Deanship of Graduate Studies

Al-Quds University

On a Generalized Fractional Calculus Operators in a Complex Domain

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M.Sc. Thesis

Jerusalem- Palestine

1438/2017

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics at Al-Quds University

1438/2017

Al-Quds University

Deanship of Graduate Studies

Graduate Studies/Mathematics

Thesis Approval

On a Generalized Fractional Calculus Operators in a Complex Domain

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Master thesis submitted and accepted, Date :7/8/2017

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Jerusalem-Palestine

1438/2017

Dedication

I dedicate this thesis to my parents, brothers and sisters who led me through my life with their light of hope and support.

I also dedicate it to my friends who touched my life with their love, passion and support.

Declaration

I certify that this thesis submitted for the degree of master, is the result of my own research, except where otherwise acknowledged, and this study has been not submitted for a higher degree to any other university or institute.

Signature:

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Date: 7/8/2017

Acknowledgement

I am thankful to all teachers in mathematics department who helped me to know that, as lifelong learners of science, I can live more productive, responsible and fulgent. Special thanks go to Dr. Ibrahim AL-Grouz for his support, understanding and giving me all needed encouragement and enthusiasm to finish this thesis successfully.

In addition, my thanks are presenting to my university presented by its head president Dr. Imad Abu Keshek for giving me all support.

Abstract

We generalized the fractional calculus in the complex plane $\mathbb{C}$ to new fractional calculus operators, namely we defined the integral and differential fractional operators. Several properties of these new operators are proved such as the boundedness and compactness. Moreover, we discussed these fractional calculus operators on the special families S, the set of normalized univalent function, and K, the set of convex function, and derived several properties on it. Finally, we derived the relations between the proposed fractional calculus operators and the Gauss hypergeometric functions for some functions.

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Introduction

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems [12]. Fractional calculus is a branch of mathematical analysis deals with integrals and derivatives of arbitrary orders. In the recent years a lot of attention has been given to fractional integral and differential operators in geometric function theory, the study of geometric properties of analytic functions, univalent functions and Riemann Mapping Theorem, etc. In fact, fractional calculus asserts that orders of integral or derivative operators can be arbitrary numbers, for instance, one could calculate the 1/2-th order integral or $\sqrt{3}$ -th order derivative of an analytic function ([1],[19]).

This thesis consists of three chapters. In the first chapter, we give the history of the fractional calculus, and we present some basic definitions and properties that are used in this theory. We define Gauss hyper-geometric function and give some examples about it. In the second chapter, we concentrate on fractional integral and fractional derivative where we present definitions and some basic properties and the relation between fractional integral and fractional derivative. The third chapter presents fractional calculus operators: we provide some definitions and give some related results, and prove some properties of there operators and show that the operators represent some special functions. Furthermore, we consider one special function in geometric function theory, that is also known as Gauss hyper-geometric function and study some of its properties in the unit disk D.

CHAPTER ONE

Introduction to Fractional Calculus

In the last few decades, fractional calculus has become an important research due to its applications appear in science, engineering, economics and applied mathematics [1].

In Section 1.1, we present the historical foreword of fractional calculus. Then, in Section 1.2, we consider some important special functions such as Gamma function and Beta function. Then, In Section 1.3, we define an analytic function and a univalent function.

1.1 Historical foreword

Most authors on this topic will cite a particular date as the birthday of Fractional Calculus. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative $\frac{D^n x}{Dx^n}$ of the function $f(x) = x$. L'Hopital's posed the question to Leibniz, what would the result be if n =1/2. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." Thus, fractional calculus was born [12].

In the following years, some famous mathematicians, such as Euler, Lagrange, Lacroix, Fourier, Liouville and Riemann, developed the theory of fractional calculus and the mathematical consequences [1]. Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann-Liouville and Grunwald-Letnikov definition.

In recent years considerable interest in fractional calculus has been stimulated by the applications it finds in different areas of applied sciences like physics and engineering, possibly including fractal phenomena. Now there are more books of proceedings and special issues of journals published that refer to the applications of fractional calculus in several scientific areas including special functions, control theory, chemical physics, stochastic processes, anomalous diffusion. Several special issues appeared in the last decade which contain selected and improved papers presented at conferences and advanced schools, concerning various applications of fractional calculus. Already since several years, there exist two international journals devoted almost exclusively to the subject of fractional calculus: Journal of Fractional Calculus (Editor-in-Chief: K. Nishimoto, Japan) started in 1992, and Fractional Calculus and Applied Analysis (Managing Editor: V. Kiryakova, Bulgaria) started in 1998. Recently the new journal Fractional Dynamic Systems has been announced to start in 2010.The authors believe that the volume of research in the area of fractional calculus will continue to grow in the forthcoming years and that it will constitute an important tool in the scientific progress of mankind [13].

1.2 Special Functions of Fractional Calculus

In this section, we concentrate on some fundamental special functions which are important in the study of the theory of fractional calculus. First, we recall Gamma function and some basic properties of this function. Second, we talk about Beta function and what is the relation between Gamma and Beta functions.

1.2.1 Gamma Function

In [19], Gamma function is a special transcendental function denoted by $\Gamma(\alpha)$ (the Euler: an Integral of the second Kind), and was first introduced by Euler to generalize the factorial to non- integer values. For $\alpha > 0$, it is defined as:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt. \qquad (1.1)$$

It follows that the Gamma function $\Gamma(\, x\,)$ is well defined and is analytic for $x > 0$, and has the following properties

i. $\Gamma(\, x) \ > 0\,.$

ii. For $x > 1\,, \Gamma(\, x+1\,) = \int_0^\infty \ t^x \ e^{-t}\text{dt}$

$$= \lim_{b\to\infty} \int_0^b t^x \ e^{-t}\text{dt} = \lim_{b\to\infty} [-t^x e^{-t}]|\begin{matrix} b \\ 0 \end{matrix} + \lim_{b\to\infty} [\, x \int_0^b t^{x-1} \ e^{-t}\text{dt}]$$

$$= x \int_0^\infty t^{x-1} \ e^{-t}\text{dt} = x\Gamma(\, x).$$

iii. $\Gamma(\, n) = (n-1)!$ for all positive integer $n \geq 2$. Also,

we have, $\Gamma(\, 1\,) = \ \int_0^\infty e^{-t}\text{dt} = 1$ (1.2)

Example 1.2.2

i. $\Gamma(\, 5) = \Gamma(\, 4+1) = 4! = 4.3.2.1 = 24.$

ii. $\Gamma\left(\frac{1}{2}\right) = \ \sqrt{\pi}\,.$

iii. $\Gamma\left(\frac{3}{2}\right) = \ \frac{\sqrt{\pi}}{\mathbf{2}}.$

Proof: By definition 2.1 we have,

$$\Gamma\left(\frac{1}{2}\right) = \ \int_0^\infty \ t^{\frac{1}{2}-1} e^{-t}\text{dt} = \int_0^\infty \ t^{-\frac{1}{2}} \, e^{-t}\text{dt}$$

Let $t\, = u^2$ then dt $= 2u$du , then

$$\Gamma\left(\frac{1}{2}\right) = \ \int_0^\infty \ u^{\frac{-2}{2}} e^{-u^2} \ 2\text{u du} = 2\int_0^\infty \ e^{-u^2}\text{dt} = 2\ \frac{\sqrt{\pi}}{2} = \ \sqrt{\pi}\,.$$

iv. $\Gamma\left(\frac{3}{2}\right) = \ \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$

1.2.2 Beta Function

Also known as the Euler Integral of the First Kind, the Beta Function is important relationship in fractional calculus. Its solution not is only defined through the use of multiple Gamma Functions, but furthermore shares a form that is characteristically similar to the Fractional Integral and Fractional Derivative of many functions. It is defined as

$$B(p\,,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,\mathrm{du}\,, \qquad \text{where } \mathrm{p,q} \in \mathbb{R}_+$$

Another property

$$B(p\,,q) = (a-b)^{-p-q+1}\int_b^a (t-b)^{p-1}(a-t)^{q-1}\,\mathrm{dt}\,, \qquad \text{where } \mathrm{p,q} \in \mathbb{R}_+ \qquad (1.3)$$

A key property of the Beta function is its relationship to the Gamma function as

$$B(p\,,q) = \frac{\Gamma(\,p)\Gamma(\,q)}{\Gamma(\,p+q)} = B(q,p)$$

when p and q are positive integers [12].

1.3 Univalent Function and Analytic Function

In this section we define the analytic function and the univalent function in open unit disk around z (the set of points whose distance from z is less than 1).

Let $D \subset \mathbb{C}$ be a domain, that is, an open and connected non-empty subset of the complex plane. A function $f\colon D \to \mathbb{C}$ is analytic at z_0 if it is complex differentiable at every point in some neighbourhood of $z_0 \in D$. We say that f is analytic on D if f is analytic at z_0 for every $z_0 \in D$. Also, an analytic function f has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!},$$

for all z in some open disk centered at z_0 [4].

Definition 1.3.1

A function $f: D \to \mathbb{C}$ is said to be univalent (or schlicht or one-to-one) in a domain $D \subset \mathbb{C}$ if $f(z_1) \neq f(z_2)$ for all points $z_1, z_2 \in D$ with $z_1 \neq z_2$. The function f is said to be locally univalent at a point $z \in D$ if it is univalent in some neighborhood of z. By Rouche's theorem if f is analytic on D, then $f'(z) \neq 0$ if and only if f is locally univalent at z [4].

Definition 1.3.2 (convex function)

If $[a, b]$ is an interval in the real line, a function $f: [a, b] \to \mathbb{R}$ is convex if for any two points x_1 and x_2 in $[a, b]$

$$f(tx_2 + (1 - t)x_1) \leq tf(x_2) + (1 - t)f(x_1)$$

whenever $0 \leq t \leq 1$. A subset $A \subset \mathbb{C}$ is convex if whenever z and w are in A, $tz + (1 - t)w$ is in A for $0 \leq t \leq 1$; that is, A is convex when for any two points in A the line segment joining the two points is also in A.

Proposition 1.3.3

A function $f: [a, b] \to \mathbb{R}$ is convex if and only if the set

$\mathcal{A} = \{(x, y) | \ a \leq x \leq b \ and \ f(x) \leq y\}$ is convex [3].

Theorem 1.3.4 [3]

A function f is univalent in D if and only if $Re\, f'(z) > 0$, $\forall\, z \in D$.

Now. In [4], we shall be concerned with the class S of functions f analytic and univalent in the unit disk $D = \{z: |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad |z| < 1.$$

The following simple examples of functions in S:

i. $f(z) = z$, the identity mapping. Clearly f is analytic in D and since $Re\, f'(z) > 0, \forall\, z \in D$, then f is univalent in D. Also $f(0) = 0$ and $f'(0) = 1$.

Hence, $f(z) \in$ S.

Some transformations for the class S

ii. Conjugation: if $f \in S$ and $g(z) = \overline{f(\bar{z})} = z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \cdots$, then $g \in S$. To show this, let $w(z) = \bar{z}$ so that $w: \mathbb{C} \to \mathbb{C}$ is clearly one-to-one. Since $g(z) = \overline{f(\bar{z})} = (w \circ f \circ w)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on D. Note that $w(z)$ is not analytic on D, the Taylor series for f is

$$z + \sum_{n=2}^{\infty} a_n z^n$$

Has radius of convergence 1. That is, the Taylor series converges to $f(z)$ for all $|z| < 1$ with the convergence uniform on every closed disk $|z| \leq r < 1$. It then follows that

$$f(\bar{z}) = \bar{z} + \sum_{n=2}^{\infty} a_n \bar{z}^n$$

and

$$\overline{f(\bar{z})} = \bar{\bar{z}} + \sum_{n=2}^{\infty} \bar{a}_n\, \bar{\bar{z}}^n = z + \sum_{n=2}^{\infty} \bar{a}_n z^n$$

Which is define an analytic function on D, has radius of convergence 1. Hence, we conclude that

$$g(z) = \overline{f(\bar{z})} = \overline{\bar{z} + a_2\bar{z}^2 + a_3\bar{z}^3 + \cdots} = z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \cdots$$

Is analytic on D with $g(0) = 0$ and $g'(0) = 1$. Thus, $g \in S$.

iii. Rotation: if $f \in S$ and $g(z) = e^{-i\theta} f\left(e^{i\theta}z\right)$, then $g \in S$. To show this, let $R(z) = e^{i\theta}z$ and $T(z) = e^{-i\theta}z$ so that $R: \mathbb{C} \to \mathbb{C}$ and $T: \mathbb{C} \to \mathbb{C}$ are clearly one-to-one. Since $g(z) = e^{-i\theta} f\left(e^{i\theta}z\right) = (T \circ f \circ R)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on D. Since

$$g'(z) = e^{-i\theta}.\, e^{i\theta}. f'\left(e^{i\theta}z\right) = f'\left(e^{i\theta}z\right)$$

We see that g is analytic on D. Furthermore, $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 1$ so that $g \in S$ as required.

We also note that the Taylor expansion of g is given by

$$g(z) = e^{-i\theta}\left(e^{i\theta}z + a_2 e^{2i\theta}z^2 + a_3 e^{3i\theta}z^3 + \cdots\right) = z + a_2 e^{i\theta}z^2 + a_3 e^{2i\theta}z^3 + \cdots.$$

iv. Range transformation: if $f \in S$ and ψ is a function analytic and univalent on the range of $f(D)$, with $\psi(0) = 0$ and

$$g(z) = \frac{(\psi \circ f)(z) - \psi(0)}{\psi'(0)},$$

then $g \in S$. To show this, let $\psi: f(D) \to \mathbb{C}$ be analytic and univalent on $f(D)$. If

$$g(z) = \frac{(\psi \circ f)(z) - \psi(0)}{\psi'(0)},$$

Then g is clearly univalent on D with $g(0) = 0$. Furthermore,

$$g'(z) = \frac{f'(z)\psi'(f(z))}{\psi'(0)}$$

So that g is analytic on D with $g'(0) = 1$. Thus, $g \in S$ as required.

We now have a subclass of S which is the class of close-to-convex functions.

A function f analytic in the unit disk is said to be close-to-convex if there is a convex function g such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \qquad \forall z \in D.$$

Let K be the set of all close-to-convex function in . If $f \in$ K and $f(z) = z + \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$, then $|a_n| \leq 1$ $(n = 2, 3, \dots)$ [7].

For example let

$f(z) = \frac{\gamma}{2} \ln \frac{1+z}{1-z} + (1-\gamma) \frac{z}{1-z}$ and take $g(z) = \frac{z}{1-z}$.

1.4 Gauss Hypergeometric Functions

We focus in this section on computing the two most commonly used hyper-geometric functions, the confluent hyper-geometric function ${}_1F_1(a;\ b;\ z\ )$ and the Gauss hyper-geometric function ${}_2F_1(a,\ b;\ c;\ z\ )$.

