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A Special Boundary Integral Equations Method For Approximate Solution of Three- Dimensional Potential Problems

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A Special Boundary Integral Equations Method For Approximate Solution of Three- Dimensional Potential Problems

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Declaration

I certify that the thesis, submitted for the degree of Master, is the result of my own research except where otherwise acknowledged, and that the thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed……………

Rania Saeed Younis Hamdan

Date: 17/06/2009

Dedication

To my father, my mother, my husband, my son, my sisters, and my brothers.

Acknowledgement

Thanks is given first to God.

I would like to express my thanks to my supervisor, Dr. Yousef Zahaykah for his help and support during all phases of my graduate study.

Also my thanks to the other members of the department of mathematics at Al-Quds University.

Special thanks go to my mother, my husband, and to my son, they have given me a lot of love and power to concentrate on my study.

Abstract

This work deals with the approximate solution of Laplace equation in three-dimensional regions.

Green's representation is of greate important in this regard. Furthermore, harmonic, subharmonic, superharmonic functions and potential layers considered here are of essential role in understanding and analyzing potential phenomena.

Special boundary integral equations are developed for solving potential problems in three-dimensional regions with arbitrary configuration of spherical cavities.

The solution on the boundary of each cavity is represented by a finite sum of spherical harmonics with unknowns coefficients. The cavity geometry is directly exploited in a new set of integral equations with special kernel functions which independently "pick out" these coefficients.

Each new equation contains only one coefficient relating to the particular cavity and so the resulting system of equations for unknown field on the boundaries of the cavities is well-conditioned.

The level of approximation in these equations depends on the number of spherical harmonics in the representation of the solution on the boundary of the cavity.

Equations corresponding to the lowest and next higher level of approximation are solved. Examples are given to demonstrate the proposed method. Moreover, this method is also applied to three-dimensional regions with slender cavities of circular-cross sections.

الملخص

في هذه الأطروحة تم تناول موضوع الحلول التقريبية لمعادلة Laplace في مناطق ثلاثية الأبعاد. إن تمثيل الاقترانات المنتمية إلى الفضاءات الاقترانية $C^2(\overline{\Omega})$ من خلال ما يعرف بـ Green's identities ذو أهمية خاصة في سياق هذه الدراسة.

كما أن مفاهيم harmonic, subharmonic, superharmonic functions و potential layers والتي تم التطرق لها ، تلعب دورًا اساسيًا في فهم ما يعرف بنظرية الـ Potential (Potential Theory).

لقد تم في هذه الأطروحة تطوير معادلات تكاملية خاصة لإيجاد حل تقريبي لمعادلة Laplace في منطقة ثلاثية الأبعاد تحوي على تجويفات كروية.

إن الحلول على المحيط لهذه التجويفات مُثِل بمتسلسلة منتهية من الـ spherical harmonics معاملاتها مجهولة.

إن الطبيعة الكروية (الهندسية) للتجويفات أدى إلى نظام من المعادلات التكاملية ذي الـ kernel الخاصة، والتي بالاعتماد عليها تم تحديد المجاهيل.

إن مستوى التقريب الناشيء يعتمد على عدد الـ spherical harmonics المأخوذ في المتسلسلة حيث تم معالجة مستويين من الدقة هما مستوى الـ low level و next level.

لقد تم اعطاء بعض الأمثلة كتطبيق لهذا الأسلوب في المعالجة، بالإضافة إلى ذلك تم تطبيق هذه الطريقة على مناطق ثلاثية الأبعاد تحوي تجويفات دقيقة ذات مقاطع دائرية.

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Introduction

One of the most important partial differential equations that occurs in applied mathematics is Laplace equation.

Many important problems require the solution of this equation such as [16], heat conduction in cooled gas turbine blades [4], flow of an inviscid compressible fluid around circular cylinders [12], and electrostatic field around gratings of charged wires [13].

Laplace equation is an example of the more general type of partial differential equations known as elliptic partial differential equations. The basic theory of these equations is presented in Gilbarg and Trudinger [7], Jost [9], Alexer [1].

Moreover the solution of boundary value problems for partial differential equations is one of the most important field of applications for integral equations, see Kress [10] and Kanwal [5].

It is the nonlinearity and the complex geometries that make analytical solution difficult to obtain. The best alternative is to seek approximate solution. Many numerical techniques were derived based on finite element, finite difference, and boundary integral methods, see [11,8,14].

The technique "special boundary integral equations…" was proposed by Baron and Caulk [2,3] for potential problems in regions with circular holes and to regions with slender cavities. The same technique was applied to three-dimensional regions with spherical cavities by Zahaykah, [17].

In [17], the potential or its outward normal derivative was assumed to be constant.

Here beside the constant case we consider also the case where the potential or its outward normal derivative is not constant.

In this work we approach the problem by formulating special boundary integral equations which take explicit account of the cavity geometry and the corresponding characteristic of the solution.

First the potential function and its normal derivative are represented by a finite series of spherical harmonics on the boundary of each cavity.

The unknown coefficients in each series are determined by a new set of integral equations with special kernel functions which independently "pick out" respective coefficient at a given cavity. Taken together, the equations at any one cavity express the coefficients, and hence the solution, on the boundary of the cavity in terms of integrals over the outer boundary and the outer cavities in the region. Because each equation contains only one coefficient at its associated cavity, the system is well-conditioned.

The outline of the thesis is as follows:

In chapter one, we outline the basic theory of the Laplace equation, [7,10]. We present some properties of harmonic functions, and we give the existence and uniqueness of the solution of the Laplace equation theorems based on Perron's method and potential layers.

In chapter two we present the method of special boundary integral equations to approximate the solution of Laplace's equation in three-dimensional regions with spherical cavities.We formulate the basic boundary integral equations.

Here we treat Dirichlet problem and we consider a general configuration of spherical cavities in a region of arbitrary shape and specify firstly boundary potential on each cavity and secondly the sum of a constant and first order harmonic. In both cases the boundary flux is taken to be a constant which leads to the so-called zeroth-order solution, or a constant plus a first order harmonic which leads to the so-called first order solution.

Also in chapter two we apply the procedure to Neumann problem, and we give some examples to demonstrate applicability of the proposed method.

Finally, in chapter three we apply this method to three-dimensional regions with slender cavities with circular-cross sections.

Chapter One

Potential Theory

As we mentioned in the introduction, one of the most important partial differential equations occurring in applied mathematics is the Laplace equation.

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = divDu = 0, \qquad (a)$$

where u is a $C^2(\Omega)$ function and Ω is a domain in $\Re^n$. Furthermore, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is called the Laplacian of u, *div* denotes the divergence of u and Du denotes the gradiant of u. Any $C^2(\Omega)$ function u that satisfies Laplace equation is called harmonic. It is called subharmonic if $\Delta u \geq 0$, and superharmonic if $\Delta u \leq 0$. A basic argument used in solving Laplace equation is the divergence theorem.

Theorem [7] (Divergence theorem)

Let Ω be a bounded domain with C^1 boundary $\partial\Omega$ and let v denote the unit outward normal to $\partial\Omega$. For any vector field w in $C^1(\overline{\Omega})$ we then have

$$\int_{\Omega} divwdx = \int_{\partial\Omega} w.vds \qquad (b)$$

Where ds indicates the (n-1) dimensional area element in $\partial\Omega$. Notice that if u is a $C^2(\overline{\Omega})$ function and we take $w = Du$ in(b) then

$$\int_{\Omega} \Delta udx = \int_{\partial\Omega} Du.vds = \int_{\partial\Omega} \frac{\partial u}{\partial v} ds \qquad (c)$$

1.1 Some Properties of Harmonic Functions

In this section we review some properties of harmonic, subharmonic and superharmonic functions.

Theorem 1.1.1 [7] (The Mean Value Theorem)

Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0 (\geq 0, \leq 0)$ in Ω. Then for any ball (centered at $\boldsymbol{y}$ with radius R) $B = B_R(\boldsymbol{y}) \subset\subset \Omega$ we have

$$u(\boldsymbol{y}) = (\leq, \geq) \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u ds \tag{1.1.2}$$

$$u(\boldsymbol{y}) = (\leq, \geq) \frac{1}{\omega_n R^n} \int_{B} u d\boldsymbol{x} \tag{1.1.3}$$

where $\omega_n = \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ is the volume of unit ball in $\Re^n$.

Proof:

Let $\rho \in (0,R)$ and apply the identity (c) to the ball $B_\rho = B_\rho(\boldsymbol{y})$, we obtain

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} ds = \int_{B_\rho} \Delta u dx = 0 (\geq, \leq 0).$$

Introducing radial and angular coordinates $r = |\boldsymbol{x} - \boldsymbol{y}|$, $\omega = \dfrac{\boldsymbol{x} - \boldsymbol{y}}{r}$ and writing $u(\boldsymbol{x}) = u(\boldsymbol{y} + r\omega)$ we have

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} ds = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}(\boldsymbol{y} + \rho\omega) ds = \rho^{n-1} \int_{|\omega|=1} \frac{\partial u}{\partial r}(\boldsymbol{y} + \rho\omega) d\omega$$

$$= \rho^{n-1} \frac{\partial}{\partial r} \int_{|\omega|=1} u(\boldsymbol{y} + \rho\omega) d\omega = \rho^{n-1} \frac{\partial}{\partial \rho} [\rho^{1-n} \int_{\partial B_\rho} u ds] = 0 (\geq, \leq 0)$$

Consequently for any $\rho \in (0,R)$ $\rho^{n-1} \int_{\partial B_\rho} u ds = (\leq, \geq) R^{n-1} \int_{\partial B_\rho} u ds$

and since $\lim_{\rho \to 0} \rho^{n-1} \int_{\partial B_\rho} u ds = n\omega_n u(\boldsymbol{y})$ the relation (1.1.2) follows.

To get the relation (1.1.3) we write the relation (1.1.2) as

$$n\omega_n\rho^{n-1}u(\boldsymbol{y}) = (\leq,\geq)\int_{\partial B_\rho} u\,ds, \quad \rho \leq R.$$

Integrate with respect to ρ from 0 to R we get

$$n\omega_n \frac{\rho^n}{n}\bigg|_0^R u(\boldsymbol{y}) = \int_0^R \int_{\partial B_\rho} u\,ds\,d\rho. \text{ Or } \omega_n R^n u(\boldsymbol{y}) = \int_B u\,d\boldsymbol{x}. \text{ Hence } u(\boldsymbol{y}) = \frac{1}{\omega_n R^n}\int_B u\,d\boldsymbol{x}. \qquad \square$$

Theorem 1.1.4 [7] (Strong Maximum and Minimum Principle)

Let $\Delta u \geq 0 (\leq 0)$ in Ω and suppose there exist a point $y \in \Omega$ for which $u(y) = \sup_\Omega u \; (\inf_\Omega u)$. Then u is constant. Consequently a harmonic function cannot assume an interior maximum or minimum value unless its constant.

Proof:

let $\Delta u \geq 0$ in Ω, $M = \sup_\Omega u$ and define $\Omega_M = \{\boldsymbol{x} \in \Omega \mid u(x) = M\}$.

By assumption Ω_M is not empty.Since u is continuous Ω_M is closed relative to Ω.

Let z be any point in Ω_M and apply the mean value inequality to subharmonic function u-M in a ball $B = B_R(z) \subset\subset \Omega$. Therefore we obtain

$0 = u(z) - M \leq \frac{1}{\omega_n R^n}\int_B (u\text{-}M)dx \leq 0$ so that u=M in $B_R(z)$. Consequently Ω_M is also open relative to Ω. Hence Ω_M=Ω.

For the superharmonic case we replace u by -u, then the result follows. □

Corollary 1.1.5 [7]

Let D be a bounded domain and let u be a nonconstant harmonic in D and continuous in $\overline{D}$. Then the maximum and minimum of u attained on the boundary.

Theorem 1.1.6 [7] (Weak maximum and minimum principle)

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $\Delta u \geq 0 (\leq 0)$ in Ω. Then provided Ω is bounded

$\sup_{\Omega} u = \sup_{\partial\Omega} u$ $(\inf_{\Omega} u = \inf_{\partial\Omega} u)$. Consequently, for harmonic function u

$\inf_{\partial\Omega} u \leq u(\mathbf{x}) \leq \sup_{\partial\Omega} u$, $\mathbf{x} \in \Omega$.

Proof:

Since $\overline{\Omega}$ is compact then u has a suprimum and an infimum in $\overline{\Omega}$. Because $\Delta u \geq 0 (\leq 0)$ in Ω and u is not constant, then by theorem 1.1.4 the suprimum and the infimum must attained at boundary points. □

Theorem 1.1.7 [7] (Uniqueness theorem)

Let $u, \mathrm{v} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\Delta u = \Delta \mathrm{v}$ in Ω and $u = \mathrm{v}$ on $\partial\Omega$. Then $u = \mathrm{v}$.

Proof:

Let $w = u - \mathrm{v}$. Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$. So by the weak maximum and minimum principle $w = 0$ in Ω and hence $u = \mathrm{v}$ in Ω. □

Remark 1.1.8 [7]

If u is harmonic and v is superharmonic agreeing on the boundary $\partial\Omega$, then by the weak maximum and minimum principle $\mathrm{v} \geq u$ in Ω.

Proof:

Let $w = u - \mathrm{v}$ then $\Delta w \geq 0$ and by theorem 1.1.6 $\sup_{\Omega} w = \sup_{\partial\Omega} w = 0$. Hence $w \leq 0$ and therefore $\mathrm{v} \geq u$. □

Theorem 1.1.9 [7] (Harnack Inequality)

Let $u \geq 0$ be harmonic function in Ω. Then for any bounded subdomain $\Omega' \subset\subset \Omega$ there exist a constant C depending only on n, Ω', and Ω such that $\sup_{\Omega'} u \leq C \inf_{\Omega} u$.

Proof:

Let $\boldsymbol{y} \in \Omega,\ B_{4R}(\boldsymbol{y}) \subset \Omega$. Then for any points $\boldsymbol{x_1}, \boldsymbol{x_2} \in B_R(\boldsymbol{y})$ we have by (1.1.3)

$$u(\boldsymbol{x_1}) = \frac{1}{\omega_n R^n} \int_{B_R(\boldsymbol{x_1})} u d\boldsymbol{x} \leq \frac{1}{\omega_n R^n} \int_{B_{2R}(\boldsymbol{y})} u d\boldsymbol{x},\ u(\boldsymbol{x_2}) = \frac{1}{\omega_n (3R)^n} \int_{B_{3R}(\boldsymbol{x_2})} u d\boldsymbol{x} \geq \frac{1}{\omega_n R^n} \int_{B_{2R}(\boldsymbol{y})} u d\boldsymbol{x}.$$

Consequently we obtain $\sup_{B_R(\boldsymbol{y})} u \leq 3^n \inf_{B_R(\boldsymbol{y})} u$.

Let $\Omega' \subset\subset \Omega$, and choose $\boldsymbol{x_1}, \boldsymbol{x_2} \in \overline{\Omega'}$ so that $u(\boldsymbol{x_1}) = \sup_{\Omega} u,\ u(\boldsymbol{x_2}) = \inf_{\Omega} u$.

Let L be closed arc such that $\boldsymbol{x_1}, \boldsymbol{x_2} \in L \subset \overline{\Omega'}$ and choose R so that

$4R < dist(L, \partial\Omega)$. So L can be covered by a finite number N of balls of radius R.

We obtain $u(\boldsymbol{x_1}) \leq 3^{nN} u(\boldsymbol{x_2})$. Thus the result holds with $C = 3^{nN}$. □

1.2 Green's Theorems

Theorem 1.2.1 [7] (Green's First Identity)

Let u, v be $C^2(\overline{\Omega})$ functions, Ω be a domain in $\Re^n$ for which the divergence theorem holds. Then

$$\int_{\Omega} v \Delta u dx + \int_{\Omega} Du.Dv\ dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} ds \qquad (1.2.2)$$

Proof:

Set w = vDu in equation (b) then

$$\int_{\Omega} div(vDu)\ dx = \int_{\partial\Omega} vDu.\nu ds.\ \text{Hence}\ \int_{\Omega} v \Delta u dx + \int_{\Omega} Du.Dv\ dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} ds$$ □

Theorem 1.2.3 [7] (Green's Second Identity)

Let u, v and Ω be as given in theorem 1.2.1 then

$$\int_{\Omega}(\mathrm{v}\Delta u - u\Delta \mathrm{v})dx = \int_{\partial\Omega}\left(\mathrm{v}\frac{\partial u}{\partial \nu} - u\frac{\partial \mathrm{v}}{\partial \nu}\right)ds \qquad (1.2.4)$$

Proof:

Interchanging u and v in equation 1.2.2 we get

$$\int_{\Omega} u\Delta \mathrm{v}\, dx + \int_{\Omega} Du.D\mathrm{v}\, dx = \int_{\partial\Omega} u\frac{\partial \mathrm{v}}{\partial \nu} ds.$$

Substracting from equation (1.2.2) then we obtain the Green's second identity

$$\int_{\Omega}(\mathrm{v}\Delta u - u\Delta \mathrm{v})dx = \int_{\partial\Omega}\left(\mathrm{v}\frac{\partial u}{\partial \nu} - u\frac{\partial \mathrm{v}}{\partial \nu}\right)ds. \qquad (1.2.5)$$

□

Laplace's equation has the radially symmetric solution r^{2-n} for $n>2$ and $\log r$ for $n=2$, r being radial distance from a fixed point.

To proceed further from equation (1.2.5) we fix a point $\boldsymbol{y}$ in Ω and introduce the normalized fundamental solution of Laplace's equation:

$$\Gamma(\boldsymbol{x}-\boldsymbol{y}) = \Gamma(|\boldsymbol{x}-\boldsymbol{y}|) = \begin{cases} \dfrac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}|, & n=2, \\ \dfrac{1}{n(2-n)\omega_n}|\boldsymbol{x}-\boldsymbol{y}|^{2-n}, & n>2. \end{cases} \qquad (1.2.6)$$

Now

$$\frac{\partial}{\partial x_i}\Gamma(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{n\omega_n}(x_i - y_i)|\boldsymbol{x}-\boldsymbol{y}|^{-n}.$$

Therefore $D\Gamma(\boldsymbol{x}-\boldsymbol{y}) = \dfrac{1}{n\omega_n}\dfrac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^n}$. Further

$$\frac{\partial^2}{\partial x_i \partial x_j}\Gamma(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{n\omega_n}\left\{|\boldsymbol{x}-\boldsymbol{y}|^2\delta_{ij} - n(x_i-y_i)(x_j-y_j)\right\}|\boldsymbol{x}-\boldsymbol{y}|^{-n-2} \qquad (1.2.7)$$

Thus $\frac{\partial^2}{\partial x_i^2}\Gamma(\boldsymbol{x}-\boldsymbol{y})=\frac{1}{n\omega_n}\left\{|\boldsymbol{x}-\boldsymbol{y}|^2-n(x_i-y_i)^2\right\}|\boldsymbol{x}-\boldsymbol{y}|^{-n-2}$

$=\frac{1}{n\omega_n}\left\{|\boldsymbol{x}-\boldsymbol{y}|^{-n}-n(x_i-y_i)^2|\boldsymbol{x}-\boldsymbol{y}|^{-n-2}\right\}$ and

$$\Delta\Gamma=\frac{1}{n\omega_n}\left\{n|\boldsymbol{x}-\boldsymbol{y}|^{-n}-n|\boldsymbol{x}-\boldsymbol{y}|^{-n}\right\}=0 \text{ if } \boldsymbol{x}\neq\boldsymbol{y}.$$

Hence if $\boldsymbol{x}\neq\boldsymbol{y}$, Γ is harmonic.

Furthermore we have the following estimates:

$$\left|\frac{\partial}{\partial x_i}\Gamma(\boldsymbol{x}-\boldsymbol{y})\right|\leq\frac{1}{n\omega_n}|\boldsymbol{x}-\boldsymbol{y}|^{1-n},\ \left|\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(\boldsymbol{x}-\boldsymbol{y})\right|\leq\frac{1}{\omega_n}|\boldsymbol{x}-\boldsymbol{y}|^{-n},$$

$$\left|D^\beta\Gamma(\boldsymbol{x}-\boldsymbol{y})\right|\leq C|\boldsymbol{x}-\boldsymbol{y}|^{2-n-|\beta|},\ C=C(n,|\beta|)$$

The singularity at $\boldsymbol{x}=\boldsymbol{y}$ prevents us from using Γ in place of v in Green's second identity (1.2.4).

Replace Ω by $\Omega-\overline{B_\rho}$ where $B_\rho=B_\rho(\boldsymbol{y})$ for sufficiently small ρ we conclude from (1.2.4) that

$$\int_{\Omega-B_\rho}\Gamma\Delta u d\boldsymbol{x}=\int_{\partial\Omega}\left(\Gamma\frac{\partial u}{\partial\nu}-u\frac{\partial\Gamma}{\partial\nu}\right)ds+\int_{\partial B_\rho}\left(\Gamma\frac{\partial u}{\partial\nu}-u\frac{\partial\Gamma}{\partial\nu}\right)ds. \qquad (1.2.8)$$

now $\int_{\partial B_\rho}\Gamma\frac{\partial u}{\partial\nu}ds=\Gamma(\rho)\int_{\partial B_\rho}\frac{\partial u}{\partial\nu}ds\leq n\omega_n\rho^{n-1}\sup_{B_\rho}|Du|\to 0$ as $\rho\to 0$

and

$$\int_{\partial B_\rho}u\frac{\partial\Gamma}{\partial\nu}ds=-\Gamma'(\rho)\int_{\partial B_\rho}u ds=\frac{-1}{n\omega_n\rho^{n-1}}\int_{\partial B_\rho}u ds\to -u(y) \text{ as } \rho\to 0.$$

Hence letting ρ tend to zero in (1.2.8) we arrive Green's representation formula

$$u(\boldsymbol{y})=\int_{\partial B_\rho}\left(u\frac{\partial\Gamma}{\partial\nu}(\boldsymbol{x}-\boldsymbol{y})-\Gamma(\boldsymbol{x}-\boldsymbol{y})\frac{\partial u}{\partial\nu}\right)ds+\int_{\partial B_\rho}\Gamma(\boldsymbol{x}-\boldsymbol{y})\Delta u d\boldsymbol{x} \qquad (\boldsymbol{y}\in\Omega) \qquad (1.2.9)$$

If u has a compact support in $\Re^n$, then (1.2.9) yields the frequently useful representation formula

$$u(\boldsymbol{y}) = \int_{\partial B_\rho} \Gamma(\boldsymbol{x} - \boldsymbol{y})\Delta u dx. \tag{1.2.10}$$

For harmonic function u, we also obtain the representation

$$u(\boldsymbol{y}) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial v}(\boldsymbol{x} - \boldsymbol{y}) - \Gamma(\boldsymbol{x} - \boldsymbol{y}) \frac{\partial u}{\partial v} \right) ds, \quad (\boldsymbol{y} \in \Omega) \tag{1.2.11}$$

This formula is called Green's formula.

