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A Generalized Definition of the Fractional Derivative with Applications in Newtonian Mechanics

Samah Mousa Khalaf Bajali

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Prepared by:

Samah Mousa Khalaf Bajali

B.Sc. Mathematics

Al-Quds University-Palestine

Supervisor: Dr. Ibrahim Algrouz

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Prepared by: Samah Mousa Khalaf Bajali

Registration No: 22011223

Supervisor: Dr. Ibrahim Alghrouz

Master thesis submitted and accepted, Date: 27/5/2023.

Names and signatures of examining committee members are as follows:

1. Head of Committee: Dr. Ibrahim Alghrouz Signature: Ibrahim
2. Internal Examiner: Dr. Jamil Abu Ismail Signature: Jamil Jamal.
3. External Examiner: Dr. Yousif Bdeir Signature:

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Dedication

I present this as a way of gratitude to my mother for, encouraging me and supporting me throughout the whole experience.

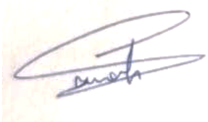
To my late father who was the reason I even started this journey.

To all people who encouraged me.

I present this to all of them.

Declaration

I certify that this thesis submitted for the degree of master is the result of my own research, except where otherwise acknowledge, and this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: 

Student Name: Samah Mousa Kalaf Bajali

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Abstract

This work studies the proposed new generalized fractional derivative (GD) definition, showing that the index law $D^{\alpha}D^{\beta}f(t) = D^{\alpha+\beta}f(t)$; $0 < \alpha, \beta \leq 1$ works for a differentiable function expanded by a Taylor series. (GD) is applied for some functions, the results are compared with Caputo fractional derivative. The solutions of some fractional differential equation are obtained via the (GD) operator. A comparison with the conformable derivative (CD) is also discussed. Newtonian Mechanics is discussed in the light of the fractional calculus.

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Chapter one

Introduction

Fractional calculus is not a new concept in math, which has applications in several fields of science such as economics [5], biology [18]. A lot of definitions for the fractional derivative have been introduced over more than four hundred years. Every definition has its own advantages and disadvantages. with Caputo (Ca) and Riemann–Liouville (R-L) definitions being the most used ones [10].

For the Caputo definition, the γ −derivative of $f(t)$, where $\gamma \in (n-1, n]$ is defined as follows

$$ {}_aD_\gamma^{Ca} f(t) = \frac{1}{\Gamma(\mathrm{n}-\gamma)} \int_a^t ((t-x)^{n-\gamma-1} \frac{d^n f(x)}{dx^n}) dx $$

For the Riemann–Liouville definition, the γ −derivative of $f(t)$, where $\gamma \in (n-1, n]$ is defined as follows

$$ {}_aD_\gamma^{R-L} f(t) = \frac{1}{\Gamma(\mathrm{n}-\gamma)} \frac{d^n \left(\int_a^t (t-x)^{n-\gamma-1} f(x)\, dx \right)}{dx^n} $$

For the conformal derivative (CD) definition of a function $f: (0, \infty) \longrightarrow \mathbb{R}$, the γ −derivative of $f(t)$ at $t > 0$, where $\gamma \in (0, 1]$ is defined as follows

$$ D_\gamma^{CD} f(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f(t+\varepsilon t^{1-\gamma}) - f(t)}{\varepsilon} $$

And if the limit exists, we have

$$ D_\gamma^{CD} f(0) = \lim_{\mathrm{t} \longrightarrow 0+} D_\gamma^{CD} f(t) $$

Or for differentiable functions,

$$D_{\gamma}^{CD} f(t) = t^{1-\gamma} \frac{df(t)}{dt}$$

The main drawback of R-L is that it does not give zero when differentiating a constant.

The main advantage of the Conformable derivative over Ca and R-L is that it simplifies the process in finding the derivative of product and quotient of functions.

As $D\ (fg) = f\ (D\ (g)) + g\ (D\ (f))$ and $D(\frac{f}{g}) = \frac{g\ (D(f)) - f\ (D(g))}{g^2}$ both are true for only the CD definition.

Also, Ca, R-L and CD definitions all do not satisfy $D^{\alpha} D^{\beta}(f) = D^{\alpha+\beta}(f)$ which is a main advantage for the generalized definition which is discussed in this thesis.

The study is based on analysing the CD definition then applying and comparing its properties with the proposed definition [1],[2],[10],[14].

In this thesis, we have chapters organised as follows. In chapter two, we study the generalized fractional derivative definition introduced by Kaabar [3] we discuss the main theorems and properties related to the definition we also apply the definition on some functions.

In chapter three we use the generalized fractional derivative operator to solve some fractional differential equations and comparing its results with other fractional operators. Furthermore, in chapter four we apply the (GD) fractional operator to the Newtonian mechanics, while trying to find the physical meaning for fractional physics.

Chapter two

Generalized Fractional Calculus

In this chapter, we study the new generalized definition of the fractional derivative introduced by Kaabar [3] which gives more accurate results than the well-known conformable derivative definition, we analyse the properties and related theorems.

2.1. Generalized Fractional Derivative

In this section we mention, discuss and complement the properties of the new generalized definition of the fractional derivative (GD) which was recently introduced by Kaabar [3]. The definition is an improvement to the known conformable fractional derivative (CD).

Definition 2.1. For a function $f\colon (0,\infty) \longrightarrow \mathbb{R}$, the γ −derivative of $f(t)$ at $t > 0$, where $\gamma \in (0,1]$ is defined as follows

$$\left(D_{\gamma}^{GD}\right)f(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\varepsilon t^{1-\gamma}\right) - f(t)}{\varepsilon}; \quad \rho \in \mathbb{R}^{+}.$$

Where Γ refers to the gamma function with $\Gamma(s) = \int_0^\infty z^{s-1}e^{-z}dz$.

If f is γ − differentiable at $(0,a)$ for some $a > 0$ and the limit exists, then at $t = 0$ the fractional derivative is defined as follows

$$\left(D_{\gamma}^{GD}\right)f(0) = \lim_{t\longrightarrow 0+} D^{GD}\, f\,(t)$$

We say that f is $\gamma\ -$ differentiable function if the generalized derivative of f of order γ exists. For ease we are using $L^{\gamma}(\,f(t))$ to denote the generalized derivative $(D_{\gamma}^{GD})\,f\,(t)$.

Remark 2.1. When $\gamma\ =\ 1$ we obtain the classical derivative definition

$$(D_{1}^{GD})f(t) = \lim_{\varepsilon\longrightarrow 0}\frac{f\,(t\ +\varepsilon\,) - f(t)}{\varepsilon}$$

Theorem 2.1. If $f(t)$ is a differentiable function then $f(t)$ is an $\gamma\ -$differentiable function, with

$$L^{\gamma}f(t) = \frac{\Gamma(\rho)}{\Gamma(\rho-\ \gamma+1)}\ t^{1-\gamma}\ \frac{d\,f(t)}{dt}\ ;\ \rho\ \in\ \mathbb{R}^{+}$$

Proof: By the definition of $L^{\gamma}f(t)$,we have

$$L^{\gamma}\,f(t) = \lim_{\varepsilon\longrightarrow 0}\frac{f\,(t+\frac{\Gamma(\rho)}{\Gamma(\rho-\gamma+1)}\varepsilon t^{1-\gamma}\,) - f(t)}{\varepsilon};\ \rho\ \in\ \mathbb{R}^{+}$$

Let $h = \frac{\Gamma(\rho)}{\Gamma(\rho-\gamma+1)}\varepsilon t^{1-\gamma}$ then $\varepsilon = \frac{\Gamma(\rho-\gamma+1)}{\Gamma(\rho)}h\ t^{\gamma-1}$,then $h\longrightarrow 0$ when $\varepsilon\longrightarrow 0$

Substituting in the definition we get

$$L^{\gamma}f(t)\ = \frac{\Gamma(\rho)}{\Gamma(\rho-\gamma\ +\ 1)}t^{1-\gamma}\lim_{h\longrightarrow 0}\frac{f\,(t\ +h\,) - f(t)}{h}$$

Hence,

$$L^{\gamma}f(t) = \frac{\Gamma(\rho)}{\Gamma(\rho-\gamma+1)}t^{1-\gamma}\frac{df(t)}{dt}\qquad \square$$

Theorem 2.2. If a function $f:[a,b] \longrightarrow \mathbb{R}$, is γ −differentiable function at $z > 0$, for some $\gamma \in (0,1]$, then f is continuous at z.

Proof: As the function is γ −differentiable function at $t = z$, the definition states

$$(L^{\gamma}(f(z)) = \lim_{\varepsilon \to 0} \frac{f\,(z + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon z^{1-\gamma}) - f(z)}{\varepsilon}$$

We can look into

$$f\left(z + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon z^{1-\gamma}\right) - f(z) = \frac{f\,(z + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon z^{1-\gamma}) - f(z)}{\varepsilon} \varepsilon$$

As $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} f\,(z + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon z^{1-\gamma}) - f(z) = L^{\gamma} f(z) * (0)$$

We assume $h = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon z^{1-\gamma}$, it's clear that $h \to 0$ as $\varepsilon \to 0$, then

$$\lim_{h \to 0} f\,(z + h) - f(z) = 0$$

$$\text{Or } \lim_{h \to 0} f\,(\mathrm{z} + \mathrm{h}) = f(z)$$

Hence, f is continuous at z. □

Theorem 2.3.

(i) $L^{\gamma} f(t) = 0$, for constant functions where $f(t) = c$.

(ii) $L^{\gamma} f(t) = \frac{\Gamma(k+1)}{\Gamma(k-\gamma+1)} t^{k-\gamma}$, for $f(t) = t^k; k \in \mathbb{R}^+$.

Proof:

Using theorem 2.1.

For (i) we have,

$$L^{\gamma} f(t) = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma}. 0 = 0$$

For (ii) we get,

$$L^{\gamma} f(t) = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} k\, t^{k-1}$$

$$L^{\gamma} f(t) = \frac{k\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\ t^{k-\gamma}$$

Taking $\rho = k$, we get

$$L^{\gamma} f(t) = \frac{k\Gamma(k)}{\Gamma(k - \gamma + 1)}\ t^{k-\gamma}$$

But $z\Gamma(z) = \Gamma(z+1)$ hence,

$$L^{\gamma} f(t) = \frac{\Gamma(k+1)}{\Gamma(k-\gamma+1)}\ t^{k-\gamma} \qquad \square$$

Remark 2.2. Taking into consideration the $\gamma -$ derivative of $f(t) = t^{\gamma}$ where $\gamma = k$,then $L^{\gamma}(t^{\gamma}) = \Gamma(k+1) = \Gamma(\gamma+1)$. Hence it is easy to see that $L^{\gamma}\left(\frac{1}{\Gamma(\gamma+1)} t^{\gamma} \right) = 1$

Theorem 2.4. Let $\gamma \in (0,1]$, $t > 0$ and $f, g\ be\ \gamma -$ differentiable functions, then

(i) $L^{\gamma}\,(a\,f + b\,g)(t) = a\,L^{\gamma}\left(f(t)\right) + b\,L^{\gamma}\left(g(t)\right)$; a, b are constants,

(ii) $L^{\gamma}(f\,g)(t) = f(t)L^{\gamma}(g(t)) + g(t)\,L^{\gamma}(f(t))$,

(iii) $L^{\gamma}(\frac{f}{g})\,(t) = \frac{g(t)\,L^{\gamma}(f)(t) - f(t)\,L^{\gamma}(g)(t)}{[g(t)]^2}$,

(iv) $L^{\gamma}\,(f \circ g)(t) = \acute{f}(g(t)\,L^{\gamma}(g(t))$, and $\acute{f}(g(t)$ exist

Proof:

Using Definition 2.1.

for convenience sometimes we would use the substitution

$$u = t + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\varepsilon t^{1-\gamma}$$

For (i) we have,

$$L^{\gamma}\,(a\,f + b\,g)(t) = \lim_{\varepsilon \to 0} \frac{(a\,f + b\,g)\,(t + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\varepsilon t^{1-\gamma}) - (a\,f + b\,g)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(a\,f)\left(t + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\varepsilon t^{1-\gamma}\right) + (b\,g)\left(t + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\varepsilon t^{1-\gamma}\right) - (a\,f)(t) - (b\,g)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(a\, f)\left(t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)} \varepsilon t^{1-\gamma}\right) - (a\, f)(t)}{\varepsilon}$$

$$+ \lim_{\varepsilon \to 0} \frac{(b\, g)\left(t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)} \varepsilon t^{1-\gamma}\right) - (b\, g)(t)}{\varepsilon}$$

$$= a\, L^{\gamma}(f)(t)\ +\ b\, L^{\gamma}(g)(t)$$

For (ii) we have,

$$L^{\gamma}(f\ g)(t) = \lim_{\varepsilon \to 0} \frac{(fg)\ (t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)} \varepsilon t^{1-\gamma}\ ) - (fg)(t)}{\varepsilon}$$