In [16]. A hypergeometric function ${}_pF_q$ is defined as follows for $a_1, \dots, a_p, b_1, \dots, b_q$, $z \in \mathbb{C}$:

$${}_pF_q\left(a_1, \dots, a_p;\ b_1, \dots, b_q; z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \dots (a_p)_j}{(b_1)_j \dots (b_q)_j} \frac{z^j}{j!}$$

Where, for some parameter > 0 , the Pochhammer symbol $(\mu)_j$ is defined as

$(\mu)_0 = 1, \quad (\mu)_j = \mu(\mu + 1) \dots (\mu + j - 1), \quad j = 1,2, \dots$

We use the basic fact about hypergeometric functions, for $n \in \mathbb{N}$,

$$(\mu)_n = \frac{\Gamma(\mu + n)}{\Gamma(\mu)}.$$

Definition 1.4.1

In [5]. The confluent hypergeometric function ${}_1F_1(a;\ b; z)$, is defined as

$${}_1F_1(a;\ b; z) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{z^j}{j!} \tag{1.4}$$

which converges for any $z \in \mathbb{C}$, and is defined for any $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

In [6]. It should be noted that ${}_1F_1(a;\ b; 0) = 1$. If Re b> Re a> 0 , then ${}_1F_1(a;\ b; z)$ can be represented as an integral

$${}_1F_1(a;\ b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt}\, t^{a-1}\, (1-t)^{b-a-1}\, dt \tag{1.5}$$

Proof. For $Re\, b > Re\, a > 0$, then

$$e^{zt} = \sum_{j=0}^{\infty} \frac{(zt)^j}{j!}$$

This implies that

$$\int_0^1 t^{a-1}\, (1-t)^{b-a-1} e^{zt}\, dt = \sum_{j=0}^{\infty} \frac{(z)^j}{j!} \int_0^1 t^{j+a-1}\, (1-t)^{b-a-1}\, dt$$

The latter integral is a beta integral which equals

$$\int_0^1 t^{j+a-1}\, (1-t)^{b-a-1}\, dt = B(j+a, b-a) = \frac{\Gamma(j+a)\Gamma(b-a)}{\Gamma(j+b)}$$

Now, we use the fact that

$$\frac{\Gamma(j+a)}{\Gamma(\mathrm{a})} = a(a+1)\ldots(a+j-1) = (a)_j\,, \quad j = 0,1,2,\ldots$$

To obtain

$$\frac{\Gamma(b)}{\Gamma(\mathrm{a})\Gamma(\mathrm{b}-\mathrm{a})}\int_0^1 t^{a-1}\,(1-t)^{b-a-1}e^{zt}\,dt = \frac{\Gamma(b)}{\Gamma(\mathrm{a})}\sum_{j=0}^{\infty}\frac{\Gamma(j+a)}{\Gamma(\mathrm{j}+\mathrm{b})}\,\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\frac{(a)_j}{(b)_j}\,\frac{z^j}{j!} = \,{}_1F_1(a;\; b; z).$$

The confluent hypergeometric function is related to various elementary and special functions using Eq.(1.4) as follows

$${}_1F_1(a;\; a; z) = \sum_{j=0}^{\infty}\frac{(a)_j}{(a)_j}\,\frac{z^j}{j!} = \sum_{j=0}^{\infty}\frac{z^j}{j!} = \; e^z$$

Definition 1.4.2

Let $, b, c\; \in \mathbb{R}$, c$\in \mathbb{C} \setminus \mathbb{Z}_0^-$ and for all z $\in D$. Define Gauss hypergeometric function by

$${}_2F_1(a,b;\; c; z) = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\,\frac{z^j}{j!} \tag{1.6}$$

$$= 1 + \frac{a.\,b}{c.\,1}\,z + \frac{a(a+1)b(b+1)}{c(c+1).\,2.1}z^2 + \cdots \tag{1.7}$$

In particular, if $a = 1, b = c$, then the series in Eq.(1.7) takes the form

$$1 + \; z + z^2 + \cdots$$

In which cases we have a polynomial as follow.

Examples 1.4.3

1) If $b = c = 1$ and $a = \frac{1}{2}$, then

$$ {}_2F_1\left(\frac{1}{2}, 1;\ 1; z\right) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2})_j (1)_j}{(1)_j} \frac{z^j}{j!} $$

$$ = 1 + \frac{\frac{1}{2}}{1} z + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2.1} z^2 + \cdots $$

$$ = 1 + \frac{z}{2} + \frac{3z^2}{4.2} + \cdots = (1 - z)^{-\frac{1}{2}} $$

.

2) Let $a = 1, b = i$ and $c = 1$, then

$$ {}_2F_1(1, i;\ 1; z) = \sum_{j=0}^{\infty} \frac{(1)_j (i)_j}{(1)_j} \frac{z^j}{j!} $$

$$ = 1 + \frac{i}{1} \mathrm{z} + \frac{\mathrm{i(i+1)}}{2.1} \mathrm{z}^2 + \frac{\mathrm{i(i+1)(i+2)}}{3.2.1} \mathrm{z}^3 + \cdots $$

$$ = 1 + i\mathrm{z} + \frac{\mathrm{(i-1)}}{2} \mathrm{z}^2 + \frac{\mathrm{(i-3)}}{6} \mathrm{z}^3 + \cdots $$

3) Consider $a = b = 1$, $c = 2$ and for $-z$, we have

$$ {}_2F_1(1, 1;\ 2; -z) = \sum_{j=0}^{\infty} \frac{(1)_j (1)_j}{(2)_j} \frac{(-z)^j}{j!} $$

$$ = 1 + \frac{1.1}{2}(-z) + \frac{1.2.1.2}{2.3} \frac{z^2}{2!} + \cdots $$

$$ = 1 - \frac{1}{2} z + \frac{1}{3} z^2 - \cdots $$

$$ = \sum_{j=0}^{\infty} (-1)^j \frac{z^j}{j+1}. $$

Now. In the following remark, we recall some properties of the Gauss hypergeometric function in unit disk which we need in the development of the work.

Remark 1.4.4

In ([12],[13]), for all z∈ D and a, b, c ∈ ℝ , c∈ $\mathbb{C} \setminus \mathbb{Z}_0^-$, then

(i) The differential of functions (1.7) defined as

$$(\ {}_2F_1(a,b;\ c;z))' = \frac{ab}{c}\ \ {}_2F_1(a+1,b+1;\ c+1;z).$$

Proof. Using Eq.(1.7) , we get

$$(\ {}_2F_1(a,b;\ c;z))'$$

$$= (1 + \frac{a.b}{c.1}\ z + \frac{a(a+1)b(b+1)}{c(c+1).2.1} z^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2).3.2.1} z^3 + ..)'$$

$$= \frac{a.b}{c.1} + \frac{a(a+1)b(b+1)}{c(c+1).2.1} 2\mathrm{z} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2).3.2.1} 3z^2 \ldots$$

$$= \frac{a.b}{c}(1\ + \frac{(a+1)(b+1)}{(c+1)}\mathrm{z} + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)}\frac{z^2}{2!} + \cdots)$$

$$= \frac{a.b}{c}\sum_{j=0}^{\infty}\frac{(a+1)_j(b+1)_j}{(c+1)_j}\ \frac{z^j}{j!}$$

$$= \frac{a.b}{c}\ \ {}_2F_1(a+1,b+1;\ c+1;z).$$

(ii) In [6]. The Euler integral representation for function (1.7) defined as

$${}_2F_1(a,b;\ c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\ (1-t)^{c-b-1}(1-zt)^{-a}\ dt$$

Where c $> b > 0$.

Proof. Suppose $z \in D$, $Re\ b > 0$ and $Re\ (c-b) > 0$, then

$$(1-tz)^{-a}=\sum_{j=0}^{\infty}\frac{(a)_j}{j!}\ t^j z^j$$

This implies that

$$\int_0^1 t^{b-1}\ (1-t)^{c-b-1}(1-zt)^{-a}\ dt=\sum_{j=0}^{\infty}\frac{(a)_j}{j!}\ z^j\int_0^1 t^{j+b-1}\ (1-t)^{c-b-1}\ dt$$

The latter integral is a beta integral which equals

$$\int_0^1 t^{j+b-1}\ (1-t)^{c-b-1}\ dt=B(j+b,c-b)=\frac{\Gamma(j+b)\Gamma(c-b)}{\Gamma(\mathrm{j+c})}$$

Now, we use the fact that

$$\frac{\Gamma(j+b)}{\Gamma(\mathrm{b})}=b(b+1)\ldots(b+j-1)=(b)_j\ ,\quad j=0,1,2,\ldots$$

To obtain

$$\frac{\Gamma(c)}{\Gamma(\mathrm{b})\Gamma(\mathrm{c-b})}\int_0^1 t^{b-1}\ (1-t)^{c-b-1}(1-zt)^{-a}\ dt=\frac{\Gamma(c)}{\Gamma(\mathrm{b})}\sum_{j=0}^{\infty}\frac{\Gamma(j+b)(a)_j}{\Gamma(\mathrm{j+c})\ j!}\ z^j$$

$$=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\ \frac{z^j}{j!}=\ {}_2F_1(a,b;\ c;z).$$

1.5 The Bergman space

Function spaces. An important class of such functions is defined next.

Definition 1.5.1

The Bergman space $\mathfrak{A}^p(D)$ for $(0<p<1)$ is the set of functions f analytic in the open unit disk $D\coloneqq\{z\colon z\in\mathbb{C};\ |z|<1\}$ with the norm $\|f\|_{\mathfrak{A}^p}^p<\infty$ defined by

$$\|f\|_{\mathfrak{A}^p}^p=\frac{1}{\pi}\int_D\ |f(z)|^p\ d\mathfrak{A}<\infty\qquad\qquad z\in D,$$

where $d\mathfrak{A}$ is known as Lebesque measure over D [9].

1.6 The Mean Value Theorem for Integrals and Dirichlet Formula

The Mean Value Theorem for integrals is a powerful tool, which can be used to prove the Fundamental Theorem of Calculus, and to obtain the average value of a function on an interval. On the other hand, its weighted version is very useful for evaluating inequalities for definite integrals.

Theorem 1.6.1.(Mean Value Theorem for integrals)

In [20]. Assume that f and g are continuous on [a, b]. If g never changes sign and is nonnegative ($g(x) \geq 0$) in [a, b], then for some c in [a, b], we have

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Theorem 1.6.2 (Leibniz Rule)

The Leibniz integral rule gives a formula for differentiation of a definite integral whose limits are functions of the differential variable,

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(w,z)dw = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}dw + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

It is sometimes known as differentiation under the integral sign [18].

Theorem 1.6.3 (Dirichlet Formula**)**

If f is continuous and $,\nu > 0$, then

$$\int_0^z (z-t)^{\mu-1}\,dt\,\int_0^t (t-x)^{\nu-1} f(z,x)dx = \int_0^z dx\int_x^z (z-t)^{\mu-1}(t-x)^{\nu-1}\,f(z,x)dt$$

[17].

CHAPTER TWO

FRACTIONAL CALCULUS

In this chapter we define the fractional integral and fractional derivative in geometric function theory on a function $f(z)$ which is analytic in simply-connected region of the complex z-plane $\mathbb{C}$.

2.1 The Fractional Integral

According to the Riemann-Liouville approach to the fractional calculus, the notion of fractional integral of order α, for a function $f(z)$ by Srivastava and Owa ([14],[15]), is as follows:

$$I_z^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(\omega)(z-\omega)^{\alpha-1}\, d\omega \qquad (2.1)$$

where $0 < \alpha \leq 1$.

As an application of the expression (2.1), we get the fractional integral of the function z^{v} as follows:

$$I_z^{\alpha} \{z^{v}\} = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} z^{v+\alpha} \quad , \qquad 0 < \alpha ; -1 < v \qquad (2.2)$$

we can show that, using Eq.(2.1), as follows:

$$I_z^{\alpha} z^{v} = \frac{1}{\Gamma(\alpha)} \int_0^z f(\omega)(z-\omega)^{\alpha-1}\, d\omega$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^z \omega^{v}(z-\omega)^{\alpha-1}\, d\omega$$

$$= \frac{1}{\Gamma(\alpha)} \; z^{\alpha-1} \int_0^z \omega^v (1-\omega/z)^{\alpha-1} \, d\omega$$

Let $\frac{\omega}{z} = t$ then $\frac{d\omega}{z} = dt$, thus as $\omega = 0$ then $t = 0$ and as

$\omega = z$ then $t = 1$, so

$$I_z^\alpha \, z^v = \frac{1}{\Gamma(\alpha)} \; z^{\alpha-1} \int_0^1 z^v \, t^v (1-t)^{\alpha-1} \, z \, dt$$

$$= \frac{1}{\Gamma(\alpha)} \; z^{\alpha+v} \int_0^1 t^v (1-t)^{\alpha-1} \; dt$$

$$= \frac{1}{\Gamma(\alpha)} \; z^{v+\alpha} \, B(v+1, \alpha)$$

$$= \frac{1}{\Gamma(\alpha)} \; z^{v+\alpha} \; \frac{\Gamma(v+1)\Gamma(\alpha)}{\Gamma(v+\alpha+1)}$$

$$= z^{v+\alpha} \; \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)}.$$

For complementation we define I_z^0 = I (identity operator), i.e. we mean $I_z^0 \, f(z) = f(z)$.

Furthermore, we can prove the following property

$$I_z^\alpha \, I_z^\beta = I_z^{\alpha+\beta} = I_z^\beta \, I_z^\alpha \quad \text{for } \alpha, \beta \geq 0 \qquad (2.3)$$

As follow

$$I_z^\alpha I_z^\beta f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-\omega)^{\alpha-1} \, I_\omega^\beta \, f(\omega) \; d\omega$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^z \frac{1}{\Gamma(\beta)} \int_0^\omega f(t)(\omega - t)^{\beta-1} \, dt \; (z-\omega)^{\alpha-1} \, d\omega$$

Using Dirichlet Formula, we get

$$I_z^{\alpha} I_z^{\beta} f(z) \;=\; \frac{1}{\Gamma(\alpha)\Gamma(\beta\,)} \int_0^{z} f(\mathrm{t})\,\mathrm{dt} \int_t^{z} (\omega - \mathrm{t})^{\beta-1}\,(\mathrm{z}-\omega)^{\alpha-1}\,\mathrm{d}\omega$$

Using Beta function formula (1.3), we have

$$I_z^{\alpha} I_z^{\beta} f(z) \;=\; \frac{1}{\Gamma(\alpha)\Gamma(\beta\,)} \int_0^{z} (\mathrm{z}-t)^{\alpha+\beta-1}\; f(\mathrm{t})\,\mathrm{dt}\; \mathrm{B}(\alpha,\beta)$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta\,)}{\Gamma(\alpha)\Gamma(\beta\,)\Gamma(\alpha+\beta)} \int_0^{z} (\mathrm{z}-t)^{\alpha+\beta-1} f(\mathrm{t})\,\mathrm{dt}$$

$$= \frac{1}{\Gamma(\alpha+\beta)} \int_0^{z} (\mathrm{z}-t)^{\alpha+\beta-1} f(\mathrm{t})\,\mathrm{dt}$$

$$= I_z^{\alpha+\beta}\left(f(z)\right).$$

Which implies the property $I_z^{\alpha}\, I_z^{\beta} \;=\; I_z^{\alpha+\beta}$ is hold.

As we proved $I_z^{\beta}\, I_z^{\alpha}(f(z)) = \; I_z^{\beta+\alpha} f(z)$, so we can conclude that

$$I_z^{\beta}\, I_z^{\alpha}(f(z)) = I_z^{\alpha}\, I_z^{\beta}\,(f(z))$$

Thus, we have

$$I_z^{\alpha+\beta} f(z) = I_z^{\alpha}\, I_z^{\beta}\left(f(z)\right) = I_z^{\beta}\, I_z^{\alpha}\left(f(z)\right) = I_z^{\beta+\alpha} f(z)$$

We shall now give an example of a fractional integral which satisfy the expression(2.2).

Example 2.1.1

Let $f(z)$ = z^3 and α =1/2, then find $I_z^{1/2}\; z^3$, by using definition (2.1).

