Transport coefficients for drifting Maxwellian plasmas: The effect of Coulomb collisions

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Transport coefficients for drifting Maxwellian plasmas: The effect of Coulomb collisions

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We derive the collisional momentum and energy transport coefficients in Maxwellian plasmas with a general drift velocity with respect to the ambient magnetic field by using two approaches, the Fokker-Planck approximation and Boltzmann collision integral. We find the transport coefficients obtained from Fokker-Planck representation are similar to those obtained by using Boltzmann collision integral approach, and both results are presented in a closed form in terms of hypergeometric functions. This has been done for drifting Maxwellian plasmas with special emphasis on Coulomb collision, i.e. inverse-square force. Also, we calculate the transport coefficients for two special cases, firstly, when the drift velocity is parallel to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\parallel}$, and zero perpendicular drift velocity), and secondly, when the drift velocity is perpendicular to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\perp}$, and zero parallel drift velocity). It is worthy to mention that, up to our knowledge, none of the derived transport coefficients for the above mentioned case are presented in closed form and in terms of hypergeometric function. https://doi.org/10.1063/1.5018318

I. INTRODUCTION

Transport equations based on an isotropic Maxwellian distribution function were first derived by Tanenbaum (1967), Burgers (1969), and reviewed by Schunk (1977). They obtained these transport equations by using Boltzmann collision integral approach and presented them in terms of the Chapman–Cowling collision integrals (Chapman and Cowling, 1970). These coefficients are valid for arbitrary temperature differences between the interacting gases, and are restricted to small relative drift velocity between the interacting gases. In this study, we removed the latter restriction and calculated transport coefficients for general drifting Maxwellian plasmas that are valid for arbitrary drift velocity differences as well as for temperature differences between the interacting plasma species. We also derived these transport coefficients for two special cases, the first one, when the drift velocity is parallel to the ambient magnetic field and the second one when the drift velocity is perpendicular to the ambient magnetic field. These coefficients are obtained by using two different approaches; Fokker-Planck approximation and Boltzmann collision integral.

This paper starts with a discussion of the theoretical formulation of Boltzmann's equation and the relevant collision terms i.e. Boltzmann collision integral and Fokker-Planck approximation. This is followed by showing the general forms of Boltzmann collision integral and Fokker-Planck approximation. Then, we derived the closed set of transport coefficients for drifting Maxwellian distribution function with emphasis on the effect of Coulomb collisions, and finally we investigated two special cases (i.e. drift velocities perpendicular and parallel to the ambient magnetic field) by using two forms of the collision terms. The last section discusses our results and future studies.

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A. Theoretical formulation

In dealing with plasma it is convenient to investigate the distribution function of these species, in general each species in the plasma is described by a separate velocity distribution function $f_s(\mathbf{r}, \boldsymbol{v}_s, \mathrm{t})$ which defined such that $f_s(\mathbf{r}, \mathbf{v}_\mathrm{s}, \mathrm{t})\, \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}_s$ represents the number density of particles of species s which at time t have positions between $\mathbf{r}$ and $\mathbf{r} + \mathrm{d}\,\mathbf{r}$ and velocities between $\boldsymbol{v}_s$ and $\boldsymbol{v}_s + \mathrm{d}\,\boldsymbol{v}_s$. The species distribution function changed with respect to time as a result of collisions and particle motions under the influence of external forces, the mathematical description of this effect is giving by Boltzmann's equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}_\mathrm{s} \cdot \nabla f_s + \left[\mathbf{G} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_\mathrm{s} \times \mathbf{B}\right)\right] \cdot \nabla_{\mathbf{v_s}} f_s = \frac{\delta f_s}{\delta t} \tag{1}$$

where $\mathrm{q_s}$, and $\mathrm{m_s}$, are the charge and mass of species s, $\mathbf{G}$ is the acceleration due to gravity, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, c is the speed of light, $\partial/\partial \mathrm{t}$ is the time derivative, ∇ is the coordinate space gradient, $\nabla_{\mathrm{v}\boldsymbol{s}}$ is the velocity space gradient, and the quantity $\delta f_s / \delta \mathrm{t}$ represents the rate of change of f_s due to the collisions, this term is given in different forms, in this study we are interested in Boltzmann collision integral and Fokker-Planck approximation forms.