The integrand in equation (1.2.11) is infinitely differentiable and also analytic with respect to $\boldsymbol{y}$, it follows that u is also analytic in Ω. Thus harmonic functions are analytic throughout their domain of definition and therefore uniquely determined by their values in any open subset.

Now suppose that $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $\Delta h = 0$ in Ω. Then again by Green's second identity (1.2.4) we obtain

$$\int_{\Omega} h \Delta u dx = \int_{\partial\Omega} (h \frac{\partial u}{\partial v} - u \frac{\partial h}{\partial v}) ds. \tag{1.2.12}$$

Writing $G=\Gamma+h$ and adding (1.2.9) and (1.2.12) we obtain a more general version of Green's representation formula:

$$u(\boldsymbol{y}) = \int_{\Omega} G \Delta u dx + \int_{\partial\Omega} (u \frac{\partial G}{\partial v} - G \frac{\partial u}{\partial v}) ds \tag{1.2.13}$$

If $G=0$ on $\partial\Omega$ we have

$$u(\boldsymbol{y}) = \int_{\Omega} G \Delta u dx + \int_{\partial\Omega} u \frac{\partial G}{\partial v} ds \tag{1.2.14}$$

and the functions $G=G(\boldsymbol{x},\boldsymbol{y})$ is called the Green's function for the domain Ω, sometimes also called the Green's function of the first kind for Ω.

By uniqueness theorem the Green's function is unique and from the formula (1.2.14) its existence implies a representation for a $C^1(\overline{\Omega})\cap C^2(\Omega)$ harmonic function in terms of its boundary values.

When the domain Ω is a ball the Green’s function can be explicity determined by the method of images and leads to the well known Poisson integral representation for harmonic function in a ball.

Let $B_R = B_R(0)$ and for $\boldsymbol{x} \in B_R$, $\boldsymbol{x} \neq 0$, let

$$\overline{\boldsymbol{x}} = \frac{R^2}{|\boldsymbol{x}|^2}\boldsymbol{x} \tag{1.2.15}$$

denote its inverse point with respect to B_R, if $\boldsymbol{x} = 0$ we take $\overline{\boldsymbol{x}} = \infty$. Then the Green's function for B_R is given by

$$G(\boldsymbol{x}-\boldsymbol{y}) = \begin{cases} \Gamma(|\boldsymbol{x}-\boldsymbol{y}|) - \Gamma(\frac{|\boldsymbol{y}|}{R}|\boldsymbol{x}-\overline{\boldsymbol{y}}|), & \boldsymbol{y} \neq 0 \\ \Gamma(|\boldsymbol{x}|) - \Gamma(R), & \boldsymbol{y} = 0 \end{cases}$$

$$= \Gamma\left(\sqrt{|\boldsymbol{x}|^2 + |\boldsymbol{y}|^2 - 2\boldsymbol{x}.\boldsymbol{y}}\right) - \Gamma\left(\sqrt{\left(\frac{|\boldsymbol{x}||\boldsymbol{y}|}{R}\right)^2 + R^2 - 2\boldsymbol{x}.\boldsymbol{y}}\right)$$

$$\text{for all } \boldsymbol{x},\boldsymbol{y} \in B_R, \boldsymbol{x} \neq \boldsymbol{y}. \tag{1.2.16}$$

The function G defined by (1.2.16) has the properties

$G(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y},\boldsymbol{x})$

$G(\boldsymbol{x},\boldsymbol{y}) \leq 0$ for $\boldsymbol{x},\boldsymbol{y} \in \overline{B_R}$.

Moreover, direct calculation shows that at $\boldsymbol{x} \in \partial B_R$ the normal derivative of G is given by

$$\frac{\partial G}{\partial \nu} = \frac{\partial G}{\partial |\boldsymbol{x}|} = \frac{R^2 - |\boldsymbol{y}|^2}{n\omega_n R} |\boldsymbol{x} - \boldsymbol{y}|^{-n} \geq 0 \qquad (1.2.18)$$

Hence if $u \in C^1(\overline{B}_R) \cap C^2(B_R)$ is harmonic, we have by $(1.2.14)$ the Poisson integral formula

$$u(\boldsymbol{y}) = \frac{R^2 - |\boldsymbol{y}|^2}{n\omega_n R} \int_{\partial B_R} \frac{u ds}{|\boldsymbol{x} - \boldsymbol{y}|^n} \qquad (1.2.19)$$

Theorem 1.2.20 [7]

Let $B = B_R(0)$ and φ be a continuous function on $\partial B.$ Then the function u defined by

$$u(\boldsymbol{x}) = \begin{cases} \dfrac{R^2 - |\boldsymbol{x}|^2}{n\omega_n R} \displaystyle\int_{\partial \mathrm{B}} \frac{\varphi(\boldsymbol{y}) ds_y}{|\boldsymbol{x} - \boldsymbol{y}|^n}, & \text{for } \boldsymbol{x} \in B \\ \varphi(\boldsymbol{x}), & \text{for } \boldsymbol{x} \in \partial B \end{cases} \qquad (1.2.21)$$

belongs to $C^0(\overline{B}) \cap C^2(B)$ and satisfies $\Delta u = 0$ in B.

Proof:

u is harmonic in B is evident from the fact that G, and hence $\dfrac{\partial G}{\partial \nu}$ is harmonic in $\boldsymbol{x}$.

To establish the continuity of u on $\partial B,$ we use the Poison formula (1.2.19) for the special case u=1 to obtain the identity

$\int_{\partial B} K(\boldsymbol{x}, \boldsymbol{y}) ds_y = 1$ for all $\boldsymbol{x} \in B$, where K is the Poisson kernel

$$K(\boldsymbol{x}, \boldsymbol{y}) = \frac{R^2 - |\boldsymbol{x}|^2}{n\omega_n R |\boldsymbol{x} - \boldsymbol{y}|^n}, \qquad \boldsymbol{x} \in B, \boldsymbol{y} \in \partial B. \qquad (1.2.22)$$

Now let $\boldsymbol{x}_0 \in \partial B$ and ε be an arbitrary positive number. Choose $\delta > 0$ so that $|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}_0)| < \varepsilon$ if $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta$ and let $|\varphi| \leq M$ on $\partial B.$ Then if

$|\boldsymbol{x}-\boldsymbol{x}_0|<\dfrac{\delta}{2}$ we have by (1.2.21) and $\int_{\partial B} K(\boldsymbol{x},\boldsymbol{y})ds_y = 1$

$$|u(\boldsymbol{x})-u(\boldsymbol{x}_0)| = \left|\int_{\partial B} K(\boldsymbol{x},\boldsymbol{y})(\varphi(\boldsymbol{x})-\varphi(\boldsymbol{x}_0))ds_{\boldsymbol{y}}\right|$$

$$\leq \int_{|\boldsymbol{y}-\boldsymbol{x}_0|\leq\delta} K(\boldsymbol{x},\boldsymbol{y})|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{x}_0)|ds_{\boldsymbol{y}} + \int_{|\boldsymbol{y}-\boldsymbol{x}_0|>\delta} K(\boldsymbol{x},\boldsymbol{y})|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{x}_0)|ds_{\boldsymbol{y}}$$

$$\leq \varepsilon + \frac{2M(R^2-|\boldsymbol{x}|^2)R^{n-2}}{(\delta/2)^n}$$

If now $|\boldsymbol{x}-\boldsymbol{x}_0|$ is sufficiently small its clear that $|u(\boldsymbol{x})-u(\boldsymbol{x}_0)|<2\varepsilon$ and hence u is continuous at $\boldsymbol{x}_0$. Consequently $u \in C^0(\overline{B})$ as required. □

Now we consider some convergence theorems.

Theorem 1.2.23 [7]

A $C^0(\Omega)$ function u is harmonic if and only if for every ball $B = B_R(\boldsymbol{y}) \subset\subset \Omega$ it satisfies the mean value property, $u(\boldsymbol{y}) = \dfrac{1}{n\omega_n R^{n-1}}\displaystyle\int_{\partial B} u ds.$

Proof:

By theorem 1.2.20 there exist for any $B \subset\subset \Omega$ a harmonic function h such that $h=u$ on ∂B. The difference $w=u-h$ will then be a function satisfying the mean value property on any ball in Ω. Consequently the maximum principle and uniqueness results apply to w since the mean value inequalities were the only properties of harmonic functions used in their derivation. Hence $w=0$ in B and consequently u must be harmonic in Ω. □

Theorem 1.2.24 [7]

The limit of a uniformly convergent sequence of harmonic functions is harmonic.

Proof:

Let $\{u_k\}$ is a sequence of harmonic functions that converges uniformly to *u*. By theorem 1.2.23 $u_k(\boldsymbol{y}) = \dfrac{1}{n\omega_n R^{n\text{-}1}} \int\limits_{\partial B} u_k dx.$ Then as $k \to \infty$ since the convergence is uniformly we have $u(\boldsymbol{y}) = \dfrac{1}{n\omega_n R^{n\text{-}1}} \int\limits_{\partial B} u dx.$

Again by theorem 1.2.23 *u* is harmonic. □

Remark 1.2.25 [7]

If $\{u_n\}$ is a sequence of harmonic functions in a bounded domain Ω with continuous boundary values $\{\varphi_n\}$ which converge uniformly on Ω to a function φ. Then by the maximum principle the sequence $\{u_n\}$ converges uniformly to a harmonic function having the boundary values φ on $\partial\Omega$.

Theorem 1.2.26 [7]

Let $\{u_n\}$ be a monotone increasing sequence of harmonic functions in a domain Ω and suppose that the sequence $\{u_n(\boldsymbol{y})\}$ is bounded for some point $\boldsymbol{y} \in \Omega$. Then the sequence converges uniformly on any subdomain $\Omega' \subset\subset \Omega$ to a harmonic function.

Proof:

The sequence $\{u_n(\boldsymbol{y})\}$ will converge, which implies that for arbitrary $\varepsilon > 0$ there is a number N such that $0 \le u_m(\boldsymbol{y}) - u_n(\boldsymbol{y}) < \varepsilon$ for all $m \ge n > N$. But by Harnack's inequality we must have $\sup\limits_{\Omega'} |u_m(\boldsymbol{x}) - u_n(\boldsymbol{x})| < C\varepsilon$ for some constant C depending on Ω' and Ω.

Consequently $\{u_n\}$ converges uniformly and by theorem (1.2.24) the limit function is harmonic. □

1.3 The Method of Subharmonic Functions

We are now approach the question of existence of solutions of the classical Dirichlet problem in arbitrary bounded domains. The treatment here will be accomplished by Perron's method of subharmonic functions which relies heavily on the maximum principle and the solvability of the Dirichlet problem in balls. The method has a number of attractive features in that it is elementary, it separates the interior existence problem from that of the boundary behavior of solutions, and it is easily extended to more general classes of second order elliptic equations. We generalize the definition of subharmonic (superharmonic) functions as follows.

Definition 1.3.1 [7]

A $C^0(\Omega)$ function u will be called subharmonic (superharmonic) in Ω if $\forall B \subset\subset \Omega$ and for every function h harmonic in B satisfying $u \le (\ge) h$ on ∂B, we also have $u \le (\ge) h$ in B.

Following we list some properties of subharmonic functions.

Corresponding results for superharmonic function are obtained by replacing u by $-u$.

(1) If u is subharmonic in a domain Ω, it satisfies the strong maximum principle in Ω, and if v is superharmonic in a bounded domain Ω with $v \ge u$ on $\partial\Omega$ then either $v \ge u$ throughout Ω or $v \equiv u$.

Proof:

For the first statement, we follow the same proof in theorem 1.1.4. For the second statement, suppose the contrary, then at some point $\boldsymbol{x}_0 \in \Omega$ we have

$(u\text{-v})(\boldsymbol{x}_0) = \sup_{\Omega}(u\text{-v}) = M \geq 0$, assume $\exists B = B(\boldsymbol{x}_0)$ such that $u\text{-v} \not\equiv M$ on ∂B.

Letting $\tilde{u}, \tilde{\mathrm{v}}$ denote the harmonic functions such that $\tilde{u} = u, \tilde{\mathrm{v}} = \mathrm{v}$ on ∂B, one find that $M \geq \sup_{\partial B}(\tilde{u} - \tilde{\mathrm{v}}) \geq (\tilde{u} - \tilde{\mathrm{v}})(\boldsymbol{x}_0) \geq (u - \mathrm{v})(\boldsymbol{x}_0) = M$ and hence the equality holds through-out. By the strong maximum principle for harmonic functions it follows that $\tilde{u} - \tilde{\mathrm{v}} \equiv M$ in B, and hence u-v $\equiv M$ on ∂B, which contradicts the choice of B. □

Definition 1.3.2 [7]

Let u be a subharmonic in Ω and B be a ball strictly contained in Ω, let $\tilde{u}$ be the harmonic function in B satisfing $\tilde{u} = u$ on ∂B. Then the **harmonic lifting** of u (in B), denoted by $U(\boldsymbol{x})$ is defined as $U(\boldsymbol{x}) = \tilde{u}(\boldsymbol{x})$ if $\boldsymbol{x} \in B$ and $U(\boldsymbol{x}) = u(\boldsymbol{x})$ if $\boldsymbol{x} \in \Omega\text{-}B$.

(2) The harmonic lifting U is also subharmonic in Ω.

Proof:

Consider an arbitrary ball $B' \subset\subset \Omega$ and let h be harmonic function in B' satisfying $h \geq U$ on $\partial B'$. Since $u \leq U$ in B' we have $u \leq h$ in B' and hence $U \leq h$ in $B'\text{-}B$.
Also since U is harmonic in B we have by the maximum principle $U \leq h$ in $B \cap B'$.
Consequently $U \leq h$ in B' and U is subharmonic in Ω. □

(3) Let $u_1, u_2, \ldots, u_N$ be subharmonic in Ω. Then the function

$u(\boldsymbol{x}) = \max\{u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), \ldots, u_N(\boldsymbol{x})\}$ is also subharmonic in Ω.

Definition 1.3.3 [7]

Let Ω be bounded and φ be a bounded function on $\partial\Omega$. A $C^0(\overline{\Omega})$ subharmonic (superharmonic) function u is called a subfunction (superfunction) relative to φ if it satisfies $u \le \varphi\ (u \ge \varphi)$ on $\partial\Omega$.

Remark 1.3.4 [7]

By the maximum principle every subfunction is less than or equal to every superfunction. In particular, constant functions $\le \inf_{\partial\Omega} \varphi\ (\ge \sup_{\partial\Omega} \varphi)$ are subfunctions (superfunctions).

Let S_φ denote the set of subfunctions relative to φ. The basic result of Perron method is contained in the following theorem.

Theorem 1.3.5 [7]

The function $u(\boldsymbol{x}) = \sup_{v\in S_\varphi} v(\boldsymbol{x})$ is harmonic in Ω.

Proof:

By the maximum principle any function $v \in S_\varphi$ satisfies $v \le \sup \varphi$ so that u is well defined. Let $\boldsymbol{y}$ be an arbitrary fixed point of Ω. By the definition of u, there exists a sequence $\{v_n\} \subset S_\varphi$ such that $v_n(\boldsymbol{y}) \to u(\boldsymbol{y})$. By replacing v_n with $\max(v_n, \inf \varphi)$ we may assume that the sequence $\{v_n\}$ is bounded. Now choose R so that the ball $B = B_R(\boldsymbol{y}) \subset\subset \Omega$ and define V_n to be the harmonic lifting of v_n in B, then $V_n \in S_\varphi, V_n(\boldsymbol{y}) \to u(\boldsymbol{y})$ and the sequence $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ converging uniformly in any ball $B_\rho(\boldsymbol{y})$ with $\rho < R$ to a function v that is harmonic in B. Clearly v $\le u$ in B and $v(\boldsymbol{y}) = u(\boldsymbol{y})$. We claim now that in fact v$=u$ in B. For suppose v(z)< u(z) at some $\boldsymbol{z} \in B$. Then there exists a function $\bar{u} \in S_{\varphi}$ such that $v(\boldsymbol{z}) < \bar{u}(\boldsymbol{z})$. Defining $w_k = \max(\bar{u}, V_{nk})$ and let W_k the harmonic lifting, we obtain before a subsequence of the sequence $\{W_k\}$ converging to a harmonic function w satisfying $v \le w \le u$ in B and $v(\boldsymbol{y}) = w(\boldsymbol{y}) = u(\boldsymbol{y})$. But then by the maximum principle we must have v=w in B. This contradicts the definition of $\bar{u}$ and hence u is harmonic in Ω. □

Definition 1.3.6 [7]

Let ξ be a point of $\partial\Omega$. Then a $C^2(\bar{\Omega})$ function $\omega = \omega_{\xi}$ is called a **barrier** at ξ relative to Ω if ω is superharmonic in Ω, $\omega > 0$ in $\bar{\Omega} - \{\xi\}$ and $\omega(\xi) = 0$. A boundary point is **regular** if there exists a barrier at that point.

Lemma1.3.7 [7]

Let u be the harmonic function defined in Ω by Perron method. If $\boldsymbol{\xi}$ is regular boundary point of Ω and φ is continuous at $\boldsymbol{\xi}$ then $u(\boldsymbol{x}) \to \varphi(\boldsymbol{\xi})$.

Proof:

Choose $\varepsilon > 0$ and let $M = \sup_{\Omega}|\varphi|$. Since $\boldsymbol{\xi}$ is a regular boundary point then there is a barrier w at $\boldsymbol{\xi}$ and by virtue of the continuity of φ there are constants δ and k such that

$|\varphi(\boldsymbol{x})\text{-}\varphi(\boldsymbol{\xi})| < \varepsilon$ if $|\boldsymbol{x}\text{-}\boldsymbol{\xi}| < \delta$ and $kw(\boldsymbol{x}) \ge 2M$ if $|\boldsymbol{x}\text{-}\boldsymbol{\xi}| \ge \delta$.

The function $\varphi(\xi) + \varepsilon + kw$ is a superfunction and $\varphi(\boldsymbol{\xi})\text{-}\varepsilon - kw$ is a subfunction relative to φ. Hence from the definition of u and the fact that every superfunction dominates every subfunction, we have in Ω $\varphi(\xi)\text{-}\varepsilon - k\,w(\boldsymbol{x}) \le u(\boldsymbol{x}) \le \varphi(\xi) + \varepsilon + k\,w(\boldsymbol{x})$ or

$|u(\boldsymbol{x}) - \varphi(\xi)| \le \varepsilon + k\,w(\boldsymbol{x})$.

Since $w(\boldsymbol{x}) \to 0$ as $\boldsymbol{x} \to \xi$, we obtain $u(\boldsymbol{x}) \to \varphi(\xi)$ as $\boldsymbol{x} \to \xi$. □

Theorem 1.3.8 [7]

The classical Dirichlet problem in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.

Proof:

If the boundary values φ are continious and the boundary $\partial\Omega$ consists of regular points, the preceding lemma states that the harmonic function provided by Perron's method solves the Dirichlet problem. Conversely, suppose that the Dirichlet problem is solvable for all continuous boundary values. Let $\boldsymbol{\xi} \in \partial\Omega$. Then the function $\varphi(\boldsymbol{x}) = |\boldsymbol{x}\text{-}\boldsymbol{\xi}|$ is continuous on $\partial\Omega$ and the harmonic function solving the Dirichlet problem in Ω with boundary values φ is obviously a barrier at $\boldsymbol{\xi}$.

Hence $\boldsymbol{\xi}$ is regular, as are all points of $\partial\Omega$. □

1.4 Potential Layers

In this section we will study briefly some of the basic boundary value problems, namely Dirichlet and Neumann problems, from the integral equations point of view. We start by defining such problems. We take the space dimension n to be 2 or 3.

Interior Dirichlet problem: Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ that satisfies $\Delta u = 0$ in Ω, and u=f on $\partial\Omega$, where f is a given continuous function.

Interior Neumann problem: Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfies $\Delta u = 0$ in Ω, and

$$\frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega,$$ where g is a given continuous function.

Exterior Dirichlet problem: Find $u \in C^2(\Re^n \setminus \overline{\Omega}) \cap C^0(\overline{\Re^n \setminus \overline{\Omega}})$ and satisfies $\Delta u = 0$ in

$\Re^n \setminus \overline{\Omega}$, and $u=f$ on $\partial\Omega$, where f is a given continuous function. For $|\boldsymbol{x}| \to \infty$ it is required that $u(\boldsymbol{x})=O(1)$, if n=2 and $u(\boldsymbol{x})= o(1)$, if n=3.