Solution:

Since $f(z) = z^3$ is analytic in the unit disk and $\alpha = 1/2$, we have:

$$I_z^{1/2}\, z^3 = \frac{1}{\Gamma(1/2)} \int_0^z f(\omega)(z-\omega)^{(1/2)-1}\, d\omega$$

$$= \frac{1}{\Gamma(1/2)} \int_0^z \omega^3 (z-\omega)^{-1/2}\, d\omega$$

using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we have

$$I_z^{1/2}\, z^3 = \frac{1}{\sqrt{\pi}}\, z^{-1/2} \int_0^z \omega^3 (1-\omega/z)^{-1/2}\, d\omega$$

Let $\frac{\omega}{z} = t$ then , $\frac{d\omega}{z} = dt$ thus as $\omega = 0$ then $t = 0$ and as $\omega = z$ then $t = 1$, so

$$I_z^{1/2}\, z^3 = \frac{1}{\sqrt{\pi}} z^{-1/2} \int_0^1 (zt)^3 (1-t)^{-1/2}\, z\, dt$$

$$= \frac{1}{\sqrt{\pi}} z^{3+1-1/2} \int_0^1 t^3 (1-t)^{-1/2}\, dt$$

$$= \frac{1}{\sqrt{\pi}}\, z^{\frac{7}{2}}\ \ B\left(4\, , \frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}}\, z^{\frac{7}{2}}\ \frac{\Gamma(4)\Gamma(1/2)}{\Gamma(4+1/2)}$$

$$= \frac{1}{\sqrt{\pi}}\, z^{\frac{7}{2}}\ \frac{3!\,\sqrt{\pi}}{\Gamma\left(\frac{9}{2}\right)}$$

$$= z^{\frac{7}{2}}\ \frac{3!}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$= \ z^{\frac{7}{2}} \ \frac{3.2.1}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$= \ z^{\frac{7}{2}} \ \frac{32}{35\sqrt{\pi}}.$$

Also, we can prove the following properties of the fractional integral as in the following theorem.

Theorem 2.1.2

Let $f(z)$ and $g(z)$ be analytic functions in simply-connected region of the complex z-plane $\mathbb{C}$ and k , λ are constants. Then

1) $I_z^{\alpha}\,(f+g)(z) = I_z^{\alpha}\,(f(z)) + I_z^{\alpha}\,(g(z))$.

2) $I_z^{\alpha}\,(\mathrm{k}f(z)) = \mathrm{k}\, I_z^{\alpha}\,(f(z))$.

3) $I_z^{\alpha}\, f(\lambda z) = \ \lambda^{\alpha}\, I_{\lambda z}^{\alpha} f(z)$.

Proof:

Using Eq. (2.1), we have

1) $$I_z^{\alpha}(f+g)(z) = \frac{1}{\Gamma(\alpha)}\int_0^z (f+g)(w)(z-w)^{\alpha-1}\,dw$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^z [f(w)+g(w)]\,(\mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw}$$

$$= \frac{1}{\Gamma(\alpha)}\left(\int_0^z f(w)\,(\mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw} + \int_0^z g(w)(\mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw}\right)$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^z f(w)\,(\mathrm{z}-\mathrm{w})^{\alpha-1}\,\mathrm{dw} + \frac{1}{\Gamma(\alpha)}\int_0^z g(w)(\mathrm{z}-\mathrm{w})^{\alpha-1}\,\mathrm{dw}$$

$$= I_z^{\alpha}(f) + I_z^{\alpha}(g).$$

2) $$I_z^{\alpha}(\mathrm{k}f(z)) = \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{z}} \mathrm{k}\, f(w)(\mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw}$$

$$= \frac{1}{\Gamma(\alpha)}\ \mathrm{k} \int_0^{\mathrm{z}} f(w)\ (\mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw}$$

$$= \mathrm{k}\ \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{z}} f(w)\ (\mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw}$$

$$= \mathrm{k}\, I_z^{\alpha} f(z)\,.$$

3) $$I_z^{\alpha} f(\lambda z) = \frac{1}{\Gamma(\alpha)} \int_0^{\lambda \mathrm{z}} f(w)\ (\lambda \mathrm{z}-\mathrm{w})^{\alpha-1}\ \mathrm{dw}$$

$$= \frac{1}{\Gamma(\alpha)}\ \lambda^{\alpha-1} \int_0^{\lambda \mathrm{z}} f(w) \left(\mathrm{z}-\frac{\mathrm{w}}{\lambda}\right)^{\alpha-1}\ \mathrm{dw}$$

Let $\frac{\mathrm{w}}{\lambda} = t$ then $\frac{\mathrm{dw}}{\lambda} = \mathrm{dt}$ thus, as $\mathrm{w} = 0$ then $\mathrm{t} = 0$ and as

$\mathrm{w} = \lambda \mathrm{z}$ then $= z$, so

$$I_z^{\alpha} f(\lambda z) = \frac{1}{\Gamma(\alpha)}\ \lambda^{\alpha-1} \int_0^{\mathrm{z}} f(\lambda t)\ (\mathrm{z}-\mathrm{t})^{\alpha-1}\, \lambda\, \mathrm{dt}$$

$$= \frac{1}{\Gamma(\alpha)}\ \lambda^{\alpha} \int_0^{\mathrm{z}} f(\lambda t)\ (\mathrm{z}-\mathrm{t})^{\alpha-1}\ \mathrm{dt}$$

$$= \lambda^{\alpha}\ I_{\lambda z}^{\alpha} f(z).$$

2.2 The Fractional Derivatives

The concept of derivative is traditionally associated to an integer; given an analytic function, we can derive it one, two, three times and so on. It can have an interest to investigate the possibility to derive a real number of times of a function. We now extend the ordinary derivative into the fractional derivative. After the notion of fractional integral, that of fractional derivative of order α, for a function $f(z)$ by Srivastava and Owa ([14], [15]), is as follows:

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^{z} f(\omega)(z-\omega)^{-\alpha}\, d\omega \qquad (2.4)$$

where $0 < \alpha \leq 1$.

By Eq.(6.19) of [2], we have

$$D_z^{\alpha} f(z) = I_z^{1-\alpha}\left(f'(z)\right) = I_z^{1-\alpha}\left(D_z^{1} f(z)\right) \text{ provided } f(0) = 0 \qquad (2.5)$$

Fundamental theorem of calculus to show that $D_z^{0}\,(f(z)) = f(z)$, Further we have

$$D_z^{\alpha}\, I_z^{\alpha}\,(f(z)) = f(z) \qquad (2.6)$$

To prove result, we use the expressions 2.4 and 2.1, as follows

$$D_z^{\alpha}\, I_z^{\alpha}\left(f(z)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^{z} I_z^{\alpha} f(\omega)(z-\omega)^{-\alpha}\, d\omega$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^{z} \frac{1}{\Gamma(\alpha)} \int_0^{\omega} f(t)(\omega - t)^{\alpha-1}\, dt\, (z-\omega)^{-\alpha}\, d\omega$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \frac{d}{dz} \int_0^{z} (z-\omega)^{-\alpha} \int_0^{\omega} f(t)(\omega - t)^{\alpha-1}\, dt\, d\omega$$

Using Dirichlet Formula, we get

$$D_z^{\alpha}\ I_z^{\alpha}\left(f(z)\right) \ = \frac{1}{\Gamma(1-\alpha\ )\Gamma(\alpha)}\frac{d}{dz}\int_0^{\mathrm{z}} f(t)\ \mathrm{dt}\int_t^{z}(\mathrm{z}-\omega)^{-\alpha}(\omega-\mathrm{t})^{\alpha-1}\ \ \mathrm{d}\omega$$

By formula (1.3), we have

$$D_z^{\alpha}\ I_z^{\alpha}\left(f(z)\right) \ = \frac{1}{\Gamma(1-\alpha\ )\Gamma(\alpha)}\frac{d}{dz}\int_0^{\mathrm{z}}(\mathrm{z}-t)^0 f(t)\ \mathrm{dt}\ \ B(\alpha,1-\alpha)$$

$$=\frac{1}{\Gamma(1)}\frac{d}{dz}\int_0^{\mathrm{z}} f(t)\ \mathrm{dt} = f(z).$$

Which implies the property $D_z^{\alpha}\ I_z^{\alpha}\left(f(z)\right) = \ f(z)$ is hold.

To show $I_z^{\alpha}\, D_z^{\alpha}\left(f(z)\right)$ we using Eq.(2.5), as follow

$$I_z^{\alpha}\, D_z^{\alpha}\left(f(z)\right) = \ \ I_z^{\alpha}\, I_z^{1-\alpha}\left(D_z^{1}\, f(z)\right) = I_z^{1}\, D_z^{1}\, f(z) = \int_0^{z} f'(w)\, dw = f(z) - f(0)$$

If $f(0) = 0$, then $I_z^{\alpha}\, D_z^{\alpha}\left(f(z)\right) = f(z)$.

If $f(0) \neq 0$, then $I_z^{\alpha}\, D_z^{\alpha}\left(f(z)\right) = f(z) - f(0)$.

And, for the expression (2.4), we get the following expression that will be used later

$$D_z^{\alpha}\ \{\mathrm{z}^{v}\} = \ \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)}\ \mathrm{z}^{v-\alpha} \qquad , \qquad 0\ <\ \alpha\ < 1\ ;\ v\ > -1 \tag{2.7}$$

It can be proved , by expression (2.4), as follow

$$D_z^{\alpha}\ \mathrm{z}^{v} = \ \frac{1}{\Gamma(1-\alpha\ )}\ \frac{d}{dz}\ \int_0^{\mathrm{z}} f(\omega)(\mathrm{z}-\omega)^{-\alpha}\ \mathrm{d}\omega$$

$$= \ \frac{1}{\Gamma(1-\alpha\ )}\ \frac{d}{dz}\ \int_0^{\mathrm{z}} \omega^{v}(\mathrm{z}-\omega)^{-\alpha}\ \mathrm{d}\omega$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} z^{-\alpha} \int_0^z \omega^v (1-\omega/z)^{-\alpha} \, d\omega$$

Let $\frac{\omega}{z} = t$ then $\frac{d\omega}{z} = dt$ thus, as $\omega = 0$ then $t = 0$ and as

$\omega = z$ then $t = 1$, so

$$D_z^\alpha \, z^v = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} z^{-\alpha} \int_0^1 z^v \, t^v (1-t)^{-\alpha} \, z \, dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} z^{v-\alpha+1} B(v+1, 1-\alpha)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} z^{v-\alpha+1} \frac{\Gamma(v+1)\Gamma(1-\alpha)}{\Gamma(v-\alpha+2)}$$

$$= \frac{1}{\Gamma(1-\alpha)} (v-\alpha+1) \, z^{v-\alpha} \frac{\Gamma(v+1)\Gamma(1-\alpha)}{\Gamma(v-\alpha+2)}$$

$$= (v-\alpha+1) \, z^{v-\alpha} \frac{\Gamma(v+1)}{(v-\alpha+1)\Gamma(v-\alpha+1)}$$

$$= z^{v-\alpha} \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} .$$

The relation between fractional integral and fractional derivative can be shown in the following theorem.

Theorem 2.2.1

Let $f(z)$ be an analytic function in simply-connected region of the complex z-plane $\mathbb{C}$. Then

$$D_z^\alpha \, f(z) = \frac{d}{dz} I_z^{1-\alpha} f(z)$$

Proof

Using the expression 2.4, we get

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^{z} f(\omega)(z-\omega)^{-\alpha}\, d\omega$$

Let $\zeta = 1-\alpha$, then

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(\zeta)} \frac{d}{dz} \int_0^{z} f(\omega)(z-\omega)^{\zeta-1}\, d\omega$$

$$= \frac{d}{dz} \frac{1}{\Gamma(\zeta)} \int_0^{z} f(\omega)(z-\omega)^{\zeta-1}\, d\omega$$

$$= \frac{d}{dz}\, I_z^{\zeta} f(z) \quad = \quad \frac{d}{dz}\, I_z^{1-\alpha} f(z)\,.$$

Now, we have some properties of the fractional derivative as in the following theorem.

Theorem 2.2.2

Let $f(z)$ and $g(z)$ be analytic functions in simply-connected region of the complex z-plane $\mathbb{C}$ and k , λ are constants. Then

1) $D_z^{\alpha}(f+g)(z) = D_z^{\alpha}(f(z)) + D_z^{\alpha}(g(z))$.

2) $D_z^{\alpha}(\mathrm{k}f(z)) = \mathrm{k}\, D_z^{\alpha} f(z)$.

3) $D_z^{\alpha} D_z^{\beta}(f(z)) = I_z^{-\alpha-\beta} f(z) = I_z^{-\beta-\alpha} f(z) = D_z^{\beta} D_z^{\alpha}(f(z))$ for $f(0) = 0$.

4) $D_z^{\alpha} f(\lambda z) = \lambda^{\alpha} D_{\lambda z}^{\alpha} f(z)$.

Proof:

Using Eq. (2.4), we get

1) $D_z^{\alpha}\,(f+g)(z) = \frac{1}{\Gamma(1-\alpha\,)}\,\frac{d}{dz}\int_0^{z}(f+g)(\omega)(z-\omega)^{-\alpha}\,d\omega$

$$= \frac{1}{\Gamma(1-\alpha\,)}\,\frac{d}{dz}\int_0^{z}[f(\omega)+\,g(\omega)](z-\omega)^{-\alpha}\,d\omega$$

$$= \frac{1}{\Gamma(1-\alpha\,)}\frac{d}{dz}\int_0^{z}f(\omega)(z-\omega)^{-\alpha}d\omega + \frac{1}{\Gamma(1-\alpha\,)}\frac{d}{dz}\int_0^{z}g(\omega)(z-\omega)^{-\alpha}d\omega$$

$$= D_z^{\alpha}\,(f) + D_z^{\alpha}\,(g).$$

2) $D_z^{\alpha}\,(kf(z)) = \frac{1}{\Gamma(1-\alpha\,)}\,\frac{d}{dz}\int_0^{z}k\,f(\omega)(z-\omega)^{-\alpha}\,d\omega$

$$= \frac{1}{\Gamma(1-\alpha\,)}\,\frac{d}{dz}\,k\int_0^{z}f(\omega)(z-\omega)^{-\alpha}\,d\omega$$

$$= \frac{k}{\Gamma(1-\alpha\,)}\,\frac{d}{dz}\int_0^{z}f(\omega)(z-\omega)^{-\alpha}\,d\omega$$

$$= k\,\frac{1}{\Gamma(1-\alpha\,)}\,\frac{d}{dz}\int_0^{z}f(\omega)(z-\omega)^{-\alpha}\,d\omega$$

$$= k\,D_z^{\alpha}\,(f).$$

3) Suppose $f(0)=0$ and apply Eq.(2.5) $D_z^{\beta} = I_z^{1-\beta}\,D_z^{1}$, we get

$$D_z^{\alpha}\,D_z^{\beta}\,f(z) = D_z^{\alpha}\,I_z^{1-\beta}\,D_z^{1}\,f(z)$$

Using Eq.(2.3), we have

$$D_z^{\alpha}\,D_z^{\beta}\,f(z) = D_z^{\alpha}\,I_z^{-\beta}\,I_z^{1}\,D_z^{1}\,f(z) = D_z^{\alpha}\,I_z^{-\beta}\,f(z)$$