B. Boltzmann collision integral

For binary elastic Coulomb collision between s and t charged particles, the appropriate collision term is the Boltzmann collision integral, which can be presented as

$$\frac{\delta f_s}{\delta t} = \sum_t \int d\mathbf{v_t} d\Omega g_{st} \sigma_{st}\,(g_{st}, \theta)\,[f_s' f_t' - f_s f_t] \tag{2}$$

where $\mathrm{d}\mathbf{v_t}$ is the velocity-space volume element of species t, $\mathrm{g_{st}}$ is the relative velocity of the colliding particles s and t, $\mathrm{d}\Omega$ is an element of solid angle in the s particle reference frame, θ is the scattering angle, the primes denote quantities evaluated after a collision, and $\sigma(\mathrm{g_{st}}, \theta)$ is the differential scattering cross-section (Goldstein, 1980; Schunk and Nagy, 2009):

$$\sigma = \frac{q_s^2 q_t^2}{64\pi^2 \varepsilon_o^2 \mu_{st}^2} \frac{1}{g^4 \sin^4\theta} \tag{3}$$

where $\mathrm{q_s}$ and $\mathrm{q_t}$ are the charges of species s and t species, respectively, $\mu_{st} = m_s m_t/(m_s + m_t)$ is the reduced mass, $\mathrm{m_t}$ and $\mathrm{m_s}$ are the masses of particles t and s, and ε_o is the permittivity of free space.

C. Fokker-Planck approximation

Sometimes Boltzmann collision integral appears to be difficult to evaluate, so that the Boltzmann collision integral (i.e. Eq. (2)) reduces to another simpler form by taking the first order of Taylor expansion of it under the assumption that a series of consecutive weak (small-angle deflection) binary collisions is a valid representation for the Coulomb interactions, the result is the Fokker-Planck approximation.

$$\frac{\delta f_s}{\delta t} = -\sum_t \nabla_v \cdot \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_o^2 m_s} \int \frac{1 g^2 - \mathbf{gg}}{g^3} \cdot \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{v_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{v_s}}\right) d^3\mathbf{v_t} \tag{4}$$

where 1 is the unity tensor, and ln Λ is the Coulomb logarithm, which is typically between 10 to 25 for space plasmas.

The moments of f_s are most conveniently defined in terms of the random or thermal velocity of the species s, $\mathbf{c_s}$, with respect to their own mean flow velocity, $\mathbf{u_s}$, as follow

$$\mathbf{c_c} = \mathbf{v_s} - \mathbf{u_s} \tag{5}$$

so that the integration over the velocity space $\mathbf{dc}_\mathrm{s} = \mathbf{dv_s}$ and the only difference being a displacement of the origin of the velocity space (Grad,1949,1958; Burgers, 1969). The advantage of it that if there are large drift velocity difference or temperature difference between interacting species, the velocity distribution function of a given species more likely to be Maxwellian about its own drift velocity than to be Maxwellian about the average velocity. Consequently, a series expansion of the species distribution function about Maxwellian will converge more rapidly if the species average drift velocity is used to define the transport properties (Schunk, 1977).

II. TRANSPORT COEFFICIENTS

The starting point for the derivation of transport coefficients for gas mixtures is Boltzmann's equation i.e. Eq. (1). The transport equations are obtained by multiplying the right hand side of the Boltzmann's equation by an appropriate function of velocity $Q_s = Q_s(\mathbf{c_s})$ and then integrating over all velocity space. The resulting transport coefficients describe the effect of collisions between different species.

If we multiply the right hand side of Eq. (1) by $Q_s = 1$, $m_s\mathbf{c_s}$, and $m_s c_s^2/2$ and integrate over velocity space, we obtain the rate of change of density, momentum and energy, and are symbolically written as $\delta n_s/\delta t$, $\delta M_s/\delta t$, and $\delta E_s/\delta t$, respectively, for species s.

For Boltzmann collision integral, the corresponding transport coefficients are given as

$$\frac{\delta Q_s}{\delta t} = \iiint d^3c_s d^3c_t d\Omega g_{st}\sigma_{st}(g_{st},\theta)\,[f_s' f_t' - f_s f_t]Q_s \tag{6}$$

Due to reversibility of elastic collision, we can interchange primed and unprimed quantities in the Eq. (6) without changing the result (Schunk and Nagy, 2009).

$$\frac{\delta Q_s}{\delta t} = \iiint d^3c_s d^3c_t d\Omega g_{st}\sigma_{st}(g_{st},\theta) f_s f_t\,[Q_s' - Q_s)] \tag{7}$$

where Q′ is a function of velocity after the collision. The evaluation of Eq. (7) is easier than that of Eq.(6), as it does not require the calculation of $f'_s f'_t$.

