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Support Points of the set of Univalent Functions

By

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Declaration:

I Certify that this submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis(or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed…………..

Wafa Badi Shaheen Al-Ashhab

Date: 31- 5 - 2004

Dedication

To my father, to my mother who resembles the spring of giving and sacrifice and to my brothers and sisters.

Acknowledgement

Praise Be To God; the Creator of the Universe. Peace Be Upon His Messenger Prophet Mohammad, and On the Prophet's Posterity and All his Companions.

Upon the completion of this thesis, I would like to extend my sincere appreciation, respect and thanks, to Dr. Ibrahim Al-Grouz (who supervised my thesis), for his efforts in giving me the best advice and counseling, and utilizing his practical experience, in order to reach at a correct, accurate, and successful study.

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Finally, I want to thank Al-Quds University for giving me the chance to obtain this Master Degree.

Abstract

Our main goal in the thesis is to study the support points of normalized univalent functions .We begin with special classes of functions in the set of normalized univalent functions such as the family of starlike functions(S^*); The family of convex functions(K) ; The family of closed convex functions(CL), then the support points of these families are studied by presenting some theorems. Later on, some families of univalent functions are discussed like Typically real functions(T) and positive real functions($\mathcal{P}$).

Considerable effort has been devoted to the study of support points of normalized univalent functions. In general many factors play a role in the determination of support points of a given class of functions. The support points of these classes(S^*,K,CL,T,$\mathcal{P}$) are identified , but this is not the case of the class of normalized univalent functions.

Some facts and properties of a function to be a support point of the family of normalized univalent functions proved at the end of this thesis.

According to this study, one might think the support points and extreme points coincide. However there is an example of extreme points which is not a support points. On the other hand there is an example of support points which is not extreme points .

الملخص

الهدف الأساسي من هذه الرسالة هو دراسة الاقترانات المساندة لمجموعة الاقترانات المركبة الأحادية والتي تحقق بعض الشروط الخاصة(S) حيث أننا بدأنا بدراسة بعض المجموعات الجزئية لهذه المجموعة مثل الاقترنات شبه النجمية(S^*) ، الاقترانات المقعرة(K)، الاقترانات القريبة من التقعر(CL) و ذلك عن طريق إثبات العديد من النظريات. كذلك درسنا مجموعات جزئية أخرى.

هنالك جهد عظيم تم تخصيصه لدراسة الاقترانات المساندة لمجموعة الاقترنات الأحادية حيث تبين أن هنالك العديد من العوامل التي تحدد كون الاقتران مساندا للمجموعة أم لا و بينا العديد من الاقترانات المساندة لبعض المجموعات الجزئية لهذه المجموعة ، و أخيرا درسنا الاقترانات المساندة للمجموعة(S) حيث أنها وحتى الآن لم يتم تحديد الاقترانات المساندة لهذه المجموعة و إنما هنالك نتائج جزئيه تدل على ذلك.

و يعتقد البعض بان الاقترانات المساندة هي نفسها الاقترانات القصوى، و هذا غير صحيح،حيث أوردنا مثالا لاقتران مساند و ليس من الاقترانات القصوى و مثالا لأحد الاقترانات القصوى و ليس اقترانا مساندا.

Contents

Introduction

Let Δ be the unit disk consists of all point z of the complex plane C of modulus $|z| < 1$. In the first chapter , we introduce the definition of univalent functions and basic results about univalent functions.

The theory of univalent functions is a classical field beginning at least as early as 1907 with the paper P. Koebe [24].

A number of survey articles have been written about the general theory of univalent functions and more specific developments and we mention [3],[7],[12],[17],[18], and [36] .

The book [17] by A.W.Goodman almost provides an encyclopedia about univalent functions and describes a very large number of results in the field, even on quite specialized topics.

We shall need the material of this chapter throughout the following chapters.

In the second chapter, we present the special families of univalent functions,

Riemann's Theorem enables many problems in general domains D to be reduced to problems in Δ. Thus the class of the corresponding conformal maps or functions univalent in Δ, acquires a special importance. We may normalize so that $f(0) = 0,\ f'(0) = 1$. Otherwise we may consider $g(z) = (f(z) - f(0))/f'(0)$ instead of f, since g is univalent if and only if f is univalent . We accordingly denote by S the class of functions $f(z) = z + \sum_{n=0}^{\infty} a_n z^n$ univalent in Δ.

We introduce the subsets of S consisting of starlike functions S^*, convex functions K, and close to convex functions CL, we also introduce the class of functions having a positive real functions $\mathcal{P}$, and the typically real functions T, and obtain similar results for these classes.