We need to add and subtract the value $(f)(t)(g)(u\ )$

$$= \lim_{\varepsilon \to 0} \frac{(f)(u\ )(g)(u\ ) - (f)(t)(g)(u\ ) + (f)(t)(g)(u\ ) - (f)(t)(\mathrm{g})(\mathrm{t})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(f)(u\ )(g)(u\ ) - (f)(t)(g)(u\ )}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{(f)(t)(g)(u\ ) - (f)(t)(\mathrm{g})(\mathrm{t})}{\varepsilon}$$

$$= (L^{\gamma}(f)(t))\ \lim_{\varepsilon \to 0} g(u\ )\ + (f)(t)\ (L^{\gamma}(g)(t))$$

but $\lim_{\varepsilon \to 0} g(u) = g(\lim_{\varepsilon \to 0} t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon t^{1-\gamma}) =\ g(t)$

$$=\ (L^{\gamma}(f)(t))\, g(t)\ + (f)(t)\ (L^{\gamma}(g)(t))$$

For (iii) we have,

$$L^{\gamma}\left(\frac{f}{g}\right)(t) = \lim_{\varepsilon\longrightarrow 0} \frac{(\frac{f}{g})\ (t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)}\varepsilon t^{1-\gamma}\ ) - (\frac{f}{g})(t)}{\varepsilon}$$

$$= \lim_{\varepsilon\longrightarrow 0} \frac{\frac{(f)\ (t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)}\varepsilon t^{1-\gamma}\ )}{(g)\ (t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)}\varepsilon t^{1-\gamma}\ )} - \frac{(f)(t)}{(g)(t)}}{\varepsilon}$$

$$= \lim_{\varepsilon\longrightarrow 0} \frac{\frac{(f)(u\ ) * (g)(t) - (g)\ (u\ )(f)(t)}{(g)\ (u\ )g(t)}}{\varepsilon}$$

Then its treated similarly the product case but using $f(t)(g)(t\ )$

$$= \lim_{\varepsilon\longrightarrow 0} \frac{(f)(u\ ) * (g)(t) - (f)(t)(g)(t\ ) + (f)(t)(g)(t\ ) - (g)\ (u\ )(f)(t)}{\varepsilon}$$

$$* \lim_{\varepsilon\longrightarrow 0} \frac{1}{(g)\ (u\ )g(t)}$$

$$= \frac{g(t)\ L^{\gamma}(f)(t) - f(t)\ L^{\gamma}(g)(t)}{[g(t)]^2}$$

For (iv) we have,

$$L^{\gamma}\ (f\ \circ g)(t) = \lim_{\varepsilon\longrightarrow 0} \frac{(f\ \circ g)\ (t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)}\varepsilon t^{1-\gamma}\ ) - (f\ \circ g)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon\longrightarrow 0} \frac{f\ (g\ (t\ + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma\ +\ 1)}\varepsilon t^{1-\gamma}\ )) - f(g(t))}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f\,(g\,(u\,)) - f(g(t))}{g(u) - g(t)} \lim_{\varepsilon \to 0} \frac{g(u) - g(t)}{\varepsilon}$$

But $\lim_{\varepsilon \to 0} g(u) = g(t)$ or $\lim_{\varepsilon \to 0} g(u) - g(t) = 0$

Hence, we could say $g(u) - g(t) = h$ with h $\longrightarrow$ 0 as ε $\longrightarrow$ 0

$$= \lim_{h \to 0} \frac{f\,(g(t) + h) - f\big(g(t)\big)}{h} * L^{\gamma}(g(t))$$

$$= \acute{f}\,\big((g)(t)\big)\, L^{\gamma}(g(t)) \qquad \square$$

Corollary 2.1. For $\gamma \in (0,1]$, $t > 0$ and f being γ −differentiable function, then

$$L^{\gamma}\left(\frac{1}{f}\right)(t) = -\frac{L^{\gamma}(f)(t)}{[f(t)]^2}$$

Proof:

Using theorem 2.4. part (iii)

We have,

$$L^{\gamma}\,(\frac{1}{f})(t) = \frac{f(t)\,L^{\gamma}(1) - (1)\,L^{\gamma}(f)(t)}{[f(t)]^2}$$

$$= \frac{f(t)\,\cdot\,0 - 1\cdot\,L^{\gamma}(f)(t)}{[f(t)]^2}$$

$$= \frac{-\,L^{\gamma}(f)(t)}{[f(t)]^2} \quad \square$$

Remark 2.3. It's easy to see that Corollary 2.1. can be generalized for the case of n power instead of power 1 to get $L^{\gamma}\left(\frac{1}{f^{n}}\right)(t)=-nf^{-n-1}(t)\cdot L^{\gamma}(f)(t)$. An easy proof can be found by induction.

Corollary 2.2. For $\gamma \in (0,1]$, $t > 0$ and f being γ −differentiable function, then,

$$L^{\gamma}[f(t)]^{2}=2\,f(t)\cdot L^{\gamma}(f)(t)$$

Proof:

Using theorem 2.4. part (ii)

We have,

$$L^{\gamma}[f(t)]^{2}= f(t)L^{\gamma}(f(t)) + f(t)\,L^{\gamma}(f(t))$$

$$=2\,f(t)\cdot L^{\gamma}(f)(t) \qquad \square$$

Remark 2.4. It's easy to see that Corollary 2.2 can be generalized for the case of n power instead of 2 to get $L^{\gamma}\,[f(t)]^{n}=n\,f^{n-1}(t)\cdot L^{\gamma}(f)(t)$. An easy proof can be found by induction.

Theorem 2.5. For the function $f(t)=t^{k}$; $k\in\mathbb{R}^{+}$, $L^{\alpha}L^{\beta}(f(t))=L^{\alpha+\beta}(f(t))$ is satisfied, where $0<\alpha,\alpha+\beta,\beta\leq 1$.

Proof:

Using theorem 2.3. part (ii) we have,

For R.H.S

$$L^{\alpha+\beta}(t^k) = \frac{\Gamma(\mathrm{k}+1)}{\Gamma(k - (\alpha+\beta) + 1)} t^{(k-(\alpha+\beta)}$$

For L.H.S

$$L^{\beta}(t^k) = \frac{\Gamma(\mathrm{k}+1)}{\Gamma(k - (\beta) + 1)} t^{(k-(\beta))}$$

$$L^{\alpha}L^{\beta}(t^k) = L^{\alpha}\frac{\Gamma(\mathrm{k}+1)}{\Gamma(k - (\beta) + 1)} t^{(k-(\beta))}$$

$$= \frac{\Gamma(\mathrm{k}+1)}{\Gamma(k - (\beta)+1)} L^{\alpha}\left(t^{\left(k-(\beta)\right)}\right)$$

$$= \frac{\Gamma(\mathrm{k}+1)}{\Gamma(k - (\beta)+1)} \left[\frac{\Gamma\left(\left(k-(\beta)\right)+1\right)}{\Gamma\left(\left(k-(\beta)\right)-(\alpha)+1\right)} t^{\left(k-(\beta)\right)-(\alpha)}\right]$$

$$= \frac{\Gamma(\mathrm{k}+1)}{\Gamma((k-(\beta)) - (\alpha)+1)} t^{\left(k-(\beta)\right)-(\alpha)}$$

$$= \frac{\Gamma(\mathrm{k}+1)}{\Gamma(k - (\alpha+\beta)+1)} t^{(k-(\alpha+\beta))}$$

= R.H.S □

Remark 2.5. For a function $f(t) = t^2$ then $L^{1/2}L^{1/2}\, t^2 = L^{1/2}\frac{\Gamma(3)}{\Gamma(2.5)}\, t^{3/2} = \frac{\Gamma(3)}{\Gamma(2)}\, t$ also $L^1\, t^2 = \frac{\Gamma(3)}{\Gamma(2)}\, t$, while in case of conformable derivative $D^{CD}_{1/2}D^{CD}_{1/2}\, t^2 = D^{CD}_{1/2}\, 2\, t^{3/2} = 2\,\frac{3}{2}\, t^{7/6}$,but $D^{CD}_{1}\, t^2 = 2\, t$.

Theorem 2.6. For a differentiable function $f(t)$ that has the expansion $f(t) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k}!}\, t^k$, then $L^{\alpha}L^{\beta}(f(t)) = L^{\alpha+\beta}\big(f(t)\big)$. satisfied, where $0 < \alpha, \alpha+\beta, \beta \leq 1$.

Proof:

Using theorem 2.4. part (i) and theorem 2.3. part (ii)we have,

For R.H.S

$$L^{\alpha+\beta}(\,f(t)) = \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\; L^{\alpha+\beta}t^{k}$$

$$= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k-\ (\alpha+\beta)+1)}t^{(k-(\alpha+\beta\ )}$$

For L.H.S

$$L^{\beta}(f(t)) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}[L^{\beta}t^{k}]$$

$$= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k-\ (\beta)+1)}t^{(k-(\beta))}$$

$$L^{\alpha}(L^{\beta}t^{k} = L^{\alpha}(\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\frac{\Gamma(\mathrm{k}+1)}{\Gamma(k\ -\ (\beta)\ +\ 1)}t^{(k-(\beta))}$$

$$= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\ \frac{\Gamma(\mathrm{k+1})}{\Gamma(k-\ (\beta)+1)}L^{\alpha}\left(t^{(k-(\beta))}\right)$$

$$= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k-\ (\beta)+1)}\left[\frac{\Gamma\left((k-(\beta))+1\right)}{\Gamma\left((k-(\beta\ ))-\ (\alpha)+1\right)}t^{(k-(\beta))-(\alpha\ )}\right]$$

$$= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma((k-(\beta\ ))-\ (\alpha)+1)}t^{(k-(\beta))-(\alpha\ )}$$

$$= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k-\ (\alpha+\beta)+1)}t^{(k-(\beta))-(\alpha\ )}$$

= R.H.S □

Remark 2.6. For a function $f(t)=e^{2t}$ $f^{(k)}(0)=2^k$ then $L^1 e^{2t}=\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}\,L^1 t^k=$ $\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k)}t^{(k\,-1)}=\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}k\,t^{(k\,-1)}=\sum_{k=0}^{\infty}\frac{2^k}{(\mathrm{k-1})!}\,t^{(k\,-1)}$ also $L^{1/2}L^{\,1/2}\,e^{2t}=$ $L^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k+0.5)}t^{(k\,-0.5)}\right)=\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}\frac{\Gamma(\mathrm{k+1})}{\Gamma(k\,)}t^{(k\,-1)}$.

While in case of conformable derivative $D_1^{CD}\,e^{2t}=\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}\,D_1^{CD}t^k=$ $\sum_{k=0}^{\infty}\frac{2^k}{\mathrm{k!}}k\,t^{(k\,-1)}=\sum_{k=0}^{\infty}\frac{2^k}{(\mathrm{k-1})!}\,t^{(k\,-1)}$,but $D_{1/2}^{CD}D_{1/2}^{CD}\,e^{2t}=D_{1/2}^{CD}\,\sum_{k=0}^{\infty}\frac{2^k}{(\mathrm{k-1})!}\,t^{(k\,-\frac{1}{2})}=$ $\sum_{k=0}^{\infty}\frac{2^k}{(\mathrm{k-1})!}\,(\mathrm{k}-\frac{1}{2})\,t^{(k\,-1)}$.

Definition 2.2. For a function $f:(0,\infty)\longrightarrow\mathbb{R}$ that is $n-$differentiable at t, the $\gamma-$derivative of $f(t)$ at $t>0$, where $\gamma\in(n,n+1]$ is defined as follows

$$L^{\gamma}\,f(t)=\lim_{\varepsilon\longrightarrow 0}\frac{f^{\lceil\gamma\rceil-1}\,(t+\frac{\Gamma(\rho)}{\Gamma(\rho-\gamma+1)}\varepsilon t^{\lceil\gamma\rceil-\gamma})-f^{\lceil\gamma\rceil-1}(t)}{\varepsilon};\ \rho\in\mathbb{R}^{+}$$

Where $\lceil\gamma\rceil$is the smallest integer greater than or equal to γ.

Remark 2.7. For function $f(t)$ which is $n-$differentiable at $t>0$ with $\gamma\in(n,n+1]$ it could be seen by applying theorem 2.1. that

$$L^{\gamma}\,f(t)=\frac{\Gamma(\rho)}{\Gamma(\rho-\gamma+1)}\,t^{\lceil\gamma\rceil-\gamma}f^{\lceil\gamma\rceil}(t).$$

Example2.1. For the function $f(t)=t^2$, $\rho=\mathrm{k}=2$

when $\gamma=1.5$ then $L^{1.5}\,t^2=\frac{\Gamma(2)}{\Gamma(3/2)}\,t^{2-1.5}f^{(2)}(t)=2\,t^{0.5}$

while when $\gamma=2$ then $L^2\,t^2=\frac{\Gamma(2)}{\Gamma(\,1)}t^{2-2}f^{(2)}(t)=2\Gamma(2)=\Gamma(3)$

Remark 2.8. For integer γ we have $\lceil \gamma \rceil = \gamma$ hence,$L^{\gamma} f(t) = f^{(\gamma)}(t)$ which is the same as the ordinary derivative.

