $$-\frac{1}{2} \leq \sigma(t) < \frac{1}{2}, \text{ for } n - \frac{1}{2} \leq t < n + \frac{1}{2}.$$

We see that in each interval $\left[n - \frac{1}{2}, n + \frac{1}{2}\right)$, equation (4.3.10) is of alternately advanced and retarded type. It is of advanced type in $\left[n - \frac{1}{2}, n\right)$ and of retarded type in $\left(n, n + \frac{1}{2}\right)$, see figure (2).

The argument $\sigma(t)$ will be of the form

$$\sigma(t)=\begin{cases} t \quad , & 0 \le t < \frac{1}{2} \\ t-1, & \frac{1}{2} \le t < \frac{3}{2} \\ t-2, & \frac{3}{2} \le t < \frac{5}{2} \\ t-3, & \frac{5}{2} \le t < \frac{7}{2} \\ & \quad \vdots \end{cases},$$

whose sketch appears in figure (2).

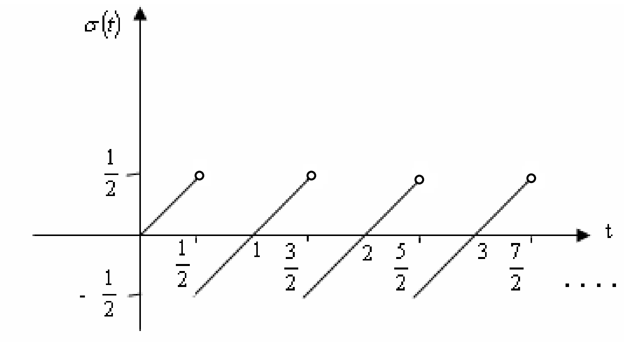

Figure (2): The graph of $\sigma(t)=t-\left[t+\frac{1}{2}\right]$

The existence and uniqueness of solution and the necessary and sufficient condition for the oscillation of all solutions of equation (4.3.10) appear in the following lemma and theorem respectively.

Lemma 4.3.2 [5]: Assume that $p, a_0 \in \Re$ and $p \neq -2$.

Then the initial value problem (4.3.10) and (4.3.4) has a unique solution $y(t)$.

Furthermore, $y(t)$ is given by

$$y(t)=[1-p(t-n)]a_n, \text{ for } t \in \left[n-\frac{1}{2}, n+\frac{1}{2}\right) \cap \Re^+, \; n \in \mathrm{N}, \tag{4.3.12}$$

where the sequence $\{a_n\}$ satisfies the equation

$$a_{n+1}=\frac{2-p}{2+p}a_n \text{, for } n=0,1,2,... \tag{4.3.13}$$

Proof: Let $y(t)$ be a solution of (4.3.10) and (4.3.4). Then in the interval $\left[n-\frac{1}{2},n+\frac{1}{2}\right)\cap\Re^+$ for any $n\in N$, (4.3.10) becomes

$$y'(t)+pa_n=0, \tag{4.3.14}$$

where we have used the notation $a_n=y(n)$ for $n\in N$. The solution of (4.3.14) with initial condition $y(n)=a_n$ is given by (4.3.12). By the continuity of the solutions as $t\to n+\frac{1}{2}$ and for $t=n-\frac{1}{2}$, (4.3.12) yields

$$y\left(n+\frac{1}{2}\right)=\left(1-\frac{1}{2}p\right)a_n \text{ and } y\left(n-\frac{1}{2}\right)=\left(1+\frac{1}{2}p\right)a_n,$$

from which (4.3.13) follows. The remaining part of the proof is similar to that of Lemma (4.3.1) and is omitted. The proof is complete.

Theorem 4.3.2 [5]: Assume that $p\in\Re$ and $p\neq-2$. Then every solution of equation (4.3.10) oscillates if and only if

$$p\in(-\infty,-2)\cup[2,\infty). \tag{4.3.15}$$

Proof: Assume that (4.3.15) holds. Then either $p<-2$ or $p\geq 2$ and in either case it follows from (4.3.13) that the sequence $\{a_n\}$ oscillates. As $y(n)=a_n$ for $n\in N$, $y(t)$ also oscillates. Conversely, assume that every solution $y(t)$ of (4.3.10) oscillates and, for the sake of contradiction, assume that

$$|p|<2. \tag{4.3.16}$$

Let $y(t)$ be the solution of (4.3.10) with $y(0)=a_0=1$. Then from (4.3.13), $a_n>0$ for $n\in N$. Hence for $t\in\left[n-\frac{1}{2},n+\frac{1}{2}\right)$ and $n\in N$, $|t-n|\leq\frac{1}{2}$, so (4.3.12) yields

$$y(t)=[1-p(t-n)]a_n\geq[1-|p|\ |t-n|]a_n\geq(1-|p|)a_n>0.$$

This contradicts the assumption that $y(t)$ oscillates and the proof is complete.

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