Generalization of the families S^* and K were introduced by M.S .Robertson in [35].The geometric characterization mentioned for close to convex functions is due to Z. Lewandowski[26,27].The class T of typically real functions was introduced and studied by W.Rogosinski[40].

In chapter three , we first present the definition of subordination and prove some initial facts about subordination.

Subordination was more formally introduced and studied by J.E.Littlewood[28] and later by Rogosinski[39], we first prove some basic properties(Littlewood 1925).

In the second part of this chapter we discuss the linear topological structure of $\mathcal{A}$, where $\mathcal{A}$ denotes the set of all functions analytic in the unit disk Δ. The topology on $\mathcal{A}$ is given by a metric which is equivalent to the topology of uniform convergence on compact subsets of Δ. We study the topological and convexity properties of the various subsets of $\mathcal{A}$. We introduce the idea of a continuous linear functional and begin our study of one of the central themes in this chapter. We also present the Krein – Milman theorem as a fundamental tool in this development.

The book[11] by N.Dunford and J.T Schwarts is a source for information about locally convex linear topological spaces, including the Krein – Milman theorem. Finally in this chapter we identify closed convex hulls and the extreme points of some classes of univalent functions.

In chapter four, we introduce some of the relationships between support points and extreme points of a compact subset $\mathcal{F}$ of $\mathcal{A}$, described at the beginning of this chapter are contained in [6].

Subclasses of functions of $\mathcal{A}$ have been studied throughout the twentieth century . However the systematic application of linear methods to study the extreme points and support points of subclasses is more recent . In the field of geometric function theory it began to play an important role starting in the 1970. In this survey we hope to convey some of the substance and flavor of the use of the linear methods in this field.

For a given compact family $\mathcal{F}$ two basic problems in the application of linear methods are:

Problem 1. Determine support points.

Problem 2. Identify geometric analytic properties of the functions in the set of support points.

Families defined by geometric analytic conditions frequently lead to integral representation which in turn yield a complete answer to Problem 1," determine support points" , represented by some theorems, as we prove in section 2.

The family S itself has not yielded to such a tractable description . However some success has been achieved in answering Problem 2 for S. In section 3 we elaborate on these remarks.

Chapter one

Elementary properties of univalent functions

The aim of this introductory chapter is to review and gather for later reference some of the general principles of complex analysis which underlie the theory of univalent functions. In many instances the statements of theorems are supported by bare indications of a proof , or by no proof at all.

1.1 Basic Principles

Let Δ be the unit disk consists of all points $z \in C$ of modulus $|z| < 1$.Its boundary is the unit circle, is denoted by $\partial\Delta$.

A function f is a complex function if its domain is a subset of a complex set. Let $z = x + iy$ .Then f can be written as $f(z) = U(x, y) + iV(x, y)$, where $U(x, y) = \text{Re } f(z)$ and $V(x, y) = \text{Im } f(z)$

Definition 1.1.1:

Let f be a complex function and write $f(z) = U(x, y) + iV(x, y)$,where $z = x + iy$.

If $z_0 = x_0 + y_0$ is a complex number then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} U(x, y) + i \lim_{(x,y)\to(x_0,y_0)} V(x, y)$$

Definition 1.1.2:

A function f is differentiable at z if $\lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists. If so, we write

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

Definition 1.1.3:

A complex function f is continuously differentiable on a domain D if $f'(z)$ is continuous on D.

Definition 1.1.4:

A complex function f is analytic on D if f is continuously differentiable on D. In particular f is analytic at a point z_0 if it is analytic in a neighborhood of z_0.

Let $\mathcal{A}$ be the set of all analytic functions on D.

The following theorem is a simple application of the definition of analytic functions.

Theorem 1.1.1:

Suppose f and g are analytic on D, then

1. $f+g, f-g, f.g$ are analytic on D.
2. if $g(z) \neq 0 \, \forall z \in D$, then $\frac{f}{g}$ is analytic on D.

Theorem 1.1.2:

If f is differentiable on a domain D and $f'(z)=0, \forall z \in D$, then f is constant.

Proof : see[9,page 37].

Lemma 1.1.1(Schwarz's lemma):

If $f(z)$ is analytic on Δ with $f(0)=0$ and $|f(z)|<1, \forall z \in \Delta$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in Δ. Moreover if $|f'(0)|=1$ or if $|f(z)|=|z|$ for some $z \neq 0$ then there is a constant $c, |c|=1$, such that $f(z)=cz$ for all z in Δ.

Proof: see [9,pages 130,131].