Theorem 2.7. (Rolle's theorem for the generalized fractional differential function). Let $a > 0$ and $f: [a, b] \longrightarrow \mathbb{R}$ be a given function which satisfies the following:

(i) f is continuous on $[a, b]$

(ii) f is γ −differentiable on (a, b) for some $\gamma \in (0, 1]$

(iii) $f(a) = f(b)$

Then, there exists $c \in (a, b)$ such that $L^{\gamma}(f(c)) = 0$.

Proof: Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there exists $c \in (a, b)$, that is a point of local extrema by extreme value theorem, and c is assumed to be a point of local minimum without loss of generality. So, we have

$$L^{\gamma}(f(c^{+})) = \lim_{\varepsilon \longrightarrow 0+} \frac{f\left(c + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon c^{1-\gamma}\right) - f(c)}{\varepsilon}$$

$$=$$

$$L^{\gamma}(f(c^{-})) = \lim_{\varepsilon \longrightarrow 0-} \frac{f\left(c + \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} \varepsilon c^{1-\gamma}\right) - f(c)}{\varepsilon}$$

However, $L^{\gamma}(f(c^{+}))$ and $L^{\gamma}(f(c^{-}))$ have opposite signs. Hence, $L^{\gamma}(f(c)) = 0$. □

Theorem 2.8. (Mean value theorem for the generalized fractional differential function). Let $a > 0$ and $f: [a, b] \longrightarrow \mathbb{R}$ be a given function which satisfies the following:

(i) f is continuous on $[a, b]$

(ii) f is γ −differentiable on (a, b) for some $\gamma \in (0, 1]$

then, there exists $c \in (a, b)$ such that

$$L^{\gamma}\big(f(c)\big) = \frac{f(b) - f(a)}{h\,b^{\gamma} - h\,a^{\gamma})}; h = \frac{1}{\Gamma(\gamma + 1)}$$

Proof: Define a function $g(t)$ by

$$g(t) = f(t) - f(a) - \left[\frac{f(b) - f(a)}{(h\,b^{\gamma} - ha^{\gamma})}\right](\,h\,t^{\gamma} - h\,a^{\gamma})$$

With $h = \frac{1}{\Gamma(\gamma+1)}$

As $g(t)$ is continuous on the interval (a, b), differentiable on the interval $[a, b]$ and $g(a) = g(b) = 0$. $g(t)$ satisfies all conditions of roll's theorem Then, there exists $c \in (a, b)$ such that $L^{\gamma}\big(g(c)\big) = 0$.

$$L^{\gamma}\big(g(t)\big) = L^{\gamma}(f(t)) - L^{\gamma}(f(a)) - \left[\frac{f(b) - f(a)}{(h\,b^{\gamma} - ha^{\gamma})}\right](\,h\,L^{\gamma}(\,t^{\gamma}) - h\,L^{\gamma}(\,a^{\gamma}))$$

But we know that $L^{\gamma}(\text{contant}) = 0$ and $L^{\gamma}(t^{\gamma}) = \Gamma(\gamma + 1)$. hence, we have

$$L^{\gamma}\big(g(t)\big) = L^{\gamma}(f(t)) - \left[\frac{f(b)-f(a)}{(h\,b^{\gamma}-ha^{\gamma})}\right](\,h\,\Gamma(\gamma + 1))$$

But $h = \frac{1}{\Gamma(\gamma+1)}$

$$L^{\gamma}\big(g(t)\big) = L^{\gamma}(f(t)) - \left[\frac{f(b) - f(a)}{(h\,b^{\gamma} - ha^{\gamma})}\right]$$

Applying roll's theorem we have,

$$L^{\gamma}\big(g(c)\big) = L^{\gamma}\big(f(c)\big) - \left[\frac{f(b) - f(a)}{(h\, b^{\gamma} -\ h a^{\gamma})}\right] = 0$$

Therefore,

$$L^{\gamma}\big(f(c)\big) = \left[\frac{f(b) - f(a)}{(h\, b^{\gamma} -\ h a^{\gamma})}\right] \quad \square$$

Theorem 2.9. (Cauchy theorem for the generalized fractional differential function). Let $a\ >\ 0$ and $f, g\colon [a, b]\ \longrightarrow\ \mathbb{R}$ be given functions which satisfy the following:

(i) f, g are continuous on $[a, b]$

(ii) f, g are γ $-$differentiable on (a, b) for some $\gamma\ \in\ (0, 1]$ and $L^{\gamma}\big(g(t)\big) \neq 0\ for\ t \in\ (a, b)$

then, there exists $c\ \in\ (a, b)$ such that

$$\frac{L^{\gamma}\big(f(c)\big)}{L^{\gamma}\big(g(c)\big)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof: Consider the function

$$H(t) = f(t) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(t) - g(a))$$

As $H(t)$ continuous on the interval $[a, b]$, differentiable on the interval (a, b) and $H(a) = H(b) = 0$. $H(t)$ satisfies all conditions of roll's theorem. Then, there exists $c\ \in\ (a, b)$ such that $L^{\gamma}\big(H(c)\big) = 0$.

$$L^{\gamma}(H(t) = L^{\gamma}(f(t)) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)L^{\gamma}\big(g(t)\big)$$

Substituting $t = c$ we get

$$L^{\gamma}(H(c) = L^{\gamma}(f(c)) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) L^{\gamma}\big(g(c)\big) = 0$$

or

$$\frac{L^{\gamma}\big(f(c)\big)}{L^{\gamma}\big(g(c)\big)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Which completes the proof. □

2.2. Generalized Fractional Integral

In this section we discuss and complement the integral associated with the generalized definition of the fractional derivative introduced by Kaabar[3].

Definition 2.2. For a continuous function $f: [a, \infty) \to \mathbb{R}$, $a \geq 0$ and $\gamma \in (0, 1]$ then the generalized fractional integral $I_{\gamma}^{a} f(t)$ exists

$$I_{\gamma}^{a} f(t) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_{a}^{t} \frac{f(x)}{x^{1-\gamma}} dx \ .$$

Example 2.2 For a function $f(t) = 2$,we have

$$I_{\gamma}^{a}\,(2) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_{a}^{t} \frac{2}{x^{1-\gamma}}\, dx$$

$$= 2 \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_{a}^{t} x^{-1+\gamma}\, dx$$

$$= \ 2\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \frac{x^{\gamma}}{\gamma} \Big|_a^t$$

$$= \ 2\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \left(\frac{t^{\gamma}}{\gamma} - \frac{a^{\gamma}}{\gamma}\right) \square$$

Remark 2.9. For the case when $a = 0$ and $\rho = \gamma$ then the fractional integral of a constant function $f(x) = c$ is $I_{\gamma}^{0}\ (c) = c\frac{1}{\Gamma(\gamma)}\left(\frac{t^{\gamma}}{\gamma}\right) = \ c\frac{t^{\gamma}}{\Gamma(\gamma+1)}$.

Example 2.3 For a function $f(t) = t^2$,we have

$$I_{\gamma}^{a}\ (t^2) = \ \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t \frac{x^2}{x^{1-\gamma}}\ dx$$

$$= \ 2\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t x^{1+\gamma}\ dx$$

$$= \ 2\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \frac{x^{2+\gamma}}{2+\gamma} \Big|_a^t$$

$$= \ 2\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \left(\frac{t^{2+\gamma}}{2+\gamma} - \frac{a^{2+\gamma}}{2+\gamma}\right) \square$$

Example 2.4 For the general case of $f(t) = t^n$ we have

$$I_{\gamma}^{a}\ (t^n) = \ \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t \frac{x^n}{x^{1-\gamma}}\ dx$$

$$= \ \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t x^{-1+\gamma+n}\ dx$$

$$= \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \frac{x^{\gamma+n}}{\gamma+n} \Big|_a^t$$

$$= \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \left(\frac{t^{\gamma+n}}{\gamma+n} - \frac{a^{\gamma+n}}{\gamma+n}\right) \square$$

Remark 2.10. For the case when $a = 0$ and $\rho = \gamma$ then fractional integral of t^n function is

$$I_\gamma^0 (t^n) = \frac{1}{\Gamma(\gamma)} \left(\frac{t^{\gamma+n}}{\gamma+n}\right)$$

Theorem 2.9. Let $f: [a, \infty) \to \mathbb{R}$ be continuous function such that $I_\gamma^a f(t)$ exists, $t > a$ and $\gamma \in (0, 1]$, then

$$L^\gamma I_\gamma f(t) = f(t)$$

Proof: As f is continuous, $I_\gamma^a f(t)$ is differentiable.

Hence, using theorem 2.1. we have

$$L^\gamma (I_\gamma f(t)) = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} \frac{d(I_\gamma f(t))}{dt}$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} \frac{d}{dt}\left(\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t \frac{f(x)}{x^{1-\gamma}} dx\right)$$

$$= t^{1-\gamma} \frac{d}{dt}\left(\int_a^t \frac{f(x)}{x^{1-\gamma}} dx\right)$$

$$= t^{1-\gamma} \frac{f(t)}{t^{1-\gamma}}$$

$$= f(t) \quad \square$$

Theorem 2.10. Let $f\colon [a,\infty) \to \mathbb{R}$ be continuous function such that $L^{\gamma} f(t)$ is continuous, $t > a$ and $\gamma \in (0,1]$, then

$$I_{\gamma} L^{\gamma} f(t) = f(t) - f(a)$$

Proof:

Using Definition 2.2. we have

$$I_{\gamma}(L^{\gamma} f(t)) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t \frac{L^{\gamma}(f(x))}{x^{1-\gamma}} dx$$

using theorem 2.1. we have

$$I_{\gamma}(L^{\gamma} f(t)) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^t \frac{1}{x^{1-\gamma}} \left(\frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} x^{1-\gamma} \frac{d\big(f(x)\big)}{dx} \right) dx$$

$$= \int_a^t \left(\frac{d\big(f(x)\big)}{dx} \right) dx$$

$$= f(x) \,\Big|_a^t$$

$$= f(t) - f(a) \quad \square$$

Theorem 2.11.(Mean value theorem for fractional integral)

Let $f\colon [a,b] \to \mathbb{R}$ be continuous function and $\gamma \in (0,1]$, then there exists $c \in [a,b]$ such that

$$\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^b \frac{f(x)}{x^{1-\gamma}} dx = f(c) \left(\frac{1}{\Gamma(\gamma + 1)} b^{\gamma} - \frac{1}{\Gamma(\gamma + 1)} a^{\gamma} \right)$$

Proof: As f is continuous on $[a, b]$ then $I_\gamma f(t)$ is continuous on $[a, b]$, γ −differentiable on (a, b) by applying the mean value theorem for fractional derivative theorem 2.8. there exists $c \in (a, b)$ such that

$$L^\gamma\big(I_\gamma f(c)\big) = \frac{I_\gamma f(b) - I_\gamma f(a)}{\left(\frac{1}{\Gamma(\gamma+1)}\, b^\gamma - \frac{1}{\Gamma(\gamma+1)}\, a^\gamma\right)}$$

But from theorem 2.9. we know that

$$L^\gamma\left(I_\gamma f(c)\right) = f(c)$$

Using definition 2.2. we have

$$I_\gamma f(b) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^b \frac{f(x)}{x^{1-\gamma}}\, dx$$

And

$$I_\gamma f(a) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^a \frac{f(x)}{x^{1-\gamma}}\, dx = 0$$

Hence, we get

$$f(c) = \frac{\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^b \frac{f(x)}{x^{1-\gamma}}\, dx}{\left(\frac{1}{\Gamma(\gamma+1)}\, b^\gamma - \frac{1}{\Gamma(\gamma+1)}\, a^\gamma\right)}$$

Or

$$\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} \int_a^b \frac{f(x)}{x^{1-\gamma}} dx = \left(\frac{1}{\Gamma(\gamma+1)}\ b^{\gamma} - \frac{1}{\Gamma(\gamma+1)}\ a^{\gamma}\right) f(c) \ \square$$

2.3. Generalized Fractional Derivative of some Functions

In this section we apply the generalized definition of the fractional derivative on some known functions.

2.3.1. Fractional Derivative of the Exponential Function

For a function $f(t) = e^{\lambda t}, \lambda \in \mathbb{C}$.