Exterior Neumann problem: Find $u \in C^2(\Re^n \setminus \overline{\Omega}) \cap C(\overline{\Re^n \setminus \overline{\Omega}})$ and satisfies

$\Delta u = 0$ in $\Re^n \setminus \overline{\Omega}$, and $\dfrac{\partial u}{\partial \nu} = g$ on $\partial\Omega$, where g is a given continuous function.

For $|\boldsymbol{x}| \to \infty$ it is required that $u(\boldsymbol{x})= o(1)$.

Theorem 1.4.1 [10]

Both the interior and the exterior Dirichlet problem have at most one solution.

Proof:

Suppose there exist two solutions to the interior Dirichlet problem u_1, u_2. So the difference $u = u_1 - u_2$ is a harmonic function continuous up to the boundary satisfy the homogeneous boundary condition $u = 0$ on $\partial\Omega$. Then from the strong maximum and minimum principle theorem we obtain $u = 0$ in Ω for the interior problem and $u = 0$ in $\Re^n \setminus \overline{\Omega}$ for the exterior problem, which implise that $u_1 = u_2$. □

Theorem 1.4.2 [10]

Two solutions of the interior Neumann problem can be differ only by a constant. The exterior Neuman problem have at most one solution.

Proof:

Let u_1, u_2 be two solutions of the interior Neumann problem and let $u = u_1 - u_2$. Then u is a harmonic function continuous up to the boundary satisfying the homogeneous boundary conditioned $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

For the interior problem, suppose u is not constant in Ω. Then there exists some closed ball B contained in Ω such that $I = \int_B |Du|^2 dx > 0$.

From the first Green's theorem applied to the interior of Ω^* of some parallel surface $\partial\Omega^* = \{x - h\nu(x) : x \in \partial\Omega\}$ with sufficiently small $h>0$ we derive

$I \le \int_{\Omega^*} |Du|^2 dx = \int_{\partial\Omega^*} u \frac{\partial u}{\partial \nu} ds.$ Letting $h \to 0$ we get $\int_{\partial\Omega^*} u \frac{\partial u}{\partial \nu} ds \to \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} ds$

so $I \le 0$, a contradiction. For the exterior problem, we observe that

$u(\boldsymbol{x}) = O\left(\frac{1}{|\boldsymbol{x}|^{n-1}}\right),\ Du(\boldsymbol{x}) = O\left(\frac{1}{|\boldsymbol{x}|^{n}}\right), |\boldsymbol{x}| \to \infty$, uniformly for all directions. Assume that $Du \ne 0$

in $\Re^n \backslash \overline{\Omega}$. Then there exist some closed ball B contained in $\Re^n \backslash \overline{\Omega}$ such that $I = \int_B |Du|^2 dx > 0$.

Appling first Green's theorem to the domain Ω^* between some parallel surface $\partial\Omega^* = \{\boldsymbol{x} + h\nu(\boldsymbol{x}) : \boldsymbol{x} \in \partial\Omega\}$ with sufficiently small $h>0$ and some sufficiently large sphere Ω_R with radius R we get $I \le \int_{\Omega^*} |Du|^2 dx = \int_{\Omega_R} u \frac{\partial u}{\partial \nu} ds - \int_{\partial\Omega^*} u \frac{\partial u}{\partial \nu} ds.$ Letting $R \to \infty$, and $h \to 0$, we arrive at the contradiction $I \le 0$. Therefore $u =$ constant in $\Re^n \backslash \overline{\Omega}$ and this constant must be zero since $u(\infty) = 0$. □

Definition 1.4.3 [10]

The functions
$$u(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y}) \Phi(\boldsymbol{x},\boldsymbol{y}) ds(\boldsymbol{y}), \qquad \boldsymbol{x} \in \Re^n \backslash \partial\Omega \tag{1.4.4}$$

and
$$v(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})} ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \Re^n \backslash \partial\Omega \tag{1.4.5}$$

where φ is a function belongs to $C(\partial\Omega)$ are called single-layer and double-layer potentials with density φ, respectively. Here Φ is the fundamental solution given by

$$\Phi(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \dfrac{1}{2\pi}\log\dfrac{1}{|\boldsymbol{x}-\boldsymbol{y}|}, & n=2 \\ \dfrac{1}{4\pi}\dfrac{1}{|\boldsymbol{x}-\boldsymbol{y}|}, & n=3. \end{cases}$$

Theorem 1.4.6 [10]

let $\varphi \in C(\partial\Omega)$ where $\partial\Omega$ is a class of C^2. Then the single-layer potential u with density φ is continuous throughout $\Re^n$. On $\partial\Omega$ there holds

$$u(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y})\Phi(\boldsymbol{x},\boldsymbol{y})ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \partial\Omega \tag{1.4.7}$$

and $$\frac{\partial u_\pm}{\partial \nu}(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y})\frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{x})}ds(\boldsymbol{y}) \mp \frac{1}{2}\varphi(x), \quad \boldsymbol{x} \in \partial\Omega \tag{1.4.8}$$

where $\dfrac{\partial u_\pm}{\partial \nu}(\boldsymbol{x}) = \lim\limits_{h\to 0}\left(\nu(\boldsymbol{x}).\ Du(\boldsymbol{x} \pm \frac{1}{2}h\nu(\boldsymbol{x}))\right),$
is to be understood in the sense of uniform convergence on $\partial\Omega$ and where the integrals exist as improper integrals. The double-layer potential v with density φ can be continuously extended from $\Re^n \backslash \overline{\Omega}$ to $\overline{\Re^n \backslash \overline{\Omega}}$ and from Ω to $\overline{\Omega}$ with limiting values

$$\mathrm{v}_\pm(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y})\frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})}ds(\boldsymbol{y}) \pm \frac{1}{2}\varphi(x), \quad \boldsymbol{x} \in \partial\Omega \tag{1.4.9}$$

where $\mathrm{v}_\pm(\boldsymbol{x}) = \lim\limits_{h\to 0} \mathrm{v}(\boldsymbol{x} \pm h\nu(\boldsymbol{x}))$ and where the integrals exist as improper integrals.

Furthermore

$$\lim_{h\to 0^+}\left\{ \frac{\partial \mathrm{v}}{\partial \nu}(\boldsymbol{x} + h\nu(\boldsymbol{x})) - \frac{\partial \mathrm{v}}{\partial \nu}(\boldsymbol{x} - h\nu(\boldsymbol{x})) \right\} = 0, \qquad \boldsymbol{x} \in \partial\Omega \tag{1.4.10}$$

uniformly on $\partial\Omega$.

Proof:

See[6].

Theorem. 1.4.11 [10]

Let $\partial\Omega$ be of class C^2. Then there exists a positive constant L such that

$|\nu(\boldsymbol{x}).(\boldsymbol{x}-\boldsymbol{y})| \leq L|(\boldsymbol{x}-\boldsymbol{y})|^2$ for all $\boldsymbol{x},\boldsymbol{y} \in \partial\Omega$

Proof:

See [6].

Theorem1.4.12 [10]

The double-layer potential $u(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})} ds(\boldsymbol{y}),\ \boldsymbol{x} \in \Omega,$ with continuous density φ is a solution of the interior Dirichlet problem provided φ is a solution of the integral equation

$$\varphi(\boldsymbol{x}) - 2\int_{\partial\Omega} \varphi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})} ds(\boldsymbol{y}) = -2f(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{1.4.13}$$

Proof:

From theorem 1.4.6 $u_-(\boldsymbol{x}) = \int_{\partial\Omega} \varphi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})} ds(\boldsymbol{y}) - \frac{1}{2}\varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega,$ which implies

$2u_-(\boldsymbol{x}) = 2\int_{\partial\Omega} \varphi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})} ds(\boldsymbol{y}) - \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega.$ Now for $\boldsymbol{x} \in \partial\Omega$ $u_-(\boldsymbol{x}) = f(\boldsymbol{x}),$ so equation 1.4.13 holds. □

Theorem 1.4.14 [10]

The single-layer potential $u(\boldsymbol{x}) = \int_{\partial\Omega} \psi(\boldsymbol{y}) \Phi(\boldsymbol{x},\boldsymbol{y}) ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega,$

with a continuous density ψ is a solution of the interior Neumann problem provided ψ is a solution of the integral equation

$$\psi(\boldsymbol{x}) + 2\int_{\partial\Omega} \psi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{x})} ds(\boldsymbol{y}) = 2g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega \tag{1.4.15}$$

Proof:

From theorem 1.4.6 $\frac{\partial u_-}{\partial \nu}(\boldsymbol{x}) = \int_{\partial\Omega} \psi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{x})} ds(\boldsymbol{y}) + \frac{1}{2}\psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega.$ Which implies

$2\frac{\partial u_-}{\partial \nu}(\boldsymbol{x}) = 2\int_{\partial\Omega} \psi(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{x})} ds(\boldsymbol{y}) + \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega.$ Now for $\boldsymbol{x} \in \partial\Omega,\ \frac{\partial u_-}{\partial \nu} = g(\boldsymbol{x})$ so

equation 1.4.15 holds. □

Theorem 1.4.16 [10]

The interior Neumann problem is solvable if and only if $\int_{\partial\Omega} g\,ds$ is equal to zero.

Proof:

Let $\partial\Omega^* = \{x - h\nu(x) : x \in \partial\Omega\}$ we apply Green's theorem to the solution u to get

$$\int_{\partial\Omega} g\,ds = \lim_{h\to 0} \int_{\partial\Omega^*} g\,ds = \lim_{h\to 0} \int_{\partial\Omega^*} \frac{\partial u}{\partial \nu} ds = 0. \qquad \square$$

Theorem 1.4.17 [10]

The solution to the Dirichlet and Neumann problem depend continuously in the maximum norm on the given data.

Proof:

See[10].

Chapter Two

Special Boundary Integral Equations for Approximate Solution of Potential Problems in Three–Dimensional Regions with spherical Cavities

In this chapter we proposed a boundary integral method for solving potentials problems in three – dimensional region with spherical cavities. Boundary quantities were expanded in spherical harmonics on the surface cavity and special boundary integral equations were introduced to determine the unknown coefficients. The outer boundary was treated in a conventional manner and in principle, all integration on the cavities are done explicitly. The theory in this chapter based on [17] .

2.1 Integral Equations [17]

Consider a three-dimensional open region Ω containing N spheres centered at the points $\boldsymbol{\xi}^{i}$, i=1,2,...,N. Let a_i be the radius of sphere i, S_i be the lateral boundary of sphere i, $\partial\Omega$ be the outer boundary of Ω. Further let φ be a harmonic functionin Ω, i.e.

$$\Delta\varphi(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega. \tag{2.1.1}$$

As we done in chapter one, using the fundamental solution $\Gamma(\boldsymbol{x},\boldsymbol{y})=\dfrac{-1}{4\pi|\boldsymbol{x}\text{-}\boldsymbol{y}|}$ $\quad \boldsymbol{x},\boldsymbol{y} \in \Omega$,

and the Green's second identity we obtain the integral representation for the potential function φ as

$$\lambda\,\varphi(\boldsymbol{y}) = \int_{\partial\Omega}(\varphi\frac{\partial\Gamma}{\partial\nu} - \Gamma\frac{\partial\varphi}{\partial\nu})ds + \sum_{i=1}^{n}\int_{S_i}(\varphi^{i}\ \frac{\partial\Gamma}{\partial\nu} - q^{i}\,\Gamma)\ ds \tag{2.1.2}$$

where $\lambda = \begin{cases} 1, & \text{if } \boldsymbol{y} \in \Omega, \\ 0, & \text{if } \boldsymbol{y} \in R^3 \setminus \overline{\Omega} \end{cases}$

and φ^{i} and q^i are values of φ and its outwardnormalderevative on thesphere i. We represent φ^{i} and q^i by thefinite sum of sphericalharmonics as

$$\varphi^{\mathrm{i}} = \sum_{n=0}^{M} \sum_{m=-n}^{n} \varphi^i_{nm} Y_n^m(\theta^i, \phi^i) \tag{2.1.3}$$

$$q^{\mathrm{i}} = \sum_{n=0}^{M} \sum_{m=-n}^{n} q^i_{nm} Y_n^m(\theta^i, \phi^i) \tag{2.1.4}$$

where φ^i_{nm}, q^i_{nm} are constants, $i = 1,2,\ldots,N$ and θ^i, ϕ^i angle centeredat $\boldsymbol{\xi^i}$ and measured relative to the positive $\boldsymbol{z}$ and $\boldsymbol{x}$ axis.

We evaluate the integral expression

$$\int_{S_i} (\varphi^i \frac{\partial \Gamma}{\partial \nu} - q^i \Gamma)\, ds.$$

let $\boldsymbol{y} \in \Omega$ and $\boldsymbol{x} \in S_i$ then

$\boldsymbol{x} - \boldsymbol{\xi^i} = (a_i \sin\theta^i \cos\phi^i, a_i \sin\theta^i \sin\phi^i, a_i \cos\theta^i)$ and

$\boldsymbol{y} - \boldsymbol{\xi^i} = (r_i \sin\psi^i \cos\gamma^i, r_i \sin\psi^i \sin\gamma^i, r_i \cos\psi^i)$ therefore

$\boldsymbol{x}\text{-}\boldsymbol{y} =$

$(a_i \sin\theta^i \cos\phi^i - r_i \sin\psi^i \cos\gamma^i, a_i \sin\theta^i \sin\phi^i - r_i \sin\psi^i \sin\gamma^i, a_i \cos\theta^i - r_i \cos\psi^i)$

and

$$|\boldsymbol{x}\text{-}\boldsymbol{y}| = \sqrt{a_i^2 + r_i^2 - 2a_i r_i \cos\bar{\gamma}^i}$$

with $\cos\bar{\gamma}^i = \cos\psi^i \cos\theta^i + \sin\psi^i \sin\theta^i \cos(\gamma^i - \phi^i)$. Hence

$$\Gamma = \frac{-1}{4\pi\sqrt{a_i^2 + r_i^2 - 2a_i r_i \cos\bar{\gamma}^i}}$$ and thus

$$\frac{\partial\Gamma}{\partial\nu}=\nabla_x\Gamma(\boldsymbol{x},\boldsymbol{y}).\nu=\frac{-1}{4\pi}\frac{\boldsymbol{x}\text{-}\boldsymbol{y}}{|\boldsymbol{x}\text{-}\boldsymbol{y}|^3}\left(\frac{\boldsymbol{x}\text{ - }\boldsymbol{\xi^i}}{a_i}\right)=\frac{-(a_i-r_i\cos\bar{\gamma}^i)}{4\pi(a_i^2+r_i^2-2a_ir_i\cos\bar{\gamma}^i)^{3/2}}$$

Moreover using spherical harmonics Γ can be expanded as [15],

$$\Gamma=\frac{-1}{4\pi R}=\frac{-1}{r_i}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\left(\frac{a_i}{r_i}\right)^{n'}\frac{1}{(2n'+1)N_{m',n'}}Y_{n'}^{m'}(\psi^i,\gamma^i)Y_{n'}^{*m'}(\theta^i,\phi^i)$$

Where $N_{m',n'}=\dfrac{4\pi}{2n'+1}\dfrac{(n'+|m'|)!}{(n'-|m'|)!}$ and $R=|\boldsymbol{x}\text{-}\boldsymbol{y}|$

Consider the points $\boldsymbol{x}\text{ - }\boldsymbol{\xi^i} = (t_i\sin\theta^i\cos\phi^i, t_i\sin\theta^i\sin\phi^i, t_i\cos\theta^i)$ that lies on the parallel sphere S_i^+ with center $\boldsymbol{\xi^i}$ and radius t_i where $t_i>a_i$.

Then as $t_i\to a_i$ the integral $\int\limits_{S_i}\varphi^i\dfrac{\partial\Gamma}{\partial\nu}ds$ is equal to $\dfrac{-\partial}{\partial t_i}\int\limits_{S_i^+}\varphi^i\Gamma ds.$

Substitute instead of φ^i from $(2.1.3)$ we get

$$\int\limits_{S_i^+}\varphi^i\Gamma ds=\sum_{n=1}^{M}\sum_{m=-n}^{n}\varphi_{nm}^i\int\limits_{S_i^+}Y_n^m(\theta^i,\phi^i)\Gamma ds=t_i^2\sum_{n=1}^{M}\sum_{m=-n}^{n}\varphi_{nm}^i\int_0^{2\pi}\int_0^{\pi}Y_n^m(\theta^i,\phi^i)\frac{-1}{4\pi R}\sin\theta^i d\theta^i d\phi^i$$

$$=\frac{-t_i^2}{r_i}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\sum_{n=1}^{M}\sum_{m=-n}^{n}Y_{n'}^{m'}(\psi^i,\gamma^i)\varphi_{nm}^i\left(\frac{t_i}{r_i}\right)^{n'}\frac{1}{(2n'+1)N_{m',n'}}\int_0^{2\pi}\int_0^{\pi}Y_n^m(\theta^i,\phi^i)Y_{n'}^{m'}(\theta^i,\phi^i).$$

$$.\sin\theta^i d\theta^i d\phi^i$$

$$=\frac{-t_i^2}{r_i}\sum_{n=1}^{M}\sum_{m=-n}^{n}\varphi_{nm}^iY_n^m(\psi^i,\gamma^i)\frac{1}{(2n+1)}\left(\frac{t_i}{r_i}\right)^n.$$

Or we can write $\int\limits_{S_i^+}\varphi^i\Gamma ds=-\sum\limits_{n=1}^{M}\sum\limits_{m=-n}^{n}\varphi_{nm}^iY_n^m(\psi^i,\gamma^i)\dfrac{t_i}{(2n+1)}\left(\dfrac{t_i}{r_i}\right)^{n+1}.$

$$=\frac{-t_i^2}{r_i}\sum_{n=1}^{M}\sum_{m=-n}^{n}\varphi_{nm}^iY_n^m(\psi^i,\gamma^i)\frac{1}{(2n+1)}\left(\frac{t_i}{r_i}\right)^n.$$

Therefore $$\int\limits_{S_i}\varphi^i\frac{\partial\Gamma}{\partial\nu}ds=\sum_{n=1}^{M}\sum_{m=-n}^{n}\varphi_{nm}^iY_n^m(\psi^i,\gamma^i)\frac{(n+2)}{(2n+1)}\left(\frac{a_i}{r_i}\right)^{n+1}. \qquad (2.1.5)$$

Similarly we substitute from $(2.1.4)$ instead of q^i to obtain

$$-\int_{S_i} q^{\mathrm{i}}\Gamma ds = -\int_{S_i}\sum_{n=0}^{M}\sum_{m=-n}^{n} q_{nm}^{i} Y_n^m(\theta^i,\phi^i)\Gamma ds = -\sum_{n=0}^{M}\sum_{m=-n}^{n} q_{nm}^{i}\int_{S_i} Y_n^m(\theta^i,\phi^i)\frac{-1}{4\pi R}ds$$

$$=\sum_{n=0}^{M}\sum_{m=-n}^{n} q_{nm}^{i}\int_{S_i} Y_n^m(\theta^i,\phi^i)\frac{1}{r_i}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\left(\frac{a_i}{r_i}\right)^{n'}\frac{1}{(2n'+1)N_{m',n'}}Y_{n'}^{m'}(\psi^i,\gamma^i)Y_{n'}^{*m'}(\theta^i,\phi^i)ds$$

$$=\frac{a_i^{\ 2}}{r_i}\sum_{n=0}^{M}\sum_{m=-n}^{n}\ \sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'}\left(\frac{a_i}{r_i}\right)^{n'}\frac{q_{nm}^{i}}{(2n'+1)N_{m',n'}}Y_{n'}^{m'}(\psi^i,\gamma^i)\int_0^{2\pi}\int_0^{\pi} Y_{n'}^{*m'}(\theta^i,\phi^i)Y_n^m(\theta^i,\phi^i).$$

$$.\sin\theta^i d\theta^i d\phi^i$$

$$=\frac{a_i^{\ 2}}{r_i}\sum_{n=0}^{M}\sum_{m=-n}^{n}\left(\frac{a_i}{r_i}\right)^{n}\frac{q_{nm}^{i}}{(2n+1)}Y_n^m(\psi^i,\gamma^i)$$

Hence $-\int_{S_i} q^{\mathrm{i}}\Gamma dS = \frac{a_i^{\ 2}}{r_i}\sum_{n=0}^{M}\sum_{m=-n}^{n}\left(\frac{a_i}{r_i}\right)^{n}\frac{q_{nm}^{i}}{(2n+1)}Y_n^m(\psi^i,\gamma^i)$

Therefore for $\boldsymbol{y}\in\Omega$ equation $(2.1.2)$ becomes

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma}{\partial\ \nu} - \Gamma\frac{\partial\,\varphi}{\partial\ \nu})da + \sum_{\mathrm{i}=1}^{\mathrm{N}}\left[\sum_{\mathrm{n}=1}^{\mathrm{M}}\sum_{\mathrm{m}=\mathrm{-n}}^{\mathrm{n}}(n+2)\varphi_{nm}^{i}\frac{a_i^{n+1}}{(2n+1)r_i^{n+1}}Y_n^m(\psi^i,\gamma^i)\right.$$

$$\left.+\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q_{nm}^{i}}{(2n+1)}a_i\left(\frac{a_i}{r_i}\right)^{n+1}Y_n^m(\psi^i,\gamma^i)\right] \qquad (2.1.6)$$

2.2 Dirichlet Problem

Next we consider the Dirichlet problem, [17], find φ such that

$$\Delta\,\varphi(\boldsymbol{x}) = 0 \qquad ,\ \boldsymbol{x}\in\Omega,$$

$$\varphi^i(\boldsymbol{x}) = \ \varphi_{00}^{\mathrm{i}} \quad ,\ \boldsymbol{x}\in S_i.$$

Where a constant potential is considered on each surface of spheres. In this case equation (2.1.6) reduced to

$\varphi(\boldsymbol{y}) =$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu}) da + \sum_{i=1}^{N} \left[\sum_{n=0}^{M} \sum_{m=-n}^{n} \frac{q_{nm}^{i}}{(2n+1)} a_i \left(\frac{a_i}{r_i} \right)^{n+1} Y_n^m(\psi^i, \gamma^i) \right]. \qquad (2.2.1)$$

Notice that this equation contains $N(M+1)^2$ unknowns q_{nm}^i. To determine these unknowns consider the following sequence of kernel functions

$$\Gamma_0^s(\boldsymbol{x}) = \frac{-1}{4\pi\left|\boldsymbol{x} - \boldsymbol{\xi}^s\right|}, \qquad \Gamma_{nm}^s(\boldsymbol{x}) = \frac{-Y_n^{*m}(\theta^s, \phi^s)}{\left|x - \xi^s\right|^n} \qquad (2.2.2)$$

$s = 1,2,\ldots,N \quad n = 1,2,\ldots,M \quad m = -n,\ldots,n$

Theorem 2.2.3 [17]

If $\boldsymbol{x} \neq \boldsymbol{\xi}^s$ then $\Delta\Gamma_{nm}^s(\boldsymbol{x}) = 0$. Thus if $\boldsymbol{x} \neq \boldsymbol{\xi}^s$ then Γ_{nm}^s is a harmonic function.