However, this integral can be evaluated by transform it from $(\mathbf{c_s},\mathbf{c_t})$ to $(\mathbf{V_c}, \mathbf{g_{st}})$, where $\mathbf{V_c}$ is the center-of-mass velocity, and $\mathbf{g_{st}}$ is the relative velocity, which are

$$\mathbf{g_{st}} = \mathbf{v_s} - \mathbf{v_t} \tag{8}$$

$$\mathbf{V_c} = \frac{m_s\mathbf{v_s} + m_t\mathbf{v_t}}{m_s + m_t} = \frac{(m_s\mathbf{c_s} + m_t\mathbf{c_t} + m_s\mathbf{u_s} + m_t\mathbf{u_t})}{(m_s + m_t)} \tag{9}$$

And the next step in evaluating the collision integral is to integrate over the solid angle $d\Omega = \sin\theta\, d\theta\, d\varphi$ by using spherical polar coordinate system in the center of mass reference frame, so we obtain

Density

$$\frac{\partial n_s}{\partial t} = 0 \tag{10}$$

Momentum

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t -\mu_{st}\iint d^3c_s d^3c_t g_{st}\mathbf{g_{st}} f_s f_t Q_{st}^{(1)} \tag{11}$$

Energy

$$\frac{\delta E_\mathbf{s}}{\delta t} = \sum_t -\mu_{st}\iint d^3c_s d^3c_t g_{st} f_s f_t\,(\mathbf{V_c}.\mathbf{g_{st}})\,Q_{st}^{(1)} \tag{12}$$

where

$$Q^{(1)} = 2\pi\int_{\theta_{\min}}^{2\pi}\sigma_{st}(g_{st},\theta)(1-\cos\theta)\sin\theta d\theta = 4\pi\left(\frac{q_s q_t}{4\pi\varepsilon_\circ\mu_{st}g^2}\right)^2 \ln\Lambda \tag{13}$$

Also, these moments can be obtained by using the Fokker-Planck approximation by multiplying it with an appropriate function of velocity $Q_s = Q_s(\mathbf{c_s})$ and integrating over all velocity space as follows:

$$\frac{\delta Q_s}{\delta t} = -\sum_t \nabla_v.\frac{q_s^2 q_t^2\ln\Lambda}{8\pi\varepsilon_\circ^2 m_s}\int\frac{1g^2 - \mathbf{gg}}{g^3}\boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial\mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial\mathbf{c_s}}\right)Q(c_s)\,d\mathbf{c_s}d\mathbf{c_t} \tag{14}$$

After integration by parts, the corresponding transport coefficients can be expressed as

Density

$$\frac{\partial n_s}{\partial t}=0 \tag{15}$$

Momentum

$$\frac{\delta \mu_s}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln \Lambda}{8\pi\varepsilon_\circ^2 n_s}\int \frac{1g^2-\mathbf{gg}}{g^3}\boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t} \tag{16}$$

Energy

$$\frac{\delta E_s}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln \Lambda}{4\pi\varepsilon_\circ^2 n_s}\int \mathbf{c_s}\frac{1g^2-\mathbf{gg}}{g^3}\boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t} \tag{17}$$

Note that for all elastic collisions the rate of change of density is zero because the particle's mass does not change.

The remaining integrals in equations (11, 12, 16, 17) can be evaluated after adopting approximate expressions for f_s and f_t. However, in this study we assume the distribution function to be drifting Maxwellian function. This assumption will be used to evaluate these integrals.

III. TRANSPORT COEFFICIENTS FOR DRIFTING MAXWELLIAN VELOCITY DISTRIBUTION FUNCTION

As noted in the last section, it is necessary to adopt approximate expression for the species velocity distribution functions, in order to evaluate the transport coefficients as presented in equations (11,12, 16,17). So we assume all colliding species in the gas have drifting Maxwellian velocity distributions function. This case is known as the 5-moment approximation because each species in the gas mixture is characterized by five parameters (i.e. species density, three components of drift velocity, and temperature).

$$f_s=\left(\frac{m_s}{2\pi k T_s}\right)^{3/2} e^{-\frac{m_s c_s^2}{2kT_s}} \tag{18}$$

$$f_t=\left(\frac{m_t}{2\pi k T_t}\right)^{3/2} e^{-\frac{m_t c_t^2}{2kT_t}} \tag{19}$$

In the following sub-sections, we will derive the transport coefficients by using firstly Boltzmann collision integral and then Fokker-Planck approximation, and finally verify that they are equivalent.