1.2 Univalent functions

Definition 1.2.1:

A complex function $f(z)$ is said to be univalent in a domain D if for any z_1, z_2 in D with $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$, or if $f(z_1) = f(z_2)$, then $z_1 = z_2$

Example 1.2.1 :

The identity function $f(z) = z$ is univalent in C..

Example 1.2.2:

The function $f(z) = z^2$ is not univalent in C..

Definition 1.2.2:

A function $f : D \to C$ is a conformal mapping if f is analytic and univalent on D.

Theorem 1.2.1(Riemann's mapping Theorem):

Let D be a simply connected domain which is a proper subset of the complex plane. Let z_0 be a given point in D .Then there is a unique function f which maps D conformally onto the unit disk and has the properties $f(z_0) = 0$ and $\arg f'(z_0) = \alpha$, where $0 \leq \alpha < 2\pi$.

Proof : see[2, page 222].

We now list some simple properties of univalent functions.

Theorem 1.2.2:

Let $f(z)$ be univalent in Δ then the addition of a constant merely translates the image domain so that $f(z) + C$ is again univalent in Δ.

Proof :

Suppose that $f(z)$ is univalent, let $g(z) = f(z) + C$,to show that $g(z)$ is univalent ,

let $g(z_1) = g(z_2)$,we must prove that $z_1 = z_2$,so $f(z_1) + C = f(z_2) + C$,which

implies $f(z_1) = f(z_2)$, since $f(z)$ is univalent ,then $z_1 = z_2$.

Hence $g(z)$ is univalent.

Theorem 1.2.3:

If $f'(z_0) = 0$ then $f(z)$ is not univalent in any neighborhood of z_0.

Proof:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 0 .$$

There exists z such that $f(z) - f(z_0) = 0$ where $z \neq z_0$, implies that $f(z) = f(z_0)$.

Hence $f(z)$ is not univalent in any neighborhood of z_0.

Theorem 1.2.4:

Let f be analytic and univalent on Δ, and let g be defined by

$$g(z) = \frac{f(z) - f(0)}{f'(0)}$$ where $f'(0) \neq 0$, then

- g is analytic and univalent in Δ.
- $g(0) = 0$ and $g'(0) = 1$.

Proof:

Clearly, $g(z)$ is analytic.

To show that $g(z)$ is univalent , let $g(z_1) = g(z_2)$,so

$$\frac{f(z_1) - f(0)}{f'(0)} = \frac{f(z_2) - f(0)}{f'(0)}$$ which implies that

$f(z_1) = f(z_2)$,since $f(z)$ is univalent, so $z_1 = z_2$.

Hence $g(z)$ is univalent.

Also $g(0)=0$ where $f'(0)\neq 0$ .Now,

$g'(z)=\dfrac{f'(z)}{f'(0)}$, therefore, $g'(0)=1$.

Theorem 1.2.5:

Let $f(z)$ be univalent in Δ then $f(z)$ must be a simple pole.

Proof:

Suppose $f(z)$ has a pole of order two at 0, $f(z)=\dfrac{g(z)}{z^2}$ where $g(z)$ is analytic

, $g(0)\neq 0$. But $f'(z)=\dfrac{z^2g'(z)-2zg(z)}{z^4}=\dfrac{zg'(z)-2g(z)}{z^3}$, which implies that

$$zg'(z)-2g(z)=z^3f'(z) \qquad (1.2.1)$$

Also, $g(z)=z^2f(z)$, so $g'(z)=z^2f'(z)+2zf(z)$, therefore $g'(0)=0$, substitute by (1.2.1) which implies that $g(0)=0$, this contradiction .

Hence $f(z)$ must be a simple pole.

Theorem 1.2.6:

Let $f(z)$ be univalent and w is a function analytic and univalent on the range of f, with $w(0)=0$ and $w'(0)=1$, then $g=w\circ f$ is univalent .

Proof :

To show that $g(z)$ is univalent, let

$g(z_1)=g(z_2)$,so

$w(f(z_1))=w(f(z_2))$ but w is univalent, which implies that

$f(z_1) = f(z_2)$, since f is univalent, then

$z_1 = z_2$, hence $g(z)$ is univalent.

Example 1.2.3:

The sum of two univalent functions need not be univalent .For example the functions

$z(1-z)^{-1}$ and $z(1+iz)^{-1}$ are univalent in Δ but the sum of $z(1-z)^{-1}$ and $z(1+iz)^{-1}$

has a derivative which vanishes at $\frac{1}{2}(1+i)$, and so the sum is not univalent in Δ.