Proof:

Let $\mathrm{r} = \left|\boldsymbol{x} - \boldsymbol{\xi}^s\right|$ and consider the Laplace equation in spherical coordinates.

$$\Delta\Gamma_{nm}^s(x) = \frac{1}{r^2 \sin\theta^s} \left[\frac{\partial}{\partial r}(r^2 \sin\theta^s \frac{\partial \Gamma_{nm}^s}{\partial r}) + \frac{\partial}{\partial \theta^s}(\sin\theta^s \frac{\partial \Gamma_{nm}^s}{\partial \theta^s}) + \frac{\partial}{\partial \phi^s}(\frac{1}{\sin\theta^s} \frac{\partial \Gamma_{nm}^s}{\partial \phi^s}) \right] \qquad (2.2.4)$$

Now $\Gamma_{nm}^s = \dfrac{-Y_n^{*m}(\theta^s, \phi^s)}{r^n} = \dfrac{-e^{-im\phi^s} \mathrm{P}_n^{|m|}(\cos\theta^s)}{r^n}$, thus

$\dfrac{\partial \Gamma_{nm}^s}{\partial r} = \dfrac{n r^{n-1} e^{-im\phi^s} \mathrm{P}_n^{|m|}(\cos\theta^s)}{r^{2n}} = \dfrac{n e^{-im\phi^s} \mathrm{P}_n^{|m|}(\cos\theta^s)}{r^{n+1}}$ and

$$r^2 \sin\theta^s \frac{\partial \Gamma_{nm}^s}{\partial r} = \frac{n e^{-im\phi^s} \sin\theta^s \mathrm{P}_n^{|m|}(\cos\theta^s)}{r^{n-1}}$$

Therefore $\dfrac{\partial}{\partial r}(r^2 \sin\theta^s \dfrac{\partial \Gamma_{nm}^s}{\partial r}) = \dfrac{-n(n-1) r^{n-2} e^{-im\phi^s} \sin\theta^s \mathrm{P}_n^{|m|}(\cos\theta^s)}{r^{2(n-1)}}$

$$= \frac{-n(n-1) e^{-im\phi^s} \sin\theta^s \mathrm{P}_n^{|m|}(\cos\theta^s)}{r^n}. \qquad (2.2.5)$$

Further $\dfrac{\partial \Gamma_{nm}^{s}}{\partial \theta^{s}} = \dfrac{-e^{-im\phi^{s}} \dfrac{d}{d\theta^{s}}\left[\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right]}{r^{n}}$. Thus

$$\sin\theta^{s} \frac{\partial \Gamma_{nm}^{s}}{\partial \theta^{s}} = \frac{-e^{-im\phi^{s}} \sin\theta^{s} \dfrac{d}{d\theta^{s}}\left[\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right]}{r^{n}} \quad \text{and hence}$$

$$\frac{\partial}{\partial\theta^{s}}(\sin\theta^{s} \frac{\partial \Gamma_{nm}^{s}}{\partial \theta^{s}})$$

$$= \frac{-e^{-im\phi^{s}} \sin\theta^{s} \dfrac{d^{2}}{d\theta^{s2}}\left[\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right]}{r^{n}} - \frac{e^{-im\phi^{s}} \cos\theta^{s} \dfrac{d}{d\theta^{s}}\left[\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right]}{r^{n}} \tag{2.2.6}$$

Finally $\dfrac{\partial \Gamma_{nm}^{s}}{\partial \phi^{s}} = \dfrac{ime^{-im\phi^{s}} \mathrm{P}_{n}^{|m|}(\cos\theta^{s})}{r^{n}}$ and therefore

$$\frac{1}{\sin\theta^{s}} \frac{\partial \Gamma_{nm}^{s}}{\partial \phi^{s}} = \frac{ime^{-im\phi^{s}} \mathrm{P}_{n}^{|m|}(\cos\theta^{s})}{\sin\theta^{s} r^{n}} \quad \text{and}$$

$$\frac{\partial}{\partial\phi^{s}}(\frac{1}{\sin\theta^{s}} \frac{\partial \Gamma_{nm}^{s}}{\partial \phi^{s}}) = \frac{m^{2}e^{-im\phi^{s}} \mathrm{P}_{n}^{|m|}(\cos\theta^{s})}{\sin\theta^{s} r^{n}} \tag{2.2.7}$$

Substitute in equation (2.2.4) from equations (2.2.5), (2.2.6) and (2.2.7) we get

$$\Delta\Gamma_{nm}^{s}(\boldsymbol{x}) = \frac{-1}{r^{2}\sin\theta^{s}}\left[\frac{e^{-im\phi^{s}} \sin\theta^{s} \dfrac{d^{2}}{d\theta^{s2}}\left(\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right)}{r^{n}} + \frac{e^{-im\phi^{s}} \cos\theta^{s} \dfrac{d}{d\theta^{s}}\left(\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right)}{r^{n}}\right.$$

$$\left. + \frac{n(n-1)e^{-im\phi^{s}} \sin\theta^{s} \mathrm{P}_{n}^{|m|}(\cos\theta^{s})}{r^{n}} - \frac{m^{2}e^{-im\phi^{s}} \mathrm{P}_{n}^{|m|}(\cos\theta^{s})}{\sin\theta^{s} r^{n}}\right]$$

$$= e^{-im\phi^{s}} \frac{1}{r^{n+2}\sin^{2}\theta^{s}}\left[\sin^{2}\theta^{s} \frac{d^{2}}{d\theta^{s2}}\left(\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right) + \sin\theta^{s}\cos\theta^{s} \frac{d}{d\theta^{s}}\left(\mathrm{P}_{n}^{|m|}(\cos\theta^{s})\right)\right.$$

$$\left. + \left(n(n-1)\sin^{2}\theta^{s} - m^{2}\right)\mathrm{P}_{n}^{|m|}(cos\theta^{s})\right] = 0 \qquad \square$$

Using the kernel functions given in equation (2.2.2) we obtain

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma_0^s}{\partial v}-\Gamma_0^s\frac{\partial\varphi}{\partial v})ds+\sum_{i=1}^{N}\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_0^s}{\partial v}-\Gamma_0^s q^i)ds=0 \tag{2.2.8}$$

and

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma_{nm}^s}{\partial v}-\Gamma_{nm}^s\frac{\partial\varphi}{\partial v})ds+\sum_{i=1}^{N}\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_{nm}^s}{\partial v}-\Gamma_{nm}^s q^i)ds=0 \tag{2.2.9}$$

This system of equations involves $N(M+1)^2$ equations.

To simplify this system of equations we evaluate the integral over S_i.

If $i \neq s$ then $\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_0^s}{\partial v}-\Gamma_0^s q^i)ds = \varphi_{00}^i\int_{S_i}\frac{\partial\Gamma_0^s}{\partial v}ds-\int_{S_i}\Gamma_0^s q^i ds=-\int_{S_i}\Gamma_0^s q^i ds$

$$=\frac{a_i^{\,2}}{r_{is}}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q_{nm}^i}{(2n+1)}\left(\frac{a_i}{r_{is}}\right)^n Y_n^m(\psi^{is},\gamma^{is})$$

Notice that we set $\int_{S_i}\frac{\partial\Gamma_0^s}{\partial v}ds$ equal to zero since Γ_0^s is harmonic function.

If $i=s$ then $\Gamma_0^s=\Gamma_0^i=\frac{-1}{4\pi a_i}$ which implies that $\frac{\partial\Gamma_0^s}{\partial v}=\frac{-1}{4\pi a^2{}_i}$. Hence

$$\int_{S_i}\varphi_{00}^i\frac{\partial\Gamma_0^s}{\partial v}ds=-\varphi_{00}^i\frac{a_i^{\,2}}{4\pi a^2{}_i}\int_0^{2\pi}\int_0^{\pi}\sin\theta^i d\theta^i d\phi^i=-\varphi_{00}^i \tag{2.2.10}$$

and $-\int_{S_i}\Gamma_0^s q^i ds=+\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q_{nm}^i a_i}{4\pi}\int_0^{2\pi}\int_0^{\pi}Y_n^m(\theta^i,\phi^i)\sin\theta^i d\theta^i d\phi^i$

$$=a_i q_{00}^i \tag{2.2.11}$$

Therefore equation (2.2.8) reduced to

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma_0^s}{\partial\nu}-\Gamma_0^s\frac{\partial\varphi}{\partial\nu})ds-\varphi_{00}^s+a_s q_{00}^s+\sum_{\substack{i=1\\ i\neq s}}^{N}\left(\frac{a_i{}^2}{r_{is}}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q_{nm}^i}{(2n+1)}\left(\frac{a_i}{r_{is}}\right)^{\mathrm{n}}Y_n^m(\psi^{is},\gamma^{is})\right)=0.$$

$$(2.2.12)$$

Analogously we simplify $\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_{nm}^s}{\partial\nu}-\Gamma_{nm}^s q^i)ds$

If $i\neq s$ then

$$\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_{nm}^s}{\partial\nu}-\Gamma_{nm}^s q^i)ds=\varphi_{00}^i\int_{S_i}\frac{\partial\Gamma_{nm}^s}{\partial\nu}ds-\int_{S_i}\Gamma_{nm}^s q^i\ ds=-\int_{S_i}\Gamma_{nm}^s q^i ds$$

Because Γ_{nm}^s is harmonic everywhere inside S_i except

for i=s

Hence $\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_{nm}^s}{\partial\nu}-\Gamma_{nm}^s q^i)ds=-\int_{S_i}\Gamma_{nm}^s q^i ds$ $\quad(2.2.13)$

If $i=s$ then

$\Gamma_{nm}^s=\Gamma_{nm}^i=\frac{-Y_n^{*m}(\theta^i,\phi^i)}{a_i^n}$ whichimplise that $\frac{\partial\Gamma_{nm}^s}{\partial\nu}=\frac{-nY_n^{*m}(\theta^i,\phi^i)}{a_i^{n+1}}$.

Thus $\int_{S_i}\varphi_{00}^i\frac{\partial\Gamma_{nm}^s}{\partial\nu}ds=-\varphi_{00}^i\int_{S_i}\frac{nY_n^{*m}(\theta^i,\phi^i)}{a_i^{n+1}}ds$

$$=\frac{-n\varphi_{00}^i}{a_i^{n+1}}a_i^2\int_0^{2\pi}\int_0^{\pi}Y_n^{*m}(\theta^i,\phi^i)\sin\theta^i d\theta^i d\phi^i=0.$$

For thesecondintegral $\int_{S_i}\Gamma_{nm}^s q^i ds$ wehave, $\int_{S_i}\Gamma_{nm}^s q^i ds=\int_{S_i}\Gamma_{nm}^i q^i ds$

$$=-\sum_{n'=0}^{M}\sum_{m'=-n'}^{n'}\frac{q_{n'm'}^i a_i^2}{a_i^n}\int_0^{2\pi}\int_0^{\pi}Y_{n'}^{m'}(\theta^i,\phi^i)Y_n^{*m}(\theta^i,\phi^i)\sin\theta^i d\theta^i d\phi^i=-q_{nm}^i a_i^{2-n}N_{m,n}$$

Thereforefor $i=s$ wehave $\int_{S_i}(\varphi_{00}^i\frac{\partial\Gamma_{nm}^s}{\partial\nu}-\Gamma_{nm}^s q^i)ds=q_{nm}^i a_i^{2-n}N_{m,n}$.

$$(2.2.14)$$

Hence equation (2.2.9) gives

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma^s_{nm}}{\partial\nu}-\Gamma^s_{nm}\frac{\partial\varphi}{\partial\nu})ds+q^s_{nm}a_s^{2-n}N_{m,n}-\sum_{\substack{i=1\\i\neq s}}^{N}\int_{S_i}\Gamma^s_{nm}q^i\,ds=0 \qquad (2.2.15)$$

Thus the system of equations (2.2.8) and (2.2.9) reduced to the following system

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma^s_0}{\partial\nu}-\Gamma^s_0\frac{\partial\varphi}{\partial\nu})ds-\varphi^s_{00}+a_s q^s_{00}+\sum_{\substack{i=1\\i\neq s}}^{N}\left(\frac{a_i{}^2}{r_{is}}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q^i_{nm}}{(2n+1)}\left(\frac{a_i}{r_{is}}\right)^n Y_n^m(\psi^{is},\gamma^{is})\right)=0 \qquad (2.2.16)$$

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma^s_{nm}}{\partial\nu}-\Gamma^s_{nm}\frac{\partial\varphi}{\partial\nu})ds+q^s_{nm}a_s^{2-n}N_{m,n}-\sum_{\substack{i=1\\i\neq s}}^{N}\int_{S_i}\Gamma^s_{nm}q^i\,ds=0 \qquad (2.2.17)$$

Now consider the Dirichlet problem if the potential on each S_i is nonconstant, say find φ such that

$$\Delta\varphi(\boldsymbol{x})=0, \qquad \boldsymbol{x}\in\Omega$$

$$\varphi^i(\boldsymbol{x})=\varphi^{\mathrm{i}}_{00}+\varphi^{\mathrm{i}}_{11}Y_1^1(\theta^i,\phi^i), \qquad \boldsymbol{x}\in S_i$$

where $\varphi^{\mathrm{i}}_{00}$ and $\varphi^{\mathrm{i}}_{11}$ are scalars. Then equation (2.1.6) reduced to

$$\varphi(\boldsymbol{y})=$$

$$\int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma}{\partial\,\nu}-\Gamma\frac{\partial\,\varphi}{\partial\,\nu})da+\sum_{\mathrm{i}=1}^{\mathrm{N}}\left[\varphi^i_{11}\frac{a_i^2}{r_i^2}Y_1^1(\psi^i,\gamma^i)+\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q^i_{nm}}{(2n+1)}a_i\,.\left(\frac{a_i}{r_i}\right)^{n+1}Y_n^m(\psi^i,\gamma^i)\right] \qquad (2.2.18)$$

Again using the sequense of kernel functions given in equation (2.2.2) we obtain

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma^s_0}{\partial\nu}+\Gamma^s_0\frac{\partial\varphi}{\partial\nu})ds+\sum_{i=1}^{N}\int_{S_i}\left(\left(\varphi^i_{00}+\varphi^{\mathrm{i}}_{11}Y_1^1(\theta^i,\phi^i)\right)\frac{\partial\Gamma^s_0}{\partial\nu}-\Gamma^s_0 q^i\right)ds=0 \qquad (2.2.19)$$

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma^s_{nm}}{\partial\nu}+\Gamma^s_{nm}\frac{\partial\varphi}{\partial\nu})ds+\sum_{i=1}^{N}\int_{S_i}\left(((\varphi^i_{00}+\varphi^{\mathrm{i}}_{11}Y_1^1(\theta^i,\phi^i))\frac{\partial\Gamma^s_{nm}}{\partial\nu}-\Gamma^s_{nm}q^i\right)ds=0 \qquad (2.2.20)$$

We proceed as in the previous case and simplify the integrals appear in equations (2.2.19) and (2.2.20).

If $i \neq s$ $\int_{S_i} \left[\left(\varphi_{00}^i + \varphi_{11}^i Y_1^1(\theta^i, \varphi^i) \right) \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q^i \right] ds$

$$= \varphi_{00}^i \int_{S_i} \frac{\partial \Gamma_0^s}{\partial v} ds + \int_{S_i} \varphi_{11}^i Y_1^1(\theta^i, \varphi^i) \frac{\partial \Gamma_0^s}{\partial v} ds - \int_{S_i} \Gamma_0^s q^i ds$$

$$= \frac{a_i^{\,2}}{r_{is}} \sum_{n=0}^{M} \sum_{m=-n}^{n} \frac{q_{nm}^i}{(2n+1)} \left(\frac{a_i}{r_{is}} \right)^n Y_n^m(\psi^{is}, \gamma^{is}) + \varphi_{11}^i \int_{S_i} Y_1^1(\theta^i, \varphi^i) \frac{\partial \Gamma_0^s}{\partial v} ds$$

Analogues to the previous case we have

$$\int_{S_i} Y_1^1(\theta^i, \varphi^i) \frac{\partial \Gamma_0^s}{\partial v} ds = \frac{\partial}{\partial t_i} \int_{S_i^+} Y_1^1(\theta^i, \varphi^i) \Gamma_0^s ds \quad \text{as } t_i \to a_i.$$

Using the spherical harmonics expansion of Γ_0^s we get

$$\frac{\partial}{\partial t_i} \int_{S_i^+} Y_1^1(\theta^i, \varphi^i) \Gamma_0^s ds =$$

$$\frac{\partial}{\partial t_i} \int_{S_i} Y_1^1(\theta^i, \varphi^i) \frac{-1}{r_{is}} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \left(\frac{t_i}{r_{is}} \right)^{n'} \frac{1}{(2n'+1) N_{m',n'}} Y_{n'}^{*m'}(\theta^i, \phi^i) Y_{n'}^{m'}(\psi^{is}, \gamma^{is}) ds$$

$$= \frac{\partial}{\partial t_i} t_i^2 \int_0^{2\pi} \int_0^{\pi} Y_1^1(\theta^i, \varphi^i) \frac{-1}{r_{is}} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \left(\frac{t_i}{r_{is}} \right)^{n'} \frac{1}{(2n'+1) N_{m',n'}} Y_{n'}^{*m'}(\theta^i, \phi^i).$$

$$Y_{n'}^{m'}(\psi^{is}, \gamma^{is}) \sin \theta^i d\theta^i d\phi^i$$

$$= \frac{\partial}{\partial t_i} \left(t_i^2 \frac{-1}{r_{is}} \frac{t_i}{r_{is}} Y_1^1(\psi^{is}, \gamma^{is}) \right) = \frac{\partial}{\partial t_i} \left(\frac{-1}{3} \frac{t_i^3}{r_{is}^2} Y_1^1(\psi^{is}, \gamma^{is}) \right) = -\frac{t_i^2}{r_{is}^2} Y_1^1(\psi^{is}, \gamma^{is})$$

Hence $\int_{S_i} Y_1^1(\theta^i, \phi^i) \frac{\partial \Gamma_0^s}{\partial v} ds = \frac{-a_i^2}{r_{is}^2} Y_1^1(\psi^{is}, \gamma^{is}).$

Therefore $\int\limits_{S_i}\left[\left(\varphi^i_{00}+\varphi^i_{11}Y^1_1(\theta^i,\varphi^i)\right)\frac{\partial \Gamma^s_0}{\partial v}-\Gamma^s_0 q^i\right]ds$

$$=\frac{a_i^{\;2}}{r_{is}}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q^i_{nm}}{(2n+1)}\left(\frac{a_i}{r_{is}}\right)^2 Y^m_n(\psi^{is},\gamma^{is})-\frac{a_i^2}{r_{is}^2}\varphi^i_{11}Y^1_1(\psi^{is},\gamma^{is}).$$

If $i=s$, $\int\limits_{S_i}\left[\left(\varphi^i_{00}+\varphi^i_{11}Y^1_1(\theta^i,\varphi^i)\right)\frac{\partial \Gamma^s_0}{\partial v}-\Gamma^s_0 q^i\right]ds$

$$=\varphi^i_{00}\int\limits_{S_i}\frac{\partial \Gamma^s_0}{\partial v}ds+\int\limits_{S_i}\varphi^i_{11}Y^1_1(\theta^i,\varphi^i)\frac{\partial \Gamma^s_0}{\partial v}ds-\int\limits_{S_i}\Gamma^s_0 q^i ds$$

$$=-\varphi^i_{00}+a_i q^i_{00}+\varphi^i_{11}\int\limits_{S_i}Y^1_1(\theta^i,\varphi^i)\frac{\partial \Gamma^s_0}{\partial v}ds, \text{ so}$$

$$\int\limits_{S_i}\left[\left(\varphi^i_{00}+\varphi^i_{11}Y^1_1(\theta^i,\varphi^i)\right)\frac{\partial \Gamma^s_0}{\partial v}-\Gamma^s_0 q^i\right]ds=-\varphi^i_{00}+a_i q^i_{00}+\varphi^i_{11}\int\limits_{S_i}Y^1_1(\theta^i,\varphi^i)\frac{\partial \Gamma^s_0}{\partial v}ds$$

Now $\frac{\partial \Gamma^s_0}{\partial v}=\frac{\partial \Gamma^i_0}{\partial v}=\frac{-1}{4\pi a_i^2}$. Hence

$$\varphi^i_{11}\int\limits_{S_i}Y^1_1(\theta^i,\varphi^i)\frac{\partial \Gamma^s_0}{\partial v}ds=\frac{-\varphi^i_{11}}{4\pi a_i^2}\int\limits_{S_i}Y^1_1(\theta^i,\varphi^i)ds=0$$