A. Boltzmann collision integral

The rate of change of the momentum and energy are obtained from equations (11) and (12) respectively, the term $f_s f_t$ can be expressed as

$$f_s f_t=n_s n_t\left(\frac{m_s}{2\pi k T_s}\right)^{3/2}\left(\frac{m_t}{2\pi k T_t}\right)^{3/2} e^{-\frac{m_s c_s^2}{2kT_s}-\frac{m_t c_t^2}{2kT_t}} \tag{20}$$

The integrations over the velocity space can be performed by introducing the following variables as follows:

$$\mathbf{c_s}=\mathbf{c}_*-\frac{m_t T_s}{m_s T_t+m_t T_s}\mathbf{g}_* \tag{21}$$

$$\mathbf{c}_t=\mathbf{c}_*+\frac{m_s T_t}{m_s T_t+m_t T_s}\mathbf{g}_* \tag{22}$$

where

$$\mathbf{c}_* = \mathbf{V_c} - \mathbf{u_c} + \frac{T_t - T_s}{m_s T_t + m_t T_s}\mu_{st}^2\left(\mathbf{\Delta u} + \mathbf{g}\right) \tag{23}$$

$$\mathbf{g}_* = -\mathbf{g} - \Delta\mathbf{u} \tag{24}$$

$$\Delta\mathbf{u} = \mathbf{u}_t - \mathbf{u}_s \tag{25}$$

$$\mathbf{u_c} = \frac{m_s\mathbf{u_s} + m_t\mathbf{u_t}}{m_s + m_t} \tag{26}$$

We also introduce

$$a^2 = \frac{2kT_sT_t}{m_sT_t + m_tT_s}; \quad \alpha^2 = \frac{2k\left(m_sT_t + m_tT_s\right)}{m_sm_t} \tag{27}$$

And

$$d\mathbf{c_s}d\mathbf{c_t} = d\mathbf{c}_*d\mathbf{g}_* \tag{28}$$

Substituting Equations from 21 to 28 into Eq.(20) and then into the expression for $\delta\mathbf{M}_s/\delta t$ (11) and $\delta E_s/\delta t$ (12) yields

$$\frac{\delta\mathbf{M_s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^3a^3\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int e^{-c_*^2/a^2}d\mathbf{c}_* \int \frac{\mathbf{g}}{g^3}e^{-g_*^2/\alpha^2}d\mathbf{g}_* \tag{29}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^3a^3\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int e^{-c_*^2/a^2}d\mathbf{c}_* \int \frac{(\mathbf{V_c}.\mathbf{g})}{g^3}e^{-g_*^2/\alpha^2}d\mathbf{g}_* \tag{30}$$

Integration with respect to d$\mathbf{c}_*$ can be evaluated immediately, using a spherical coordinate system (Gaussian integral)

$$\int e^{-c_*^2/a^2}d\mathbf{c}_* = \pi^{3/2}a^3 \tag{31}$$

It remains to carry out the integrations with respect to d$\mathbf{g}_*$. We expressed $\mathbf{g}_*$in terms of $\mathbf{g}$ and $\mathbf{u}$

$$g_*^2 = g^2 + 2g\Delta g\Delta uco + (\Delta u)^2 \tag{32}$$

where θ is the angle between **g** and **v**.

With these changes, Eq. (29) and Eq. (30) become

$$\frac{\delta\mathbf{M_s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^{3/2}\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int \frac{\mathbf{g}}{g^3}e^{-\left(g^2+2g\Delta g\Delta uco+(\Delta u)^2\right)/\alpha^2}d\mathbf{g} \tag{33}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^{3/2}\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int \frac{(\mathbf{V_c}.\mathbf{g})}{g^3}e^{-\left(g^2+2g\Delta g\Delta uco+(\Delta u)^2\right)/\alpha^2}d\mathbf{g} \tag{34}$$

Schunk (1977) calculated these remaining integral over g by expanded the exponential terms with cosθ, and he assumed a small relative drifts between the interacting gases (i.e., when the drift velocity differences are much smaller than thermal speeds), so that Schunk (1977) neglected the exponential term of $(\Delta u/\alpha)^2$, and finally the transport coefficients expressed in terms of the so-called Chapman–Cowling collision integrals (Chapman and Cowling, 1970).