Proof:

Let $f(z)$ the sum of two functions $z(1-z)^{-1}$ and $z(1+iz)^{-1}$,so

$f(z) = \frac{z}{1-z} + \frac{z}{1+iz}$. But

$$f'(z) = \frac{(1+iz)^2 + (1-z)^2}{(1-z)^2(1-iz)^2} = \frac{1+2iz - z^2 + 1 - 2z + z^2}{(1-z)^2(1-iz)^2} = \frac{2-2(1-i)z}{(1-z)^2(1-iz)^2}.$$

So $f'(z) = 0$, which implies that $z = \frac{1}{2}(1+i)$ .Since $|z| = \left|\frac{1}{2}(1+i)\right| < 1$, then , by

theorem (1.2.3), $f(z)$ is not univalent, hence the sum of two univalent functions need not be univalent.

Definition 1.2.3:

Let A be a set, A is called a convex set if the line segment joining any two points of A is contained in A, that is to say, if $x_1, x_2 \in \mathrm{A}$ then $tx_1 + (1-t)x_2 \in \mathrm{A}, \quad t \in [0,1]$.

Also, we say that the function f is convex if and only if $f(\Delta)$ is a convex set.

Theorem 1.2.7:

Suppose that D is a convex domain, if h is analytic in D and satisfies $\operatorname{Re} h'(z) > 0$ ($z \in D$) then h is univalent in D.

This theorem special case of theorem (1.2.8).

Lemma 1.2.1:

Suppose that f is analytic in Δ and satisfies $\operatorname{Re}[f'(z)/g'(z)] > 0$ $(|z| < 1)$ where g is analytic and univalent in Δ and $g(\Delta)$ is convex. Then f is univalent in Δ.

Proof:

Let $h = f \circ g^{-1}$ and $D = g(\Delta)$, then h is analytic in the convex domain D.

$h(w) = f(g^{-1}(w)), \quad w \in D.$

Then $h'(w) = \dfrac{f'(g^{-1}(w))}{g'(g^{-1}(w))} = \dfrac{f'(z)}{g'(z)}$, where $w = g(z)$

So $\operatorname{Re}\{h'(w)\} > 0$ in D. Thus h is univalent ,by theorem (1.2.7), since $f = h \circ g$ and g is univalent in Δ this shows that f is univalent in Δ .

Theorem 1.2.8(The Noshiro-Warchawski Theorem):

Suppose that for some real α we have $\operatorname{Re}(e^{i\alpha} f'(z)) > 0$

for all z in a convex domain D, then $f(z)$ is univalent in D .

Proof:

Let z_1 and z_2 be any two distinct points in D.Then $f(z)$ is defined on the linear segment joining z_1 to z_2,we integrate $f'(z)$ along the line segment

$L: z = z_1 + t(z_2 - z_1), \quad 0 \le t \le 1$. Since D is convex, L lies in D. Now

$dz = (z_2 - z_1)dt$, hence

$$f(z_2) - f(z_1) = \int_L f'(z)dz = \int_0^1 f'(z)(z_2 - z_1)dt,$$

$$f(z_2) - f(z_1) = (z_2 - z_1)e^{-i\alpha} \int_0^1 e^{i\alpha} f'(z)dt.$$

But the integral is not zero because it has a positive real part.

Hence, $f(z_1) \neq f(z_2)$ and $f(z)$ is univalent in D.

Chapter Two

Special families of univalent functions

In this chapter we introduce the special families of univalent functions, normalized univalent functions, starlike functions, convex functions, close to convex functions, Typically real functions,positive real function, and obtain similar results for these classes.

2.1 Normalized univalent functions

Definition 2.1.1(normalized univalent functions S):

S is defined to be the set of functions that are analytic and univalent in the unit disk and satisfy the normalization conditions $f(0)=0$ and $f'(0)=1$.

Now let $f \in S$ then, using Taylor series expansion at $z=0$, we have

$f(z)=\sum_{n=0}^{\infty} a_n z^n \,(|z|<1)$ and as $a_0 = f(0)=0$ and $a_1 = f'(0)=1$,in fact, this power

has the form $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$. (2.1.1)

Examples of functions in S :

Example 2.1.1:

Clearly the identity mapping, $f(z)=z$ is analytic in Δ, univalent and $f(0)=0$

, $f'(0)=1$

so $f(z)=z$ is in S .

Example 2.1.2:

The Koebe function $k(z)=\dfrac{z}{(1-z)^2}$, $z \in \Delta$,is in S.

Proof:

To show that $k(z) \in \mathrm{S}$,we must show that $k(z)$ is analytic in Δ,but this is clear, since $z \in \Delta$.