Therefore for $i=s$ we have

$$\int\limits_{S_i}\left[\left(\varphi^i_{00}+\varphi^i_{11}Y^1_1(\theta^i,\varphi^i)\right)\frac{\partial \Gamma^s_0}{\partial v}-\Gamma^s_0 q^i\right]ds=-\varphi^i_{00}+a_i q^i_{00}$$

Thus equation (2.2.19) reduced to

$$\int\limits_{\partial\Omega}\left[\varphi\frac{\partial \Gamma^s_0}{\partial v}-\Gamma^s_0\frac{\partial \varphi}{\partial v}\right]ds-\varphi^s_{00}+a_s q^s_{00}+\sum_{\substack{i=1\\ i\neq s}}^{N}-\frac{a_i^2}{r_{is}^2}\varphi^i_{11}Y^1_1(\psi^{is},\gamma^{is})$$
$$+\frac{a_i^2}{r_{is}}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{q^i_{nm}}{(2n+1)}\left(\frac{a_i}{r_{is}}\right)^2 Y^m_n(\psi^{is},\gamma^{is})=0 \qquad (2.2.21)$$

Regarding Γ^s_{nm} weget

If $i \neq s$, $\int\limits_{S_i} \left[\left(\varphi^i_{00} + \varphi^i_{11} Y^1_1(\theta^i, \varphi^i) \right) \frac{\partial \Gamma^s_{nm}}{\partial v} - \Gamma^s_{nm} q^i \right] ds$

$$= \varphi^i_{00} \int\limits_{S_i} \frac{\partial \Gamma^s_{nm}}{\partial v} ds + \varphi^i_{11} \int\limits_{S_i} Y^1_1(\theta^i, \varphi^i) \frac{\partial \Gamma^s_{nm}}{\partial v} ds - \int\limits_{S_i} \Gamma^s_{nm} q^i ds$$

$$= -\int\limits_{S_i} \Gamma^s_{nm} q^i ds + \varphi^i_{11} \int\limits_{S_i} Y^1_1(\theta^i, \varphi^i) \frac{\partial \Gamma^s_{nm}}{\partial v} ds$$

If $i = s$ then

$\Gamma^i_{nm} = \Gamma^s_{nm} = \frac{-Y^{*m}_n(\theta^s, \phi^s)}{a^n_i}$. Thus $\frac{\partial \Gamma^s_{nm}}{\partial v} = \frac{\partial \Gamma^i_{nm}}{\partial v} = \frac{-nY^{*m}_n(\theta^s, \phi^s)}{a^{n+1}_i}$ Thererfore

$$\int\limits_{S_i} \left[\left(\varphi^i_{00} + \varphi^i_{11} Y^1_1(\theta^i, \varphi^i) \right) \frac{\partial \Gamma^s_{nm}}{\partial v} - \Gamma^s_{nm} q^i \right] ds$$

$$= \varphi^i_{00} \int\limits_{S_i} \frac{\partial \Gamma^s_{nm}}{\partial v} ds + \varphi^i_{11} \int\limits_{S_i} Y^1_1(\theta^i, \varphi^i) \frac{\partial \Gamma^s_{nm}}{\partial v} ds - \int\limits_{S_i} \Gamma^s_{nm} q^i ds$$

$$= q^i_{nm} a^{2-n}_i N_{m,n} - \varphi^i_{11} \frac{n}{a^{n+1}_i} \int\limits_{S_i} Y^1_1(\theta^i, \varphi^i) Y^{*m}_n(\theta^i, \varphi^i) ds$$

$$= q^i_{nm} a^{2-n}_i N_{m,n} - \varphi^i_{11} \frac{n}{a^{n-1}_i} \int\limits_0^{2\pi} \int\limits_0^{\pi} Y^1_1(\theta^i, \varphi^i) Y^{*m}_n(\theta^i, \varphi^i) \sin\theta^i d\theta^i d\phi^i$$

$$= q^i_{nm} a^{2-n}_i N_{m,n} - \varphi^i_{11} N_{1,1} = q^i_{nm} a^{2-n}_i N_{m,n} - \varphi^i_{11} \frac{8\pi}{3}$$

Hence equation (2.2.20) reduced to

$$\int\limits_{\partial\Omega} (\varphi \frac{\partial \Gamma^s_{nm}}{\partial v} + \Gamma^s_{nm} \frac{\partial \varphi}{\partial v}) dS + q^s_{nm} a^{2-n}_s N_{m,n} - \varphi^s_{11} \frac{8\pi}{3} -$$

$$\sum_{\substack{i=1 \\ i \neq s}}^{N} \left(\int\limits_{S_i} \Gamma^s_{nm} q^i ds - \varphi^i_{11} \int\limits_{S_i} Y^1_1(\theta^i, \varphi^i) \frac{\partial \Gamma^s_{nm}}{\partial v} ds \right) = 0. \qquad (2.2.22)$$

2.3 Neumann Problem

We can apply the ideas of spherical harmonics to Neumann problem too. We treat the case where a constant normal derivative is prescribed on the boundary of the spheres.

That is we can consider the problem, [17], find φ such that

$$\Delta\varphi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega,$$

$$q^i(\boldsymbol{x}) = q^i_{00}, \quad \boldsymbol{x} \in \mathrm{S}_i .$$

Thus equation $(2.1.6)$, reduced to

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega} (\varphi \frac{\partial \Gamma}{\partial v} - \Gamma \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \left[q^i_{00} \frac{a_i^2}{r_i} + \sum_{n=1}^{M} \sum_{m=-n}^{n} (n+2)\varphi^i_{nm} \frac{a_i^{n+1}}{(2n+1) r_i^{n+1}} Y_n^m(\psi^i, \gamma^i) \right] \tag{2.3.1}$$

Substitute the kernel in equation (2.2.2) instead of Γ in equation (2.1.2) we end with

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q^i_{00}) ds = 0, \tag{2.3.2}$$

and $$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s \frac{\partial \varphi}{\partial v}) ds \quad + \sum_{i=1}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s q^i_{00}) ds = 0, \tag{2.3.3}$$

where $s = 1,2,\ldots,N, \quad n = 1,2,\ldots,M \quad ; m = -n,\ldots,n$

Analogous to the Dirichlet problem the system of equations (2.3.2) and (2.3.3) consists of $N(M+1)^2$ equations. Again we simplify the integrals over S_i that appear in this system.

We consider $\int_{S_i} (\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q^i_{00}) ds.$ Then

$$\int_{S_i} (\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q^i_{00}) ds = \int_{S_i} \varphi^i \frac{\partial \Gamma_0^s}{\partial v} ds - q^i_{00} \int_{S_i} \Gamma_0^s ds$$

Hence in the case $i \neq s$ we

have

$$\int_{S_i} (\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q_{00}^i) ds =$$

$$\sum_{n=1}^{M} \sum_{m=-n}^{n} \frac{(n+2)\varphi_{nm}^i a_i^{n+1}}{(2n+1) r_{is}^{n+1}} Y_n^m(\psi^{is}, \gamma^{is}) + \frac{a_i^2}{r_{is}} q_{00}^i \tag{2.3.4}$$

If $i = s$ then $\Gamma_0^s = \Gamma_0^i = \dfrac{-1}{4\pi a_i}$ and $\dfrac{\partial \Gamma_0^s}{\partial v} = \dfrac{-1}{4\pi a_i^2}$

Therefore $\int_{S_i} \varphi^i \dfrac{\partial \Gamma_0^s}{\partial v} ds = \int_{S_i} \varphi^i \dfrac{\partial \Gamma_0^i}{\partial v} ds = \int_{S_i} \varphi^i \dfrac{-1}{4\pi a_i^2} ds$

$$= \frac{-1}{4\pi a_i^2} a_i^2 \sum_{n=1}^{M} \sum_{m=-n}^{n} \int_0^{2\pi} \int_0^{\pi} \varphi_{nm}^i Y_n^m(\theta^i, \phi^i) \sin\theta^i d\theta^i d\phi^i = -\varphi_{00}^s$$

and $$\int_{S_i} (\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q_{00}^i) ds = -\varphi_{00}^s + a_s q_{00}^s \tag{2.3.5}$$

Thus $$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s q_{00}^i) ds = \int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \frac{\partial \varphi}{\partial v}) ds$$

$$-\varphi_{00}^s + a_s q_{00}^s + \sum_{\substack{i=1 \\ i \neq s}}^{N} \left(\sum_{n=1}^{M} \sum_{m=-n}^{n} \frac{(n+2)\varphi_{nm}^i a_i^{n+1}}{(2n+1) r_{is}^{n+1}} Y_n^m(\psi^{is}, \gamma^{is}) + \frac{a_i^2}{r_{is}} q_{00}^i \right) = 0 \tag{2.3.6}$$

We evaluate $\displaystyle\int_{S_i} \left(\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s q_{00}^i \right) ds$

If $i = s$ then $\Gamma_{nm}^s = \Gamma_{nm}^i = \dfrac{-Y_n^{*m}(\theta^i, \phi^i)}{a_i^n} \Rightarrow \dfrac{\partial \Gamma_{nm}^s}{\partial v} = \dfrac{-nY_n^{*m}(\theta^i, \phi^i)}{a_i^{n+1}}$

and $\displaystyle\int_{S_i} \Gamma_{nm}^s q_{00}^i ds = q_{00}^i a_i^2 \int_0^{2\pi} \int_0^{\pi} \Gamma_{nm}^s \sin\theta^i d\theta^i d\phi^i = 0.$

For $\displaystyle\int_{S_i} \varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} ds$ we have $\displaystyle\int_{S_i} \varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} ds = -\int_{S_i} \varphi^i \frac{nY_n^{*m}(\theta^i, \phi^i)}{a_i^{n+1}} ds$

$$= \frac{-n}{a_i^{n+1}} a_i^2 \int_0^{2\pi}\int_0^{\pi} \sum_{n=0}^{M} \sum_{m=-n}^{n} \varphi_{nm}^i Y_n^m(\theta^i,\phi^i) Y_n^{*m}(\theta^i,\phi^i) \sin\theta^i d\theta^i d\phi^i = \frac{-n}{a_i^{n-1}} \varphi_{nm}^i N_{m,n}$$

So if $i = s,\ \int_{S_i} (\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s q_{00}^i) ds = \frac{-n}{a_s^{n-1}} \varphi_{nm}^s N_{m,n}$

Hence $\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s q_{00}^i) ds = 0 \Leftrightarrow$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s \frac{\partial \varphi}{\partial v}) ds - \frac{n}{a_s^{n-1}} \varphi_{nm}^s N_{m,n} + \sum_{\substack{i=1\\ i\neq s}}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s q_{00}^i) ds = 0 \tag{2.3.7}$$

Therefore the system in (2.3.2) and (2.3.3) becomes

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \frac{\partial \varphi}{\partial v}) ds - \varphi_{00}^s + a_s q_{00}^s +$$

$$\sum_{\substack{i=1\\ i\neq s}}^{N} \left(\sum_{n=1}^{M} \sum_{m=-n}^{n} \frac{(n+2)\varphi_{nm}^i a_i^{n+1}}{(2n+1) r_{is}^{n+1}} Y_n^m(\psi^{is},\gamma^{is}) + \frac{a_i^2}{r_{is}} q_{00}^i \right) = 0, \tag{2.3.8}$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s \frac{\partial \varphi}{\partial v}) ds - \frac{n}{a_s^{n-1}} \varphi_{nm}^s N_{m,n} + \sum_{\substack{i=1\\ i\neq s}}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s q_{00}^i) ds = 0 \tag{2.3.9}$$

Where $s = 1,2,\ldots,N,\quad n = 1,2,\ldots,M,\qquad m = -n,\ldots,n$

Next we consider the Neumann problem, find $\varphi(\boldsymbol{x})$ such that

$$\Delta\varphi(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega.$$

$$q^i(\boldsymbol{x}) = q_{00}^i + q_{11}^i Y_1^1(\theta^i,\phi^i), \qquad \boldsymbol{x} \in S_i$$

Where q_{00}^i and q_{11}^i are scalars. Then equation (2.1.6) reduced to

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega} (\varphi \frac{\partial \Gamma}{\partial v} - \Gamma \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \left(q_{00}^{i} \frac{a_i^2}{r_i} + \frac{q_{11}^{i}}{3} \frac{a_i^3}{r_i^2} Y_1^1(\psi^i, \gamma^i) \right.$$

$$\left. + \sum_{n=1}^{M} \sum_{m=-n}^{n} (n+2) \varphi_{nm}^{i} \frac{a_i^{n+1}}{(2n+1) r_i^{n+1}} Y_n^m(\psi^i, \gamma^i) \right).$$

Substitute the kernels given in equation (2.2.2) in equation (2.1.2) we obtain

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \int_{S_i} \left(\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \left(q_{00}^i + q_{11}^i Y_1^1(\theta^i, \phi^i) \right) \right) ds = 0$$

and

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s \frac{\partial \varphi}{\partial v}) ds \quad + \sum_{i=1}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{nm}^s}{\partial v} - \Gamma_{nm}^s \left(q_{00}^i + q_{11}^i Y_1^1(\theta^i, \phi^i) \right)) ds = 0$$

where $s = 1,2,\ldots,N, \quad n = 1,2,\ldots,M \,,\; m = -n,\ldots,n$

Want to simplify this system of equations. Let us evaluate the integral over S_i

First we consider

$$\int_{S_i} \left(\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \left(q_{00}^i + q_{11}^i Y_1^1(\theta^i, \phi^i) \right) \right) ds$$

If $i \neq s$ then $\displaystyle\int_{S_i} \left(\varphi^i \frac{\partial \Gamma_0^s}{\partial v} - \Gamma_0^s \left(q_{00}^i + q_{11}^i Y_1^1(\theta^i, \phi^i) \right) \right) ds$

$$= \int_{S_i} \varphi^i \frac{\partial \Gamma_0^s}{\partial v} ds - q_{00}^i \int_{S_i} \Gamma_0^s ds - q_{11}^i \int_{S_i} \Gamma_0^s Y_1^1(\theta^i, \phi^i) ds$$

$$= -a_i^2 \int_0^{2\pi} \int_0^{\pi} \frac{1}{r_{is}} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left(\frac{a_i}{r_i} \right)^{n'} \frac{Y_1^1(\theta^i, \phi^i)}{(2n'+1) N_{m',n'}} Y_{n'}^{*m'}(\theta^i, \phi^i) Y_{n'}^{m'}(\psi^{is}, \gamma^{is}) \; \sin\theta^i \mathrm{d}\theta^i \mathrm{d}\phi^i$$

Therefore $n' = m' = 1$ and

Now $\int\limits_{S_i} \Gamma_0^s Y_1^1(\theta^i, \phi^i) ds = \int\limits_{S_i} \frac{-Y_1^1(\theta^i, \phi^i)}{4\pi|\boldsymbol{x} - \boldsymbol{\xi}^s|} ds$

$$= \frac{-a_i^2}{r_{is}} \frac{a_i}{r_i} \frac{Y_1^1(\psi^{is}, \gamma^{is})}{3.\frac{8\pi}{3}} \int_0^{2\pi}\int_0^{\pi} \sin\ \theta^i d\theta^i d\varphi = \frac{-a_i^3}{2r_{is}^2}\ Y_1^1(\psi^{is}, \gamma^{is}).$$

$$-\int\limits_{S_i} \Gamma_0^s q_{00}^i ds = \frac{a_i^2}{r_{is}}\ q_{00}^i$$

So if $i \neq s$ we have

$$\int\limits_{S_i} \left(\varphi^i \frac{\partial \Gamma_0^s}{\partial \nu} - \Gamma_0^s \left(q_{00}^i + q_{11}^i Y_1^1(\theta^i, \phi^i) \right) \right) ds =$$

$$\sum_{n=1}^{M} \sum_{m=-n}^{n} \frac{(n+2)\varphi_{nm}^i a_i^{n+1}}{(2n+1) r_{is}^{n+1}}\ Y_n^m(\psi^{is}, \gamma^{is}) + \frac{a_i^2}{r_{is}}\ q_{00}^i + \frac{a_i^3}{2r_{is}^2} Y_1^1(\psi^{is}, \gamma^{is}) q_{11}^i.$$

If $i = s$ then

$$\Gamma_0^s = \Gamma_0^i = \frac{-1}{4\pi a_i} \text{ and } \frac{\partial \Gamma_0^s}{\partial \nu} = \frac{-1}{4\pi a_i^{\ 2}}.$$

$$\int\limits_{S_i} \varphi^i \frac{\partial \Gamma_0^s}{\partial \nu} ds = \int\limits_{S_i} \varphi^i \frac{\partial \Gamma_0^i}{\partial \nu} ds = \int\limits_{S_i} \varphi^i \frac{-1}{4\pi a_i^2} ds = -\varphi_{00}^i$$

$$\int\limits_{S_i} \left(\varphi^i \frac{\partial \Gamma_0^s}{\partial \nu} - \Gamma_0^s \left(q_{00}^i + q_{11}^i Y_1^1(\theta^i, \phi^i) \right) \right) ds = -\varphi_{00}^s + a_s q_{00}^s$$

Thus

$$\int\limits_{\partial\Omega} \left(\varphi \frac{\partial \Gamma_0^s}{\partial \nu} - \Gamma_0^s \frac{\partial \varphi}{\partial \nu} \right) ds + \sum_{i=1}^{N} \int\limits_{S_i} \left(\varphi^i \frac{\partial \Gamma_0^s}{\partial \nu} - \Gamma_0^s q^i \right) ds =$$

$$\int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma_0^s}{\partial\,\nu}-\Gamma_0^s\,\frac{\partial\,\varphi}{\partial\,\nu})ds-\varphi_{00}^s+a_s\;\;q_{00}^s+$$

$$\sum_{\substack{i=1\\ i\neq s}}^{N}\left(\sum_{n=1}^{M}\sum_{m=-n}^{n}\frac{(n+2)\varphi_{nm}^{i}a_i^{n+1}}{(2n+1)r_{is}^{n+1}}\;Y_n^m(\psi^{is},\gamma^{is})+\frac{a_i^2}{r_{is}}\;\;q_{00}^i\;+\frac{1}{2}\frac{a_i^3}{r_{is}^2}Y_1^1(\psi^{is},\gamma^{is})\right)=0$$

Furthermore, we have

$$\int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma_{nm}^s}{\partial\,\nu}-\Gamma_{nm}^s\,\frac{\partial\,\varphi}{\partial\,\nu})ds+\sum_{i=1}^{N}\int_{S_i}(\varphi^i\,\frac{\partial\,\Gamma_{nm}^s}{\partial\,\nu}-\Gamma_{nm}^s q^i)ds=$$

$$\int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma_{nm}^s}{\partial\,\nu}-\Gamma_{nm}^s\,\frac{\partial\,\varphi}{\partial\,\nu})ds-\frac{n}{a_s^{n-1}}\varphi_{nm}^s N_{m,n}+\frac{8\pi}{3}a_s q_{11}^s+$$

$$\sum_{\substack{i=1\\ i\neq s}}^{N}\int_{S_i}\left(\varphi^i\,\frac{\partial\,\Gamma_{nm}^s}{\partial\,\nu}-\Gamma_{nm}^s q_{00}^i-\Gamma_{nm}^s Y_1^1(\theta^i,\phi^i)q_{11}^i\right)ds=0.$$

2.4 Levels of Approximations [17]

In general the same normal derivatives (2.1.4) on each sphere can be represented by any finite number of spherical harmonics. It is reasonable to expect that the accuracy of the solution to improve as this number is increased.

The simplest representation is $q^{i}=q_{00}^{i}$.

In this case equations (2.2.1) and (2.2.16) reduced to what we called zeroth-order equations

$$\varphi(y)=\int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma}{\partial\,\nu}-\Gamma\,\frac{\partial\,\varphi}{\partial\,\nu})ds+\sum_{i=1}^{N}\;q_{00}^i\;\frac{a_i^2}{r_i}\qquad(2.4.1)$$

and $$\int_{\partial\Omega}(\varphi\frac{\partial\,\Gamma_0^s}{\partial\,\nu}-\Gamma_0^s\,\frac{\partial\varphi}{\partial\,\nu})ds-\varphi_{00}^s+a_s q_{00}^s+\sum_{\substack{i=1\\ i\neq s}}^{N}q_{00}^i\;\frac{a_i^2}{r_{is}}=0\qquad(2.4.2)$$

The solution of equations $(2.4.1)$ and $(2.4.2)$ is called the zeroth-order solution.