By Taylor theorem $e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k$

Hence, we have for $\gamma \in (0,1]$

$$L^{\gamma}(e^{\lambda t}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\ L^{\gamma}(\ t^k)$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\frac{\Gamma(k+1)}{\Gamma(k - \gamma + 1)} t^{k-\gamma}\right)$$

$$= D_{\gamma}^{Ca} e^{\lambda t} \ \square$$

Remark 2.11. For $\gamma = \frac{1}{2}$, the γ – Generalized derivative of the exponential $e^{\lambda t}$

$$L^{\frac{1}{2}}(e^{\lambda t}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} t^{k-\frac{1}{2}} \right)$$

2.3.2. Fractional Derivative of Sine Function

For a function $f(t) = sin\ \omega t$

We know that the sine function could be written in complex as

$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

Then, we have for $\gamma \in (0,1]$

$$L^{\gamma}(\sin(\omega t)) = \frac{1}{2i}\left(L^{\gamma}(e^{i\omega t}) - L^{\gamma}(e^{-i\omega t})\right)$$

$$= \frac{1}{2i}\left(D_{\gamma}^{Ca}(e^{i\omega t}) - D_{\gamma}^{Ca}(e^{-i\omega t})\right)$$

$$= D_{\gamma}^{Ca}\frac{1}{2i}\left((e^{i\omega t}) - (e^{-i\omega t})\right)$$

$$= D^{Ca}\,(\sin(\omega t))\ \square$$

Remark 2.12. For $\gamma = \frac{1}{2}$, the γ – Generalized derivative of the exponential $(\sin(\omega t))$ reduces to

$$L^{\frac{1}{2}}(\sin(\omega t)) = \frac{1}{2i}\left[\sum_{k=0}^{\infty} \frac{(i\omega)^k}{k!}\left(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} t^{k-\frac{1}{2}}\right) - \sum_{k=0}^{\infty} \frac{(-i\omega)^k}{k!}\left(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} t^{k-\frac{1}{2}}\right)\right]$$

2.3.3. Fractional Derivative of Cosine Function $f(t) = cos\ \omega t$

We know that the cosine function could be written in complex as

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

Then, we have for $\gamma \in (0, 1]$

$$L^{\gamma}(\cos(\omega t)) = \frac{1}{2}\left(L^{\gamma}(e^{i\omega t}) + L^{\gamma}(e^{-i\omega t})\right)$$

$$= \tfrac{1}{2}\left(D_{\gamma}^{Ca}(e^{i\omega t}) + D_{\gamma}^{Ca}(e^{-i\omega t})\right)$$

$$= D_{\gamma}^{Ca}\tfrac{1}{2}\left((e^{i\omega t}) + (e^{-i\omega t})\right)$$

$$= D_{\gamma}^{Ca}\,(\cos(\omega t))\ \square$$

Remark 2.13. For $\gamma = \frac{1}{2}$, the γ – Generalized derivative of the exponential $(\cos(\omega t))$ reduces to

$$L^{\frac{1}{2}}(\cos(\omega t)) = \frac{1}{2}[\sum_{k=0}^{\infty}\frac{(i\omega)^k}{k!}(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)}t^{k-\frac{1}{2}}) + \sum_{k=0}^{\infty}\frac{(-i\omega)^k}{k!}(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)}t^{k-\frac{1}{2}})]$$

Remark 2.14. We can find the fractional derivative of hyperbolic functions since its related to the trigonometric ones as $\sinh(t)) = -i\ sin(it)$ and $\cosh(t)) = cos(it)$. it's easy to see that the fractional derivative is as follows:

$L^{\gamma}\ (cosh(t)) = L^{\gamma}\ (cos(it))$

$$L^{\gamma}\ (\sinh(t)) = L^{\gamma}(-i\ sin(it))$$

For $\gamma = \frac{1}{2}$, we have

$$L^{\gamma}\ (\,cosh(t)) = \frac{1}{2}[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)}t^{k-\frac{1}{2}}) + \sum_{k=0}^{\infty}\frac{1}{k!}(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)}t^{k-\frac{1}{2}})]$$

$$L^{\gamma}\ (\sinh(t)) = \frac{1}{2i}[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)}t^{k-\frac{1}{2}}) - \sum_{k=0}^{\infty}\frac{1}{k!}(\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)}t^{k-\frac{1}{2}})]$$

Chapter three

Generalized Fractional Differential Equations

In this chapter we solve many types of fractional differential equations using the generalized derivative operator.

3.1. Generalized First Order Fractional Differential Equations

In this section we take in consideration some types of first order fractional differential equations.

3.1.1. Generalized Linear Fractional Differential Equations

In this section we solve some linear γ $-$Order fractional differential by first transforming the fractional equation to ordinary differential equation. Many different examples are used to demonstrate the ideas.

3.1.1.1. General Solution of Linear Equation

Theorem 3.1. For the linear first order fractional differential equation of the form as follows,

$$L^{\gamma} y(x) + g(x)y(x) = h(x)$$

with $\gamma \in (0, 1]$ and $g(x), h(x)\ y(x)$ are differentiable functions has the general solution

$$y(x) = \frac{1}{s(x)}\left[\int s(x) * H(x)dx + c\right]$$

where $s(x) = e^{\int G(x)dx}$,

$$G(x) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} x^{\gamma-1} * g(x) \quad ,$$

$$H(x) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} x^{\gamma-1} * h(x)$$

Proof: First we use the theorem 2.1. to get

$$L^{\gamma} y(x) + g(x)y(x) = h(x)$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} x^{1-\gamma} \frac{d\, y(x)}{dx} + g(x)y(x) = h(x)$$

Multiplying both sides with $\frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} x^{\gamma-1}$ we get,

$$\frac{d\, y(x)}{dx} + G(x)y(x) = H(x)$$

Where $G(x) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} x^{\gamma-1} * g(x)$,

$$H(x) = \frac{\Gamma(\rho - \gamma + 1)}{\Gamma(\rho)} x^{\gamma-1} * h(x)$$

Multiplying both sides with $s(x) = e^{\int G(x)dx}$ we get,

$$s(x) * \frac{d\,y(x)}{dx} + s(x) * G(x)y(x) = s(x) * H(x)$$

which can be written as

$$\frac{d}{dx}[s(x) * y(x)] = s(x) * H(x)$$

By integrating both sides, we get

$$[s * y(x)] = \int s * H(x) \; + \; c$$

Hence, it is easy to see that,

$$y(x) = \frac{1}{s}\,[\int s * H(x)dx + c] \;\square$$

Remark 3.1: In case of an initial value problem with initial condition $y(a) = b$. The constant c is calculated based on the initial condition

Example 3.1: Let's consider the initial value problem

$$L^{\frac{1}{2}}y(x) + y(x) = 1\,, \qquad y(0) = 0.$$

Solution: Using theorem 2.1. we get

$$L^{\frac{1}{2}}y(x) + y(x) = 1$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho + \frac{1}{2})}x^{\frac{1}{2}}\,\frac{d\,y(x)}{dx} + y(x) = 1$$

For $\rho = \gamma = 0.5$ we have,

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(1)} x^{\frac{1}{2}} \frac{d\,y(x)}{dx} + y(x) = 1$$

Or

$$\frac{d\,y(x)}{dx} + \frac{x^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})} y(x) = \frac{x^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}$$

The integrating factor $s(x) = e^{\frac{1}{\Gamma(\frac{1}{2})}\int x^{\frac{-1}{2}} dx} = e^{\frac{2}{\Gamma(\frac{1}{2})} x^{0.5}}$

Hence,

$$y(x) = \frac{1}{e^{\frac{2}{\Gamma(\frac{1}{2})} x^{0.5}}} \, [\int e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} x^{0.5}} * 1dx + c]$$

$$= e^{\frac{-2}{\Gamma(\frac{1}{2})} x^{0.5}} \quad [I + c]$$

$$I = \int e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} x^{0.5}} dx = \int e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} u} * 2u \, du$$

using the substitution $u = x^{0.5}$, $du = 0.5\, x^{-0.5} dx = 0.5\, u^{-1} dx$

Using the integration by parts method, we get

$$I = \frac{2u}{2/\Gamma\left(\frac{1}{2}\right)} \, e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} u} - \frac{2}{[2/\Gamma\left(\frac{1}{2}\right)]^2} \, e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} u}$$

$$= \frac{2x^{0.5}}{2/\Gamma\left(\frac{1}{2}\right)} \, e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} x^{0.5}} - \frac{2}{[2/\Gamma\left(\frac{1}{2}\right)]^2} \, e^{\frac{2}{\Gamma\left(\frac{1}{2}\right)} x^{0.5}}$$

Hence,

$$y(x) = e^{\frac{-2}{\Gamma(\frac{1}{2})}x^{0.5}} \left[x^{0.5}\,\Gamma\left(\frac{1}{2}\right) e^{\frac{2}{\Gamma(\frac{1}{2})}x^{0.5}} - \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2} e^{\frac{2}{\Gamma(\frac{1}{2})}x^{0.5}} + c \right]$$

$$= \left[x^{0.5}\,\Gamma\left(\frac{1}{2}\right) - \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2} + c e^{\frac{-2}{\Gamma(\frac{1}{2})}x^{0.5}} \right]$$

The constant c is calculated based on the initial condition $y(0) = 0$

$$y(0) = 0 = \left[- \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2} + c \right] \text{ then } c = \frac{[\Gamma\left(\frac{1}{2}\right)]^2}{2}$$

Hence $\quad y(x) = x^{0.5}\,\Gamma\left(\frac{1}{2}\right) - \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2} + \frac{[\Gamma\left(\frac{1}{2}\right)]^2}{2} e^{\frac{-2}{\Gamma(\frac{1}{2})}x^{0.5}}$ □

Example 3.2: Solve the differential equation

$$L^{\frac{1}{2}} y(x) + y(x) = x^2 + \frac{2}{\Gamma(2.5)} x^{3/2}$$

Solution: Using theorem 2.1. we get

$$L^{\frac{1}{2}} y(x) + y(x) = x^2 + \frac{2}{\Gamma(2.5)} x^{3/2}$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho + \frac{1}{2})} x^{\frac{1}{2}} \frac{d\,y(x)}{dx} + y(x) = x^2 + \frac{2}{\Gamma(2.5)} x^{3/2}$$

For $\rho = \gamma = 0.5$ we have,

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(1)} x^{\frac{1}{2}} \frac{d\,y(x)}{dx} + y(x) = x^2 + \frac{2}{\Gamma(2.5)} x^{3/2}$$

or

$$\frac{d\,y(x)}{dx} + \frac{x^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} y(x) = \frac{x^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2}\right)} + \frac{2}{\Gamma\left(\frac{1}{2}\right)*\Gamma(2.5)} x$$

The integrating factor $s(x) = e^{\frac{1}{\Gamma(\frac{1}{2})}\int x^{\frac{-1}{2}} dx} = e^{\frac{2}{\Gamma(\frac{1}{2})} x^{0.5}}$

Hence,

$$y(x) = \frac{1}{e^{\frac{2}{\Gamma(\frac{1}{2})} x^{0.5}}} \left[\int e^{\frac{2}{\Gamma(\frac{1}{2})} x^{0.5}} * \left(\frac{x^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2}\right)} + \frac{2x}{\Gamma\left(\frac{1}{2}\right) * \Gamma(2.5)} \right) dx + c \right]$$

$$= e^{\frac{-2}{\Gamma(\frac{1}{2})} x^{0.5}} [I + c]$$

To find the value of I, substituting $u = x^{1/2}$, $du = 0.5x^{-0.5}dx = 0.5u^{-1}dx$ we get,

$$I = \int e^{\frac{2}{\Gamma(\frac{1}{2})} u} \left(\frac{u^3}{\Gamma\left(\frac{1}{2}\right)} + \frac{2u^2}{\Gamma\left(\frac{1}{2}\right) * \Gamma(2.5)} \right) 2u\, du$$

Which is easily solved using integrating by parts method to get, known that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ and\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$,

$$I = e^{\frac{2}{\sqrt{\pi}} u} \left(u^4 - 2\sqrt{\pi}\, u^3 + 3\pi u^2 - 3\pi^{3/2} u + \frac{3}{2}\pi^2 \right)$$
$$+ e^{\frac{2}{\sqrt{\pi}} u} \left(\frac{8}{3}\pi^{\frac{3}{2}} u^3 - 4\pi^2 u^2 + 4\pi^{\frac{5}{2}} u - 2\pi^3 \right)$$

and write in terms of x to get,

$$y(x) = \left(x^2 - 2\sqrt{\pi}\, x^{3/2} + 3\pi x - 3\pi^{3/2}\sqrt{x} + \frac{3}{2}\pi^2 \right)$$
$$+ \left(\frac{8}{3}\pi^{\frac{3}{2}} x^{3/2} - 4\pi^2 x + 4\pi^{\frac{5}{2}}\sqrt{x} - 2\pi^3 \right) + c e^{\frac{-2}{\Gamma(\frac{1}{2})} x^{0.5}} \quad \square$$

3.1.1.2. Using Taylor series

Another way to simplify the way to find the solution of the fractional differential equation is to substitute the function with its Taylor series expansion.