Also, $k(0)=0$ .Now, $k'(z)=\dfrac{1+z}{(1-z)^3}$, therefore, $k'(0)=1$

It remains to show that $k(z)$ is univalent. To do this, let

$k(z_1)=k(z_2)$ so

$\dfrac{z_1}{(1-z_1)^2}=\dfrac{z_2}{(1-z_2)^2}$ which implies that

$(1-z_1)^2 z_2=(1-z_2)^2 z_1$ and thus $z_2-2z_1z_2+z_1^{\,2}z_2=z_1-2z_1z_2+z_2^{\,2}z_1$ which implies $z_2-z_1+z_1^{\,2}z_2-z_2^{\,2}z_1=0$ and hence

$(z_2-z_1)(1-z_1z_2)=0$, but $1-z_1z_2\neq 0$

so $z_1-z_2=0$, $z_1=z_2$ and hence $k(z)$ is univalent in Δ.

Hence $k(z)\in \mathrm{S}$.

Example 2.1.3:

$f(z)=\dfrac{z}{1-z}$, which maps Δ conformally onto the half – plane $\mathrm{Re}\{w\}>-\dfrac{1}{2}$, is in S.

Example 2.1.4:

$f(z)=\dfrac{z}{1-z^2}$, which maps Δ onto the entire plane minus the two half-lines

$\dfrac{1}{2}\le x<\infty$ and $-\infty<x\le-\dfrac{1}{2}$, is in S.

Example 2.1.5:

$f(z)=\frac{1}{2}\log\left[\frac{(1+z)}{(1-z)}\right]$, which maps Δ onto the horizontal strip $-\pi/4<\operatorname{Im}\{w\}<\pi/4$,

is in S.

Definition 2.1.2:

Let Σ denote the set of functions that are analytic and univalent in $\{ z : 0<|z|<1 \}$ and have a simple pole at z =0 with the residue 1. The Laurent series expansion of a function in Σ has the form

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n \qquad (0<|z|<1). \qquad (2.1.2)$$

We note that the subset of Σ for which $g(z)\neq 0$ when $0<|z|<1$ is one to one correspondence with S through the relation $f=\frac{1}{g}$.

Theorem 2.1.1(Area theorem):

If $g\in\Sigma$ and g has the representation (2.1.2) then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

proof: See[20, pages 2,3].

Corollary 2.1.1:

If $g\in\Sigma$ and g is given by (2.1.2) then $|b_1|\leq 1$ .Moreover, if $|b_1|=1$ then $g(z)=1/z+b_0+b_1 z$ and any function of this form belongs to Σ when $|b_1|\leq 1$.

Proof :see[20, pages 3,4].

To help us constructing more examples of functions in S ,we can easily prove the following theorem.

Theorem 2.1.2:

If $f \in$ S and f is given by (2.1.1) then each of the following functions is also in S :

1. $e^{-i\theta} f(e^{i\theta} z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\theta} z^n$, θ real,

2. $\overline{f(\overline{z})} = z + \sum_{n=2}^{\infty} \overline{a_n} z^n$.

3. $\frac{1}{t} f(tz) = z + \sum_{n=2}^{\infty} a_n t^{n-1} z^n$, $0 < t \le 1$

4. $\left[f(z^k)\right]^{1/k} = z + \frac{a_2}{k} z^{k+1} + \frac{1}{2k^2}(2ka_3 - (k-1)a_2{}^2) z^{2k+1} + \ldots..$, where k is a positive integer.

Proof: see[17, page 18].

Lemma 2.1.1:

Let $f \in$S then the function $g(z) = \left[f(z^n)\right]^{\frac{1}{n}}$ belongs to S , for any natural number $n = 2,3,\ldots\ldots$.

Proof: see [20, page 4].

Theorem 2.1.3(De Branges' theorem):

Let $f(z) \in$ S and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ then $|a_n| \le n$ for $n = 2,3,\ldots..$

Proof: See[33, pages 24-26].

Theorem 2.1.4(koebe covering theorem):

The range of every function of class S contains the disk $\{w : |w| < \frac{1}{4}\}$.

Proof: see[20, page 6].

Theorem 2.1.5(distortion theorem):

If $f \in S$ and $|z| < 1$ then

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \qquad (2.1.3)$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \qquad (2.1.4)$$

Proof:

Suppose that $|\mu| < 1$. the function g defined by

$g(z) = f(\frac{z+\mu}{1+\bar{\mu}z})$ is analytic and univalent in Δ. The power series for g begins

$g(z) = b_0 + b_1 z + b_2 z^2 + ...$, where $b_n = \frac{g^n(0)}{n!}$.

$b_0 = f(\mu)$, $b_1 = f'(\mu)(1-|\mu|^2)$ and

$b_2 = \frac{1}{2} f''(\mu)(1-|\mu|^2)^2 - \bar{\mu} f'(\mu)(1-|\mu|^2)$.