The next level of approximation corresponds to take M=1 in equation (2.1.4). Then equations (2.2.1) and (2.2.16) reduced to

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega} (\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu})ds + \sum_{i=1}^{N} a_i \left[q_{00}^i \frac{a_i}{r_i} Y_0^0(\psi^i,\gamma^i) + \frac{q_{1-1}^i}{3}\left(\frac{a_i}{r_i}\right)^2 Y_1^{-1}(\psi^i,\gamma^i) \right.$$

$$\left. + \frac{q_{10}^i}{3}\left(\frac{a_i}{r_i}\right)^2 Y_1^0(\psi^i,\gamma^i) + \frac{q_{11}^i}{3}\left(\frac{a_i}{r_i}\right)^2 Y_1^1(\psi^i,\gamma^i) \right] \qquad (2.4.3)$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial \nu} - \Gamma_0^s \frac{\partial \varphi}{\partial \nu})ds - \varphi_{00}^s + a_s q_{00}^s + \sum_{\substack{i=1\\ i\neq s}}^{N} \frac{a_i^2}{r_{is}} \left[q_{00}^i Y_0^0(\psi^{is},\gamma^{is}) + \frac{q_{1-1}^i}{3}\frac{a_i}{r_{is}} Y_1^{-1}(\psi^{is},\gamma^{is}) \right.$$

$$\left. + \frac{q_{10}^i}{3}\frac{a_i}{r_{is}} Y_1^0(\psi^{is},\gamma^{is}) + \frac{q_{11}^i}{3}\frac{a_i}{r_{is}} Y_1^1(\psi^{is},\gamma^{is}) \right] = 0 \qquad (2.4.4)$$

Taking $n = 1$ in equation $(2.2.17)$, then m = -1,0,1 and we obtain

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{1-1}^s}{\partial \nu} - \Gamma_{1-1}^s \frac{\partial \varphi}{\partial \nu})ds + q_{1-1}^s \frac{8\pi a_s}{3} - \sum_{\substack{i=1\\ i\neq s}}^{N} \int_{S_i} \Gamma_{1-1}^s q^i ds = 0 \qquad (2.4.5)$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{10}^s}{\partial \nu} - \Gamma_{10}^s \frac{\partial \varphi}{\partial \nu})ds + q_{10}^s \frac{4\pi a_s}{3} - \sum_{\substack{i=1\\ i\neq s}}^{N} \int_{S_i} \Gamma_{10}^s q^i ds = 0 \qquad (2.4.6)$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{11}^s}{\partial \nu} - \Gamma_{11}^s \frac{\partial \varphi}{\partial \nu})ds + q_{11}^s \frac{8\pi a_s}{3} - \sum_{\substack{i=1\\ i\neq s}}^{N} \int_{S_i} \Gamma_{11}^s q^i ds = 0 \qquad (2.4.7)$$

Equations (2.4.3) to (2.4.7) are called the first order equations and their solution is called the first order solution.

To evaluate $-\int_{S_i} \Gamma_{1\alpha}^s q^i ds$ where $\alpha = -1,0,1,$ notice that

$$-\int_{S_i} \Gamma_{1\alpha}^s q^i ds = \int_{S_i} \frac{Y_1^{*\alpha}(\theta^s,\phi^s)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|} \sum_{n=0}^{M}\sum_{m=-n}^{n} q_{nm}^i Y_n^m(\theta^i,\phi^i)ds$$

$$= a_i^2 \sum_{n=0}^{M}\sum_{m=-n}^{n} q_{nm}^i \int_0^{2\pi}\int_0^{\pi} \frac{Y_1^{*\alpha}(\theta^s,\phi^s) Y_n^m(\theta^i,\phi^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|} \sin\theta^i d\theta^i d\phi^i$$

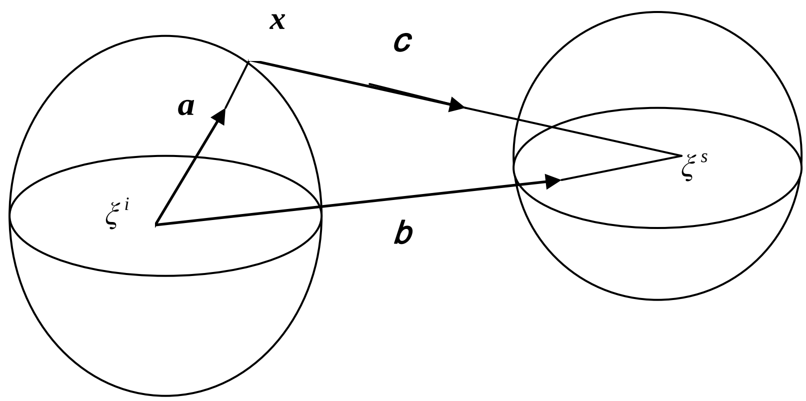

Figure 1

From the figure we can find a relation between θ^s, ϕ^s. Now the spherical coordinates of

$\boldsymbol{x}$ and $\boldsymbol{\xi^s}$ are given as $\boldsymbol{x} = (a_i, \theta^i, \phi^i)$, $\boldsymbol{\xi^s} = (r_{is}, \psi^{is}, \gamma^{is})$,

thus $\boldsymbol{a} = (a_i \sin\theta^i \cos\phi^i, a_i \sin\theta^i \sin\phi^i, a_i \cos\theta^i)$

$\boldsymbol{b} = (r_{is} \sin\psi^{is} \cos\gamma^{is}, r_{is} \sin\psi^{is} \sin\gamma^{is}, r_{is} \cos\psi^{is})$

and $\boldsymbol{c} = (|\boldsymbol{x} - \boldsymbol{\xi^s}| \sin\theta^s \cos\phi^s, |\boldsymbol{x} - \boldsymbol{\xi^s}| \sin\theta^s \sin\phi^s, |\boldsymbol{x} - \boldsymbol{\xi^s}| \cos\theta^s)$

since $\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a}$, then

$$|\boldsymbol{x} - \boldsymbol{\xi^s}| \sin\theta^s \cos\phi^s = r_{is} \sin\psi^{is} \cos\gamma^{is} - a_i \sin\theta^i \cos\phi^i,$$

$$|\boldsymbol{x} - \boldsymbol{\xi^s}| \sin\theta^s \sin\phi^s = r_{is} \sin\psi^{is} \sin\gamma^{is} - a_i \sin\theta^i \sin\phi^i,$$

$$|\boldsymbol{x} - \boldsymbol{\xi^s}| \cos\theta^s = r_{is} \cos\psi^{is} - a_i \cos\theta^i.$$

Hence

$$\frac{\sin\theta^s \cos\phi^s}{|\boldsymbol{x} - \boldsymbol{\xi^s}|} = \frac{r_{is} \sin\psi^{is} \cos\gamma^{is} - a_i \sin\theta^i \cos\phi^i}{|\boldsymbol{x} - \boldsymbol{\xi^s}|^2}, \tag{2.4.8}$$

$$\frac{\sin\theta^s \sin\phi^s}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|} = \frac{r_{is}\sin\psi^{is}\sin\gamma^{is} - a_i \sin\theta^i \sin\phi^i}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2}, \tag{2.4.9}$$

$$\frac{\cos\theta^s}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|} = \frac{r_{is}\cos\psi^{is} - a_i\cos\theta^i}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2}. \tag{2.4.10}$$

Define I_α to be $I_\alpha = \int_0^{2\pi}\int_0^{\pi} \frac{Y_1^{*\alpha}(\theta^s,\phi^s) Y_n^m(\theta^i,\phi^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|} \sin\theta^i d\theta^i d\phi^i \quad \alpha = -1,0,1$

and notice that $Y_1^{*\alpha}(\theta,\phi) = (-1)^\alpha Y_1^{-\alpha}(\theta,\phi)$

then $\qquad I_{-1} = r_{is}\sin\psi^{is}\cos\gamma^{is} \int_0^{2\pi}\int_0^{\pi} \frac{e^{im\phi^i} P_n^{|m|}(\cos\theta^i)\sin\theta^i}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i$

$$- a_i \int_0^{2\pi}\int_0^{\pi} \frac{\sin^2\theta^i e^{im\phi^i} \cos\phi^i P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i$$

$$+ i\left[r_{is}\sin\psi^{is}\sin\gamma^{is} \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta^i e^{im\phi^i} P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i \right.$$

$$\left. - a_i \int_0^{2\pi}\int_0^{\pi} \frac{\sin^2\theta^i e^{im\phi^i} \sin\phi^i P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i \right] \tag{2.4.11}$$

$$I_0 = r_{is}\cos\psi^{is} \int_0^{2\pi}\int_0^{\pi} \frac{e^{im\phi^i} P_n^{|m|}(\cos\theta^i)\sin\theta^i}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i$$

$$- a_i \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta^i \cos\theta^i e^{im\phi^i} P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i \tag{2.4.12}$$

and $I_1 = r_{is}\sin\psi^{is}\cos\gamma^{is} \int_0^{2\pi}\int_0^{\pi} \frac{e^{im\phi^i} P_n^{|m|}(\cos\theta^i)\sin\theta^i}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i$

$$- a_i \int_0^{2\pi}\int_0^{\pi} \frac{\sin^2\theta^i e^{im\phi^i} \cos\phi^i P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2} d\theta^i d\phi^i$$

$$-i\left[r_{is}\sin\psi^{is}\sin\gamma^{is}\int_0^{2\pi}\int_0^{\pi}\frac{\sin\theta^i e^{im\phi^i}P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2}d\theta^i d\phi^i\right.$$

$$\left.-\,a_i\int_0^{2\pi}\int_0^{\pi}\frac{\sin^2\theta^i e^{im\phi^i}\sin\phi^i P_n^{|m|}(\cos\theta^i)}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2}d\theta^i d\phi^i\right] \tag{2.4.13}$$

Now $\dfrac{1}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2}=\dfrac{1}{a_i^2+r_{is}^2-2a_ir_{is}\cos\bar{\gamma}^i}=\dfrac{1}{a_i^2+r_{is}^2}+\dfrac{2a_ir_{is}}{\left(a_i^2+r_{is}^2\right)^2}\cos\bar{\gamma}^i+O(\varepsilon^2)$

where $\varepsilon=\dfrac{2a_ir_{is}}{a_i^2+r_{is}^2}$ and $\varepsilon<1$, see[15].

Substitute the value of $\dfrac{1}{\left|\boldsymbol{x}-\boldsymbol{\xi}^s\right|^2}$ in the integral I_α where $\alpha=-1,0,1$ we get

$$-\int_{S_i}\Gamma_{1-1}^s q^i ds=q_{00}^i\left[\sin\psi^{is}\cos\gamma^{is}\frac{4\pi a_i^2 r_{is}}{a_i^2+r_{is}^2}-\sin\psi^{is}\cos\gamma^{is}\frac{8\pi a_i^4 r_{is}}{3\left(a_i^2+r_{is}^2\right)^2}\right.$$

$$\left.+i\left[\sin\psi^{is}\sin\gamma^{is}\frac{4\pi a_i^2 r_{is}}{a_i^2+r_{is}^2}-\sin\psi^{is}\sin\gamma^{is}\frac{8\pi a_i^4 r_{is}}{3\left(a_i^2+r_{is}^2\right)^2}\right]\right]+$$

$$q_{10}^i\left[\frac{8\pi}{3}\cos\psi^{is}\sin\psi^{is}\cos\gamma^{is}\frac{a_i^3 r_{is}^2}{\left(a_i^2+r_{is}^2\right)^2}+i\left[\frac{8\pi}{3}\cos\psi^{is}\sin\psi^{is}\sin\gamma^{is}\frac{a_i^3 r_{is}^2}{\left(a_i^2+r_{is}^2\right)^2}\right]\right]$$

$$+q_{11}^i\left[\cos\gamma^{is}\sin^2\psi^{is}e^{i\gamma^{is}}\frac{8\pi a_i^3 r_{is}^2}{3\left(a_i^2+r_{is}^2\right)^2}-\frac{4\pi a_i^3}{3\left(a_i^2+r_{is}^2\right)}\right.$$

$$-\int_{S_i}\Gamma_{10}^s q^i ds=q_{00}^i\left[\cos\psi^{is}\frac{4\pi a_i^2 r_{is}}{a_i^2+r_{is}^2}-\cos\psi^{is}\frac{8\pi a_i^4 r_{is}}{3\left(a_i^2+r_{is}^2\right)^2}\right]+q_{10}^i\left[\frac{8\pi}{3}\cos^2\psi^{is}\frac{a_i^3 r_{is}^2}{\left(a_i^2+r_{is}^2\right)^2}\right.$$

$$\left.-\frac{4\pi a_i^3}{3\left(a_i^2+r_{is}^2\right)}\right]+q_{11}^i\left[\cos\psi^{is}\sin\psi^{is}e^{i\gamma^{is}}\frac{8\pi a_i^3 r_{is}^2}{3\left(a_i^2+r_{is}^2\right)^2}\right]+$$

$$q_{1-1}^i\left[\cos\psi^{is}\sin\psi^{is}e^{-i\gamma^{is}}\frac{8\pi a_i^3 r_{is}^2}{3\left(a_i^2+r_{is}^2\right)^2}\right]$$

$$-\int_{S_i}\Gamma_{11}^{s}q^{i}ds = q_{00}^{i}\left[\sin\psi^{is}\cos\gamma^{is}\frac{4\pi a_i^2 r_{is}}{a_i^2+r_{is}^2}-\sin\psi^{is}\cos\gamma^{is}\frac{8\pi a_i^4 r_{is}}{3\left(a_i^2+r_{is}^2\right)^2}\right.$$

$$\left.-i\left[\sin\psi^{is}\sin\gamma^{is}\frac{4\pi a_i^2 r_{is}}{a_i^2+r_{is}^2}-\sin\psi^{is}\sin\gamma^{is}\frac{8\pi a_i^4 r_{is}}{3\left(a_i^2+r_{is}^2\right)^2}\right]\right]+$$

$$q_{10}^{i}\left[\frac{8\pi}{3}\cos\psi^{is}\sin\psi^{is}\cos\gamma^{is}\frac{a_i^3 r_{is}^2}{\left(a_i^2+r_{is}^2\right)^2}-i\left[\frac{8\pi}{3}\cos\psi^{is}\sin\psi^{is}\sin\gamma^{is}\frac{a_i^3 r_{is}^2}{\left(a_i^2+r_{is}^2\right)^2}\right]\right]$$

$$+q_{11}^{i}\left[\cos\gamma^{is}\sin^2\psi^{is}e^{i\gamma^{is}}\frac{8\pi a_i^3 r_{is}^2}{3\left(a_i^2+r_{is}^2\right)^2}-\frac{4\pi a_i^3}{3\left(a_i^2+r_{is}^2\right)}\right.$$

$$\left.-i\left[\frac{8\pi}{3}\sin^2\psi^{is}\sin\gamma^{is}e^{i\gamma^{is}}\frac{a_i^3 r_{is}^2}{\left(a_i^2+r_{is}^2\right)^2}\right]\right]+q_{1-1}^{i}\left[\cos\gamma^{is}\sin^2\psi^{is}e^{-i\gamma^{is}}\frac{8\pi a_i^3 r_{is}^2}{3\left(a_i^2+r_{is}^2\right)}\right.$$

The first level of approximation according to Neumann problem corresponds to take $\varphi^{\mathrm{i}} = \varphi_{00}^{i}$. Then equations (2.3.1) and (2.3.8) reduced to

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega}\left(\varphi\frac{\partial\Gamma}{\partial\nu} - \Gamma\frac{\partial\varphi}{\partial\nu}\right)ds + \sum_{i=1}^{N} q_{00}^{i}\frac{a_i^2}{r_i} \qquad (2.4.14)$$

$$\int_{\partial\Omega}\left(\varphi\frac{\partial\Gamma_0^s}{\partial\nu} - \Gamma_0^s\frac{\partial\varphi}{\partial\nu}\right)ds - \varphi_{00}^{s} + a_s q_{00}^{s} + \sum_{\substack{i=1\\ i\neq s}}^{N}\frac{a_i^2}{r_{is}}q_{00}^{i} = 0 \qquad (2.4.15)$$

These equations are called the zeroth-order equations and their solution is called the zeroth–order solution.

The next level of approximation corresponds to set M=1 in equation (2.1.3). Thus equations (2.1.6) and (2.3.8) reduced to

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega} (\varphi \frac{\partial \Gamma}{\partial v} \text{-}\Gamma \frac{\partial \varphi}{\partial v}) ds + \sum_{i=1}^{N} \left[q_{00}^{i} \frac{a_i^2}{r_i} + \left[\varphi_{1-1}^{i} \frac{a_i^2}{r_i^2} Y_1^{-1}(\psi^i, \gamma^i) + \right.\right.$$

$$\left.\left. \varphi_{10}^{i} \frac{a_i^2}{r_i^2} Y_1^{0}(\psi^i, \gamma^i) + \varphi_{11}^{i} \frac{a_i^2}{r_i^2} Y_1^{1}(\psi^i, \gamma^i) \right]\right], \tag{2.4.16}$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_0^s}{\partial v} \text{-}\Gamma_0^s \frac{\partial \varphi}{\partial v}) ds - \varphi_{00}^s + a_s q_{00}^s + \sum_{\substack{i=1 \\ i \neq s}}^{N} \left[q_{00}^{i} \frac{a_i^2}{r_{is}} + \varphi_{1-1}^{i} \frac{a_i^2}{r_{is}^2} Y_1^{-1}(\psi^{is}, \gamma^{is}) \right.$$

$$\left. + \varphi_{10}^{i} \frac{a_i^2}{r_{is}^2} Y_1^{0}(\psi^{is}, \gamma^{is}) + \varphi_{11}^{i} \frac{a_i^2}{r_{is}^2} Y_1^{1}(\psi^{is}, \gamma^{is}) \right] = 0. \tag{2.4.17}$$

If we take $n = 1$ in equation $(2.3.9)$ we obtain

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{1-1}^s}{\partial v} \text{-}\Gamma_{1-1}^s \frac{\partial \varphi}{\partial v}) ds + \frac{8\pi}{3} \varphi_{1-1}^s + \sum_{\substack{i=1 \\ i \neq s}}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{1-1}^s}{\partial v} \text{-}\Gamma_{1-1}^s q_{00}^i) ds = 0, \tag{2.4.18}$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{10}^s}{\partial v} \text{-}\Gamma_{10}^s \frac{\partial \varphi}{\partial v}) ds + \frac{4\pi}{3} \varphi_{10}^s + \sum_{\substack{i=1 \\ i \neq s}}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{10}^s}{\partial v} \text{-}\Gamma_{10}^s q_{00}^i) ds = 0, \tag{2.4.19}$$

$$\int_{\partial\Omega} (\varphi \frac{\partial \Gamma_{11}^s}{\partial v} \text{-}\Gamma_{11}^s \frac{\partial \varphi}{\partial v}) ds + \frac{8\pi}{3} \varphi_{11}^s + \sum_{\substack{i=1 \\ i \neq s}}^{N} \int_{S_i} (\varphi^i \frac{\partial \Gamma_{11}^s}{\partial v} \text{-}\Gamma_{11}^s q_{00}^i) ds = 0, \tag{2.4.20}$$

Similar to Dirichlet case we evaluate the integral $\int_{S_i} (\varphi^i \frac{\partial \Gamma_{1\beta}^s}{\partial v} \text{-}\Gamma_{1\beta}^s q_{00}^i) ds = 0.$

To achieve this, let $\beta = \text{-}1,0,1.$ Then

$$\text{-}\int_{S_i} \Gamma_{1\beta}^s q_{00}^i \, ds = \text{-}q_{00}^i \int_{S_i} \Gamma_{1\beta}^s \, ds = \text{-}q_{00}^i \int_{S_i} \frac{-Y_1^{*\beta}(\theta^s, \phi^s)}{\left|\boldsymbol{x} - \boldsymbol{\xi}^s\right|} ds.$$

Using equations (2.4.8), (2.4.9), and (2.4.10) we get to order $O(\varepsilon^2)$,

$$\text{-}\int_{S_i} \Gamma_{1-1}^s q_{00}^i \, ds = -\left(\frac{4\pi q_{00}^i a_i^2 r_{is} \sin\psi^{is}}{a_i^2 + r_{is}^2} \right) \left[\frac{2a_i^2 \cos\gamma^{is}}{3\left(a_i^2 + r_{is}^2\right)} - \cos\gamma^{is} - i\left(\sin\gamma^{is} - \frac{2a_i^2 \sin\gamma^{is}}{3\left(a_i^2 + r_{is}^2\right)} \right) \right],$$

$$\text{-}\int_{S_i} \Gamma_{10}^s q_{00}^i \, ds = -\left(\frac{4\pi q_{00}^i a_i^2 r_{is}}{a_i^2 + r_{is}^2} \right) \left(\frac{2a}{3\left(a_i^2 + r_{is}^2\right)} - 1 \right),$$

$$-\int_{S_i} \Gamma_{11}^s q_{00}^i \, ds = -\left(\frac{4\pi q_{00}^i a_i^2 r_{is} \sin\psi^{is}}{a_i^2 + r_{is}^2}\right)\left[\frac{2a_i^2 \cos\gamma^{is}}{3\left(a_i^2 + r_{is}^2\right)} - \cos\gamma^{is} + i\left(\sin\gamma^{is} - \frac{2a_i^2 \sin\gamma^{is}}{3\left(a_i^2 + r_{is}^2\right)}\right)\right].$$

For the integral $\int_{S_i} \varphi^i \frac{\partial \Gamma_{1\beta}^s}{\partial v} ds$ we have

$$\int_{S_i} \varphi^i \frac{\partial \Gamma_{1\beta}^s}{\partial v} ds = \sum_{n=0}^{K}\sum_{m=-n}^{n} \varphi_{nm}^i \int_{S_i} Y_n^m(\theta^i,\phi^i) \frac{\partial}{\partial v}\left(\frac{Y_1^{*\beta}(\theta^s,\phi^s)}{\left|x-\xi^s\right|}\right) ds$$

$$= \sum_{n=0}^{K}\sum_{m=-n}^{n} \varphi_{nm}^i \left.\frac{\partial}{\partial t_i}\right|_{t_i=a_i}\left(\int_{S_i} \frac{Y_n^m(\theta^i,\phi^i)Y_1^{*\beta}(\theta^s,\phi^s)}{\left|x-\xi^s\right|} ds\right) \text{ where}$$

$$\boldsymbol{x} = (t_i \sin\theta^i \cos\phi^i, t_i \sin\theta^i \sin\phi^i, t_i \cos\theta^i)$$

$$= a_i^2 \sum_{n=0}^{K}\sum_{m=-n}^{n} \varphi_{nm}^i \left.\frac{\partial}{\partial t_i}\right|_{t_i=a_i}\left(\int_0^{2\pi}\int_0^{\pi} \frac{Y_n^m(\theta^i,\phi^i)Y_1^{*\beta}(\theta^s,\phi^s)}{\left|x-\xi^s\right|} \sin\theta^i d\theta^i d\phi^i\right)$$