Example 3.3: Solve the initial value problem $L^{\frac{1}{2}}y(x) = e^{kx}$, $\qquad y(0) = 0.$

Solution: As $e^{kx} = \sum_{n=0}^{\infty} \frac{k^n}{n!} x^n$.The fractional differential equation can be simplified using theorem 2.1. we get

$$L^{\frac{1}{2}}y(x) = e^{kx}$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho + \frac{1}{2})} x^{\frac{1}{2}} \frac{d\, y(x)}{dx} = \sum_{n=0}^{\infty} \frac{k^n}{n!} x^n$$

$$\frac{d\, y(x)}{dx} = \frac{\Gamma(\rho + \frac{1}{2})}{\Gamma(\rho)} \sum_{n=0}^{\infty} \frac{k^n}{n!} x^{n-\frac{1}{2}}$$

$$\int dy(x) = \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho)} \sum_{n=0}^{\infty} \frac{k^n}{n!} \int x^{n-\frac{1}{2}}\, dx$$

$$y(x) = \frac{\Gamma(\rho + \frac{1}{2})}{\Gamma(\rho)} \sum_{n=0}^{\infty} \frac{k^n}{n!} \left(\frac{x^{n+\frac{1}{2}}}{n + \frac{1}{2}}\right) + c$$

$$y(x) = \sum_{n=0}^{\infty} \frac{k^n}{n!} \frac{\Gamma(\rho + \frac{1}{2})}{\Gamma(\rho)} \left(\frac{x^{n+\frac{1}{2}}}{n + \frac{1}{2}}\right) + c$$

Letting $\rho = n + \frac{1}{2}$, we get

$$y(x) = \sum_{n=0}^{\infty} \frac{k^n}{n!} \frac{\Gamma(n+\ 1)}{\Gamma(n+\frac{1}{2})} (\frac{x^{n+\frac{1}{2}}}{n+\frac{1}{2}}) + c$$

$$y(x) = \sum_{n=0}^{\infty} \frac{k^n}{\Gamma(n+\frac{3}{2})} x^{n+\frac{1}{2}} + c$$

$y(0) = 0$ implies that $c = 0$, we get

$$y(x) = \sum_{n=0}^{\infty} \frac{k^n}{\Gamma(n+\frac{3}{2})} x^{n+\frac{1}{2}} \quad \square$$

Example 3.4: Solve the initial value problem $L^{\frac{1}{2}}y(x) = x^2 \sin(x)$, $\qquad y(0) = 0.$

Solution: As $\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ .The fractional differential equation can be simplified using theorem 2.1. we get

$$L^{\frac{1}{2}}y(x) = x^2 \sin(x)$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho+\frac{1}{2})} x^{\frac{1}{2}} \frac{d\,y(x)}{dx} = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{(2n+1)!}$$

$$\frac{d\,y(x)}{dx} = \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho)} \sum_{n=0}^{\infty} \frac{x^{2n+\frac{5}{2}}}{(2n+1)!}$$

$$\int dy(x) = \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho)} \sum_{n=0}^{\infty} \int \frac{x^{2n+\frac{5}{2}}}{(2n+1)!}\, dx$$

$$y(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho)} (\frac{x^{2n+\frac{7}{2}}}{(2n+\frac{7}{2})(2n+1)!}) + c$$

Letting $\rho\ = 2\,n + \frac{7}{2}$, we get

$$y(x) = \sum_{n=0}^{\infty} \frac{\Gamma(2\,n+4)}{\Gamma(2\,n+\frac{7}{2})}\ (\frac{x^{2n+\frac{7}{2}}}{(2n+\frac{7}{2})(2n+1)!}) + c$$

$$y(x) = \sum_{n=0}^{\infty} \frac{(2n+3)!}{\Gamma(2\,n+\frac{9}{2})}\ (\frac{x^{2n+\frac{7}{2}}}{(2n+1)!}) + c$$

$$y(x) = \sum_{n=0}^{\infty} \frac{(2n+3)(2n+2)}{\Gamma(2\,n+\frac{9}{2})}\ x^{2n+\frac{7}{2}} + c$$

$y(0) = 0$ implies that $c = 0$, we get

$$y(x) = \sum_{n=0}^{\infty} \frac{(2n+3)(2n+2)}{\Gamma(2\,n+\frac{9}{2})}\ x^{2n+\frac{7}{2}} \quad \square$$

3.1.2. Generalized non-Linear Riccati Fractional Differential Equations

In this section we look into some non-linear Riccati fractional differential equation by first transforming the fractional equation to non-linear ordinary differential equation. Many different examples are used to demonstrate the ideas.

This section is dedicated not to how the equations are solved but rather comparing the solution at different values of x with the conformable derivative results.

The general form of Riccati equation is

$$L^{\gamma} y(x) = a(x) + b(x) * y + c(x) * y^2$$

Where $a(x), b(x), c(x)$ are continuous γ −differentiable functions, with $\gamma \in (0,1]$.

If one particular solution y_1 can be found, then the general solution of the Riccati equation is obtained as

$y = y_1 + u$ with $\acute{u} = [b(x) + 2c(x)\, y_1]u + c(x)\, u^2$

which is a Bernoulli equation and is easily solved with the substitution $u = \frac{1}{s}$

Example 3.5: Solve the fractional Riccati differential equation

$$L^{\gamma} y(x) + y(x)^2 = 1\,, \qquad y(0) = 0\,, 0 < \gamma \leq 1.$$

Solution: Using theorem 2.1. we get

$$L^{\gamma} y(x) + y^2(x) = 1$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} x^{1-\gamma} \frac{d\, y(x)}{dx} + y^2(x) = 1$$

For $\rho = \gamma = 0.75$ the equation becomes

$$\frac{\Gamma(0.75)}{\Gamma(1)}\, x^{0.25} \frac{d\, y(x)}{dx} + y^2(x) = 1, y(0) = 0$$

As $\Gamma(0.75) = 1.2254167$, $\Gamma(1) = 1$ using MATLAB we get the solution

$$y(x) = \tanh\left(\frac{4 * 10^7}{36762501} x^{0.75}\right)$$

For $\gamma = \rho = 0.9$ the equation becomes

$$\frac{\Gamma(0.9)}{\Gamma(1)}\ x^{0.1}\ \frac{d\,y(x)}{dx} + y^2(x) = \ 1,\ \ y(0) = 0$$

As $\Gamma(0.9) = 1.0686287$, $\Gamma(1) = 1$ using MATLAB we get the solution

$$y(x) = \tanh\,(\frac{10^8}{96176583}x^{0.9})$$

Using the conformable derivative definition, we get

$$x^{1-\gamma}\ \frac{d\,y(x)}{dx} + y^2(x) = \ 1, y(0) = 0$$

For $\rho = \gamma = 0.75$ the equation becomes

$$x^{0.25}\ \frac{d\,y(x)}{dx} + y^2(x) = \ 1, y(0) = 0$$

Using MATLAB, we get the solution

$$y(x) = \tanh\,(\frac{4}{3}x^{0.75})$$

For $\rho = \gamma = 0.9$ the equation becomes

$$x^{0.1}\ \frac{d\,y(x)}{dx} + y^2(x) = \ 1, y(0) = 0$$

Using MATLAB, we get the solution

$$y(x) = \tanh\,(\frac{10}{9}x^{0.9})$$

X	GD	CD
0	0	0
0.1	0.191109	0.232758
0.2	0.314388	0.378887
0.3	0.414521	0.493351
0.4	0.498470	0.585395
0.6	0.630212	0.720640
0.8	0.726082	0.810287
1	0.796171	0.870062

Table1: comparison of GD results with CD at $\gamma = 0.75$

X	BPM [17]	EHPM [9]
0	0	0
0.2	0.30996891	0.3214
0.4	0.48162749	0.5077
0.6	0.59777979	0.6259
0.8	0.67884745	0.7028
1	0.73684181	0.7542

Table 2: comparison of the results of different methods at $\gamma = 0.75$

Remark 3.2: We could easily see from Tables 1 and 2 that the values obtained from (GD) method are in good agreement with the results from the Bernstein polynomial method (BPM) and enhanced homotopy perturbation method (EHPM).

However, the conformable (CD) method although easier to use than the previous methods its results are less accurate than the (GD) method.

X	GD	CD
0	0	0
0.1	0.130155	0.138975
0.2	0.239518	0.255255
0.3	0.338002	0.359213
0.4	0.426664	0.451906
0.6	0.576062	0.605387
0.8	0.691369	0.720626
1	0.777791	0.804455

Table 3: comparison of GD results with CD at $\gamma = 0.9$ □

Example 3.4: Solve the fractional Riccati differential equation

$$L^{\gamma} y(x) - 2 * y(x) + y^2(x) = 1 , \qquad y(0) = 0 , 0 < \gamma \leq 1.$$

Solution: Using theorem 2.1. we get

$$L^{\gamma} y(x) - 2 * y(x) + y^2(x) = 1$$

$$\frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} x^{1-\gamma} \frac{d\, y(x)}{dx} - 2 * y(x) + y^2(x) = 1$$

For $\rho = \gamma = 0.75$ the equation becomes

$$\frac{\Gamma(0.75)}{\Gamma(1)} x^{0.25} \frac{d\, y(x)}{dx} - 2 * y(x) + y^2(x) = 1, y(0) = 0$$

As $\Gamma(0.75) = 1.2254167, \Gamma(1) = 1$ using MATLAB we get the solution

$$y(x) = \sqrt{2} * \tanh\left(\sqrt{2}\frac{4 * 10^7}{36762501} x^{\frac{3}{4}} - \tanh^{-1}(\frac{1}{\sqrt{2}})\right) + 1$$

Using the conformable derivative definition, we get

$$x^{1-\gamma} \frac{d\, y(x)}{dx} - 2 * y(x) + y^2(x) = 1, y(0) = 0$$

For $\rho = \gamma = 0.75$ the equation becomes

$$x^{0.25} \frac{d\, y(x)}{dx} - 2 * y(x) + y^2(x) = 1, y(0) = 0$$

using MATLAB, we get the solution

$$y(x) = \sqrt{2} * \tanh\left(\sqrt{2}\frac{4}{3} x^{\frac{3}{4}} - \tanh^{-1}(\frac{1}{\sqrt{2}})\right) + 1$$

X	GD	CD
0	0	0
0.1	0.232741	0.296344
0.2	0.437253	0.565564
0.3	0.643518	0.835261
0.4	0.848667	1.09467
0.6	1.23489	1.5423
0.8	1.56165	1.86687
1	1.81546	2.07957

Table 4: comparison of GD results with CD at $\gamma = 0.75$ □

3.2. Generalized Second Order Fractional Differential Equations

In this section we briefly look into second order fractional differential equations. We also study the solution to the homogeneous fractional equation with constant coefficient.

Defention3.2 The general form of the linear second order fractional differential equation is as follows,

$$L^{\gamma} L^{\gamma} y + A(x)\, L^{\gamma} y + B(x)\, y = C(x)$$

With $A(x), B(x), C(x)$ are $\gamma -$ differential functions, with $\gamma \in (0, 1]$.

Defention3.3 The fractional Wronskian of two functions $f(x)$ and $g(x)$ is defined by

$$W_\gamma(f(x), g(x)) = \begin{vmatrix} f(x) & g(x) \\ L^\gamma(f(x)) & L^\gamma(g(x)) \end{vmatrix}$$

$$= f(x) * L^\gamma\big(g(x)\big) - g(x) * L^\gamma(f(x))$$

Remark 3.3 Two functions $f(x), g(x)$ are linearly dependent if and only if their fractional Wronskian is identically zero.

We are going to study the case of Homogeneous fractional equation with constant coefficient which has the general form

$$L^\gamma L^\gamma y + a\, L^\gamma y + b\, y = 0$$

With a, b being constants.

We start with considering the function $e^{c\,(\frac{1}{\Gamma(\gamma+1)}x^\gamma)}$ because this exponential function has the γ fractional derivative is a constant multiple of the exponential itself. We consider the solution

$$y = e^{c\,(\frac{1}{\Gamma(\gamma+1)}x^\gamma)}$$

Which has the γ $-$derivative below

$$L^\gamma\left(e^{c\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}\right) = ce^{c\,(\frac{1}{\Gamma(\gamma+1)}x^\gamma)}$$

and

$$L^\gamma L^\gamma\left(e^{c\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}\right) = c^2 e^{c\,(\frac{1}{\Gamma(\gamma+1)}x^\gamma)}$$

Substituting these values in the equation we get,

$$[c^2 + ca + b]e^{c\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)} = 0$$

But we know that the exponential function is never zero then

$$c^2 + ca + b = 0$$

Case 1: The roots c_1, c_2 of this equation are distinct reals if $a^2 - 4b > 0$

In this case the two solutions

$$y_1 = e^{c_1\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}, y_2 = e^{c_2\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}$$

Since the ratio

$$\frac{y_1}{y_2} = \frac{e^{c_1\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}}{e^{c_2\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}} = e^{(c_1-c_2)\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)} \neq constant\ as\ (c_1 - c_2) \neq 0$$

Hence the two solutions are linearly independent and the general solution in this case is as follows

$$y = s_1 e^{c_1\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)} + s_2 e^{c_2\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}$$

s_1, s_2 are arbitrary constants.