Also apply Eq.(2.6) $D_z^{\alpha} = I_z^{1-\alpha}\,D_z^{1}$, we get

$$D_z^{\alpha}\, D_z^{\beta}\, f(z) = I_z^{1-\alpha}\, D_z^{1}\, I_z^{-\beta}\, f(z) = I_z^{-\alpha}\, I_z^{-\beta}\, f(z) = I_z^{-\alpha-\beta}\, f(z).$$

Thus

$$D_z^{\alpha}\, D_z^{\beta}\, (f(z)) = I_z^{-\alpha-\beta}\, f(z) = I_z^{-\beta-\alpha}\, f(z) = D_z^{\beta}\, D_z^{\alpha}\, (f(z))$$

Note that, to determine $I_z^{-\gamma}$ we refer to [10].

$$I_z^{-\gamma}\, f(z) = I_z^{1-\gamma}\, D_z^{1}\, (f(z)) \text{ provided } f(0) = 0.$$

If $f(0) \neq 0$, then

$$D_z^{\alpha}\, D_z^{\beta}\, (f(z)) = D_z^{\alpha}\, I_z^{1-\beta}\, D_z^{1}\, (f(z)) = D_z^{\alpha}\, I_z^{1-\beta}\, (f'(z))$$

$$= D_z^{\alpha} \left(\frac{1}{\Gamma(1-\beta)} \int_0^{z} f'(\omega)(z-\omega)^{-\beta}\, d\omega \right)$$

Integrating by part, we get

$$D_z^{\alpha}\, D_z^{\beta}\, f(z) = D_z^{\alpha}\, \frac{1}{\Gamma(1-\beta)} \left(\frac{(z-\omega)^{1-\beta} f'(\omega)}{(1-\beta)} \Big|_0^{z} - \frac{1}{(1-\beta)} \int_0^{z} f''(\omega)(z-\omega)^{1-\beta} d\omega \right)$$

$$= D_z^{\alpha} \left(\frac{(z)^{1-\beta} f'(0)}{\Gamma(2-\beta)} \right) - D_z^{\alpha} \left(\frac{1}{\Gamma(2-\beta)} \int_0^{z} f''(\omega)(z-\omega)^{1-\beta} d\omega \right)$$

$$= I_z^{1-\alpha}\, D_z^{1} \left(\frac{(z)^{1-\beta} f'(0)}{\Gamma(2-\beta)} \right) - I_z^{1-\alpha}\, D_z^{1}\, I_z^{2-\beta}\, (f''(z))$$

$$= I_z^{1-\alpha} \left(\frac{(z)^{-\beta} f'(0)}{\Gamma(1-\beta)} \right) - I_z^{1-\alpha}\, D_z^{1}\, I_z^{1}\, I_z^{1-\beta}\, (f''(z))$$

$$= \frac{f'(0)}{\Gamma(1-\beta)\Gamma(1-\alpha)} \int_0^{z} (z-\omega)^{-\alpha} \omega^{-\beta}\, d\omega - I_z^{1-\alpha}\, I_z^{1-\beta}\, (f''(z))$$

$$= \frac{f'(0) \quad z^{-\alpha}}{\Gamma(1-\beta)\Gamma(1-\alpha)} \int_0^z (1-\frac{\omega}{z})^{-\alpha} \omega^{-\beta}\, d\omega - I_z^{2-\alpha-\beta}\,(f''(\mathrm{z}))$$

Let $\frac{\omega}{z} = t$ then $\frac{d\omega}{z} = dt$ thus, as $\omega = 0$ then $t = 0$ and as

$\omega = z$ then $t = 1$, so

$$D_z^{\alpha}\, D_z^{\beta}\, f(z) = \frac{f'(0) \quad z^{-\alpha-\beta+1}}{\Gamma(1-\beta)\Gamma(1-\alpha)} \int_0^1 (1-t)^{-\alpha} t^{-\beta}\, dt - I_z^{2-\alpha-\beta}\,(f''(\mathrm{z}))$$

$$= \frac{f'(0) \quad z^{-\alpha-\beta+1}}{\Gamma(1-\beta)\Gamma(1-\alpha)} \frac{\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(1-\alpha-\beta)} - I_z^{2-\alpha-\beta}\,(f''(\mathrm{z}))$$

$$= \frac{f'(0) \quad z^{-\alpha-\beta+1}}{\Gamma(1-\alpha-\beta)} - I_z^{2-\alpha-\beta}\,(f''(\mathrm{z}))$$

Hence for $(0) \neq 0$, then

$$D_z^{\alpha}\, D_z^{\beta}\, f(z) = \frac{f'(0) \quad z^{1-\alpha-\beta}}{\Gamma(1-\alpha-\beta)} - I_z^{2-\alpha-\beta}\left(f''(\mathrm{z})\right).$$

4) $$D_z^{\alpha}\, f(\lambda z) = \frac{1}{\Gamma(\alpha)} \frac{d}{dz} \int_0^{\lambda z} f(\omega)(\lambda z - \omega)^{\alpha-1}\, d\omega$$

$$= \frac{1}{\Gamma(\alpha)} \frac{d}{dz}\, \lambda^{\alpha-1} \int_0^{\lambda z} f(\omega) \left(z - \frac{\omega}{\lambda}\right)^{\alpha-1} d\omega$$

Let $\frac{\omega}{\lambda} = t$ then $\frac{d\omega}{\lambda} = dt$ thus, as $\omega = 0$ then $t = 0$ and as

$\omega = \lambda z$ then $t = z$, so

$$D_z^{\alpha}\, f(\lambda z) = \frac{1}{\Gamma(\alpha)} \frac{d}{dz}\, \lambda^{\alpha-1} \int_0^{z} f(\lambda t)(z-t)^{\alpha-1} \lambda\, dt$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{d}{dz} \int_0^z f(\lambda t)(z-t)^{\alpha-1}\, dt$$

$$= \lambda^{\alpha} D_{\lambda z}^{\alpha} f(z).$$

We shall illustrate the above theorem by the following example.

Example 2.2.3

Consider $\alpha = 3/2$ and a function $f(z) = z^2 + 5(\sqrt{z})^3 - 1$, then find $D_z^{3/2} f(z)$.

Solution:

Since f is analytic and by previous theorem, we get

$$D_z^{3/2}\left(z^2 + 5(\sqrt{z})^3 - 1\right) = D_z^{3/2}(z^2) + D_z^{3/2}\left(5(\sqrt{z})^3\right) - D_z^{3/2} 1$$

Using expression (2.8), so

$$D_z^{3/2} f(z) = \frac{\Gamma(2+1)}{\Gamma\left(2-\frac{3}{2}+1\right)} z^{\left(2-\frac{3}{2}\right)} + 5\, \frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}-\frac{3}{2}+1\right)} z^{\left(\frac{3}{2}-\frac{3}{2}\right)} - \frac{\Gamma(0+1)}{\Gamma\left(0-\frac{3}{2}+1\right)} z^{0-\frac{3}{2}}$$

$$= \frac{2!}{\Gamma\left(\frac{3}{2}\right)} z^{\left(\frac{1}{2}\right)} + 5\, \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(1)} z^{(0)} - \frac{\Gamma(1)}{\Gamma(-1/2)} z^{(-3/2)}$$

$$= \frac{2}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} z^{\left(\frac{1}{2}\right)} + 5\, \frac{\frac{3}{2}\Gamma(\frac{3}{2})}{1} - \frac{1}{-2\sqrt{\pi}} z^{(-3/2)}$$

$$= \frac{2}{\frac{1}{2}\sqrt{\pi}} z^{\left(\frac{1}{2}\right)} + 5\, \frac{3\sqrt{\pi}}{2.2} - \frac{1}{-2\sqrt{\pi}} z^{(-3/2)}$$

$$= \frac{4}{\sqrt{\pi}} z^{(\frac{1}{2})} + \frac{15\sqrt{\pi}}{4} + \frac{1}{2\sqrt{\pi}} z^{(-3/2)}$$

$$= \frac{1}{2\sqrt{\pi}}\left(8\, z^{1/2} + 15\,\pi + z^{-3/2}\right).$$

CHAPTER THREE

FRACTIONAL CALCULUS OPERATORS

In this chapter some basic definitions and results are given about new fractional calculus operators, [9].

3.1 Fractional Integral Operator

In this section, we provide some definitions and give some related results in the present work and in the future work. We prove some properties of the new fractional integral operator, for instance: the boundedness, compactness in the Bergman space and study two further examples.

Definition 3.1.1

Let $f(z)$ be analytic in a simple connected region, for all $z \in D$, containing the origin and $(0 < \alpha \le 1), (0 < \beta \le 1)$. Then the fractional integral operator $£_{z}^{\alpha,\beta}$ is given by:

$$£_{z}^{\alpha,\beta} f(z) := \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_{0}^{z} \frac{t^{\beta-1} f(t)}{(z-t)^{1-\alpha+\beta}} \, dt$$

And if $\alpha = \beta$, then

$$£_{z}^{\alpha,\alpha} f(z) = f(z).$$

In the following theorems, we consider to show that the operator in definition 3.1.1 is bounded and compact in the space $\mathfrak{A}^{p}(D)$.

From now on, we denote $\mathcal{A}$ to be the set of all analytic functions in the unit disk D with $f(0) = 0$ and $f'(0) = 1$.

Theorem 3.1.2.(Boundedness)

Let $f \in \mathcal{A}$ on unit disk D. Then for all $z \in D$ the operator $£_{z}^{\alpha,\beta}: \mathfrak{A}^p \to \mathfrak{A}^p$ is a bounded operator and

$$\left\| £_{z}^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^{p} \leq \| f(z) \|_{\mathfrak{A}^p}^{p}$$

Proof. Suppose that $f(z) \in \mathfrak{A}^p$, then it follows that

$$\left\| £_{z}^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^{p} = \frac{1}{\pi} \int_D \left| £_{z}^{\alpha,\beta} f(z) \right|^p d\mathfrak{A}$$

$$= \frac{1}{\pi} \int_D \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z \frac{t^{\beta-1} f(t)}{(z-t)^{1-\alpha+\beta}} dt \right|^p d\mathfrak{A}$$

$$= \frac{1}{\pi} \int_D \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} z^{\alpha-\beta-1} \int_0^z t^{\beta-1} (1-\frac{t}{z})^{\alpha-\beta-1} f(t)\, dt \right|^p d\mathfrak{A}$$

By setting $\frac{t}{z} = u$ we get $\frac{dt}{z} = du$, so as $t = 0$ then $u = 0$, and as $t = z$ then $u = 1$, we obtain

$$\left\| £_{z}^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^{p} = \frac{1}{\pi} \int_D \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 u^{\beta-1} (1-u)^{\alpha-\beta-1} f(uz)\, du \right|^p d\mathfrak{A}$$

Using theorem 1.6.1 and Beta function, we have

$$\left\| £_{z}^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^{p} \leq \frac{1}{\pi} \int_D \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \mathrm{B}(\beta, \alpha-\beta) f(z) \right|^p d\mathfrak{A}$$

$$= \frac{1}{\pi} \int_D \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} f(z) \right|^p d\mathfrak{A}$$

$$= \frac{1}{\pi} \int_D | f(z)|^p \, d\mathfrak{A} = \| f(z) \|^p.$$

This complete the proof as $\left\|£_z^{\alpha,\beta} f(z)\right\|_{\mathfrak{A}^p}^p \leq \|f(z)\|_{\mathfrak{A}^p}^p$ for all $z \in D$.

Theorem 3.1.3.(Compactness)

Let $f \in \mathcal{A}$. Then $£_z^{\alpha,\beta}: \mathfrak{A}^p \to \mathfrak{A}^p$ is compact.

Proof. Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $\mathfrak{A}^p$ and that $f_n \to 0$ uniformly on $\overline{D}$ as $n \to \infty$. Then

$$\left\|£_z^{\alpha,\beta} f_n(z)\right\|_{\mathfrak{A}^p}^p = sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|£_z^{\alpha,\beta} f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha}\int_0^z t^{\beta-1}(z-t)^{\alpha-\beta-1} f_n(t)\, dt\right|^p d\mathfrak{A}\right\} \qquad (3.1)$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} z^{\alpha-\beta-1}\int_0^z t^{\beta-1}(1-\frac{t}{z})^{\alpha-\beta-1} f_n(t)\, dt\right|^p d\mathfrak{A}\right\}$$

Set $\frac{t}{z} = u$ then $\frac{dt}{z} = du$, so as $t = 0$ then $u = 0$, and as $t = z$ then $u = 1$, thus we obtain

$$\left\|£_z^{\alpha,\beta} f_n\right\|_{\mathfrak{A}^p}^p = sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\alpha-\beta-1} f_n(uz)\, du\right|^p d\mathfrak{A}\right\}$$

Using theorem 1.6.1 and Beta function, we have

$$\left\|£_z^{\alpha,\beta} f_n\right\|_{\mathfrak{A}^p}^p \leq sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \mathrm{B}(\beta,\alpha-\beta)\, f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D |f_n(z)|^p\, d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\|f_n(z)\|^p = \|f_n\|^p.$$

Since $f_n \to 0$ on $\overline{D}$, we obtain $\|f_n\|_{\mathfrak{A}^p} \to 0$, and by putting $n \to \infty$ in Eq.(3.1), we have that $\lim_{n\to\infty}\left\|£_z^{\alpha,\beta} f_n\right\|_{\mathfrak{A}^p}^p = 0$. Hence, the compactness of the operator $£_z^{\alpha,\beta}$ follows.

Now, we are ready to prove some properties of fractional integral operator $£_z^{\alpha,\beta}$ in open unit disk D.