Define J_β to be $J_\beta = \left.\frac{\partial}{\partial t_i}\right|_{t_i=a_i}\left(\int_0^{2\pi}\int_0^{\pi} \frac{Y_n^m(\theta^i,\phi^i)Y_1^{*\beta}(\theta^s,\phi^s)}{\left|x-\xi^s\right|} \sin\theta^i d\theta^i d\phi^i\right)$ then

$J_{-1} = \left.\frac{\partial}{\partial t_i}\right|_{t_i=a_i}(I_{-1});\ J_0 = \left.\frac{\partial}{\partial t_i}\right|_{t_i=a_i}(I_0);\ J_1 = \left.\frac{\partial}{\partial t_i}\right|_{t_i=a_i}(I_1)$, where I_{-1}, I_0, I_1

are given in equations (2.4.11), (2.4.12), and (2.4.13). Therefore

$$\int_{S_i} \varphi^i \frac{\partial \Gamma_{1-1}^s}{\partial v} ds = a_i^2\left[\varphi_{1-1}^i\left(\left(r_{is}^2 \sin^2\psi^{is} \cos\gamma^{is} e^{-i\gamma^{is}} \frac{8\pi(r_{is}^2 - 3a_i^2)}{3\left(a_i^2 + r_{is}^2\right)^3} - \frac{4\pi(r_{is}^2 - a_i^2)}{3\left(a_i^2 + r_{is}^2\right)^2}\right)\right.\right.$$

$$\left.\left.-i\left(r_{is}^2 \sin^2\psi^{is} \sin\gamma^{is} e^{-i\gamma^{is}} \frac{-8\pi(r_{is}^2 - 3a_i^2)}{3\left(a_i^2 + r_{is}^2\right)^3} - i\frac{4\pi(r_{is}^2 - a_i^2)}{3\left(a_i^2 + r_{is}^2\right)^2}\right)\right) + \varphi_{10}^i.\right.$$

$$.\left(\left(\frac{8\pi(r_{is}^2 - a_i^2)}{3\left(a_i^2 + r_{is}^2\right)^2} r_{is}^2 \sin\psi^{is} \cos\psi^{is} \cos\gamma^{is}\right)\right.$$

$$\left.+i\left(r_{is}^2 \sin\psi^{is} \cos\psi^{is} \sin\gamma^{is} e^{-i\gamma^{is}} \frac{-8\pi(r_{is}^2 - a_i^2)}{3\left(a_i^2 + r_{is}^2\right)^2}\right)\right)$$

$$+\varphi_{11}^{i}\left(\left(r_{is}^{2}\sin^{2}\psi^{is}\cos\gamma^{is}e^{i\gamma^{is}}\frac{8\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}-\frac{4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right.$$

$$\left.\left.-i\left(\frac{-8\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}r_{is}^{2}\sin^{2}\psi^{is}\sin\gamma^{is}e^{i\gamma^{is}}-i\frac{4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right)\right].$$

$$\int_{S_i}\varphi^{i}\frac{\partial\Gamma_{10}^{s}}{\partial v}ds=a_{i}^{2}\left[\varphi_{1-1}^{i}\left(2r_{is}^{2}\sin\psi^{is}\cos\psi^{is}e^{-i\gamma^{is}}\frac{4\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}\right)+\right.$$

$$\varphi_{10}^{i}\left(\frac{-4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}+r_{is}\frac{8\pi(r_{is}^{2}-3a_{i}^{3})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}\cos^{2}\psi^{is}\right)+$$

$$\left.\varphi_{11}^{i}\left(2r_{is}^{2}\sin\psi^{is}\cos\psi^{is}e^{i\gamma^{is}}\frac{4\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}\right)\right].$$

$$\int_{S_i}\varphi^{i}\frac{\partial\Gamma_{11}^{s}}{\partial v}ds=a_{i}^{2}\left[\varphi_{1-1}^{i}\left(\left(r_{is}^{2}\sin^{2}\psi^{is}\cos\gamma^{is}e^{-i\gamma^{is}}\frac{8\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}-\frac{4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right.\right.$$

$$\left.+i\left(r_{is}^{2}\sin^{2}\psi^{is}\sin\gamma^{is}e^{-i\gamma^{is}}\frac{-8\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}-i\frac{4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right)+\varphi_{10}^{i}\cdot$$

$$\cdot\left(\left(\frac{8\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}r_{is}^{2}\sin\psi^{is}\cos\psi^{is}\cos\gamma^{is}\right)\right.$$

$$\left.+i\left(r_{is}^{2}\sin\psi^{is}\cos\psi^{is}\sin\gamma^{is}e^{-i\gamma^{is}}\frac{-8\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right)$$

$$+\varphi_{11}^{i}\left(\left(r_{is}^{2}\sin^{2}\psi^{is}\cos\gamma^{is}e^{i\gamma^{is}}\frac{8\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}-\frac{4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right.$$

$$\left.\left.+i\left(\frac{-8\pi(r_{is}^{2}-3a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{3}}r_{is}^{2}\sin^{2}\psi^{is}\sin\gamma^{is}e^{i\gamma^{is}}-i\frac{4\pi(r_{is}^{2}-a_{i}^{2})}{3\left(a_{i}^{2}+r_{is}^{2}\right)^{2}}\right)\right)\right].$$

2.5 Applications

Example 1 [17]

Let us solve the Dirichlet problem for the exterior of the unit sphere.

That is, find φ ($\boldsymbol{x}$) such that

$$\begin{aligned} &\Delta\varphi(\boldsymbol{x}) = 0 && \boldsymbol{x} \in \Omega \\ &\varphi(\boldsymbol{x}) = \varphi_{00} = \text{constant} && \boldsymbol{x} \in S \end{aligned}$$

where Ω is the exterior of the unit sphere.

The zeroth order equations (2.4.1) and (2.4.2) reduced to

$$\varphi(\boldsymbol{y}) - \frac{q_{00}}{r} = 0 \tag{2.5.1}$$

$$-\varphi_{00} + q_{00} = 0 \tag{2.5.2}$$

Hence $\varphi(\boldsymbol{y}) = \dfrac{\varphi_{00}}{r}$ which agrees with the solution obtained by Stackgold [15]

The fist order equations (2.4.3)-(2.4.7) reduced to equations same as (2.5.1), and (2.5.2).

Equations (2.4.5)-(2.4.7) implies that $q_{1-1} = q_{10} = q_{11} = 0$.

Thus equations (2.4.3) and (2.4.4) implies

$$\varphi(\boldsymbol{y}) - \frac{q_{00}}{r} = 0,$$

$$-\varphi_{00} + q_{00} = 0,$$

and we obtain the same solution as the zeroth-order equations.

Example 2 (A single sphere in a half–space) [17]

In this example we consider a single sphere of radius a with center a distance d ($a < d$) below the surface of the half – space $-\infty < \boldsymbol{x_1} < \infty$, $-\infty < \boldsymbol{x_2} < \infty$, , $\boldsymbol{x_3} < 0$. The potential on the sphere is considered to be constant φ_{00}, while on the surface of the half–space is taken zero.

We consider the first order equations (2.4.3)–(2.4.7), applied to the sphere and to its image with respect to the plane $\boldsymbol{x_3} = 0$. .

We denote the quantities associated with the image sphere by a prime. From the geometry of the problem we have $q'_{1-1} = q_{1-1} = 0 = q_{11} = q'_{11}$, further since the potential φ is vanishing on the surface $\boldsymbol{x_3} = 0$, we obtain $q'_{00} = -q'_{00}, q'_{10} = q_{10}$. We enclose the sphere and its image by a large sphere of radius R_1 and we let $R_1 \to \infty$ so that the integrals over $\partial\Omega$ in equations (2.4.3)-(2.4.7) vanish.

Since $\psi^{is} = 0$ and $q_{1-1} = 0 = q_{11}$ then equations $(2.4.5)$-$(2.4.7)$ are satisfied. Equation $(2.4.4)$ reduced to

$$-\varphi_{00} + aq_{00} + \frac{a^2}{2d}(q_{00} + \frac{q_{10}}{3}\frac{a}{2d}) = 0 \tag{2.5.3}$$

Setting $\varepsilon_1 = \dfrac{a}{2d}\left(\varepsilon_1 < \dfrac{1}{2}\right)$ we get

$$-\varphi_{00} + aq_{00} + a\varepsilon_1(q_{00} + \frac{q_{10}}{3}\varepsilon_1) = 0$$

or $$aq_{00}(1+\varepsilon_1) + q_{10}\frac{a\varepsilon_1{}^2}{3} = \varphi_{00} \tag{2.5.4}$$

Now equation (2.4.6) reduced to

$$q_{10}\frac{4\pi a}{3} - \int_S \Gamma_{10} q\, ds = 0.$$

But

$$-\int_S \Gamma_{10} q\, ds = 2q_{00}\left(\frac{2\pi a^2.2d}{a^2+4d^2} - \frac{a^4.(2d)}{\left(a^2+4d^2\right)^2}\frac{4\pi}{3}\right) + \frac{2}{3}q_{10}\left(\frac{4\pi a^3.4d^2}{\left(a^2+4d^2\right)^2} - \frac{2\pi a^3}{a^2+4d^2}\right)$$

$$= -q_{00}\left(\frac{8\pi a^2 d}{a^2+4d^2} - \frac{16\pi a^4 d}{3\left(a^2+4d^2\right)^2}\right) + q_{10}\left(\frac{32\pi a^3 d^2}{3\left(a^2+4d^2\right)^2} - \frac{4\pi a^3}{3\left(a^2+4d^2\right)}\right)$$

$$= q_{00}\frac{8\pi a^2 d}{a^2+4d^2}\left(1-\frac{2a^2}{3\left(a^2+4d^2\right)}\right)+q_{10}\left(\frac{4\pi a}{3}+\frac{4\pi a^3}{3\left(a^2+4d^2\right)}\left(\frac{8d^2}{\left(a^2+4d^2\right)}-1\right)\right)=0$$

$$\text{or } q_{00}\frac{a\varepsilon_1}{\varepsilon_1^2+1}\left(1-\frac{2}{3}\frac{\varepsilon_1^2}{\varepsilon_1^2+1}\right)+q_{10}\left(\frac{-a}{3}+\frac{a\varepsilon_1^2}{3\left(\varepsilon_1^2+1\right)^2}\left(\varepsilon_1^2-1\right)\right)=0. \qquad (2.5.5)$$

From equation $(2.5.4)$ we have $-q_{10}=\frac{3\varphi_{00}}{\varepsilon_1^2 a}+q_{00}\frac{3(1-\varepsilon_1)}{\varepsilon_1^2}$,

and from equation $(2.5.5)$

$$-q_{00}\frac{a\varepsilon_1}{\varepsilon_1^2+1}\left(1-\frac{2\varepsilon_1^2}{3\left(\varepsilon_1^2+1\right)}\right)-\left(\frac{-3\varphi_{00}}{\varepsilon_1^2 a}-q_{00}\frac{3(1-\varepsilon_1)}{\varepsilon_1^2}\right)\left(\frac{-a}{3}-\frac{a\varepsilon_1^2}{3\left(\varepsilon_1^2+1\right)^2}\left(\varepsilon_1^2-1\right)\right)=0.$$

Thus

$$q_{00}\frac{a\varepsilon_1}{\varepsilon_1^2+1}\left(1-\frac{2\varepsilon_1^2}{3\left(\varepsilon_1^2+1\right)}\right)-\left(\frac{-a}{3}+\frac{a\varepsilon_1^2\left(\varepsilon_1^2-1\right)}{3\left(\varepsilon_1^2+1\right)^2}\left(\frac{3(1-\varepsilon_1)}{\varepsilon_1^2}\right)\right)=\frac{3\varphi_{00}}{\varepsilon_1^2 a}$$

$$\text{Hence } q_{00}=\frac{\dfrac{\varphi_{00}}{\varepsilon_1^2 a}\left(-a+\dfrac{a\varepsilon_1^2\left(\varepsilon_1^2-1\right)}{\left(\varepsilon_1^2+1\right)^2}\right)}{\dfrac{a\varepsilon_1}{\varepsilon_1^2+1}\left(1-\dfrac{2\varepsilon_1^2}{3\left(\varepsilon_1^2+1\right)}\right)-\left(-a+\dfrac{a\varepsilon_1^2\left(\varepsilon_1^2-1\right)}{\left(\varepsilon_1^2+1\right)^2}\left(\dfrac{(1-\varepsilon_1)}{\varepsilon_1^2}\right)\right)}$$

and the first order solution is given by q_{00}, q_{10}

The corresponding zeroth-order solution of the zeroth-order equations $(2.4.1)$ and $(2.4.2)$ is given by

$$q_{00}=\frac{\varphi_{00}}{a}\frac{1}{\varepsilon_1-1}.$$

Example 3 (An infinite row of spheres in a half – space) [17]

Consider an infinite row of identical spheres in a half –space. Suppose that each sphere has a radius *a* and that the centers of the spheres are uniformly equal distributed along an axis parallel to $\boldsymbol{x}_3$ at a distance *d* (*d* > *a*) belong $\boldsymbol{x}_3=0$. Further suppose the distance between two successive centers is *L* distance apart. Assume that the potential boundary of each sphere is constant φ_{00} and that the potential on the surface of the half–space ($\boldsymbol{x_3} = 0$) is zero. In order to solve this problem, we consider a finite number of spheres namely 2*N* + 1 sphere. We completely reflect the problem about the plane $\boldsymbol{x_3} = 0$ so that the boundary condition on $\boldsymbol{x_3} = 0$ is satisfied identically. For conveince we index the spheres by an integer *n* which takes the values from –*N* to *N*.

Let the quantities associated with the image spheres be denoted by a prime.

From the untisymmetry we have

$$q_{00}'^{n} = -q_{00}^{n}; \quad q_{10}'^{n} = q_{10}^{n}$$

$$q_{11}'^{n} = q_{11}^{n} = 0 = q_{1-1}'^{n} = q_{1-1}^{n}.$$

Also from the symmetry about the center sphere (*n*=0) we have

$$q_{00}^{-n} = q_{00}^{n}; \quad q_{10}^{-n} = q_{10}^{n} \qquad q_{11}^{-n} = q_{11}^{n} = 0 \ \text{ and } \ q_{1-1}^{-n} = q_{1-1}^{n} = 0$$

As we did in the case of a single sphere, we enclose all the spheres by a sphere of radius $R_1 \to \infty$ so that the surface integrals in equation (2.4.3) to (2.4.7) vanish in the limit. Next identity sphere *S* in these equations with *n*=0 then from the symmetry of the problem equations (2.4.5) and (2.4.7) are satisfied identically and equations (2.4.4), (2.4.6) reduced to

$$-\varphi_{00} + aq_{00} + \frac{a^2}{2d}(-q_{00} - \frac{q_{10}a}{3.2d}) - 2\sum_{n=1}^{N}\left[\frac{-a^2}{(nL)}q_{00}^{n} + \frac{a^2}{\sqrt{(nL)^2 + 4d^2}}.\right.$$

$$\left.\left(q_{00}^{n} + \frac{q_{10}^{n}.2ad}{3.\left((nL)^2 + 4d^2\right)}\right)\right] = 0 \tag{2.5.6}$$

$$q_{10}\,\frac{4\pi a}{3} + q_{00}\left(\frac{-8\pi a^2 d}{a^2 + 4d^2} + \frac{16\pi a^4 d}{3\left(a^2 + 4d^2\right)}\right) + q_{10}\left(\frac{4\pi a^3}{3\left(a^2 + 4d^2\right)} - \frac{32\pi a^3 d^2}{3\left(a^2 + 4d^2\right)^2}\right)$$

$$-2\sum_{n=1}^{N}\left[-q_{10}^{n}\frac{4\pi a^3}{3\left(a^2+(nL)^2\right)}-q_{00}^{n}\left(\frac{4\pi a^2\sqrt{(nL)^2+4d^2}}{a^2+(nL)^2+4d^2}\cdot\frac{2d}{\sqrt{(nL)^2+4d^2}}\right.\right.$$

$$\left.+\frac{8\pi a^4}{3d\left(a^2+(nL)^2+4d^2\right)^2}\right)-q_{10}^{n}\left(\frac{(4\pi/3)a^3}{\left(a^2+(nL)^2+4d^2\right)}\right.$$

$$\left.\left.-\frac{(8\pi/3)a^3\left((nL)^2+4d^2\right)}{3d\left(a^2+(nL)^2+4d^2\right)^2}\frac{4d^2}{(nL)^2+4d^2}\right)\right]=0. \qquad (2.5.7)$$

Where we have dropped the superscript when n=0. Now pass to the limit $N\to\infty$ and note that $q_{00}^{n}\to q_{00}$, $q_{10}^{n}\to q_{10}$ weget

$$-\varphi_{00}+q_{00}\left[\frac{a^2+2ad}{2d}-2\sum_{n=1}^{\infty}\left(\frac{-a^2}{(nL)}+\frac{a^2}{\sqrt{(nL)^2+4d^2}}\right)\right]+q_{10}\left(\frac{-a^3}{12d^2}-2\sum_{n=1}^{\infty}\frac{2a^3d}{3\left((nL)^2+4d^2\right)^{3/2}}\right)$$

$$=0 \qquad (2.5.8)$$

$$q_{00}\left[\frac{-8\pi a^2 d}{a^2+4d^2}+\frac{16\pi a^4 d}{3\left(a^2+4d^2\right)}+2\sum_{n=1}^{\infty}\left(\frac{-4\pi a^2}{a^2+(nL)^2+4d^2}+\frac{8\pi a^4}{3d\left(a^2+(nL)^2+4d^2\right)^2}\right)\right]$$

$$+q_{10}\left[\frac{4\pi a^3}{3\left(a^2+4d^2\right)}-\frac{32\pi a^3 d^2}{3\left(a^2+4d^2\right)^2}-\frac{4\pi a}{3}+2\sum_{n=1}^{\infty}\left(\frac{4\pi a^3}{3\left(a^2+(nL)^2\right)}\right.\right.$$

$$\left.\left.+\frac{4\pi a^3}{3\left(a^2+(nL)^2+4d^2\right)}-\frac{8\pi a^3\left(4d^2\right)}{3d\left(a^2+(nL)^2+4d^2\right)^2}\right)\right]=0 \qquad (2.5.9)$$

Example 4 (A single sphere in a half – space) [17]

Consider a sphere of radius a kept at a constant normal flux is placed in a lower half-space with a distance d (d > a) below the surface of the half - space. Further assume that q is equal to zero at the surface of the half –sphere and equal to constant q_{00} on the surface of the sphere.

Consider the image of the sphere and enclose it with the original sphere by sphere of radius R_1 so that as $R_1\to\infty$ the integrals over $\partial\Omega$ in equations (2.4.17)-(2.4.20) vanish.

Now in order to have zero flux on the surface of the half -space the flux on the image sphere is taking to be $-q_{00}$ and we take $\varphi'_{00} = -\varphi_{00}$, $\varphi'_{10} = -\varphi_{10}$, $\varphi'_{1-1} = \varphi_{1-1} = \varphi'_{11} = \varphi_{11} = 0$.

Hence the flux depend only on ψ and r.

Thus equation (2.4.17) reduced to

$$-\varphi_{00} + aq_{00} + \frac{a^2}{2d} q_{00} + \varphi_{10}\frac{a^2}{(2d)^2} = 0 \qquad (2.5.10)$$

And equation (2.4.19) reduced to

$$\frac{4\pi}{3}\varphi_{10} + \int_S \varphi \frac{\partial \Gamma_{10}}{\partial \nu} ds - \int_S \Gamma_{10} q_{00} ds = 0.$$

Or

$$\frac{4\pi}{3}\varphi_{10} + a^2\varphi_{10}\left(\frac{-4\pi}{3}\frac{4d^2 - a^2}{(a^2+4d^2)^2} + \frac{8\pi}{3}(2d)\frac{4d^2-3a^2}{(a^2+4d^2)^3}\right)$$

$$+\frac{4\pi q_{00} a^2 (2d)}{a^2+4d^2}\left(\frac{2}{3(a^2+4d^2)} - 1\right) = 0 \qquad (2.5.11)$$

Thus φ_{00} and φ_{10} determine the potintial on the surface of the sphere.

Example 5

In this example we consider Dirichlet problem for the exterior of the unit sphere. That is find φ such that

$$\begin{aligned} \Delta\varphi(\boldsymbol{x}) &= 0 && , \boldsymbol{x} \in \Omega \\ \varphi(\boldsymbol{x}) &= \varphi_{00} + \varphi_{11} Y_1^1(\theta,\phi) && , \boldsymbol{x} \in S \end{aligned}$$

Where Ω is the exterior of the unit ball and S is the surface of the ball. Taking $q = q_{00}$, we drop the superscripts since we have only one ball, then equations (2.2.18) and (2.2. 21) reduced to

$$\varphi(\boldsymbol{y}) = \frac{1}{r^2}\varphi_{11}Y_1^1(\psi,\gamma) + \frac{q_{00}}{r} \tag{2.5.12}$$

and

$$-\varphi_{00} + q_{00} = 0. \tag{2.5.13}$$

Hence

$$\varphi(\boldsymbol{y}) = \frac{1}{r^2}\varphi_{11}Y_1^1(\psi,\gamma) + \frac{\varphi_{00}}{r}$$

Chapter Three

Special Boundary Integral Equation for Approximate Solution of Potential Problems in Three-Dimensional Regions with Slender Cavities of Circular Cross–section

In this chapter we consider potential problems in general three–dimensional regions with slender internal cavities of circular cross–section. We assume that the surface potential and its normal derivative are locally axis-symmetric. With this assumption, the surface integrals on a cavity boundary can be reduced to contour integrals a long the center line of the cavity. The solution at any point a long the cavity is determined by a special integral equation formed by locating the fundamental solution at point in the center of the cavity. The theory in this chapter based on [3].