In this study, we removed the latter restriction and calculated transport coefficients by using other strategy in which we introduced new variables as follow:

$$\mathbf{v} = \frac{\mathbf{g}}{\alpha}; \mathbf{V} = \frac{2\Delta\mathbf{u}}{\alpha} \tag{35}$$

The momentum and energy exchange collision terms reduce to

$$\frac{\delta\mathbf{M_s}}{\delta t} = -\sum_t \frac{n_tq_s^2q_t^2\ln\Lambda}{2\pi^{3/2}\varepsilon_\circ^2\mu_{st}\alpha^3}e^{-\Delta u^2/\alpha^2}\int_0^\pi\int_0^\infty e^{-\left(v^2+vV\cos\theta\right)}\alpha\sin\theta\cos\theta dvd\theta \tag{36}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{n_tq_s^2q_t^2\ln\Lambda}{2\pi^{3/2}\varepsilon_\circ^2\mu_{st}\alpha^3}e^{-\Delta u^2/\alpha^2}\int_0^\pi\int_0^\infty \frac{1}{m_s+m_t}\left(m_sc_s + m_tc_t - m_t\left(u_s - u_t\right)\right)e^{-\left(v^2+vV\cos\theta\right)} \times\alpha\sin\theta\cos\theta dvd\theta \tag{37}$$

This integral can be evaluated by using the technique Maclaurin series expansion for the exponential terms with cosθ, and then express them in terms of the hypergeometric function, so with these changes the final expressions for the coefficients are

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st}\alpha^2} e^{-\Delta u^2/\alpha^2} F\left(\frac{1}{2};\left(\frac{\Delta \mathbf{u}}{2}\right)^2\right) \tag{38}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 m_s m_t \alpha^2} e^{-\Delta u^2/\alpha^2} \left(3k\left(T_t - T_s\right) - m_t(\mathbf{u_s} - \mathbf{u_t}) F\left(\frac{1}{2};\left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right) \tag{39}$$

where $F\left(\frac{1}{2};\left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)$ is hypergeometric function(Lebedev, 1965; Koepf, 2014).

B. Fokker-Planck approximation

The first step in evaluating the transport coefficient for momentum by using the Fokker Planck approximation is the derivation of f_s *and* f_t with respect to $\mathbf{c_s}$, $\mathbf{c_t}$ respectively as follows

$$\frac{\partial f_s}{\partial \mathbf{c_s}} = -\frac{2\mathbf{c_s}}{a_s^2} f_s \tag{40}$$

$$\frac{\partial f_t}{\partial \mathbf{c_t}} = -\frac{2\mathbf{c}_t}{a_t^2} f_t \tag{41}$$

Where

$$a_s^2 = \frac{2kT_s}{m_s}; \quad a_t^2 = \frac{2kT_t}{m_t} \tag{42}$$

Then

$$\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right) = \frac{-2 f_s f_t}{m_s m_t}\left(\frac{m_s \mathbf{c}_t}{a_t^2} - \frac{m_t \mathbf{c_s}}{a_s^2}\right) \tag{43}$$

When this term is substituted into Equation (16) and use is made of the relations

$$\frac{1g^2 - \mathbf{gg}}{g^3} \boldsymbol{.} \left(\frac{m_s \mathbf{c}_t}{a_t^2} - \frac{m_t \mathbf{c_s}}{a_s^2}\right) = \frac{1}{(m_s + m_t)}\frac{\mathbf{g}}{g^3} \tag{44}$$

the result is

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st}} \iint d^3c_s d^3c_t g_{st} \mathbf{g_{st}} f_s f_t \tag{45}$$

This is the same as we obtained from Boltzmann collision integral (i.e. Eq. (11)), so the final expression is

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st}\alpha^2} e^{-\Delta u^2/\alpha^2} F\left(\frac{1}{2};\left(\frac{\Delta \mathbf{u}}{2}\right)^2\right) \tag{46}$$

Similarly, the energy coefficient $\delta \mathrm{E}_s/\delta t$ can be calculated as we did for the momentum coefficient $\delta \mathbf{M}_s/\delta t$. We obtained approximately similar results as those obtained from Boltzmann collision integral.