Proposition 3.1.4

Let $f, g \in \mathcal{A}$ and $a, b \in \mathbb{C}$, then for all $z \in D$

$$£_z^{\alpha,\beta}(af + bg) = a\, £_z^{\alpha,\beta} f + b\, £_z^{\alpha,\beta} g.$$

Proof:

Using definition 3.1.1, we have

$$£_z^{\alpha,\beta}(af + bg) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, z^{1-\alpha} \int_0^z t^{\beta-1}\,(z-t)^{\alpha-\beta-1}[af(t) + bg(t)]\; dt$$

$$= \frac{\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}\left(\int_0^z t^{\beta-1}\,(z-t)^{\alpha-\beta-1} af(t)dt + \int_0^z t^{\beta-1}\,(z-t)^{\alpha-\beta-1}\, bg(t)\; dt\right)$$

$$= \frac{\Gamma(\alpha)\, z^{1-\alpha}\; a}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z \frac{t^{\beta-1} f(t)}{(z-t)^{1-\alpha+\beta}}\, dt + \frac{\Gamma(\alpha)\, z^{1-\alpha}\; b}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z \frac{t^{\beta-1} g(t)}{(z-t)^{1-\alpha+\beta}}\, dt$$

$$= \mathrm{a}\; \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z \frac{t^{\beta-1} f(t)}{(z-t)^{1-\alpha+\beta}}\, dt + \mathrm{b}\, \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z \frac{t^{\beta-1} g(t)}{(z-t)^{1-\alpha+\beta}}\; dt$$

$$= a\, £_z^{\alpha,\beta} f + b\, £_z^{\alpha,\beta} g.$$

Proposition 3.1.5

Let $f \in \mathcal{A}$. For all $z \in D$ and for some $0 < \alpha \leq 1$, $0 < \beta \leq 1$, we have

$$£_z^{\alpha,\beta}\, £_z^{\beta,\alpha} f(z) = f(z)$$

And

$$£_z^{\alpha,\beta}\, £_z^{\beta,\mu} f(z) = £_z^{\alpha,\mu} f(z)$$

Proof. Using definition 3.1.1, we have

$$£_z^{\alpha,\beta}\, £_z^{\beta,\alpha} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, z^{1-\alpha} \int_0^z t^{\beta-1} (z-t)^{\alpha-\beta-1}\, £_z^{\beta,\alpha} f(t)\ dt$$

$$= \frac{\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z t^{\beta-1} (z-t)^{\alpha-\beta-1} \left(\frac{\Gamma(\beta)\, t^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^t \frac{w^{\alpha-1} f(w)}{(t-w)^{1-\beta+\alpha}}\ dw\right) dt$$

$$= \frac{z^{1-\alpha}}{\Gamma(\beta-\alpha)\Gamma(\alpha-\beta)} \int_0^z (z-t)^{\alpha-\beta-1} \int_0^t w^{\alpha-1} (t-w)^{\beta-\alpha-1} f(w)\, dw\, dt$$

$$= \frac{z^{1-\alpha}}{\Gamma(\beta-\alpha)\Gamma(\alpha-\beta)} \int_0^z (z-t)^{\alpha-\beta-1}\ t^{\beta-\alpha-1} \int_0^t w^{\alpha-1} \left(1-\frac{w}{t}\right)^{\beta-\alpha-1} f(w)\, dw\, dt$$

Set $\frac{w}{t} = u$ then $\frac{dw}{t} = du$, so as $w = 0$ then $u = 0$, and as $w = t$ then $u = 1$, thus $£_z^{\alpha,\beta} £_z^{\beta,\alpha} f(z)$ is equal

$$\frac{z^{1-\alpha}}{\Gamma(\beta-\alpha)\Gamma(\alpha-\beta)} \int_0^z (z-t)^{\alpha-\beta-1}\ t^{\beta-\alpha-1} t^{\alpha} \int_0^1 u^{\alpha-1} (1-u)^{\beta-\alpha-1} f(tu)\, du\, dt$$

Using theorem 1.6.1 and Beta function with $|tu| < 1$, we have

$$£_z^{\alpha,\beta} £_z^{\beta,\alpha} f(z) = \frac{z^{1-\alpha}}{\Gamma(\beta-\alpha)\Gamma(\alpha-\beta)} \int_0^z (z-t)^{\alpha-\beta-1}\ t^{\beta-1}\, \mathrm{B}(\alpha,\beta-\alpha) f(t) dt$$

$$= \frac{\mathrm{z}^{1-\alpha}}{\Gamma(\beta-\alpha)\Gamma(\alpha-\beta)} \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)} \int_0^z (z-t)^{\alpha-\beta-1}\, t^{\beta-1} f(t)\, dt$$

$$= \frac{\Gamma(\alpha)\ \mathrm{z}^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z (z-t)^{\alpha-\beta-1}\, t^{\beta-1} f(t)\, dt$$

$$= \frac{\Gamma(\alpha)\ \mathrm{z}^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \mathrm{z}^{\alpha-\beta-1} \int_0^z t^{\beta-1} \left(1-\frac{t}{z}\right)^{\alpha-\beta-1} f(t)\, dt$$

Set $\frac{t}{z} = v$ then $\frac{dt}{z} = dv$, so as $t = 0$ then $v = 0$, and as $t = z$ then $v = 1$, thus we obtain

$$£_{\mathrm{z}}^{\alpha,\beta}\, £_{\mathrm{z}}^{\beta,\alpha} f(z) = \frac{\Gamma(\alpha)\ \mathrm{z}^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \mathrm{z}^{\alpha-\beta-1}\mathrm{z}^{\beta-1} z \int_0^1 v^{\beta-1} (1-v)^{\alpha-\beta-1}\ f(zv)\, dv$$

Using theorem 1.6.1 and Beta function with $|zv| < 1$, we get

$$£_{\mathrm{z}}^{\alpha,\beta} £_{\mathrm{z}}^{\beta,\alpha} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ B(\beta, \alpha-\beta)\, f(z)$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ \frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\ f(z) = f(z)\,.$$

To prove $£_{\mathrm{z}}^{\alpha,\beta}\, £_{\mathrm{z}}^{\beta,\mu} f(z) = £_{\mathrm{z}}^{\alpha,\mu} f(z)$, we have

$$£_{\mathrm{z}}^{\alpha,\beta}\, £_{\mathrm{z}}^{\beta,\mu} f(z) = \frac{\Gamma(\alpha)\ \mathrm{z}^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z t^{\beta-1} (z-t)^{\alpha-\beta-1}\ £_{\mathrm{z}}^{\beta,\mu}(f(t))dt$$

$$= \frac{\Gamma(\alpha)\ \mathrm{z}^{1-\alpha}}{\Gamma(\alpha-\beta)} \int_0^z t^{\beta-1} (z-t)^{\alpha-\beta-1}\ \frac{\mathrm{t}^{1-\beta}}{\Gamma(\mu)\Gamma(\beta-\mu)} \int_0^t w^{\mu-1}(t-w)^{\beta-\mu-1} f(w)\, dwdt$$

$$= \frac{\Gamma(\alpha)\ \mathrm{z}^{1-\alpha}}{\Gamma(\mu)\Gamma(\alpha-\beta)\Gamma(\beta-\mu)} \int_0^z (z-t)^{\alpha-\beta-1}\, dt \int_0^t w^{\mu-1}\, (t-w)^{\beta-\mu-1} f(w)\, dw$$

Using Dirichlet formula and Beta function formula (1.3), thus $£_{\mathrm{z}}^{\alpha,\beta}£_{\mathrm{z}}^{\beta,\mu} f(z)$ is equal

$$\frac{\Gamma(\alpha)\ z^{1-\alpha}}{\Gamma(\mu)\Gamma(\alpha-\beta)\Gamma(\beta-\mu)}\int_0^z w^{\mu-1}f(w)\,dw\int_w^z(z-t)^{\alpha-\beta-1}\,(t-w)^{\beta-\mu-1}\,dt$$

$$=\frac{\Gamma(\alpha)\ z^{1-\alpha}}{\Gamma(\mu)\Gamma(\alpha-\beta)\Gamma(\beta-\mu)}\int_0^z w^{\mu-1}f(w)\,dw\,\frac{B(\alpha-\beta,\beta-\mu)}{(z-w)^{\mu-\alpha+1}}$$

$$=\frac{\Gamma(\alpha)\ z^{1-\alpha}}{\Gamma(\mu)\Gamma(\alpha-\beta)\Gamma(\beta-\mu)}\int_0^z w^{\mu-1}(z-w)^{\alpha-\mu-1}f(w)\,dw\,\frac{\Gamma(\alpha-\beta)\Gamma(\beta-\mu)}{\Gamma(\alpha-\mu)}$$

$$=\frac{\Gamma(\alpha)\ z^{1-\alpha}}{\Gamma(\mu)\Gamma(\alpha-\mu)}\int_0^z w^{\mu-1}(z-w)^{\alpha-\mu-1}f(w)\,dw$$

$$=\ \pounds_z^{\alpha,\mu}f(z).$$

Proposition 3.1.6

$$\pounds_z^{\alpha,\beta}f(z)=\frac{\Gamma(\alpha)}{\Gamma(\beta)}\ z^{1-\alpha}I_z^{\alpha-\beta}\ (z^{\beta-1}f(z))$$

Proof:

Using definition 3.1.1, we get

$$\pounds_z^{\alpha,\beta}f(\mathrm{z})=\ \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ \mathrm{z}^{1-\alpha}\ \int_0^z t^{\beta-1}\ (z-t)^{\alpha-\beta-1}f(t)\ dt$$

$$=\ \frac{\Gamma(\alpha)}{\Gamma(\beta))}\ \mathrm{z}^{1-\alpha}\ \frac{1}{\Gamma(\alpha-\beta)}\int_0^z t^{\beta-1}\ (z-t)^{\alpha-\beta-1}f(t)\ dt$$

By Eq. (2.1), we get

$$\pounds_z^{\alpha,\beta}f(z)=\frac{\Gamma(\alpha)}{\Gamma(\beta)}\ z^{1-\alpha}I_z^{\alpha-\beta}\ z^{\beta-1}f(z).$$

3.2 Fractional Differential Operator

In this section, we provide some definitions and give some related results in the present work and in the future work. We prove some properties of the new fractional differential operator, for instance: the boundedness, compactness in the Bergman space and study two further examples.

Definition 3.2.1

Let $f(z)$ be analytic in a simple connected region, for all $z \in D$, containing the origin and $(0 < \alpha \leq 1), (0 < \beta \leq 1)$. Then we define the fractional differential operator $\mathcal{Z}_z^{\alpha,\beta}$ as follows:

$$\mathcal{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z \frac{t^{\alpha-1} f(t)}{(z-t)^{\alpha-\beta}}\, dt$$

In particular if $\alpha = \beta$, we have

$$\mathcal{Z}_z^{\alpha,\alpha} f(z) = f(z).$$

We can show that using definition 3.2.1 as follow

$$\mathcal{Z}_z^{\alpha,\alpha} f(z) = \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\alpha)\, \Gamma(1-\alpha+\alpha)} \frac{d}{dz} \int_0^z \frac{t^{\alpha-1} f(t)}{(z-t)^{\alpha-\alpha}}\, dt$$

$$= z^{1-\alpha} \frac{d}{dz} \int_0^z t^{\alpha-1} f(t)\, dt$$

$$= z^{1-\alpha} z^{\alpha-1} f(z) = f(z).$$

Now, we are ready to prove some properties of fractional differential operator $\mathcal{Z}_z^{\alpha,\beta}$ in open unit disk D.

Proposition 3.2.2

Let $f, g \in \mathcal{A}$ and $a, b \in \mathbb{C}$, then for all $z \in D$

$$\mathfrak{Z}_z^{\alpha,\beta}(af + bg) = a\, \mathfrak{Z}_z^{\alpha,\beta} f + b\, \mathfrak{Z}_z^{\alpha,\beta} g.$$

Proof

Using definition 3.2.1, we get

$$\mathfrak{Z}_z^{\alpha,\beta}(af + bg) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z \frac{t^{\alpha-1}\, [a\, f(t) + bg(t)]}{(z-t)^{\alpha-\beta}}\, dt$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \left\{ \int_0^z \frac{t^{\alpha-1}\, [a\, f(t)]}{(z-t)^{\alpha-\beta}} + \int_0^z \frac{t^{\alpha-1}\, [bg(t)]}{(z-t)^{\alpha-\beta}} \right\} dt$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}\ a}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z \frac{t^{\alpha-1} f(t)}{(z-t)^{\alpha-\beta}} + \frac{\Gamma(\beta)\, z^{1-\beta}\ b}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z \frac{t^{\alpha-1} g(t)}{(z-t)^{\alpha-\beta}} dt$$

$$= a\, \mathfrak{Z}_z^{\alpha,\beta} f + b\, \mathfrak{Z}_z^{\alpha,\beta} g.$$

Proposition 3.2.3

Let $\in \mathcal{A}$, then

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} \frac{d}{dz} I_z^{1-\alpha+\beta}\, z^{\alpha-1} f(z)$$

Proof. Using definition 3.2.1 and theorem 2.2.1, we get

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z t^{\alpha-1}\, (z-t)^{\beta-\alpha} f(t)\ dt$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \frac{d}{dz} \frac{1}{\Gamma(1-\alpha+\beta)} \int_0^z (z-t)^{\beta-\alpha} f(t)\, t^{\alpha-1}\ dt$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \frac{d}{dz}\ I_z^{1-\alpha+\beta}\ z^{\alpha-1} f(z).$$

Proposition 3.2.4

Let $f \in \mathcal{A}$, $0 < \alpha \le 1$ and $0 < \beta \le 1$, then

$$\text{ƻ}_z^{\alpha,\beta}\, \pounds_z^{\alpha,\beta} f(z) = f(z) \qquad \text{and} \qquad \pounds_z^{\alpha,\beta}\, \text{ƻ}_z^{\alpha,\beta} f(z) = f(z) \text{ if } f(0) = 0$$

are holds true for all $z \in D$.

Proof. By definitions 3.1.1 and 3.2.1, we obtain

$$\text{ƻ}_z^{\alpha,\beta}\, \pounds_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\ z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z t^{\alpha-1} (z-t)^{\beta-\alpha}\, \pounds_z^{\alpha,\beta} f(t) dt$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z t^{\alpha-1} (z-t)^{\beta-\alpha} \frac{\Gamma(\alpha)\, t^{1-\alpha}}{\Gamma(\beta)\, \Gamma(\alpha-\beta)} \int_0^t \frac{w^{\beta-1} f(w)}{(t-w)^{1-\alpha+\beta}} dw dt$$

$$= \frac{z^{1-\beta}}{\Gamma(\alpha-\beta)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z (z-t)^{\beta-\alpha}\ dt \int_0^t w^{\beta-1} (t-w)^{\alpha-\beta-1} f(w)\, dw$$

Using Dirichlet formula, we get $\text{ƻ}_z^{\alpha,\beta}\, \pounds_z^{\alpha,\beta} f(z)$ is equal

$$\frac{z^{1-\beta}}{\Gamma(\alpha-\beta)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z w^{\beta-1} f(w)\, dw \int_w^z (z-t)^{\beta-\alpha} (t-w)^{\alpha-\beta-1}\, dt$$

Using Beta function formula 1.3, we have

$$\text{ƻ}_z^{\alpha,\beta}\, \pounds_z^{\alpha,\beta} f(z) = \frac{z^{1-\beta}}{\Gamma(\alpha-\beta)\, \Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z w^{\beta-1} f(w)\, dw\ \mathrm{B}(\alpha-\beta, \beta-\alpha+1)$$

$$= \frac{z^{1-\beta}}{\Gamma(\alpha-\beta)\, \Gamma(1-\alpha+\beta)}\ \frac{\Gamma(\alpha-\beta)\, \Gamma(1-\alpha+\beta)}{\Gamma(1)} \frac{d}{dz} \int_0^z w^{\beta-1} f(w)\, dw$$

$$= z^{1-\beta}\, z^{\beta-1} f(z) = f(z).$$

To prove $\pounds_z^{\alpha,\beta}\, \text{ƻ}_z^{\alpha,\beta} f(z)$, using proposition 3.1.6 and proposition 3.2.3, we obtain

$$£_z^{\alpha,\beta}\, Ƶ_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)} I^{\alpha-\beta}(z^{\beta-1}\, Ƶ_z^{\alpha,\beta} f(z))$$

$$= \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)} I^{\alpha-\beta}\left(z^{\beta-1}\, \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)}\, \frac{d}{dz} I^{1-\alpha+\beta}\left(z^{\beta-1}\, f(z)\right)\right)$$

Using theorem 2.2.1, we get

$$£_z^{\alpha,\beta}\, Ƶ_z^{\alpha,\beta} f(z) = z^{1-\alpha}\, I^{\alpha-\beta}(D^{\alpha-\beta}\left(z^{\alpha-1}\, f(z)\right)$$

By Eq.(2.5), $D^{\alpha-\beta} = I^{1+\beta-\alpha} D^1(z^{\alpha-1}\, f(z))$, so

$$£_z^{\alpha,\beta}\, Ƶ_z^{\alpha,\beta} f(z) = z^{1-\alpha}\, I^{\alpha-\beta}(I^{1+\beta-\alpha} D^1\left(z^{\alpha-1}\, f(z)\right)$$

By property 2.3 and $(0) = 0$, we get

$$£_z^{\alpha,\beta}\, Ƶ_z^{\alpha,\beta} f(z) = z^{1-\alpha}\, I^1 D^1\left(z^{\alpha-1}\, f(z)\right) = z^{1-\alpha}\, z^{\alpha-1}\, f(z) = f(z).$$

Theorem 3.2.5

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$Ƶ_z^{\alpha,\beta} f(z) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} z^n = \sum_{n=0}^{\infty} A_n\, z^n$$

Proof:

Using proposition 3.2.3, we have

$$Ƶ_z^{\alpha,\beta} z^n = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta}\, \frac{d}{dz} I_z^{1-\alpha+\beta}\, (z^{\alpha-1} z^n)$$

Using theorem 2.2.1, we get

$$\mathcal{Z}_z^{\alpha,\beta} z^n = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} D_z^{\alpha-\beta}\,(z^{\alpha-1} z^n)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} D_z^{\alpha-\beta}\,(z^{\alpha+n-1})$$

By Eq. (2.7), we get

$$\mathcal{Z}_z^{\alpha,\beta} z^n = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} \frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha-1-\alpha+\beta+1)} z^{\alpha+n-1-\alpha+\beta}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} z^{n+\beta-1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} z^n$$

thus

$$\mathcal{Z}_z^{\alpha,\beta} a_n z^n = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} z^n = \sum_{n=0}^{\infty} A_n\, z^n$$

where

$$A_n = a_n \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \qquad (3.2)$$

Proposition 3.2.6

Let $f \in \mathcal{A}$. For all $z \in D$ and for some $0 < \alpha \le 1$, $0 < \beta \le 1$, we have

$$\mathcal{Z}_z^{\alpha,\beta}\, \mathcal{Z}_z^{\beta,\alpha} f(z) = f(z) \quad \text{if} \quad f(0) = 0$$

And

$$\mathfrak{Z}_z^{\alpha,\beta}\,\mathfrak{Z}_z^{\beta,\mu}\,f(z) = \mathfrak{Z}_z^{\alpha,\mu} f(z)$$

Proof.