The function $(g - b_0)/b_1$ belongs to S and so theorem(2.1.3)implies that $\frac{|b_2|}{|b_1|} \leq 2$,

and so $|b_2| \leq 2|b_1|$ .This coefficient inequality is equivalent to

$$\left|\frac{1}{2} f''(\mu)(1-|\mu|^2)^2 - \bar{\mu} f'(\mu)(1-|\mu|^2\right| \leq 2\left|f'(\mu)(1-|\mu|^2)\right|$$

multiply by 2, we get

$$\left|f''(\mu)(1-|\mu|^2)^2 - 2\overline{\mu}f'(\mu)(1-|\mu|^2\right| \le 4\left|f'(\mu)(1-|\mu|^2)\right|.$$

Divided by $f'(\mu)(1-|\mu|^2)^2$, we get

$$\left|\frac{f''(\mu)}{f'(\mu)} - 2\frac{\overline{\mu}}{1-|\mu|^2}\right| \le \left|\frac{4}{1-|\mu|^2}\right| \quad .$$

multiply by μ we get

$$\left|\frac{\mu f''(\mu)}{f'(\mu)} - \frac{2|\mu|^2}{1-|\mu|^2}\right| \le \frac{4|\mu|}{1-|\mu|^2} .$$ which implies that

$$\frac{-4|\mu|}{1-|\mu|^2} \le \frac{\mu f''(\mu)}{f'(\mu)} - \frac{2|\mu|^2}{1-|\mu|^2} \le \frac{4|\mu|}{1-|\mu|^2}$$

$$\frac{2|\mu|(|\mu|-2)}{1-|\mu|^2} \le \frac{\mu f''(\mu)}{f'(\mu)} \le \frac{2|\mu|(|\mu|+2)}{1-|\mu|^2}$$

$$\frac{2|\mu|(|\mu|-2)}{1-|\mu|^2} \le \mathrm{Re}(\frac{\mu f''(\mu)}{f'(\mu)}) \le \frac{2|\mu|(|\mu|+2)}{1-|\mu|^2} \quad . \qquad (2.1.5)$$

Therefore, inequality (2.1.5) holds whenever $f \in S$ and $|\mu| < 1$.

If $\mu = \mathrm{P}e^{i\theta}$ then

$$\frac{\partial}{\partial \mathrm{P}}\log|f'(\mu)| = \frac{\partial}{\partial \mathrm{P}}\mathrm{Re}\log f'(\mu) = e^{i\theta}\,\mathrm{Re}\frac{f''(\mu)}{f'(\mu)} = \frac{1}{\mathrm{P}}\mathrm{Re}\left(\frac{\mu f''(\mu)}{f'(\mu)}\right)$$

so the inequality(2.1.5)is as same as

$$\frac{2(\mathrm{P}-2)}{1-\mathrm{P}^2} \le \frac{\partial}{\partial \mathrm{P}}\log|f'(\mu)| \le \frac{2(\mathrm{P}+2)}{1-\mathrm{P}^2}$$

If we integrate this inequality from 0 to r we get

$$\log\left(\frac{1-r}{(1+r)^3}\right) \le \log|f'(re^{i\theta})| \le \log\left(\frac{1+r}{(1-r)^3}\right).$$

And therefore

$$\frac{1-r}{(1+r)^3} \le \left|f'(re^{i\theta})\right| \le \frac{1+r}{(1-r)^3}.$$

This proves(2.1.4)

suppose that $f \in$ S and $z = re^{i\theta}$ $(0 < r < 1)$ if γ is the closed line segment from 0 to z then

$$|f(z)| = \left|\int_\gamma f'(\mu)d\mu\right| = \left|\int_0^r f'(\mathrm{P}e^{i\theta})d\mathrm{P}\right| \le \int_0^r \left|f'(\mathrm{P}e^{i\theta})d\mathrm{P}\right| \le \int_0^r \frac{1+\mathrm{P}}{(1-\mathrm{P})^3}d\mathrm{P} = \frac{r}{(1-r)^2}$$

Here we have used the upper bound in (2.1.4).This proves the upper bound in (2.1.3)

In order to prove the lower bound in (2.1.3) suppose that $f(re^{i\theta}) = \mathrm{R}e^{i\phi}$ if $R \ge \frac{1}{4}$ the inequality immediately follows; so we may assume that $R < \frac{1}{4}$. Because of theorem (2.1.4), if $\Gamma = \{w : w = te^{i\phi}, 0 \le t \le R\}$ then $f(\Delta) \supset \Gamma$. Hence. $\gamma = f^{-1}(\Gamma)$ is a curve in Δ with the endpoints 0 and $re^{i\theta}$,using the lower bound in (2.1.4) we find that

$$\left|f(re^{i\theta})\right| = R = \int_\Gamma |dw| = \int_\gamma |f'(\mu)d\mu|$$

$$\ge \int_\gamma |f'(\mu)|d|\mu| \ge \int_0^r \frac{1-u}{(1+u)^3}du = \frac{r}{(1+r)^2}.$$