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 m_s m_t \alpha^2} \left[e^{-\Delta u^2/\alpha^2} \left(3k\left(T_t - T_s\right) - m_t(\mathbf{u_s} - \mathbf{u_t}) F\left(\frac{1}{2};\left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right) + \left(\frac{m_s \mathbf{c}_s}{2} + m_t \mathbf{c}_s + m_s \mathbf{c}_t\right)\right] \tag{47}$$

The comparison between the results of Eq. (47) and Eq. (39) produce similar results and the little difference due to the Fokker-Planck approximation which obtained from expanding the Boltzmann collision integral and taking first terms in the Taylor series and neglect the other terms.

C. Special cases

1) ($\mathbf{u}_{\parallel}=0$, i.e. $\mathbf{u}=\mathbf{u}_{\perp}$), zero drift velocity parallel to the ambient magnetic field, and the drift velocity is perpendicular to the ambient magnetic field, the transport coefficients equations reduce to:

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 \mu_{st} \alpha^2} e^{-\Delta u_\perp^2/\alpha^2} F\left(\frac{1}{2}; \left(\frac{\Delta \mathbf{u}_\perp}{2}\right)^2\right) \tag{48}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 m_s m_t \alpha^2} \left[e^{-\Delta u_\perp^2/\alpha^2} \left(3k\,(T_t - T_s) - m_t(\mathbf{u_{s\perp}} - \mathbf{u_{t\perp}}) F\left(\frac{1}{2}; \left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right); \right.$$
$$\left. + \left(\frac{m_s \mathbf{c}_s}{2} + m_t \mathbf{c}_s + m_s \mathbf{c}_t\right) \right] \tag{49}$$

2) ($\mathbf{u}_{\perp}$=0, i.e. $\mathbf{u}=\mathbf{u}_{\parallel}$), zero drift velocity component perpendicular to the ambient magnetic field, and the drift velocity is parallel to the ambient magnetic field, the transport coefficients take the form:

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 \mu_{st} \alpha^2} e^{-\Delta u_\parallel^2/\alpha^2} F\left(\frac{1}{2}; \left(\frac{\Delta \mathbf{u}_\parallel}{2}\right)^2\right) \tag{50}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 m_s m_t \alpha^2} \left[e^{-\Delta u_\perp^2/\alpha^2} \left(3k\,(T_t - T_s) - m_t \left(\mathbf{u_{s\parallel}} - \mathbf{u_{t_\parallel}}\right) F\left(\frac{1}{2}; \left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right) \right.$$
$$\left. + \left(\frac{m_s \mathbf{c}_s}{2} + m_t \mathbf{c}_s + m_s \mathbf{c}_t\right) \right] \tag{51}$$

These coefficients derived by using Fokker-Planck approximation are, nearly, similar to the results obtained by using Boltzmann collision integral approach.

IV. RESULTS AND DISCUSSIONS

For temperature isotropic plasmas, we obtained the transport coefficients (density, momentum, and energy) based on a drifting Maxwellian velocity distribution functions with drift velocity $\mathbf{u}$ with respect to the ambient magnetic field (i.e. $\mathbf{u}=\mathbf{u}_{\parallel}+\mathbf{u}_{\perp}$) by using Boltzmann collision integral, and Fokker Planck approximation. The final results are presented in the closed form in terms of hypergeometric functions. The two approaches produce approximately similar results.

We extended the work of Schunk (1977) and calculated the transport coefficients by using Boltzmann collision integral for two special cases where the relative drift is either parallel or perpendicular to the magnetic field, which are the two most common cases in astronomy and space physics. Then we investigated the previously problem by using another approach, Fokker Planck approximation, we obtained nearly similar results. The transport coefficients are presented in the form of hypergeometric functions. These results can be further generalized to an inverse power force interaction.

Finally, it should be noted that we derived the closed set of the collisional momentum and energy transport coefficients, however Chapman and Cowling (1970) calculated these coefficients approximately and for special case i.e. when the drift velocity differences between the various species are much smaller than typical thermal speeds, and they performed approximation for some specific collision processes.

Similarly, Jubeh and Barghouthi (2017) derived the above transport coefficients for bi-Maxwellian drifting plasma with special emphasis on the effect of Coulomb collisions. In an on-going study we are interested to derive, in closed form, the velocity diffusion coefficients for both cases, Maxwellian and bi-Maxwellian drifting plasma, and provide them in terms of Hypergeometric functions. These diffusion coefficients are going to be very useful to the solar and polar wind communities, especially in modeling the plasma behavior in these regions.

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