

**HOMOGENEOUS SYSTEMS OF LINEAR
DIFFERENTIAL EQUATIONS WITH CONSTANT
OR PERIODIC COEFFICIENTS**

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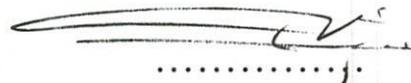
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DEDICATION

**TO
MY
PARENTS**

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ABSTRACT

We study the general properties of linear systems of first-order differential equations, where the coefficients of the unknown variables are continuous functions. Furthermore we study the n -th order homogeneous linear differential equation.

Several analytic methods for determining the general solution of the homogeneous system with constant and periodic coefficients are studied.

We use basic results in linear algebra for determining the general solution.

Furthermore, as an application, Riccati Differential Equation is investigated. Lyapunov transformation and reducible systems are used for solving the periodic system.

Finally, we study the stability of perturbed linear systems; where we remark that the stability of any nonzero solution of the homogeneous linear equation is determined by the stability of the zero solution.

Introduction

Studying simultaneous differential equations plays a major role in solving many problems in Control Theory, Optimization, Fluid Dynamics, and Electrical Engineering.

In chapter one we study the general properties of general linear systems of first-order differential equations, whose coefficients are continuous functions. Furthermore we solve n -th order homogeneous linear differential equation by converting such equation to a system involving n first-order ordinary differential equations.

In chapter two, we turn our attention to the special case of linear systems of ordinary differential equations with constant coefficients. First we study the diagonal system and generalize this method to any linear system with constant coefficients. We use the Jordan Canonical form in the case where eigenvalues are repeated. Furthermore we give different methods for determining the matrix exponentials as solutions of linear homogeneous systems. Finally, these methods are used to solve the matrix Riccati equation.

In chapter three, we use elementary results in linear algebra for solving systems of homogeneous ordinary linear differential equations with constant coefficients.

In chapter four, Lyapunov transformation and reducible systems are used for solving the periodic system.

In the last chapter the stability of linear systems with both constant and periodic coefficients is investigated.

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CHAPTER ONE

SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

In section (1.1) we will give a general discussion of n -dimensional first-order system of ordinary differential equations. Such a system arise naturally in problems involving several dependent variables each of which is a function of a single independent variable. We will denote the independent variable by t , and let $x_1, x_2, x_3, \dots, x_n$ represent dependent variables which are functions of t .

In section (1.2) we will apply the result obtained in section (1.1) to a consideration of the n -th order linear differential equation.

1.1 Linear System Of First-Order Differential Equations

Consider the homogeneous system of linear first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n.\end{aligned}\tag{1.1}$$

If the vector X and the matrix A are defined by

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}$$

where $a_{ij}(t)$ are continuous real-valued functions for t on an closed interval I of real numbers, for $i, j = 1, 2, \dots, n$. Then system (1.1) can be written as

$$\frac{dX}{dt} = A(t)X\tag{1.2}$$

Also if

$$F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

where $f_i(t)$ are continuous functions of $t \in I$.

Then the non-homogeneous system can be written as

$$\frac{dX}{dt} = AX + F(t) \quad (1.3)$$

Definition 1.1.1: A solution vector on an interval I is any column

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

whose entries are differentiable functions satisfying system (1.3) on the interval I . Let t_0 denote a point in an interval I and

$$X(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}, \text{ and } X_0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix},$$

where γ_i are given constants, for $i=1, 2, \dots, n$. Then the problem of finding the solution of (1.3) subject to the initial condition $X(t_0) = X_0$ is called an **initial-value problem**.

Theorem 1.1.2: Let X_1, X_2, \dots, X_k be a set of solution vectors of the homogeneous system (1.2) on the interval I , then the linear combination $X = c_1X_1 + c_2X_2 + \dots + c_kX_k$ is also a solution of (1.2) on the interval I , where c_i are arbitrary constants for $i=1, 2, \dots, k$.

Proof : Let X_1, X_2, \dots, X_k be a set of solution vectors of the homogeneous system (1.2) where A is an $n \times n$ matrix. Then (1.2) can be written as

$$\frac{dX}{dt} - A(t)X = \mathbf{0} \quad (1.4)$$

Let

$X = c_1 X_1 + \dots + c_k X_k$ where c_1, \dots, c_k are arbitrary constants, then

$$\frac{dX}{dt} = c_1 \frac{dX_1}{dt} + c_2 \frac{dX_2}{dt} + \dots + c_k \frac{dX_k}{dt} \quad (1.5)$$