Theorem 2.1.6 (Prawitz inequality):

Suppose that $f \in$ S and $M(r,f) = \max_{|z|=r}|f(z)|$.If $0 < r < 1$ and $p > 0$ then

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\mathrm{P}} d\theta \le p\int_0^r \frac{M^{\mathrm{P}}(u,f)}{u}du.$$

As a result of this theorem, if $f \in$ S and $f(z)$ has the power series(2.1.1),then

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|d\theta < \frac{r}{1-r} \qquad (0 < r < 1).$$

Proof : see[20, pages 8,9].

2.1 Starlike Functions

Definition 2.2.1:

A set D is called starlike with respect to the point w_0 provided that

$w_0 + t(w - w_0) \in D$ whenever $w \in D$ and $0 \le t \le 1$.

Definition 2.2.2(starlike functions S^*):

A function $f \in S$ is called a starlike if the image of f is starlike with respect to 0.

Let S^* be the set of all starlike functions in S.

Lemma 2.2.1:

If $f \in S^*$ and $0 < r < 1$ then $f(\{z : |z| < r\})$ is starlike with respect to 0 .

Proof: see[20, page 11].

The following theorem showes the relation between the families S and S^*.

Theorem 2.2.1:

$f \in S^*$ if and only if $f \in S$ and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (|z| < 1) . \qquad (2.2.1)$$

Proof:

Suppose that $f \in S^*$ and $0 < r < 1$,by lemma(2.2.1), $f(\{z : |z| < r\})$ is starlike and consequently argument $f(z)$ may be defined locally and becomes a non- decreasing function of θ where $z = re^{i\theta}$ since

$\log f(z) = \ln f(z) + i \arg f(z)$ and so

$\operatorname{Im} \log f(z) = \arg f(z)$, if $z = re^{i\theta}$ then $dz = iz$ and therefore

$$\frac{\partial}{\partial \theta} \operatorname{Im} \log f(z) = \frac{\partial}{\partial \theta} \arg f(z), \text{ but}$$

$$\frac{\partial}{\partial \theta} \arg f(z) = \operatorname{Im} \frac{f'(z)}{f(z)} iz = \operatorname{Re} \frac{zf'(z)}{f(z)} > 0.$$

The converse assertion follows from the fact (2.2.1) is precisely the assertion that the vector $f(re^{i\theta})$ turns with an increasing argument as θ increases. This implies that $D_r = f(\{z : |z| < r\})$ is starlike for every r $(0 < r < 1)$ .Since $f(\Delta) = \bigcup_{0<r<1} D_r$ we conclude that $f(\Delta)$ is starlike.

Example 2.2.1:

The koebe function $k(z) = \dfrac{z}{(1-z)^2}$ is in S^*

Proof:

In example(2.1.2),we proved that $k(z)$ is in S ,to show that $k(z)$ is in S^*,as in theorem(2.2.1) it is enough to show that $\operatorname{Re}\left(\dfrac{zk'(z)}{k(z)}\right) > 0$. To do this

$$k'(z) = \frac{1+z}{(1-z)^3}$$

$$\frac{zk'(z)}{k(z)} = z\frac{1+z}{(1-z)^3} \div \frac{z}{(1-z)^2} = \frac{1+z}{1-z}$$

$$\frac{1+z}{1-z} \times \frac{1-\bar{z}}{1-\bar{z}} = \frac{1-\bar{z}+z-|z|^2}{1-(\bar{z}+z)+|z|^2}$$

Let $z = re^{i\theta}$ which implies that $\dfrac{2ir\sin\theta - r^2 + 1}{1 + r^2 - 2r\cos\theta} = \dfrac{1+z}{1-z}$

$$\mathrm{Re}\left(\frac{zk'(z)}{k(z)}\right)=\frac{1-r^2}{1+r^2-2r\cos\theta}>0.$$ Hence, $k\in S^*$.

Example 2.2.2:

The function $f(z)=\frac{1}{4}\log k'(z)$ is in S^*, where $k(z)$ is koebe function.