Using proposition 3.2.3, we have

$$\mathfrak{Z}_z^{\alpha,\beta}\,\mathfrak{Z}_z^{\beta,\alpha}\,f(z) = \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}\,\frac{d}{dz}I_z^{1-\beta+\alpha}\,z^{\beta-1}f(z)\right)$$

Using theorem 2.2.1, we get

$$\mathfrak{Z}_z^{\alpha,\beta}\,\mathfrak{Z}_z^{\beta,\alpha}\,f(z) = \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}D^{\beta-\alpha}(z^{\beta-1}f(z))\right)$$

By Eq.(2.5), $D^{\beta-\alpha} = I^{1+\alpha-\beta}D^1$, then

$$\mathfrak{Z}_z^{\alpha,\beta}\,\mathfrak{Z}_z^{\beta,\alpha}\,f(z) = \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}I^{1+\alpha-\beta}D^1(z^{\beta-1}f(z))\right)$$

By Eq. (2.3), we have

$$\mathfrak{Z}_z^{\alpha,\beta}\,\mathfrak{Z}_z^{\beta,\alpha}\,f(z) = \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}I^{\alpha-\beta}I^1D^1(z^{\beta-1}f(z))\right)$$

For $f(0) = 0$ then $I^1D^1f(z) = f(z)$, we have

$$\mathfrak{Z}_z^{\alpha,\beta}\,\mathfrak{Z}_z^{\beta,\alpha}\,f(z) = \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}I^{\alpha-\beta}(z^{\beta-1}f(z))\right)$$

$$= \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}\frac{1}{\Gamma(\alpha-\beta)}\int_0^z (z-t)^{\alpha-\beta-1}\,t^{\beta-1}f(t)dt\right)$$

$$= \mathfrak{Z}_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\,z^{1-\alpha}\,z^{\alpha-\beta-1}\int_0^z (1-\frac{t}{z})^{\alpha-\beta-1}\,t^{\beta-1}f(t)dt\right)$$

Set $\frac{t}{z} = u$ then $\frac{dt}{z} = du$, so as $t = 0$ then $u = 0$, and as $t = z$ then $u = 1$, thus we obtain

$$ƻ_z^{\alpha,\beta}\, ƻ_z^{\beta,\alpha} f(z) = ƻ_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\int_0^1 (1-u)^{\alpha-\beta-1}\, u^{\beta-1} f(zu)du\right)$$

Using theorem 1.6.1 and Beta function, we have

$$ƻ_z^{\alpha,\beta}\, ƻ_z^{\beta,\alpha} f(z) = ƻ_z^{\alpha,\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, f(z)B(\alpha-\beta,\beta)\right)$$

$$= ƻ_z^{\alpha,\beta}\big(\, f(z)\big) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta}\,\frac{d}{dz} I_z^{1-\alpha+\beta}\, z^{\alpha-1} f(z)$$

Repeat the same steps, we get

$$ƻ_z^{\alpha,\beta}\, ƻ_z^{\beta,\alpha} f(z) = f(z) \quad \text{if} \quad f(0) = 0$$

Now, we want to show

$$ƻ_z^{\alpha,\beta}\, ƻ_z^{\beta,\mu} f(z) = ƻ_z^{\alpha,\mu} f(z)$$

By using proposition 3.2.5, then

$$ƻ_z^{\alpha,\beta}\, ƻ_z^{\beta,\mu} f(z) = \sum_{n=0}^{\infty} A_n \;\; ƻ_z^{\beta,\mu} z^n$$

$$= \sum_{n=0}^{\infty} A_n\, \frac{\Gamma(\mu)}{\Gamma(\beta)}\,\frac{\Gamma(n+\beta)}{\Gamma(n+\mu)}\, z^n$$

By Eq. (3.2), we have

$$ƻ_z^{\alpha,\beta}\, ƻ_z^{\beta,\mu} f(z) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\beta)}{\Gamma(\alpha)}\,\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}\,\frac{\Gamma(\mu)}{\Gamma(\beta)}\,\frac{\Gamma(n+\beta)}{\Gamma(n+\mu)}\, z^n$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(\mu)}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+\mu)} \; z^n$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+\mu)} a_n z^n = \; \mathfrak{Z}_z^{\alpha,\mu} f(z).$$

In the following theorems, we consider to show that the operator 3.2.1 is bounded and compact in the space $\mathfrak{A}^p(D)$.

Theorem 3.2.7.(Boundedness)

Let $f \in \mathcal{A}$ on unit disk D. Then for all $z \in D$ the operator $\mathfrak{Z}_z^{\alpha,\beta}: \mathfrak{A}^p \to \mathfrak{A}^p$ is a bounded operator and

$$\left\| \mathfrak{Z}_z^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^p \leq \; \| f(z) \|_{\mathfrak{A}^p}^p$$

Proof. Suppose that $(z) \in \mathfrak{A}^p$, $f(0) = 0$ and using proposition 3.2.3, we obtain

$$\left\| \mathfrak{Z}_z^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^p = \frac{1}{\pi} \int_D \left| \mathfrak{Z}_z^{\alpha,\beta} f(z) \right|^p \, d\mathfrak{A}$$

$$= \frac{1}{\pi} \int_D \left| \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \frac{d}{dz} I_z^{1-\alpha+\beta} \; z^{\alpha-1} f(z) \right|^p \, d\mathfrak{A}$$

Using theorem 2.2.1 and Eq. (2.5) $D^{\alpha-\beta} = I^{1-\alpha+\beta} D^1$, we have

$$\left\| \mathfrak{Z}_z^{\alpha,\beta} f(z) \right\|_{\mathfrak{A}^p}^p = \frac{1}{\pi} \int_D \left| \frac{\Gamma(\beta)\;\, z^{1-\beta}}{\Gamma(\alpha)} \; D_z^{\alpha-\beta} z^{\alpha-1} f(z) \right|^p d\mathfrak{A}$$

$$= \frac{1}{\pi} \int_D \left| \frac{\Gamma(\beta)\;\, z^{1-\beta}}{\Gamma(\alpha)} \; I^{1-\alpha+\beta} D^1 z^{\alpha-1} f(z) \right|^p d\mathfrak{A}$$

Using Eq. (2.1), Eq.(2.3) and for $f(0) = 0$, we get

$$\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f(z)\right\|_{\mathfrak{A}^p}^{p} = \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\ z^{1-\beta}}{\Gamma(\alpha)}\ I^{\beta-\alpha} I^{1} D^{1} z^{\alpha-1} f(z)\right|^{p} d\mathfrak{A}$$

$$= \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\ z^{1-\beta}}{\Gamma(\alpha)}\ I^{\beta-\alpha}\left(z^{\alpha-1} f(z)\right)\right|^{p} d\mathfrak{A}$$

$$= \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\ z^{1-\beta}}{\Gamma(\alpha)}\ \frac{1}{\Gamma(\beta-\alpha)}\int_0^z (z-t)^{\beta-\alpha-1} t^{\alpha-1}\, f(t)dt\right|^{p} d\mathfrak{A}$$

$$= \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\ z^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\ z^{\beta-\alpha-1}\int_0^z \left(1-\frac{t}{z}\right)^{\beta-\alpha-1} t^{\alpha-1}\, f(t)dt\right|^{p} d\mathfrak{A}$$

By setting $\frac{t}{z} = u$ we get $\frac{dt}{z} = du$, so as $t = 0$ then $u = 0$, and as $t = z$ then $u = 1$, thus we obtain

$$\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f(z)\right\|_{\mathfrak{A}^p}^{p} = \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\ \int_0^1 (1-u)^{\beta-\alpha-1}\, u^{\alpha-1}\, f(zu)du\right|^{p} d\mathfrak{A}$$

By theorem 1.6.1 and Beta function with $|zu| < 1$, we get

$$\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f(z)\right\|_{\mathfrak{A}^p}^{p} \le \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} B(\alpha,\beta-\alpha) f(z)\right|^{p} d\mathfrak{A}$$

$$= \frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\ f(z)\right|^{p} d\mathfrak{A}$$

$$= \frac{1}{\pi}\int_D |f(z)|^{p}\, d\mathfrak{A} = \|f(z)\|^{p} .$$

Hence, the fractional differential operator is bounded, i.e. $\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f(z)\right\|_{\mathfrak{A}^p}^{p} \le \|f(z)\|_{\mathfrak{A}^p}^{p}$.

Theorem 3.2.8.(Compactness)

Let $f \in \mathcal{A}$ on D. Then $\mathfrak{Q}_z^{\alpha,\beta}: \mathfrak{A}^p \to \mathfrak{A}^p$ is compact.

Proof. Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of the functions in $\mathfrak{A}^p$ and that $f_n \to 0$ uniformly on $\overline{D}$ as $\to \infty$, $f(0) = 0$ and using proposition 3.2.3, then

$$\left\|\mathfrak{Q}_z^{\alpha,\beta} f_n(z)\right\|_{\mathfrak{A}^p}^p = sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\mathfrak{Q}_z^{\alpha,\beta} f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)}\frac{d}{dz} I_z^{1-\alpha+\beta} z^{\alpha-1} f_n(z)\right|^p d\mathfrak{A}\right\}$$

Using theorem 2.2.1, we get

$$\left\|\mathfrak{Q}_z^{\alpha,\beta} f_n(z)\right\|_{\mathfrak{A}^p}^p = sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} D_z^{\alpha-\beta} z^{\alpha-1} f_n(z)\right|^p d\mathfrak{A}\right\}$$

By Eq.(2.5) $D_z^{\alpha-\beta} = I_z^{1-\alpha+\beta} D_z^1$ and using Eq. (2.3), we have

$$\left\|\mathfrak{Q}_z^{\alpha,\beta} f_n(z)\right\|_{\mathfrak{A}^p}^p = sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} I_z^{1-\alpha+\beta} D_z^1 z^{\alpha-1} f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} I_z^{\beta-\alpha} I_z^1 D_z^1 z^{\alpha-1} f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} I_z^{\beta-\alpha}\left(z^{\alpha-1} f_n(z)\right)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\; z^{1-\beta}}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta-\alpha)}\int_0^z (z-t)^{\beta-\alpha-1} t^{\alpha-1} f_n(t)dt\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)\; z^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta-\alpha)} z^{\beta-\alpha-1}\int_0^z (1-\frac{t}{z})^{\beta-\alpha-1} t^{\alpha-1} f_n(t)dt\right|^p d\mathfrak{A}\right\}$$

Set $\frac{t}{z} = u$ then $\frac{dt}{z} = du$, so as $t = 0$ then $u = 0$, and as $t = z$ then $u = 1$, thus we obtain

$$\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f_n\right\|_{\mathfrak{A}^p}^{p} = sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 (1-u)^{\beta-\alpha-1}\, u^{\alpha-1} f_n(zu)du\right|^p d\mathfrak{A}\right\}$$

By theorem 1.6.1 and Beta function with $|zu| < 1$, we obtain

$$\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f_n\right\|_{\mathfrak{A}^p}^{p} \leq sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} B(\alpha,\beta-\alpha) f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\, f_n(z)\right|^p d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\left\{\frac{1}{\pi}\int_D |\, f_n(z)\,|^p \, d\mathfrak{A}\right\}$$

$$= sup_{z\in D}\|f_n(z)\|^p = \|f_n\|^p.$$

Since $f_n \to 0$ on $\overline{D}$, thus we obtain $\|f_n\|_{\mathfrak{A}^p} \to 0$, and by putting $n \to \infty$ in Eq.(3.1), we have that $\lim_{n\to\infty}\left\|\mathfrak{Z}_{z}^{\alpha,\beta} f_n\right\|_{\mathfrak{A}^p}^{p} = 0$. Hence, the compactness of the operator $\mathfrak{Z}_{z}^{\alpha,\beta}$ follows.

3.3 New Operators And Special Functions

In this section we show that the operators $£_{z}^{\alpha,\beta}$, $\mathfrak{Z}_{z}^{\alpha,\beta}$ represents some special functions. In the next section as in section 1.3 we consider one special functions in geometric function theory, that is also known as Gauss hypergeometric function and study some of their properties in the unit disk D. In [4], we know that if $f(z)$ is given by 3.3.1 which is a member in the class of univalent functions $\mathcal{S}$, then $|a_m| \leq m, \quad m = \{2,3,\ldots\}$. Furthermore, if $f(z)$ given by 3.3.1 is in the class of convex functions $\mathcal{K}$, then

$|a_m| \leq 1, \quad m = \{1,2,3,\ldots\}$.

Now, we consider to find the upper bounded of the operators $£_z^{\alpha,\beta}$, $ƻ_z^{\alpha,\beta}$ of univalent and convex functions.