Case 2: The roots c_1, c_2 are equal reals then we have

$$y_1 = e^{c_1\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}$$

And we can find a second linearly independent solution in the form of

$$y_2 = \frac{1}{\Gamma(\gamma+1)}x^\gamma e^{c_1\left(\frac{1}{\Gamma(\gamma+1)}x^\gamma\right)}$$

The general solution in this case is

$$y = \left(s_1 + s_2 \, \frac{1}{\Gamma(\gamma + 1)} x^{\gamma} \right) \, e^{c_1 \, (\frac{1}{\Gamma(\gamma+1)} x^{\gamma})}$$

Case 3: The roots c_1, c_2 are distinct complex numbers which can be written as $m \pm i\, n$ then we have the two real solutions

$$y_1 = e^{m \, (\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} \cos\left(n \, \frac{1}{\Gamma(\gamma + 1)} x^{\gamma}\right)$$

$$y_2 = e^{m \, (\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} \sin\left(n \, \frac{1}{\Gamma(\gamma + 1)} x^{\gamma}\right)$$

With the general solution being

$$y = \; s_1 y_1 + \; s_2 y_2$$

Example 3.4 Solve the following homogeneous fractional equation

$$L^{\gamma} L^{\gamma} y + 4 \, L^{\gamma} y + 3 \, y = 0$$

Solution: The equation $c^2 + 4c + 3 = 0$ has the two distinct solutions

$$c = -1, -3$$

Then the general solution is

$$y = s_1 e^{-1 \, (\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} + s_2 e^{-3 \, (\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} \; \square$$

Example 3.5 Solve the following homogeneous fractional equation

$$L^{\gamma} L^{\gamma} y + 4\, L^{\gamma} y + 4\, y = 0$$

Solution: The equation $c^2 + 4c + 4 = 0$ has one solution

$$c = -2$$

Then the general solution is

$$y = \left(s_1 + s_2\, \frac{1}{\Gamma(\gamma + 1)} x^{\gamma}\right) e^{-2\,(\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} \square$$

Example 3.6 Solve the following homogeneous fractional equation

$$L^{\gamma} L^{\gamma} y + 2\, L^{\gamma} y + 2\, y = 0$$

Solution: The equation $c^2 + 2c + 2 = 0$ has the two complex solutions

$$c = -1 \pm i$$

the general solution is

$$y = s_1 e^{-1\,(\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} \cos\left(\frac{1}{\Gamma(\gamma + 1)} x^{\gamma}\right) + s_2 e^{-1\,(\frac{1}{\Gamma(\gamma+1)} x^{\gamma})} \sin\left(\frac{1}{\Gamma(\gamma + 1)} x^{\gamma}\right) \square$$

Defention3.4. The general form of the linear nth order fractional differential equation is as follows,

$$(L^{\gamma})^{n}y + A_{n}(x)(L^{\gamma})^{n-1}y \dots + A_{2}(x)\, L^{\gamma}y + A_{1}(x)\, y = A_{0}(x)$$

With $A_n, A_{n-1}, \dots, A_0$ are $\gamma -$ differential functions.

Which is treated after transforming the fractional equation to ordinary differential equation.

Chapter four

Fractional Newtonian Mechanics

In this chapter we study the Newtonian Mechanics in a new light based on the new generalized definition of the fractional derivative. We look into different topics for the one-dimensional case including the equations of motion at constant velocity and constant acceleration, free falling objects, linear momentum and fractional kinetic energy. We also looked into the two-dimensional motion.

4.1. Introduction

A possible mechanical interpretation of the half-derivative can be given in terms of Abel's solution to the classical tautochrone problem. A tautochrone or isochrone curve which is a Greek term that means equal time is the curve for which the time that an object take sliding disregarding friction to the lowest point in the curve is independent of the object starting point on the curve.

Motion of an object involves its displacement from one place in space and time to another. The displacement Δx is defined as the change in position, and is given by

$$\Delta x = x_f - x_i$$

Whereas the average velocity $\bar{v}$ during a time interval Δt is given by

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

And when we take the limit of the average velocity as the time interval Δt becomes infinitely small we get the instantaneous velocity v and is given by

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \acute{x}$$

While the instantaneous acceleration a is given by

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \acute{v}$$

For a known physical value s we denote the fractional one as $\hat{s}$.

Looking into these values considering the generalized fractional derivative in theorem 2.1. with $\gamma \in (0,1]$ it is easy to see that the γ −velocity

$$\hat{v}(t) = L^{\gamma}(x(t)) = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} \frac{d\,x(t)}{dt}$$

While the $\gamma -$ acceleration

$$\hat{a}(t) = L^{\gamma}(v(t)) = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} \frac{d\,v(t)}{dt}$$

Or

$$\hat{a}(t) = L^{\gamma}(L^{\gamma}(x(t)) = \frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} \frac{d}{dt}\left(\frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)} t^{1-\gamma} \frac{d\,x(t)}{dt}\right)$$

$$= \left(\frac{\Gamma(\rho)}{\Gamma(\rho - \gamma + 1)}\right)^2 t^{2-2\gamma} \left(\frac{d^{\,2}x(t)}{dt^2}\right)$$

we are only interested in the case where $\rho = \gamma$.

4.2. One-dimensional motion

We will discuss the motion of objects in one dimension in the light of the generalized fractional derivative. We will discuss the case when $\rho = \gamma$

Newtons second law states that $F = m * a$ which becomes

$$\hat{F} = m * \hat{a} = m * \Gamma(\gamma)\, t^{1-\gamma} \; \frac{d\, v(t)}{dt}$$

or

$$\hat{F} = \Gamma(\gamma)\, t^{1-\gamma}\, (m * \frac{d\, v(t)}{dt}) = \Gamma(\gamma)\, t^{1-\gamma}(F)$$

Let us consider the case where the fractional force is constant for example $(\hat{F} = 3N)$ for an object with $1kg$ mass

$$\frac{3}{\Gamma(\gamma)}\, t^{-1+\gamma} = \frac{d\, v(t)}{dt}$$

or

$$\hat{v}(t) = \int_0^t \frac{3}{\Gamma(\gamma)}\, \tau^{-1+\gamma}\, d\tau = \frac{3}{\Gamma(\gamma)} \frac{t^{\gamma}}{\gamma} = \frac{3\, t^{\gamma}}{\Gamma(\gamma + 1)}$$

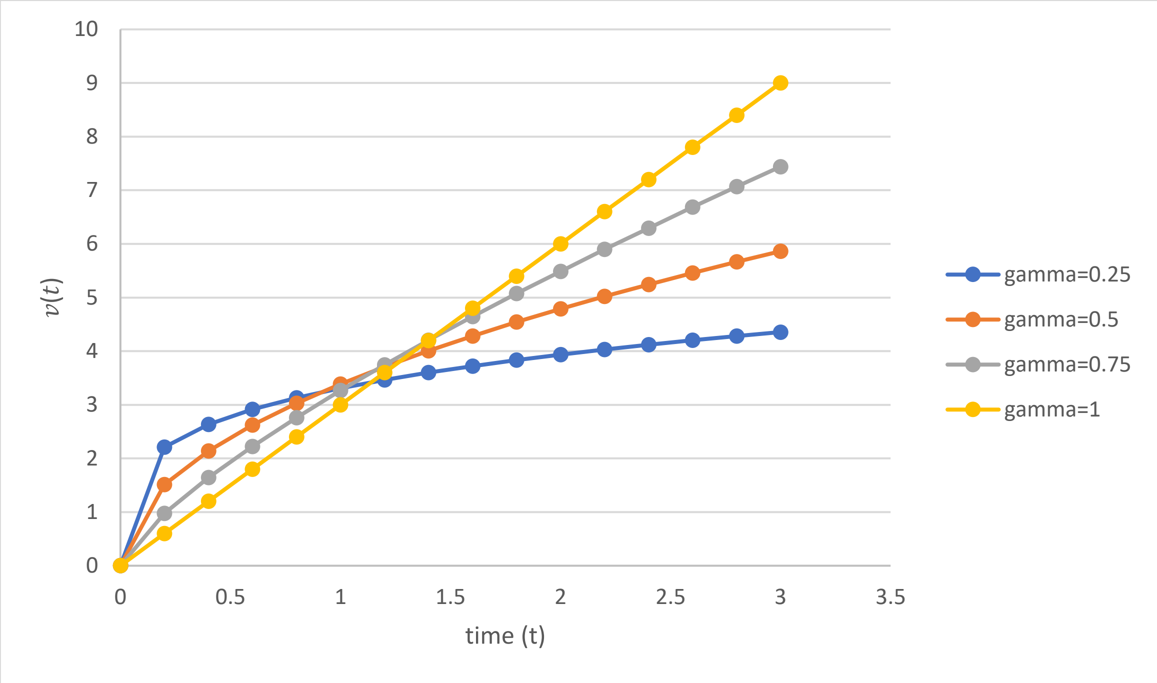

Figure 1: Graph of $\hat{v}(t)$ vs. t for different values of gamma

We can see from the figure that for gamma = 1 the constant force yields a constant acceleration.

4.2.1. Constant velocity motion

An object is moving with constant velocity, it means that the instantaneous velocity at any point $(t = c\ )$ in a time interval say $[a, b]$ is the same value as the average velocity over the entire time interval. In other words, the acceleration is equal to zero.

From the definition of the fractional velocity, we have

$$\hat{v}(t) = L^{\gamma}(x(t))$$

Multiplying both side by I_γ from the left then using theorem 2.10 and remark 2.8, we have

$$\hat{x}(t) - \hat{x}(0) = \hat{v}\,\frac{t^\gamma}{\Gamma(\gamma+1)} \qquad 4.1$$

Example 4.1 Let us consider an object moving with constant velocity $\hat{v} = 3$ from a starting point (assume $x(0) = 0$), then the equation of motion for this case

$$\hat{x}(t) = 3\,\frac{t^\gamma}{\Gamma(\gamma+1)}$$

Values of γ	$\hat{x}(t)$
0.25	$3\,\frac{t^{0.25}}{\Gamma(1.25)}$
1/3	$3\,\frac{t^{1/3}}{\Gamma(4/3)}$
0.5	$3\,\frac{t^{0.5}}{\Gamma(1.5)}$
2/3	$3\,\frac{t^{2/3}}{\Gamma(5/3)}$
0.75	$3\,\frac{t^{0.75}}{\Gamma(1.75)}$
1	$3t$