Now system (1.4) becomes as

$$\begin{aligned} & c_1 \frac{dX_1}{dt} + c_2 \frac{dX_2}{dt} + \dots + c_k \frac{dX_k}{dt} - A(t)c_1 X_1 - A(t)c_2 X_2 - \dots - A(t)c_k X_k \\ &= \left(c_1 \frac{dX_1}{dt} - A(t)c_1 X_1 \right) + \left(c_2 \frac{dX_2}{dt} - A(t)c_2 X_2 \right) + \dots + \left(c_k \frac{dX_k}{dt} - A(t)c_k X_k \right) \\ &= c_1 \left(\frac{dX_1}{dt} - A(t)X_1 \right) + c_2 \left(\frac{dX_2}{dt} - A(t)X_2 \right) + \dots + c_k \left(\frac{dX_k}{dt} - A(t)X_k \right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \dots + c_k \cdot 0 = 0. \end{aligned}$$

Hence the linear combination $X = c_1 X_1 + \dots + c_k X_k$ is also a solution on I . ■

Definition 1.1.3: Let X_1, X_2, \dots, X_k be a set of solution vectors of the homogeneous system (1.2) on an interval I . We say that this set is **linearly dependent** on the interval if there exist constants

c_1, c_2, \dots, c_k not all zero, such that $c_1 X_1 + \dots + c_k X_k = 0$ for every $t \in I$. If the set of vectors is not linearly dependent on the interval, it is said to be **linearly independent**.

Theorem 1.1.4 : Let X_1, X_2, \dots, X_n be n solution vectors of the homogeneous system (1.2) on an interval I . A necessary and sufficient condition for the set of solution to be linearly independent is that the Wronskian $W(X_1, X_2, \dots, X_n) \neq 0$ for every $t \in I$.

Proof : Assume that X_1, X_2, \dots, X_n are linearly dependent on I .

Let X be a nontrivial solution of (1.2), so there exist constants $\alpha_1, \dots, \alpha_n$, not all zero, such that

$$X = \sum_{j=1}^n \alpha_j X_j \quad \text{or} \quad X = \Phi C, \quad (1.6)$$

where Φ is a fundamental matrix for (1.2), and $C = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$.

Let $t = \tau \in I$. Then we have

$$X(\tau) = \Phi(\tau)C \quad (1.7)$$

This system of equations has a unique solution for any choice of $X(\tau)$.

Hence, $\det(\Phi(\tau)) \neq 0$. that $\det(\Phi(t)) \neq 0$ for any $t \in I$.

i.e $W(X_1, X_2, \dots, X_n) \neq 0$ for any $t \in I$.

Conversely, let Φ be a solution of the matrix equation (1.2) and assume that $\det(\Phi(t)) \neq 0$ for all $t \in I$. Then the column of Φ , X_1, \dots, X_n , are linearly independent for all $t \in I$. Hence, Φ is a fundamental matrix of (1.2). ■

Definition 1.1.5 : (i) Any set X_1, X_2, \dots, X_n of n linearly independent solution vectors of the homogeneous system (1.2) on an interval I is said to be a **fundamental set of solutions** on I .

(ii) For a non-homogeneous system a **particular solution** X_p on an interval I is any vector, free of arbitrary parameters, whose entries are functions satisfying the system (1.3).

Lemma 1.1.6: There exists a unique solution of the initial-value problem

$$\frac{dX}{dt} = A(t)X, \quad X(t_0) = I_n \text{ for } t_0 \in I.$$

Proof: We shall construct the solution by means of

Let

$$X_0(t) = I_n,$$

and

$$X_k(t) = I_n + \int_0^t A(u)X_{k-1}(u)du, \quad k = 1, 2, \dots \quad (1.8)$$

Then

$$\begin{aligned} X_k(t) = I_n + \int_0^t A(u)du + \int_0^t A(t_1)dt_1 \int_0^{t_1} A(t_2)dt_2 \\ + \dots + \int_0^t A(t_1)dt_1 \int_0^{t_1} A(t_2)dt_2 \dots \int_0^{t_{k-1}} A(t_k)dt_k. \end{aligned} \quad (1.9)$$

The convergence of the sequence (1.8) is equivalent to the convergence of the series

$$\begin{aligned} X_0 + (X_1(t) - X_0) + (X_2(t) - X_1(t)) + (X_3(t) - X_2(t)) \\ + \dots + (X_k(t) - X_{k-1}(t)) + \dots \end{aligned} \quad (1.10)$$

By induction, for the terms of this series, we have the estimate

$$\|X_m(t) - X_{m-1}(t)\| \leq \frac{1}{m!} \left(\int_{t_0}^t \|A(u)\| du \right)^m, \quad m = 1, 2, \dots, \quad \text{for } t \in I.$$

For $m = 1$ the estimate is true.