Theorem 3.3.2

For extension the operator 3.1.1 in unit disk, let $f(\mathrm{z}) = \sum_{m=0}^{\infty} a_m z^m$ belongs to class of analytic functions $\mathcal{A}$, then

$$£_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} \mathrm{B}(\alpha-\beta, \beta+m) a_m z^m$$

Proof. For all $z \in D$, using definition 3.1.1, we obtain

$$£_z^{\alpha,\beta} f(z) = \sum_{m=0}^{\infty} £_z^{\alpha,\beta}\{a_m z^m\}$$

$$= \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z s^{\beta-1}\, (z-s)^{\alpha-\beta-1} \sum_{m=0}^{\infty} \{a_m s^m\} ds$$

$$= \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, z^{\alpha-\beta-1} \int_0^z s^{\beta-1}\, (1-\frac{s}{z})^{\alpha-\beta-1} \sum_{m=0}^{\infty} \{a_m s^m\} ds$$

Let $\frac{s}{z} = t$, then $\frac{ds}{z} = dt$. So as $s = 0$ then $t = 0$, and as $s = z$ then $t = 1$, we get

$$£_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)\, z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, z^{\alpha-1} \int_0^1 t^{\beta-1}\, (1-t)^{\alpha-\beta-1} \sum_{m=0}^{\infty} \{a_m (zt)^m\}\, dt$$

Using theorem 1.6.1 and Beta function with $|zt| < 1$, we have

$$£_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} a_m z^m \int_0^1 t^{m+\beta-1}\, (1-t)^{\alpha-\beta-1}\, dt$$

Where we can change the order of integration and summation since the series $\sum_{m=0}^{\infty}\{a_m z^m t^m\}$ is uniformly convergent in open unit disk D for $0 \le t \le 1$ and the integral $\int_0^1 \left|t^{\beta-1}\,(1-t)^{\alpha-\beta-1}\right| dt$ is convergent as long as $0 \le \alpha \le 1$ and $0 \le \beta \le 1$, then we have

$$£_{z}^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} a_m z^m \int_0^1 t^{m+\beta-1}\,(1-t)^{\alpha-\beta-1}\,dt$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} \mathrm{B}(m+\beta, \alpha-\beta)\; a_m z^m.$$

Hence, we arrive at the desired result.

Theorem 3.3.3

For extension the operator 3.2.1 in unit disk, let $f(\mathrm{z}) = \sum_{m=0}^{\infty} a_m z^m$ belongs to class of analytic functions $\mathcal{A}$, then

$$\mathcal{Z}_{z}^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \sum_{m=1}^{\infty} \mathrm{B}(\alpha+m, \beta-\alpha+1)\; a_m (m+\beta) z^m$$

Proof. For all $z \in D$, using definition 3.2.1, we obtain

$$\mathcal{Z}_{z}^{\alpha,\beta} f(z) = \sum_{m=0}^{\infty} \mathcal{Z}_{z}^{\alpha,\beta}\{a_m z^m\}$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z t^{\alpha-1}\,(z-t)^{\beta-\alpha} \sum_{m=0}^{\infty} a_m t^m \;\mathrm{dt}$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz}\; z^{\beta-\alpha} \int_0^z t^{\alpha-1}\,(1-\frac{t}{z})^{\beta-\alpha} \sum_{m=0}^{\infty} a_m t^m \;\mathrm{dt}$$

Let $\frac{t}{z} = u$, then $\frac{dt}{z} = du$. So as $t = 0$ then $u = 0$, and as $t = z$ then $u = 1$, we get

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz}\, z^{\beta} \int_0^1 u^{\alpha-1} (1-u)^{\beta-\alpha} \sum_{m=0}^{\infty} a_m (zu)^m \,\mathrm{du}$$

Using theorem 1.6.1 and Beta function with $|zu| < 1$, we have

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz} \sum_{m=0}^{\infty} a_m\, z^{\beta+m} \int_0^1 u^{m+\alpha-1} (1-u)^{\beta-\alpha} \,\mathrm{du}$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \sum_{m=0}^{\infty} a_m \frac{d}{dz} z^{\beta+m}\; B(\alpha+\mathrm{m}, \beta-\alpha+1)$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \sum_{m=1}^{\infty} a_m\, (\beta+m) z^{\beta+m-1}\; B(\alpha+\mathrm{m}, \beta-\alpha+1)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \sum_{m=1}^{\infty} B(\alpha+\mathrm{m}, \beta-\alpha+1)\; a_m\, (\beta+m) z^m.$$

This completes the proof of theorem 3.3.3 for differentiation.

Theorem 3.3.4 (Univalence)

Let $f \in \mathcal{S}$, then

$$\left| \pounds_z^{\alpha,\beta} f(z) \right| \le r(\, {}_2F_1(1,\beta;\alpha;r))'$$

Proof. By assuming that the function $f(z)$ given by definition 3.3.1 in the class $\mathcal{S}$, then by using proposition 3.2.3, we have

$$\pounds_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\, z^{1-\alpha} I_z^{\alpha-\beta} \left\{ \sum_{m=1}^{\infty} a_m\, z^{m+\beta-1} \right\}, a_1 = 1$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)}\, z^{1-\alpha} I_z^{\alpha-\beta} \sum_{m=0}^{\infty} a_{m+1}\, z^{m+\beta}$$

Using Eq. (2.2), we have

$$£_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\, z^{1-\alpha} \sum_{m=0}^{\infty} \frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+1)}\, a_{m+1}\, z^{m+\beta+\alpha-\beta}$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+1)}\, a_{m+1}\, z^{m+1}$$

Using properties of Gauss hypergeometric function in 1.4, we have

$$£_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{(\beta+1)_m\, \Gamma(\beta+1)}{(\alpha+1)_m\, \Gamma(\alpha+1)}\, a_{m+1}\, z^{m+1}$$

$$= \frac{\Gamma(\alpha)\, \beta\, \Gamma(\beta)}{\Gamma(\beta)\, \alpha\, \Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m}\, a_{m+1}\, z^{m+1}$$

$$= \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m}\, a_{m+1}\, z^{m+1}$$

Now, subsequently

$$\left| £_z^{\alpha,\beta} f(z) \right| = \left| \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m}\, a_{m+1}\, z^{m+1} \right|$$

By triangle inequality

$$\left| £_z^{\alpha,\beta} f(z) \right| \leq \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m}\, |a_{m+1}|\, |z^{m+1}|$$

$$\leq \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m}\, (m+1) r^{m+1}, |z| = r$$

Since $(2)_m = (m+1)!$ and

$$m + 1 = \frac{(m+1)!}{m!}$$

then

$$\left| £_z^{\alpha,\beta} f(z) \right| = r\frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m} \frac{(2)_m}{m!} r^m$$

$$= r\frac{\beta}{\alpha} \ {}_2F_1(2, \beta+1;\ \alpha+1; r)$$

Using Remark 1.4.4, we have

$$\left| £_z^{\alpha,\beta} f(z) \right| = r\frac{\beta}{\alpha} \frac{\alpha}{\beta} \left({}_2F_1(1, \beta, ; \alpha; r) \right)' = r \left({}_2F_1(1, \beta;\ \alpha; r) \right)'.$$

Now, by applying the last assertion on $Ƨ_z^{\alpha,\beta}$ we conclude

$$\left| Ƨ_z^{\alpha,\beta} f(z) \right| \le r \frac{\alpha}{\beta} \left\{ {}_2F_1(2, \alpha+1; \beta+1; r) \right\}$$

To prove this, let $f(\mathrm{z}) = \sum_{m=0}^{\infty} a_m z^m$ in the class $\mathcal{S}$, then

$$Ƨ_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z t^{\alpha-1} (z-t)^{\beta-\alpha} \sum_{m=1}^{\infty} a_m t^m \ \mathrm{dt},\ a_1 = 1$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z (z-t)^{\beta-\alpha} \sum_{m=1}^{\infty} a_m t^{m+\alpha-1} \ \mathrm{dt}$$

Using Eq. (2.4), we have

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)}\ D_z^{\alpha-\beta}\left\{\sum_{m=1}^{\infty} a_m z^{m+\alpha-1}\right\}$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)}\ D_z^{\alpha-\beta}\left\{\sum_{m=0}^{\infty} a_{m+1} z^{m+\alpha}\right\}$$

Using Eq.(2.7), we get

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{\Gamma(m+\alpha-\alpha+\beta+1)}\ a_{m+1} z^{m+\alpha-\alpha+\beta}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{\Gamma(m+\beta+1)}\ a_{m+1} z^{m+1}$$

By properties of Gauss hypergeometric function in 1.4, we have

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m\, \Gamma(\alpha+1)}{(\beta+1)_m\, \Gamma(\beta+1)}\ a_{m+1} z^{m+1}$$

Thus,

$$\left|\mathfrak{Z}_z^{\alpha,\beta} f(z)\right| = \left|\frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m\, \Gamma(\alpha+1)}{(\beta+1)_m\, \Gamma(\beta+1)}\ a_{m+1} z^{m+1}\right|$$

$$\leq \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\beta+1)_m}\ |a_{m+1}||z^{m+1}|$$

Since $|a_{m+1}| \leq m+1$ and $|z| = r$

$$\left|\mathfrak{Z}_z^{\alpha,\beta} f(z)\right| \leq \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\beta+1)_m}\ (m+1) r^{m+1}$$

$$= r\ \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\beta+1)_m}\ \frac{(2)_m}{m!} r^m$$

$$= r\ \frac{\alpha}{\beta}\ \{\ {}_2F_1(2, \alpha+1; \beta+1; r)\}.$$

Remark 3.3.5

We note that, this series in above theorem are absolutely convergent for all $z \in D$, so that represented as the analytic functions and holds true for property of Gauss hypergeometric function in open unit disk D (see [8]; p.28; Ch.1; Eq.(1.6.11)).

Next, we are interested to find the upper bound for inequality involving the hypergeometric function, which is given in the following theorem.

Theorem 3.3.6 (convexity)

Let the function $f(z)$ belongs to class of convex functions $\mathcal{K}$. Then

$$\left| \pounds_{\mathrm{z}}^{\alpha,\beta} f(z) \right| \leq r \frac{\beta}{\alpha} \{\ {}_2F_1(1, \beta+1; \alpha+1; r)\}$$

Proof. By imposing that $f(z) \in \mathcal{K}$ and using the same method in the previous theorem and proposition 3.1.6, we obtain

$$\pounds_{\mathrm{z}}^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\ z^{1-\alpha} I_z^{\alpha-\beta} \left\{ \sum_{m=1}^{\infty} a_m\ z^{m+\beta-1} \right\},$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)}\ z^{1-\alpha} \sum_{m=1}^{\infty} a_m\ I_z^{\alpha-\beta}\ z^{m+\beta-1}$$

Using Eq. (2.2), we have

$$\pounds_{\mathrm{z}}^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\ z^{1-\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}} \Gamma(\beta+1)}{(\alpha+1)_{\mathrm{m}} \Gamma(\alpha+1)}\ a_{m+1}\ z^{m+\beta-1+\alpha-\beta+1}$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}\, \beta\, \Gamma(\beta)}{(\alpha+1)_{\mathrm{m}}\, \alpha\, \Gamma(\alpha)}\, a_{m+1}\, z^{m+1}$$

$$= \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}}{(\alpha+1)_{\mathrm{m}}}\, a_{m+1}\, z^{m+1} \tag{3.3}$$

Now, subsequently

$$\left| £_{\mathrm{z}}^{\alpha,\beta} f(z) \right| = \left| \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}}{(\alpha+1)_{\mathrm{m}}}\, a_{m+1}\, z^{m+1} \right|$$

By triangle inequality and $|a_{m+1}| \leq 1$, we have

$$\left| £_{\mathrm{z}}^{\alpha,\beta} f(z) \right| \leq \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}}{(\alpha+1)_{\mathrm{m}}}\, |a_{m+1}|\, |z^{m+1}|$$

$$\leq r\, \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}}{(\alpha+1)_{\mathrm{m}}}\, (1)\, r^{m}, |z| = r$$

$$= r\, \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}}{(\alpha+1)_{\mathrm{m}}} \frac{(1)_m}{m!}\, r^{m}$$

$$= r\, \frac{\beta}{\alpha} \left\{ {}_{2}F_{1}(1, \beta+1; \alpha+1; r) \right\}.$$

For all $z \in D$.

For differentiation method we can show that as for $f(z) \in \mathcal{K}$, then

$$\left| \mathfrak{Z}_{\mathrm{z}}^{\alpha,\beta} f(z) \right| \leq r \frac{\alpha}{\beta} \left\{ {}_{2}F_{1}(1, \alpha+1; \beta+1; r) \right\}$$

Proof. Let $f(\mathrm{z}) = \sum_{m=0}^{\infty} a_m z^m$ be in the class $\mathcal{K}$, then

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz}\int_0^z t^{\alpha-1}\,(z-t)^{\beta-\alpha} \sum_{m=1}^{\infty} a_m t^m \; \mathrm{dt},$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \left(\frac{1}{\Gamma(1-\alpha+\beta)} \frac{d}{dz}\int_0^z (z-t)^{\beta-\alpha} \sum_{m=1}^{\infty} a_m t^{m+\alpha-1} \; \mathrm{dt} \right)$$

$$= \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)}\; D_z^{\alpha-\beta} \left\{ \sum_{m=1}^{\infty} a_m z^{m+\alpha-1} \right\} = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \left\{ \sum_{m=1}^{\infty} a_m D_z^{\alpha-\beta} z^{m+\alpha-1} \right\}$$

Using Eq.(2.7), we get

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) \; = \frac{\Gamma(\beta)\, z^{1-\beta}}{\Gamma(\alpha)} \sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha)}{\Gamma(m+\alpha-1-\alpha+\beta+1)} \; a_m z^{m+\alpha-1-\alpha+\beta}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha)}{\Gamma(m+\beta)} \; a_m z^{m+\beta-1} z^{1-\beta}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{\Gamma(m+\beta+1)} \; a_{m+1} z^{m+1}$$

By properties of Gauss hypergeometric function in 1.4, we get

$$\mathfrak{Z}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m\, \Gamma(\alpha+1)}{(\beta+1)_m\, \Gamma(\beta+1)} \; a_{m+1} z^{m+1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m \alpha\, \Gamma(\alpha)}{(\beta+1)_m\, \beta\, \Gamma(\beta)} \; a_{m+1} z^{m+1}$$

$$= \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\beta+1)_m} \; a_{m+1} z^{m+1} \qquad (3.4)$$

Thus,

$$\left| \text{ƻ}_{\mathrm{z}}^{\alpha,\beta} f(z) \right| = \left| \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_{\mathrm{m}}}{(\beta+1)_{\mathrm{m}}} \; a_{m+1} z^{m+1} \right|$$

$$\leq \; \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_{\mathrm{m}}}{(\beta+1)_{\mathrm{m}}} \; |a_{m+1}||z^{m+1}|$$

Since $|a_{m+1}| \leq 1$ and $|z| = r$

$$\left| \text{ƻ}_{\mathrm{z}}^{\alpha,\beta} f(z) \right| \leq \; \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_{\mathrm{m}}}{(\beta+1)_{\mathrm{m}}} \; (1) r^{m+1}$$

$$= r \; \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_{\mathrm{m}}}{(\beta+1)_{\mathrm{m}}} \; \frac{(1)_m}{m!} r^m$$

$$= \; r \frac{\alpha}{\beta} \left\{ \, {}_2F_1(1, \alpha+1; \beta+1; r) \right\}.$$

Theorem 3.3.7

Let $f(z) \in \mathcal{K}$, then

$$\left| £_{\mathrm{z}}^{\alpha,\beta} f(z) \right| \leq \frac{r}{\mathrm{B}(\beta, \alpha-\beta)} \int_0^1 s^{\beta} \; (1-s)^{\alpha-\beta-1} \; (1-rs)^{-1} \, ds$$

Proof. Suppose that, $f(z) \in \mathcal{K}$ on D, and by Eq.(3.3), we have

$$£_{\mathrm{z}}^{\alpha,\beta} f(z) = \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_{\mathrm{m}}}{(\alpha+1)_{\mathrm{m}}} \; a_{m+1} \, z^{m+1}$$

Since $|z| = r$ and $|a_{m+1}| \leq 1$, then

$$\left| £_{\mathrm{z}}^{\alpha,\beta} f(z) \right| \; \leq \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta+1)_m}{(\alpha+1)_m} \; r^{m+1}$$