Table 5: $\hat{x}(t)$ at different values of γ

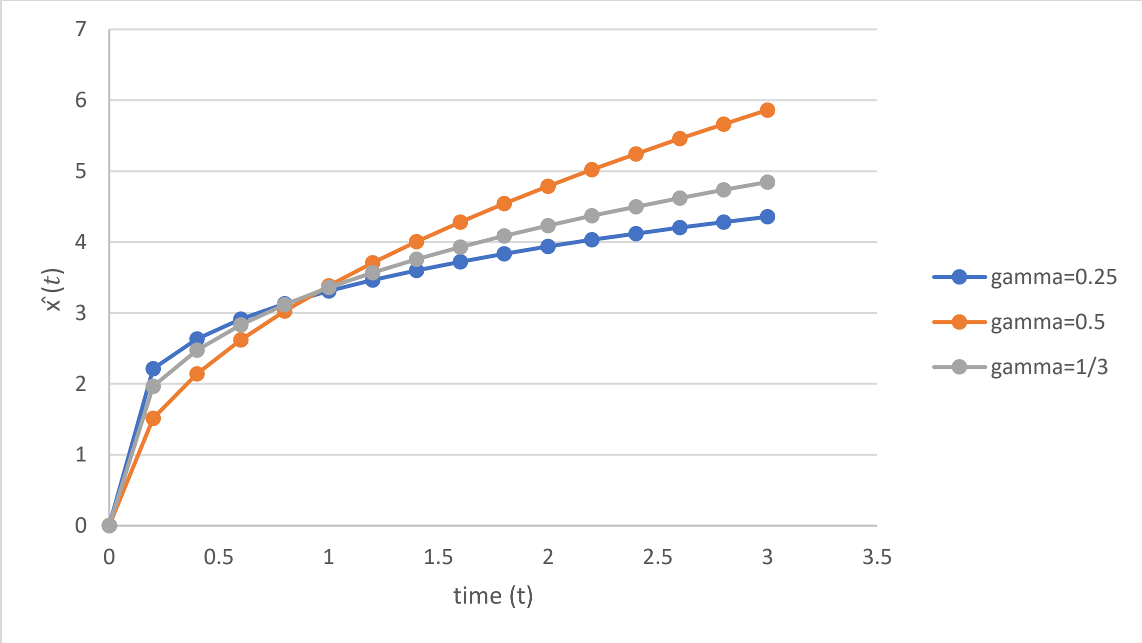

Figure 2(a): Graph of $\hat{x}(t)$ vs t for $\gamma = 0.25, \frac{1}{3}, 0.5$

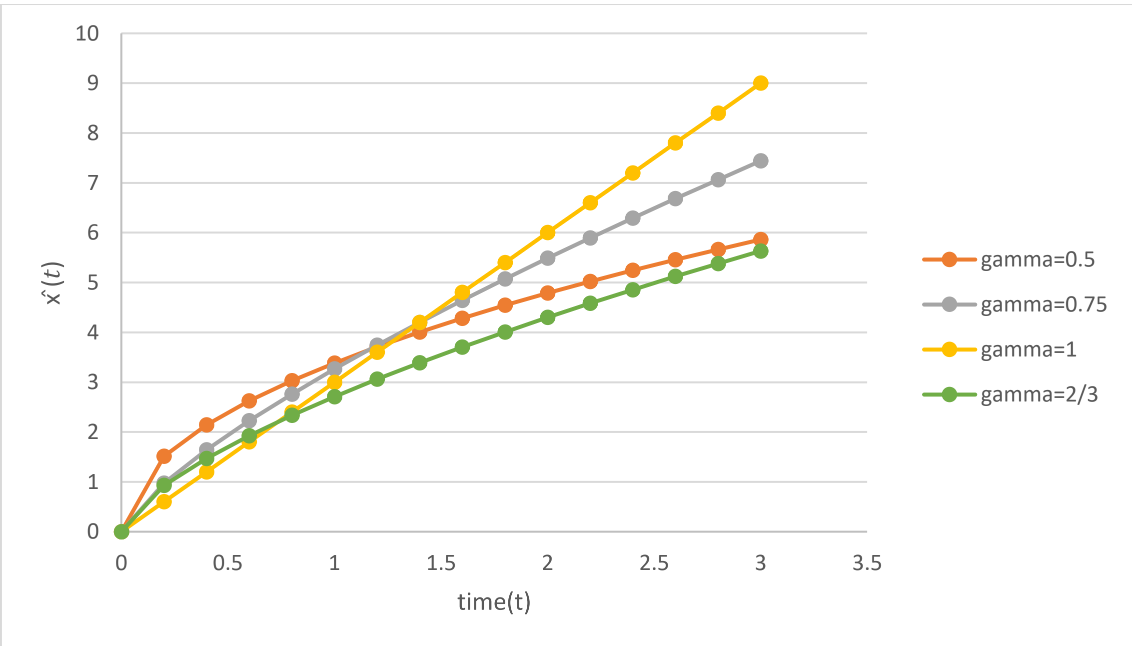

Figure 2(b): Graph of $\hat{x}(t)$ vs t for $\gamma = 0.5, \frac{2}{3}, 0.75, 1.$

We can see from the figure that for gamma = 1 is the classical case. □

4.2.2. Constant acceleration motion

An object is moving with constant acceleration, it means that the instantaneous acceleration at any point $(t = c\ )$ in a time interval say $[a, b]$ is the same value as the average acceleration over the entire time interval. In other words, the velocity increases or decreases at the same rate throughout the motion.

The plot of the normal a versus t gives a horizontal line while the plot of normal v versus t gives a straight line with either positive, zero, or negative slope.

Because the average acceleration equals the instantaneous acceleration when a is constant, we can write $a = \bar{a}$,we can choose initial time to be zero for convenience.

This case is actually the generalization of constant velocity

From the definition of the fractional acceleration, we have

$$\hat{a}(t) = L^{\gamma}(v(t))$$

Multiplying both side by I_{γ} from the left then using theorem 2.10 and remark 2.8, we have

$$\hat{v}(t) - \hat{v}(0) = \hat{a}\,\frac{t^{\gamma}}{\Gamma(\gamma+1)} \qquad 4.2$$

Remark 4.1. When $\gamma = 1$ the equation 4.2 reduces to the known newton equation

$$v(t) - v(0) = at$$

Applying the definition of the fractional velocity to equation 4.2, we have

$$\hat{v}(t) = L^{\gamma}\big(x(t)\big) = \hat{v}(0) + \hat{a}\,\frac{t^{\gamma}}{\Gamma(\gamma+1)}$$

Multiplying both side by I_{γ} from the left then using theorem 2.10 and remark2.9, we have

$$\hat{x}(t) - \hat{x}(0) = \hat{v}(0)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{\hat{a}}{\Gamma(\gamma+1)}\frac{1}{\Gamma(\gamma)}\frac{t^{2\gamma}}{2\gamma} \qquad 4.3$$

Remark 4.2. When $\gamma = 1$, equation 4.2 reduces to the known newton equation

$$x(t) - x(0) = v(0)t + a\frac{t^{2}}{2}$$

Remark 4.3. We can see that when we substitute $a = 0$ in equation 4.3 then we would get equation 4.1.

Now if we want to Solve for $\hat{v}(t)$ from $\hat{x}$ and $\hat{a}$ without knowing the time it needed we first solve the equation 4.2 for t we get

$$\frac{\hat{v}(t) - \hat{v}(0)}{\hat{a}}\Gamma(\gamma+1) = t^{\gamma}$$

then we substitute it in equation 4.3

$$\hat{x}(t) - \hat{x}(0) = \hat{v}(0)\frac{1}{\Gamma(\gamma+1)}\left(\frac{\hat{v}(t) - \hat{v}(0)}{\hat{a}}\Gamma(\gamma+1)\right) + \frac{\hat{a}}{\Gamma(\gamma+1)}\frac{1}{\Gamma(\gamma)}\frac{1}{2\gamma}\left(\frac{\hat{v}(t) - \hat{v}(0)}{\hat{a}}\Gamma(\gamma+1)\right)^{2}$$

$$\hat{x}(t) - \hat{x}(0) = \hat{v}(0)\left(\frac{\hat{v}(t) - \hat{v}(0)}{\hat{a}}\right) + \frac{\Gamma(\gamma+1)}{\hat{\mathrm{a}}}\frac{1}{\Gamma(\gamma)}\frac{1}{2\gamma}\left(\hat{v}(t) - \hat{v}(0)\right)^2 \qquad 4.4$$

For the case when $\gamma = 1$ we get the

$$x(t) - x(0) = v(0)\left(\frac{v(t) - v(0)}{a}\right) + \frac{1}{\mathrm{a}}\frac{1}{2}\left(v(t) - v(0)\right)^2$$

Or

$$2a(x(t) - x(0)) = v(t)^2 - v(0)^2$$

Which coincides with newtons thirds law.

Example 4.2 Let us consider an object moving from rest (zero initial velocity) with constant γ −acceleration of 3 from a starting point (assume $\hat{x}(0) = 0$), then the equations of motion for this case

$$\hat{v}(t) = 3\frac{t^\gamma}{\Gamma(\gamma+1)}$$

$$\hat{x}(t) = \frac{3}{\Gamma(\gamma+1)}\frac{1}{\Gamma(\gamma)}\frac{t^{2\gamma}}{2\gamma}$$

γ	$\hat{v}(t)$	$\hat{x}(t)$
0.5	$3\frac{t^{0.5}}{\Gamma(1.5)}$	$\frac{3}{\Gamma(1.5)\Gamma(0.5)}t$
0.75	$3\frac{t^{0.75}}{\Gamma(1.75)}$	$\frac{2}{\Gamma(1.75)\Gamma(0.75)}t^{1.5}$
1	$3t$	$\frac{3}{2}t^2$

Table 6: $\hat{x}(t)$, $\hat{v}(t)$ at different values of γ

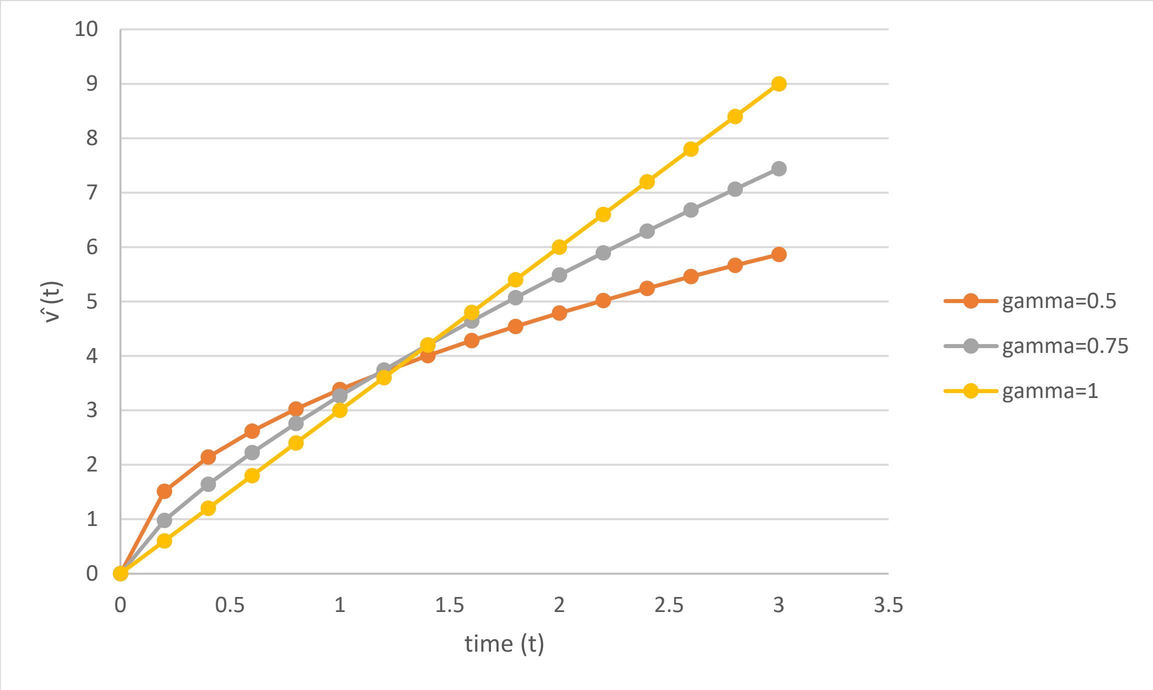

Figure 3(a): Graph of $\hat{v}(t)$ vs t for $\gamma = 0.5, 0.75, 1$

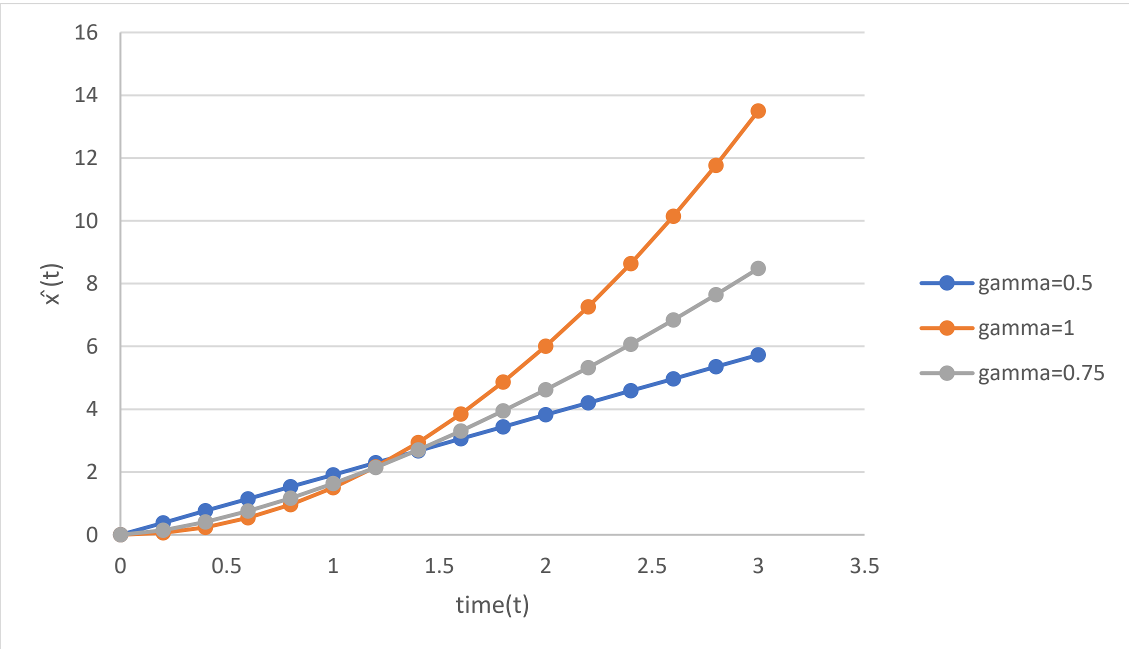

Figure 3(b): Graph of $\hat{x}(t)$ vs t for $\gamma = 0.5, 1, 2$ □

4.2.3. Free fall motion

A freely falling object is any object moving freely under the influence of gravity only, regardless of its initial motion (whether it has initial velocity or not), also we neglect the air resistance. Objects thrown upward or downward and those released from rest are all considered freely falling.

This is a special case of constant downward acceleration motion of 9.8 m/s^2 .