Assume it is true for m , want to show it is true for $m + 1$.

$$\begin{aligned} \|X_{m+1}(t) - X_m(t)\| &= \left\| \int_{t_0}^t A(t_1) dt_1 \int_{t_0}^{t_1} A(t_2) dt_2 \cdots \int_{t_0}^{t_m} A(t_{m+1}) dt_{m+1} \right\| \\ &\leq \left| \int_{t_0}^t A(t_1) dt_1 \frac{1}{m!} \left(\int_{t_0}^t \|A(u)\| du \right)^m \right|. \end{aligned}$$

By substitute

$$W = \left| \int_{t_0}^{t_1} \|A(u)\| du \right|$$

we get

$$\|X_{m+1}(t) - X_m(t)\| \leq \frac{1}{m+1!} \left(\int_{t_0}^t \|A(u)\| du \right)^{m+1},$$

so the estimate is true for $m + 1$.

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\int_{t_0}^t \|A(u)\| du \right)^k = \exp \left| \int_{t_0}^t \|A(u)\| du \right|,$$

This implies its converges for (1.10).

Denoting the sum of series (1.10) by $X(t)$ we obtain

$$X(t) = I_n + \int_0^t A(u) du + \sum_{k=2}^{\infty} \left(\int_0^t A(t_1) dt_1 \cdots \int_0^{t_{k-1}} A(t_k) dt_k \right) \quad (1.11)$$

Differentiate the series (1.11) term by term, we obtain the same series multiplied on the left by $A(t)$. ■

Theorem 1.1.7: There exists a fundamental set of solutions for the homogeneous system (1.2) on an interval I .

Proof: Let e_1, e_2, \dots, e_n be n -dimensional vectors such that

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ place.}$$

Since

$$\sum_{j=1}^n c_j e_j = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$$

if and only if $c_j = 0$ for $j = 1, 2, \dots, n$.

Hence e_1, e_2, \dots, e_n are linearly independent. For any $t_0 \in I$, Let $X_1(t), X_2(t), \dots, X_n(t)$ be the solutions of (1.2) satisfying $X_j(t_0) = e_j$, $j = 1, 2, \dots, n$. These are all distinct, since each satisfies distinct initial values. Furthermore are linearly independent, For

$$X(t) = \sum_{j=1}^n c_j X_j(t) = \mathbf{0} \text{ for } t \in I.$$

With the c_j not all zero, then

$$X(t_0) = \sum_{j=1}^n c_j X_j(t_0) = \sum_{j=1}^n c_j e_j = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

But this implies $c_1 = \dots = c_n = 0$, which is contradiction.

Therefore $X_1(t), X_2(t), \dots, X_n(t)$ are linearly independent, and form a fundamental set of solutions of (1.2) on I . ■

Theorem 1.1.8: Let X_1, X_2, \dots, X_n be a set of solution vectors of the homogeneous system (1.2) on I and let X_p be any solution vector of the non-homogenous system (1.3) on I then

$X = c_1 X_1 + c_2 X_2 + \dots + c_k X_n + X_p$ is a solution of (1.3) on I .

Proof: Suppose X and X_p are both solutions of

$$\frac{dX}{dt} = A(t)X + F(t) \tag{1.12}$$

if we define the function U by $U(t) = X(t) - X_p(t)$ then

$$\begin{aligned}
\frac{d}{dt}(U(t)) &= \frac{d}{dt}(X(t)) - \frac{d}{dt}(X_p(t)) \\
&= (A(t)X(t) + F(t)) - (A(t)X_p + F(t)) \\
&= A(t)(X(t) - X_p(t)) + (F(t) - F(t)) \\
&= A(t)(X(t) - X_p(t)) \\
&= A(t)U(t).
\end{aligned}$$

Thus $U(t)$ is a solution of the homogenous system.

That is

$$U(t) = c_1 X_1 + c_2 X_2 + \cdots + c_k X_n.$$

Hence

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_k X_n + X_p$$

is a solution of the non-homogeneous system (1.12). ■

Definition 1.1.9: Let X_p be a given solution of the non-homogeneous system (1.8) on I , and let $X = c_1 X_1 + c_2 X_2 + \cdots + c_k X_k + X_p$ denote the **general solution** on the same interval of the corresponding homogeneous system (1.2). The **general solution of the non-homogeneous** system on the interval is defined to be $X = X_c + X_p$. The general solution X_c of the homogeneous System (1.2) is called the **complementary solution** of the non-homogeneous system.

Let X_1, X_2, \dots, X_n be a fundamental set of solutions of the homogeneous system (1.2) on an interval I then its general solution on the interval I is $X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$.

Thus if

$$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{nn} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \dots, X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}.$$

Then

$$X = \begin{pmatrix} c_1 x_{11} + c_2 x_{12} + \cdots + c_n x_{1n} \\ c_1 x_{21} + c_2 x_{22} + \cdots + c_n x_{2n} \\ \vdots \\ c_1 x_{n1} + c_2 x_{n2} + \cdots + c_n x_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$