3.1 Integral Equations [3]

As in chapter two we consider a three–dimensional open region Ω containing n slender cavities with variable circular cross – section. We notice the following definitions:

C_i is the space curve along the center of cavity i where $i=1,2,...,n$.

$a_i(s)$ is the radius of the cavity i circular cross-section at a particular arc length S along C_i.

$\boldsymbol{r}_i(S)$ is the position vector to points C_i.

$\boldsymbol{\lambda}_i(S)=\dfrac{\partial \boldsymbol{r}_i}{\partial s}$ the unit tangent vector to C_i.

k_i is the local curvature of C_i.

S_i is the lateral boundary of cavity i.

$\partial\Omega$ is the outer boundary of region Ω.

Let φ be a regular harmonic function in Ω, a point $\boldsymbol{x}$ will be stand for the triple (x_1, x_2, x_3).

let $\Gamma(\boldsymbol{x},\boldsymbol{y}) = \dfrac{-1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$.

If φ is a $C^2(\overline{\Omega})$ harmonic function then by Green's representation formula we have

$$\lambda\,\varphi(\boldsymbol{y}) = \int_{\partial\Omega} \left(\varphi\frac{\partial\Gamma}{\partial\nu} - \Gamma\frac{\partial\varphi}{\partial\nu}\right)da + \sum_{i=1}^{n}\int_{\partial S_i}\left(\varphi\frac{\partial\Gamma}{\partial\nu} - \Gamma\frac{\partial\varphi}{\partial\nu}\right)dS \tag{3.1.1}$$

where $\lambda = \begin{cases} 1, & \text{if } \boldsymbol{y}\in\Omega \\ 0, & \text{if } \boldsymbol{y}\in R^3\setminus\overline{\Omega} \end{cases}$

Assume that both the surface potential φ and its normal derivative on S_i are axi-symmetric about C_i so that

$$\text{Let}\quad \varphi = \varphi_i(s)\ ,\frac{\partial\varphi}{\partial\nu} = q_i(s) \quad \text{on } S_i. \tag{3.1.2}$$

The integrals on S_i in equation (3.1.1) then reduced to contour integrals along C_i as follows. Let θ represent the polar angle in the cross-section of the cavity so that the coordinates (s,θ) span the lateral surface S_i. In terms of these coordinates the element of area da on S_i is

$$\begin{aligned} da &= a_i[(a_i')^2 + (1-a_ik_i\cos\theta)^2]^{\frac{1}{2}}\,d\theta ds \\ &\approx a_i[(a_i')^2 + 1^2]^{\frac{1}{2}}\,d\theta ds = a_i\sigma_i\,d\theta\,ds \end{aligned} \tag{3.1.3}$$

where $a_i' = \dfrac{\partial a_i}{\partial s}$; $\sigma_i = [(a_i')^2 + 1^2]^{\frac{1}{2}}$

also we assumed that $a_ik_i \ll 1$.

Let ν be the inward unit normal to S_α, then

$$\frac{\partial \Gamma}{\partial \nu} = \nabla \Gamma . \nu = \frac{\partial \Gamma}{\partial r}(\frac{\boldsymbol{r}}{r}) . \nu = \frac{\boldsymbol{r} . \nu}{4\pi \, r^3}$$

where $r = |\boldsymbol{x} - \boldsymbol{y}|$. $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$

So that with (3.1.2) and (3.1.3) the surface integral may be expanded in the form

$$\int_{S_i} (\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu}) da = \int_{C_i} \int_0^{2\pi} [\varphi_i \frac{\boldsymbol{r} . \nu}{4\pi r^3} + \frac{1}{4\pi \, r} q_i] a_i \sigma_i \, d\theta ds$$

$$= \frac{1}{4\pi} \int_{C_i} a_i \varphi_i \sigma_i \int_0^{2\pi} \frac{\boldsymbol{r} . \nu}{r^3} d\theta ds + \frac{1}{4\pi} \int_{C_i} a_i q_i \sigma_i \int_0^{2\pi} \frac{1}{r} d\theta \, ds$$

From the following figure

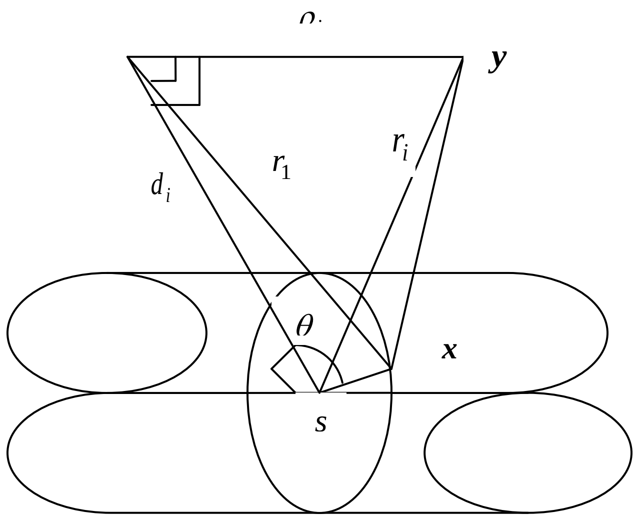

Figure 2

We get

$$\rho_i = |(\boldsymbol{r}_i - y) . \boldsymbol{\lambda}_i|, \qquad r_i = |\boldsymbol{r}_i - y|, \qquad d_i^2 = r_i^2 - \rho_i^2,$$

$$b_i^2 = \rho_i^2 + (a_i + d_i)^2, \qquad \text{and} \;\; k_i^2 = \frac{4 a_i d_i}{b_i^2} \tag{3.1.4}$$

Now $(r_1)^2 = r^2 - \rho_i^2 = a_i^2 + d_i^2 - 2a_i d_i \cos\theta$, (Law of cosines).

Or $\quad r^2 = \rho_i^2 + a_i^2 + d_i^2 - 2a_i d_i \cos\theta$ (3.1.5)

Thus $r^2 = b_i^2 \left(\dfrac{\rho_i^2 + a_i^2 + d_i^2 - 2a_i d_i \cos\theta}{b_i^2} \right)$

$$= b_i^2 \left(\frac{\rho_i^2 + a_i^2 + d_i^2 + 2a_i d_i - 2a_i d_i - 2a_i d_i \cos\theta}{\rho_i^2 + (a_i + d_i)^2} \right)$$

$$= b_i^2 \left(1 - \frac{2a_i d_i (1+\cos\theta)}{\rho_i^2 + (a_i + d_\alpha)^2} \right) = b_i^2 \left(1 - \frac{4a_i d_i}{b_i^2} \frac{(1+\cos\theta)}{2} \right)$$

$$= b_i^2 \left(1 - k_i^2 \frac{(1+\cos\theta)}{2} \right) = b_i^2 \left(1 - k_i^2 \cos^2 \frac{\theta}{2} \right)$$

Hence $\quad r^2 = b_i^2 \left(1 - k_i^2 \cos^2 \dfrac{\theta}{2} \right).$ (3.1.6)

So that

$$\int_0^{2\pi} \frac{d\theta}{r} = \int_0^{2\pi} \frac{d\theta}{b_i \left(1 - k_i^2 \cos^2 \frac{\theta}{2} \right)^{\frac{1}{2}}} = 2\int_0^{\pi} \frac{d\theta}{b_i \left(1 - k_i^2 \sin^2 \frac{\theta}{2} \right)^{\frac{1}{2}}} = \frac{4}{b_i} \int_0^{\pi/2} \frac{d\theta}{\left(1 - k_i^2 \sin^2 \theta\right)^{\frac{1}{2}}} = \frac{4}{b_i} K(k_i)$$

where $K(k_i) = \displaystyle\int_0^{\pi/2} \frac{d\theta}{\left(1 - k_i^2 \sin^2 \theta\right)^{\frac{1}{2}}}$,

and

$$\int_0^{2\pi} \frac{\boldsymbol{r}.\nu}{r^3} d\theta = \frac{2}{b_i^3} \int_0^{\pi} \frac{d_i \cos\theta - a_i}{\left(1 - k_i^2 \cos^2 \frac{\theta}{2} \right)^{\frac{3}{2}}} d\theta.$$

After some calculations we end with

Then

$$\int_0^{2\pi} \frac{\boldsymbol{r}.\nu}{r^3} d\theta = \frac{2}{a_i b_i} [-K(k_i)] + \frac{1}{a_i b_i} \left[\left(2 - \frac{4a_i^2}{b_i^2} - k_i^2 \right) E(K_i) \right]$$

where $E(K_i) = \int_0^{\pi/2} \frac{d\theta}{\left(1-k_i^2 \sin^2\theta\right)^{\frac{3}{2}}}$

Hence

$$\int_{S_i} (\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu}) da = \frac{1}{4\pi} \int_{C_i} a_i \varphi_i \sigma_i \int_0^{2\pi} \frac{\boldsymbol{r}.\nu}{4\pi\, r^3} d\theta ds + \frac{1}{4\pi} \int_{C_i} a_i q_i \sigma_i \int_0^{2\pi} \frac{1}{r} d\theta ds$$

$$= \int_{C_i} \left\{ \varphi_i \frac{\sigma_i}{4\pi b_i} \left[-2K(k_i) + \left(2 - \frac{4a_i^2}{b_i^2} - k_i^2 \right) E(K_i) \right] - q_i \left(\frac{-a_i \sigma_i}{\pi b_i} K(k_i) \right) \right\} ds$$

$$= \int_{C_i} (\varphi_i H_i - q_i G_i) ds \tag{3.1.7}$$

Where $G_i = G_i(\boldsymbol{r}_i(s), \boldsymbol{y}) = \frac{-\sigma_i a_i}{\pi b_i} K(k_i)$

$$H_i = H_i(\boldsymbol{r}_i(s), \boldsymbol{y}) = \frac{\sigma_i}{4\pi b_i} \left[-2K(k_i) + \left(2 - \frac{4a_i^2}{b_i^2} - k_i^2 \right) E(K_i) \right]$$

Therefore for $\boldsymbol{y} \in \Omega$,

$$\varphi(\boldsymbol{y}) = \int_{\partial\Omega} (\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu}) da + \sum_{i=1}^{n} \int_{S_i} (\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu}) da =$$

$$\int_{\partial\Omega} \left(\varphi \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial \varphi}{\partial \nu}\right) da + \sum_{i=1}^{n} \int_{C_i} (\varphi_i H_i - q_i G_i) ds \tag{3.1.8}$$

We introduce the special kernel $\Gamma_i = \Gamma(\boldsymbol{x}, \boldsymbol{r}_i) = \frac{-1}{4\pi |\boldsymbol{x} - \boldsymbol{r}_i|}$ to determine either φ_i or q_i along the cavities.

Now if $\boldsymbol{x} \neq \boldsymbol{r}_i$ then $\Delta \Gamma_i = 0$. Hence for every point on C_i which is located by $\boldsymbol{r}_i$

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma_j}{\partial\nu}-\Gamma_j\frac{\partial\varphi}{\partial\nu})da+\sum_{i=1}^{n}\int_{S_i}(\varphi\frac{\partial\Gamma_j}{\partial\nu}-\Gamma_j\frac{\partial\varphi}{\partial\nu})da=0 \quad (j=1,2,...,n) \tag{3.1.9}$$

Because of (3.1.2) the integrals on S_i may be reduced to contour integrals on C_i so (3.1.9) reduced to

$$\int_{\partial\Omega}(\varphi\frac{\partial\Gamma_j}{\partial\nu}-\Gamma_j\frac{\partial\varphi}{\partial\nu})da+\sum_{i=1}^{n}\int_{C_i}(\varphi_i H_{ij}-q_i G_{ij})ds=0 \qquad (j=1,2,\ldots,n) \tag{3.1.10}$$

where $H_{ij}=H_{ij}(\boldsymbol{r}_i,\boldsymbol{r}_j)=\dfrac{\sigma_{ij}}{4\pi b_{ij}}\left[-2K(k_{ij})+\left(2-\dfrac{4a_{ij}^2}{b_{ij}^2}-k_{ij}^2\right)E(K_{ij})\right]$

$$G_{ij}=G_{ij}(\boldsymbol{r}_i,\boldsymbol{r}_j)=\frac{-a_i\sigma_i}{\pi b_{ij}}K(k_{ij}) \tag{3.1.11}$$

And corresponding to (3.1.2) we have

$$\rho_{ij}=\left|(\boldsymbol{r}_i-\boldsymbol{r}_j).\vec{\lambda}_i\right|, \qquad r_{ij}=\left|\boldsymbol{r}_i-\boldsymbol{r}_j\right|, \qquad d_{ij}^2=r_{ij}^2-\rho_{ij}^2,$$

$$b_{ij}^2=\rho_{ij}^2+(a_i+d_{ij})^2, \quad \text{and} \quad k_{ij}^2=\frac{4a_i d_{ij}}{b_{ij}^2}$$

3.2 Coaxial Tori [3]

Now want to apply the integral equation (1.1.4) to collection of coaxial toroidal cavities in unbounded region. In this case all the curves C_i are circles centered on a common axis of symmetry.

Define the following:

R_i the radius of C_i $(i=1,2,\ldots,n)$

Z_i the point where the plane containing C_i intersects the axis of symmetry

Let $\dfrac{a_i}{R_i}<<1$ (3.2.1)

$$\varepsilon_{ij}=\frac{a_i}{\sqrt{(R_i-R_j)^2+(Z_i-Z_j)^2}}<<1 \tag{3.2.2}$$

These assumptions state that because of axially symmetric, the separation between cavities is large compared with $a_\alpha . \varphi_i$ and q_i are constants on each cavity and for an unbounded region (3.1.9) is reduced to

$$\sum_{i=1}^{n}\varphi_i \int_{S_i}\frac{\partial\Gamma_j}{\partial\nu}\text{da-}\, q_i\int_{S_i}\Gamma_j = 0 \quad . \qquad (j=1,2,\ldots,n) \tag{3.2.3}$$

In this equation the integrals depend on the geometry only. Since Γ_j is a regular harmonic function inside every cavity except S_i we have

$$\sum_{i=1}^{n}\varphi_i \int_{S_i}\frac{\partial\Gamma_j}{\partial\nu}da=\varphi_j \tag{3.2.4}$$

The resulting system is

$$\varphi_j=a_j q_j Log\left(\frac{8R_j}{a_j}\right)+\sum_{\substack{i=1\\ i\neq j}}^{n} a_i q_i\left(\frac{R_i}{R_j}\right)^{\frac{1}{2}}\xi_{ij}K(\xi_{ij}) \quad (j=1,2,\ldots,n) \tag{3.2.5}$$

where $$\xi_{ij}^2=\frac{4R_iR_j}{(R_i\text{-}R_j)^2+(Z_i\text{-}Z_j)^2} \tag{3.2.6}$$

Example 1 [3]

Consider two identical tori with opposite potentials $\pm\varphi_0$ separated by a distance *2d*. The corresponding flux q_1, q_2 are equal and opposite says $\pm q_0$

Thus $a_1=a_2=a,$ $\qquad R_1=R_2=R$

$\varphi_1=\varphi_2=\varphi_0,$ $\qquad q_1=-q_2=q_0$

sothat (3.2.5) and (3.2.6) will be

$$\varphi_1=a_1q_1Log\frac{8R_1}{a_1}+a_2q_2\left(\left(\frac{R_2}{R_1}\right)^{1/2}\xi_{21}K(\xi_{21})\right)$$

$$\varphi_0 = aq_0 Log\frac{8R}{a} - aq_0\xi K(\xi) \Rightarrow \ \frac{\varphi_0}{aq_0} = \text{Log}\frac{8\text{R}}{\text{a}} - \xi\text{K}(\xi) \quad \text{where}$$

$$\xi^2 = \xi_{21}^2 = \xi_{12}^2 = \frac{4R^2}{\left((2R)^2 + (2d)^2\right)} = \frac{R^2}{R^2 + d^2} = \frac{R^2 + d^2 - d^2}{R^2 + d^2} = 1 - \frac{d^2}{R^2 + d^2} = 1 - \frac{1}{1 + \left(\frac{R}{d}\right)^2}$$

as $\frac{d}{R} \to \infty \quad \frac{R}{d} \to 0$ so $\xi^2 \to 0 \quad \xi \to 0$

Thus $\frac{\varphi_0}{aq_0} = Log\frac{8R}{a}$

Note that $\varphi_2 = a_2 q_2 Log\frac{8R_2}{a_2} + a_1 q_1\left(\left(\frac{R_2}{R_1}\right)^{\frac{1}{2}} \xi_{12} K(\xi_{12})\right)$

$$\Rightarrow -\varphi_0 = -aq_0 Log\frac{8R}{a} + aq_0\xi K(\xi) \ \Rightarrow \ \varphi_0 = aq_0 Log\frac{8R}{a} - aq_0\xi K(\xi)$$

Or $\frac{\varphi_0}{aq_0} = Log\frac{8R}{a} - \xi K(\xi)$ Where $\xi^2 = 1 - \frac{1}{1 + \left(\frac{R}{d}\right)^2}$

So we obtain the same result.

Example 2 [3]

Consider a torodial cavity located in the midplane of a uniform slap of thikness *d* has zero potential on each surface. The solution of this problem can be obtained by solving the equivalent problem of an infinite series of parallel tori assigned with alternating potentials and separated by a distance *d*. The solution relating the flux *q* to the potential φ on the torus is

$$\frac{\varphi}{aq} = Log\left(\frac{8R}{a}\right) + 2\sum_{n=1}^{\infty}(-1)^n\left(\xi_n^{-1}\text{K}\left(\xi_n^{-1}\right)\right) \qquad \text{where} \quad \xi_n^2 = 1 + \left(\frac{nd}{2R}\right)^2$$

To show this assume there exist *r* tori under our torus, so there exist *r* tori above it, now

$$\varphi = aq\log\frac{8R}{a} + 2\left(a(-q)\xi_1' K(\xi_1') + \ldots\ldots \pm aq\xi_r' k(\xi_r')\right)$$

$$= aq\log\frac{8R}{a} + 2\sum_{n=1}^{r}(-1)^n aq\xi_n' K(\xi_n')$$

Where $\xi_n'^2 = \frac{4R^2}{(2R)^2 + (nd)^2} \Rightarrow \frac{1}{\xi_n'^2} = \frac{(2R)^2 + (nd)^2}{4R^2} = 1 + \left(\frac{nd}{2R}\right)^2$

Set $\xi_{\text{n}} = \frac{1}{\xi_n'}$ then $\varphi = aq\log\frac{8R}{a} + 2\sum_{n=1}^{r}(-1)^n aq\xi_n^{-1} K(\xi_n^{-1})$

Or $\frac{\varphi}{aq} = \log\frac{8R}{a} + 2\sum_{n=1}^{r}(-1)^n \xi_n^{-1} K(\xi_n^{-1})$

So as $r \to \infty$ we have

$$\frac{\varphi}{aq} = \log\frac{8R}{a} + 2\sum_{n=1}^{\infty}(-1)^n \xi_n^{-1} K(\xi_n^{-1})$$

where $\xi_{\text{n}}^2 = 1 + (\frac{nd}{2r})^2$

As $\frac{d}{R} \to \infty$, $\frac{\varphi}{aq}$ approaches that for single torus in unbounded region, and as R increases, we have $\frac{d}{R} \to 0$ $\frac{\varphi}{aq} \to Log\frac{2d}{\pi a}$.

References

[1] S. Axler, D. Bourdon and W. Ramay. Harmonic Functions Theory. Springer-Verlag, 2001.

[2] M. R. Barone and D. A. Caulk. Special Boundary Integral Equations For Approximate Solution Of Laplace's Equation In Two-Dimensional Regions with Circular Holes. Q. J. Mech. Appl. Math. Vol.xxxiv,Pt.3,1981.

[3] M. R. Barone and D. A. Caulk. Special Boundary Integral Equations For Approximate Solution Of Laplace's Equation In Three Dimensional Region with Slender Cavities Of Circular Cross-section. IMA Journal of Applied Mathematics, 35 (311-325), 1985.

[4] F. J. Bayley, J. M. Owen and A. B. Turner. Heat Transfer. Barnes and Noble, New York, 1972.

[5] R. P. Canwal. Linear Integral Equations Theory and Technique. Academic Press, 1971.

[6] D. Colton and R. Kress. Integral Equation Methods in Scaltering theory. Wiley, New York, 1983.

[7] D. Gilbarg and N. S. Trudinger. Elliptic Partial Differential Equations of Second Order. Springer-Verlag Berlin Heidelberg 2001.

[8] M. A. Jaswon and G. T. Symm. Integral Equation Methods in Potential Theory and Electrostatics. Academic Press, 1977.

[9] J. Jost. Partial Differential Equations. Springer-Verlag, 2007

[10] R. Kress. Linear Integral Equations. Springer-Verlag, 1989.

[11] T. A. Kruse and F.J. Rizzo, eds. Boundary Integral Equation Method: Computational Applications in Applied Mechanics. ASME, New York, 1975.

[12] L. M. Milne-Thomson. Theortical Hydrodynamics. Macmillan, London, 5th edition, 1968.

[13] H. W. Richmond. Proc. Lond. Math.Soc., 22(389-403). 1923.

[14] M. N. O. Sadiku. Numerical Techniques in Electromagnetics. Second Edition, CRC Press, 2001.

[15] I. Stakgold. Boundary Value Problems Of Mathematical Physics. Vol.ii, MacMillan, 1968.

[16] J. L. Throne. Plastics Processing Engineering. Marcel Dekker, Inc.,New York, 1979.

[17] Y. Zahaykah. Special Boundary Integral Equations For Approximate Solution Of Laplace's Equation In Three-Dimensional Regions. Master Thesis. Jordan, 1989.