The equations of motion for this case are

$$\hat{v}(t) - \hat{v}(0) = -9.8\frac{t^{\gamma}}{\Gamma(\gamma+1)} \qquad 4.5$$

$$\hat{x}(t) - \hat{x}(0) = \hat{v}(0)\frac{t^{\gamma}}{\Gamma(\gamma+1)} - \frac{9.8}{\Gamma(\gamma+1)}\frac{1}{\Gamma(\gamma)}\frac{t^{2\gamma}}{2\gamma} \qquad 4.6$$

Example 4.3 Let us consider an object thrown upward with initial velocity $\hat{v}(0) = 15m/s$ .We can assume the hand of the person as a reference point (assume $\hat{x}(0) = 0$), then the equations of motion for this case

$$\hat{v}(t) = -9.8\frac{t^{\gamma}}{\Gamma(\gamma+1)} + 15$$

$$\hat{x}(t) = 15\frac{t^{\gamma}}{\Gamma(\gamma+1)} - \frac{9.8}{\Gamma(\gamma+1)}\frac{1}{\Gamma(\gamma)}\frac{t^{2\gamma}}{2\gamma}$$

γ	$\hat{v}(t)$	$\hat{x}(t)$
0.5	$-9.8\frac{t^{0.5}}{\Gamma(1.5)}+15$	$15\frac{t^{0.5}}{\Gamma(1.5)}-\frac{9.8}{\Gamma(1.5)\Gamma(0.5)}t$
0.75	$-9.8\frac{t^{0.75}}{\Gamma(1.75)}+15$	$15\frac{t^{0.75}}{\Gamma(1.75)}-\frac{19.3}{3\,\Gamma(1.75)\Gamma(0.75)}t^{1.5}$
1	$-9.8t+15$	$15t-\frac{9.8}{2}t^{2}$

Table 7: $\hat{x}(t)$, $\hat{v}(t)$ at different values of γ

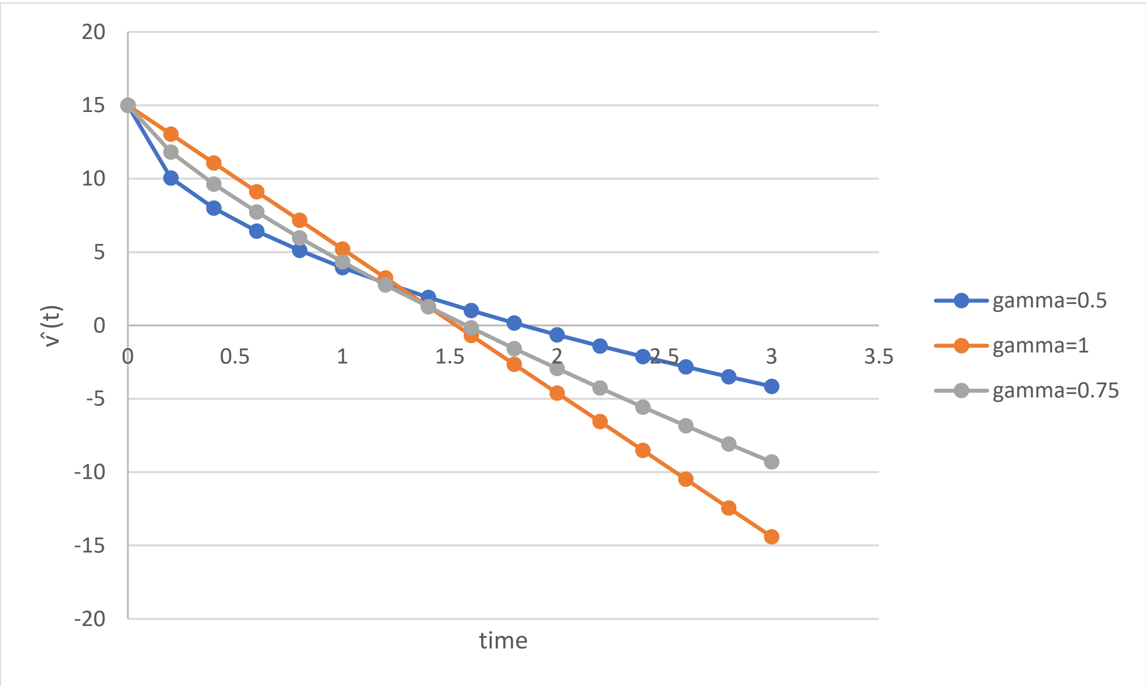

Figure 4(a): Graph of $\hat{v}(t)$ vs t for $\gamma = 0.5, 0.75,1$

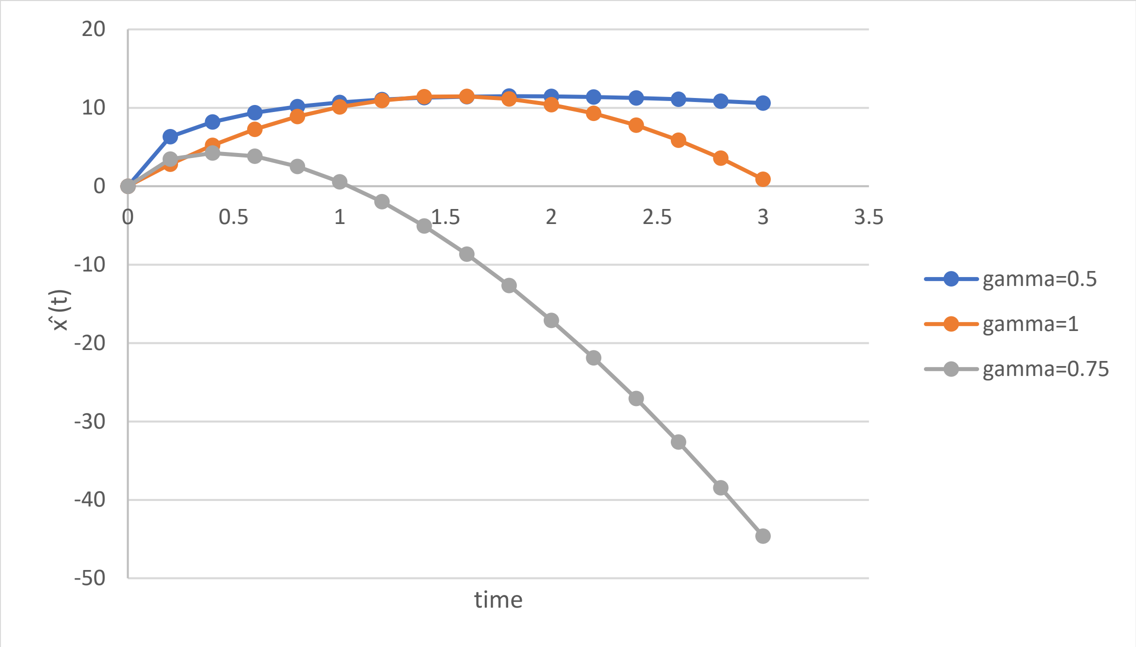

Figure 4(b): Graph of $\hat{x}(t)$ vs t for $\gamma = 0.5, 0.75, 1$□

4.3. Linear momentum, Work and Kinetic energy

Linear momentum is a physical quantity that is directly proportional to the object's mass (m) and velocity (v). Therefore, the greater an object's mass or its velocity, the greater its momentum.

Linear momentum (p) is defined as follows

$$p = m * v$$

We can define the fractional linear momentum as

$$\hat{p} = m * (\hat{v})$$

or

$$\hat{p} = m * (\Gamma(\gamma)\, t^{1-\gamma} \ \frac{d\, x(t)}{dt})$$

Work (w) is a measure of energy transfer that occurs when an external force is applied over an object hence it moves a certain distance. In the case where the force is constant, work is easily computed by multiplying the length of the path (s) by the component of the force acting along the path.

The work is defined as follows

$$w = F * s * cos\theta$$

Where θ is the angle between the objects path and the applied force.

We can define the fractional work as

$$\widehat{w} = (\hat{F}\ ) * s * cos\theta$$

Or

$$\widehat{w} = (\Gamma(\gamma\ )\, t^{1-\gamma} \ \ F) * s * cos\theta$$

Example 4.4 let us consider an object being pulled through a path $5m$ long with a constant force $F = 50N$ that is inclined at an angle $\theta = 30^{\circ}$. Then the fractional work applied by this force is

$$\widehat{w} = (50\, \Gamma(\gamma\ )\, t^{1-\gamma} \ \ ) * 6 * cos30 \quad \square$$

Kinetic energy (k) is a form of energy that an object has as a consequence of its motion. If work is done on an object by applying a net force, the object speeds up hence it gains kinetic energy. Kinetic energy is a property depends on the object motion and its mass and is defined as follows

$$k = 0.5\, m\, v^2$$

We can define the fractional Kinetic energy as

$$\hat{k} = 0.5\, m\ (\ \hat{v}\,)^2$$

or

$$\hat{k} = 0.5\, m\ (\,\Gamma(\gamma)\, t^{1-\gamma}\ \frac{d\,x(t)}{dt})^2$$

It’s also known that linear momentum and kinetic energy are related as follows

$$k = \frac{p^2}{2m}$$

It still holds for the fractional relation.

Example 4.5 In example 4.1 if the object has a $1kg$ mass, we have

$$\hat{p} = (3\,\Gamma(\gamma)\, t^{1-\gamma}\ )$$

$$\hat{k} = 0.5\,(3\,\Gamma(\gamma)\, t^{1-\gamma}\,)^2\ \square$$

Remark 4.4 The case where $\gamma = 1$ we get the classical Newtonian formulas.

4.4. Two-dimensional motion

In this section we study the motion of objects in both the x- and y-directions simultaneously under a constant acceleration. An important special case of this two-dimensional motion is called projectile motion. We neglect the effects of air resistance and the rotation of Earth, the path of a projectile in Earth's gravity field is curved in the shape of a parabola.

An important fact about projectile motion is that the horizontal and vertical motions don't affect each other (they're independent).

Constant acceleration motion equations developed in section 4.2.2. can be applied separately for the x-direction motion and the y-direction motion. With the difference that the initial velocity has two components, not just one as seen in figure 5.

We assume that at $t = 0$ the projectile leaves the origin with an initial velocity v_0. If the velocity vector makes an angle θ (the projection angle) with the horizontal line, we have

$$v_{x0} = v_0 \cos\theta \quad and \quad v_{y0} = v_0 \sin\theta$$

where v_{x0} is the initial velocity in the x −direction and v_{y0} is the initial velocity in the y −direction.

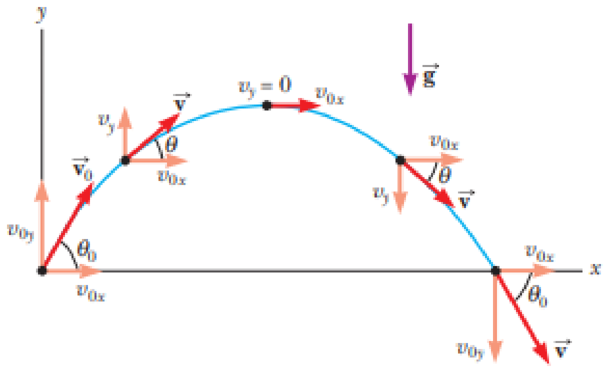

Figure 5: The parabolic trajectory of a particle that leaves the origin with a velocity of v_0

Because we neglected air resistance this means that acceleration component in the x-direction is zero ($a_x = 0$), hence the projectile's velocity component along the x −direction remains constant. The acceleration component in the y −direction is the acceleration of gravity ($a_x = -g$).

We look into 4.2,4.3,4.4 equation in section 4.2.2.

For the equation of the horizontal motion (in x −direction), we have

$$v_x(t) - v_{x0} = 0$$

$$x(t) - x_0 = v_{x0}\frac{t^\gamma}{\Gamma(\gamma+1)}$$

For the equation of the vertical motion (in y −direction), we have

$$v_y(t) - v_{y0} = -9.8\frac{t^\gamma}{\Gamma(\gamma+1)}$$

$$y(t) - y_0 = v_{y0}\frac{t^\gamma}{\Gamma(\gamma+1)} - \frac{9.8}{\Gamma(\gamma+1)}\frac{1}{\Gamma(\gamma)}\frac{t^{2\gamma}}{2\gamma}$$

$$y(t) - y_0 = v_{y0}\left(\frac{v_y(t) - v_{y0}}{9.8}\right) - \frac{\Gamma(\gamma+1)}{9.8}\frac{1}{\Gamma(\gamma)}\frac{1}{2\gamma}\left(v_{y0}(t) - v_{y0}\right)^2$$

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تعريف معمم للاشتقاق الكسري مع تطبيقات في ميكانيكا نيوتن

إعداد: سماح موسى خلف بجالي

إشراف: الدكتور إبراهيم الغروز

ملخص الرسالة

يدرس هذا العمل التعريف الجديد المعمم للاشتقاق الكسري (GD)، والذي يوضح أن قانون المؤشر $D^{\alpha}D^{\beta}f(t) = D^{\alpha+\beta}f(t);\ 0 < \alpha,\beta \leq 1$ تنطبق على جميع الاقترانات القابلة للتفاضل موسعة بواسطة سلسلة تايلور. يتم تطبيق (GD) على بعض الاقترانات ومقارنة النتائج مع مشتقة Caputo الكسرية. يتم الحصول على حلول بعض المعادلات التفاضلية الكسرية عبر عامل التشغيل (GD). كما تمت مناقشة النتائج مع تعريف (CD) للاشتقاق الكسري. كما تمت مناقشة ميكانيكا نيوتن في ضوء حساب التفاضل والتكامل الكسري.