Using the definition of $(a)_m$, we get

$$\left| £_z^{\alpha,\beta} f(z) \right| \leq \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{\Gamma(m+\beta+1)\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(\beta+1)} r^{m+1}$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+1)} r^{m+1}$$

Multiplying and dividing by $\Gamma(\alpha-\beta)$, also we add and subtract β in denominator to get the Beta function

$$£_z^{\alpha,\beta} f(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\beta+m+\alpha+1-\beta)} r^{m+1}$$

Where $m+\beta > 0$, $\alpha-\beta > 0$, $\alpha > \beta > 0$ and since

$$£_z^{\alpha,\beta} f(z) = \frac{\mathrm{r}\,\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} \mathrm{B}(m+\beta+1, \alpha-\beta)\ r^{m+1}$$

$$= \frac{\mathrm{r}\,\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} \int_0^1 s^{m+\beta} (1-s)^{\alpha-\beta-1}\, ds\, r^m$$

then it follows that

$$\left| £_z^{\alpha,\beta} f(z) \right| = \left| \frac{\mathrm{r}\,\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 s^{\beta} (1-s)^{\alpha-\beta-1} \left\{ \sum_{m=0}^{\infty} (rs)^m \right\} ds \right|$$

$$\leq \frac{\mathrm{r}\,\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 s^{\beta} (1-s)^{\alpha-\beta-1} \frac{1}{1-rs}\, ds$$

$$= \frac{\mathrm{r}}{\mathrm{B}(\beta, \alpha-\beta)} \int_0^1 s^{\beta} (1-s)^{\alpha-\beta-1} (1-rs)^{-1}\, ds.$$

For differential operator it follows as

$$\left| \mathcal{Z}_z^{\alpha,\beta} f(z) \right| \leq \frac{r}{B(\alpha, \beta-\alpha)} \int_0^1 s^{\alpha}\ (1-s)^{\beta-\alpha}\ (1-rs)^{-1} ds$$

Proof. Suppose $f(z) \in \mathcal{K}$, and by Eq.(3.4), we have

$$\mathcal{Z}_z^{\alpha,\beta} f(z) = \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\beta+1)_m}\ a_{m+1} z^{m+1}$$

Since $|z| = r$ and $|a_{m+1}| \leq 1$, then

$$\left| \mathcal{Z}_z^{\alpha,\beta} f(z) \right| \leq \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\beta+1)_m}\ r^{m+1}$$

Using the definition of $(a)_m$, we get

$$\left| \mathcal{Z}_z^{\alpha,\beta} f(z) \right| \leq \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)\Gamma(\beta+1)}{\Gamma(m+\beta+1)\Gamma(\alpha+1)}\ r^{m+1}$$

$$= \frac{\alpha}{\beta} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)\ \beta\ \Gamma(\beta)}{\Gamma(m+\beta+1)\ \alpha\ \Gamma(\alpha)}\ r^{m+1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{\Gamma(m+\beta+1)}\ r^{m+1}$$

Multiplying and dividing by $\Gamma(\beta-\alpha)$, also we add and subtract α in denominator to get the Beta function

$$\left| \mathcal{Z}_z^{\alpha,\beta} f(z) \right| = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)\Gamma(\beta-\alpha)}{\Gamma(m+\alpha+1+\beta-\alpha)}\ r^{m+1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \sum_{m=0}^{\infty} B(m+\alpha+1, \beta-\alpha)\ r^{m+1}$$

$$= \frac{r\,\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \sum_{m=0}^{\infty} \int_0^1 \mathrm{s}^{\mathrm{m}+\alpha}\ (1-s)^{\beta-\alpha-1} ds\, r^m$$

$$\leq \frac{r\,\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \sum_{m=0}^{\infty} \int_0^1 \mathrm{s}^{\alpha}\ (1-s)^{\beta-\alpha-1} ds\, (rs)^m$$

$$= \frac{r}{\mathrm{B}(\alpha,\beta-\alpha)} \int_0^1 \mathrm{s}^{\alpha}\ (1-s)^{\beta-\alpha-1} \sum_{m=0}^{\infty} (rs)^m\, ds$$

$$= \frac{r}{\mathrm{B}(\alpha,\beta-\alpha)} \int_0^1 \mathrm{s}^{\alpha}\ (1-s)^{\beta-\alpha-1} \frac{1}{1-rs} ds$$

$$= \frac{r}{\mathrm{B}(\alpha,\beta-\alpha)} \int_0^1 \mathrm{s}^{\alpha}\ (1-s)^{\beta-\alpha}\ (1-rs)^{-1} ds.$$

In the next we provide some examples.

Example 3.3.8

Let $f(z) = z^v$, for all $z \in D$, then

$$£_z^{\alpha,\beta}\{z^v\} = \frac{\Gamma(\alpha)\Gamma(v+\beta)}{\Gamma(\beta)\Gamma(v+\alpha)}\ z^v.$$

Solution. By consider the operator 3.1.1, we obtain

$$£_z^{\alpha,\beta}\{z^v\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{1-\alpha} \int_0^z \frac{t^{\beta-1}\ t^v}{(z-t)^{1-\alpha+\beta}}\ dt$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{1-\alpha}\ z^{\alpha-\beta-1} \int_0^z t^{\beta+v-1}\ (1-\frac{t}{z})^{\alpha-\beta-1}\ dt$$

Where $0 < \alpha \leq 1, 0 < \beta \leq 1$ and $0 < \alpha - \beta < 1$ and $z \in D, v \in \mathbb{R}$. Set $\frac{t}{z} = u$ then $\frac{dt}{z} = du$, as $t = 0$, then $u = 0$ and as $t = z$, then $u = 1$ and using Beta function, we have

$$\pounds_{z}^{\alpha,\beta}\{z^{v}\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{v} \int_{0}^{1} u^{\beta+v-1}\,(1-u)^{\alpha-\beta-1}\ du$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{v}\ B(\beta+v, \alpha-\beta)$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ \frac{\Gamma(\beta+v)\Gamma(\alpha-\beta)}{\Gamma(\alpha+v)}\ z^{v}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta+v)}{\Gamma(\beta)\Gamma(\alpha+v)}\ z^{v}\,.$$

Examples 3.3.9

Let $f(z) = e^{z} = \sum_{v=0}^{\infty} \frac{z^{v}}{v!}$ for all $z \in D$, then

(1) $\pounds_{z}^{\alpha,\beta}\{e^{z}\} = {}_{1}F_{1}(\beta;\ \alpha; z), \quad (0 < \beta < \alpha).$

(2) $\mathfrak{Z}_{z}^{\alpha,\beta}\{e^{z}\} = {}_{1}F_{1}(\alpha;\ \beta; z), \quad (0 < \alpha < \beta).$

Solution. In this example we follow same as methods in a previous Example and we obtain

$$(1) \qquad \pounds_{z}^{\alpha,\beta}\{e^{z}\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{1-\alpha} \int_{0}^{z} t^{\beta-1}\,(z-t)^{\alpha-\beta-1}\, e^{t}\ dt$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{1-\alpha}\, z^{\alpha-\beta-1} \int_{0}^{z} t^{\beta-1}\,(1-\frac{t}{z})^{\alpha-\beta-1}\, e^{t}\ dt$$

Set $\frac{t}{z} = u$ then $\frac{dt}{z} = du$, as $t = 0$, then $u = 0$ and as $t = z$, then $u = 1$, thus we have

$$£_z^{\alpha,\beta}\{e^z\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{-\beta+\beta-1+1}\int_0^1 u^{\beta-1}(1-u)^{\alpha-\beta-1}\, e^{zu}du$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\alpha-\beta-1}\, e^{zu}du$$

Hence, by Eq. (1.5), we get

$$£_z^{\alpha,\beta}\{e^z\} = \ {}_1F_1(\beta;\ \alpha; z).$$

(2) $ƻ_z^{\alpha,\beta}\{e^z\} = \ {}_1F_1(\alpha;\ \beta; z)$

Using proposition 3.2.3, we get

$$ƻ_z^{\alpha,\beta}\{e^z\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\ z^{1-\beta}\frac{d}{dz}I_z^{1-\alpha+\beta}\,(z^{\alpha-1}e^z)$$

By theorem 2.2.1, we have

$$ƻ_z^{\alpha,\beta}\{e^z\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\ z^{1-\beta}D_z^{\alpha-\beta}\,(z^{\alpha-1}e^z)$$

By Eq.(2.5) $D_z^{\alpha-\beta} = I_z^{1-\alpha+\beta}\, D_z^1$, we get

$$ƻ_z^{\alpha,\beta}\{e^z\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\ z^{1-\beta}I_z^{1-\alpha+\beta}\, D_z^1\,(z^{\alpha-1}e^z)$$

Using Eq.(2.3), we get

$$ƻ_z^{\alpha,\beta}\{e^z\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\ z^{1-\beta}I_z^{-\alpha+\beta}\, I_z^1\, D_z^1\,(z^{\alpha-1}e^z)$$

For $f(0) = 0$, then

$$ƻ_z^{\alpha,\beta}\{e^z\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\ z^{1-\beta}I_z^{\beta-\alpha}\,(z^{\alpha-1}e^z)$$

Using Eq. (2.1), then

$$\mathcal{Z}_{\mathrm{z}}^{\alpha,\beta}\{e^{z}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, \mathrm{z}^{1-\beta} \frac{1}{\Gamma(\beta-\alpha)} \int_0^{\mathrm{z}} \mathrm{t}^{\alpha-1} (\mathrm{z}-t)^{\beta-\alpha-1}\, e^{t} \mathrm{d}t$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\, \mathrm{z}^{1-\beta} \mathrm{z}^{\beta-\alpha-1} \int_0^{\mathrm{z}} \mathrm{t}^{\alpha-1} \left(1-\frac{t}{z}\right)^{\beta-\alpha-1} e^{t}\, \mathrm{d}t$$

Set $\frac{t}{z} = u$ then $\frac{dt}{z} = du$, as $t = 0$, then $u = 0$ and as $t = z$, then $u = 1$, thus we have

$$\mathcal{Z}_{\mathrm{z}}^{\alpha,\beta}\{e^{z}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 u^{\alpha-1} (1-u)^{\beta-\alpha-1}\, e^{zu} du$$

Hence, by Eq. (1.5), we get

$$\mathcal{Z}_{\mathrm{z}}^{\alpha,\beta}\{e^{z}\} = {}_1F_1(\alpha;\ \beta; z).$$

Remark. We note that for any $z \in \mathbb{C}$ ${}_1F_1(\alpha;\ \beta; z)$ is convergent and it has an integral representation (see [8], p.29, Eq.1.6.15). In any case, may be can represent the operator $£_{\mathrm{z}}^{\alpha,\beta}$ in example 3.3.8 as the following integral

$$£_{\mathrm{z}}^{\alpha,\beta}\{e^{z}\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 u^{\beta-1}\, (1-u)^{\alpha-\beta-1}\ e^{uz}\, du$$

Or, by using the fact of the beta function

$$\mathrm{B}(\beta, \alpha-\beta) = \frac{\Gamma(\beta)\Gamma(\alpha-\beta\,)}{\Gamma(\alpha)}$$

yields

$$£_{\mathrm{z}}^{\alpha,\beta}\{e^{z}\} = \frac{1}{\mathrm{B}(\beta, \alpha-\beta)} \int_0^1 u^{\beta-1}\, (1-u)^{\alpha-\beta-1}\ e^{uz}\, du, (0 < \beta < \alpha).$$

Examples 3.3.10

Let $0 < \alpha \leq 1, 0 < \beta \leq 1$ and $|z| < 1$, then

(1) $\mathcal{L}_z^{\alpha,\beta}\{(1-z)^{-v}\} = {}_2F_1(v,\beta;\ \alpha;z)$

(2) $\mathcal{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = {}_2F_1(v,\alpha;\ \beta;z)$

Where

$$(1-z)^{-v} = 1 + vz + \frac{v(v+1)}{2!}z^2 + \frac{v(v+1)(v+2)}{3!}z^3 + \cdots$$

Solution. By direct calculations, we obtain

(1) $$\mathcal{L}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\ z^{1-\alpha}\int_0^z s^{\beta-1}\,(z-s)^{\alpha-\beta-1}\,(1-s)^{-v}\ ds$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha+\alpha-\beta-1}\int_0^z s^{\beta-1}(1-\frac{s}{z})^{\alpha-\beta-1}(1-s)^{-v}\,ds$$

Set $\frac{s}{z} = u$ then $\frac{ds}{z} = du$, as $s = 0$, then $u = 0$ and as $s = z$, then $u = 1$, thus we have

$$\mathcal{L}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\alpha-\beta-1}(1-zu)^{-v}\,du$$

$$= {}_2F_1(v,\beta;\ \alpha;z).$$

(2) $\mathcal{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = {}_2F_1(v,\alpha;\ \beta;z)$

Using proposition 3.2.3, we get

$$\mathcal{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\ z^{1-\beta}\frac{d}{dz}I_z^{1-\alpha+\beta}\,(z^{\alpha-1}(1-z)^{-v})$$

By theorem 2.2.1, we have

$$\mathfrak{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} D_z^{\alpha-\beta}\,(z^{\alpha-1}(1-z)^{-v})$$

By Eq.(2.5) $D_z^{\alpha-\beta} = I_z^{1-\alpha+\beta} D_z^1$, we get

$$\mathfrak{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} I_z^{1-\alpha+\beta}\, D_z^1\,(z^{\alpha-1}(1-z)^{-v})$$

Using Eq.(2.3), we have

$$\mathfrak{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} I_z^{\beta-\alpha} I_z^1 D_z^1\,(z^{\alpha-1}(1-z)^{-v})$$

For $f(0) = 0$, then

$$\mathfrak{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta} I_z^{\beta-\alpha}(z^{\alpha-1}(1-z)^{-v})$$

Using Eq. (2.1), then

$$\mathfrak{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, z^{1-\beta}\, \frac{1}{\Gamma(\beta-\alpha)} \int_0^z s^{\alpha-1}(z-s)^{\beta-\alpha-1}\,(1-s)^{-v} ds$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\, z^{1-\beta} z^{\beta-\alpha-1} \int_0^z s^{\alpha-1}\left(1-\frac{s}{z}\right)^{\beta-\alpha-1} (1-s)^{-v}\, ds$$

Set $\frac{s}{z} = u$ then $\frac{ds}{z} = du$, as $s = 0$, then $u = 0$ and as $s = z$, then $u = 1$, thus we have

$$\mathfrak{Z}_z^{\alpha,\beta}\{(1-z)^{-v}\} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 u^{\beta-1}(1-u)^{\beta-\alpha-1}(1-zu)^{-v}\, du$$

$$= {}_2F_1(v,\alpha;\ \beta; z).$$

Therefore the proof is complete.

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تعميم عامل التفاضل والتكامل الكسري في الفضاء العقدي

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ملخص

في هذه الدراسة تم تعميم التفاضل والتكامل الكسري في الفضاء العقدي باستخدام تعريف خاصية التكامل الكسري وخاصية التفاضل الكسري والتوصل لنتائج يمكن تعميمها.

عدة خصائص تم إثباتها لهذين العاملين مثل الحصر والضغط بالإضافة إلى انه تم مناقشة عوامل التفاضل والتكامل على بعض الصفوف الجزئية واشتقاق بعض الخصائص منها، وبالنهاية توصلنا إلى العلاقة لبعض الاقترانات بين عوامل التفاضل والتكامل واقتران الجاوس جيومترك.