Theorem 2.2.2:

Let $f(z)$ be in S^*. Then

$$\frac{r}{(1+r)^2}\le|f(z)|\le\frac{r}{(1-r)^2},$$
$$\frac{1-r}{(1+r)^3}\le|f'(z)|\le\frac{1+r}{(1-r)^3},$$

and for each $k\ge 2$

$$\left|f^{(k)}(z)\right|\le\frac{k!(k+r)}{(1-r)^{k+2}}.$$

Proof : see[17, page 118].

2.3 Positive real part functions

<u>Definition 2.3.1</u> (positive real part functions):

Let $\mathcal{P}$ denote the set of functions p that are analytic in Δ and satisfy

$\mathrm{Re}\,p(z)>0 \quad (|z|<1)$ and $p(0)=1$.

This family is directly related to S^* and to several other classes of univalent functions.

Example 2.3.1:

The function $p(z)=\frac{1+z}{1-z}=1+2\sum_{n=1}^{\infty}z^n$ is in $\mathcal{P}$.

proof:

To show that $p(z) \in \mathcal{P}$, we must show that $p(0)=1$ and $\operatorname{Re} p(z) > 0$. clearly, from the definition of $p(z)$, $p(0)=1$ .Also, $\operatorname{Re} p(z) > 0$, we proved that as in example(2.2.1).

Hence $p(z) \in \mathcal{P}$.

To construct more examples of functions in $\mathcal{P}$, we can use the following theorem.

Theorem 2.3.1:

Suppose that $f(z), f_1(z)$ and $f_2(z)$ are in $\mathcal{P}$ then the function $g(z)$ is also in $\mathcal{P}$, where:

1. $g(z) = f(e^{i\theta} z)$, where θ is any real number.
2. $g(z) = [f(z)]^t$, or $g(z) = f(tz)$ $\qquad -1 \le t \le 1$
3. $g(z) = 1/f(z)$
4. $g(z) = [f_1(z)]^{t_1} [f_2(z)]^{t_2}$ $\qquad 0 \le t_1, t_2, \quad t_1 + t_2 \le 1$
5. $g(z) = \dfrac{1}{a}\left[f(\dfrac{z+\lambda}{1+\bar{\lambda} z}) - bi \right]$ $\qquad$ if $f(\lambda) = a + bi, \quad |\lambda| < 1$
6. $g(z) = \dfrac{f(z) + ib}{1 + ibf(z)}$ $\qquad$ where b is real.

Theorem 2.3.2:

Let $f(z)$ be in $\mathcal{P}$ and $z = re^{i\theta}$ then

$$\frac{1-r}{1+r} \le |f(z)| \le \frac{1+r}{1-r}$$

and

$$|f'(z)| \le \frac{2}{(1-r)^2}.$$

Equality occurs for suitable z if and only if $f(z) = p(e^{i\theta} z)$, where $p(z) = \frac{1+z}{1-z}$.

Proof : see[17, pages 82,83].

Are there any relations between the class $\mathcal{P}$ and the univalent functions?.The following theorem gives us some idea.

Theorem 2.3.3:

Let $f'(z)$ be in $\mathcal{P}$, then $f(z)$ is univalent in Δ.

Proof:

Clearly, from the definition of $\mathcal{P}$. Also $\operatorname{Re} f'(z) > 0$, using theorem (1.2.8).Hence $f(z)$ is univalent in Δ.

Example 2.3.2:

Let $f(z) = -z - 2\ln(1-z)$ then $f'(z) = \frac{1+z}{1-z}$,but in example(2.3.1),we showed that $f'(z) \in \mathcal{P}$, so $f(z)$ is univalent in Δ.

Theorem 2.3.4:

Let $p \in \mathcal{P}$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (|z| < 1)$ then $|p_n| \le 2$ for $n = 1,2,......$.

Proof:

The function $\phi = (p-1)/(p+1)$ is analytic in Δ ,

$|\phi(z)| < 1 \quad (|z| < 1)$

and $\phi(z) = \dfrac{1 + \sum_{n=1}^{\infty} p_n z^n - 1}{1 + \sum_{n=1}^{\infty} p_n z^n + 1} = \dfrac{p_1 z + p_2 z^2 + ...}{2 + p_1 z + p_2 z^2 + ...}$,so

$\phi(0) = 0$.

Therefore $\phi(z) = zw(z)$ where w is analytic in Δ and $|w(z)| \leq 1$ .This implies that

$|w(0)| \leq 1$ and since

$$w(z) = \frac{\phi(z)}{z} = \frac{p(z)-1}{z(p(z)+1)} = \frac{p_1 z + p_2 z^2 + ...}{2z + p_1 z^2 + p_2 z^3 + ...}$$

$$= \frac{z}{z}(\frac{p_1 + p_2 z + ...}{2 + p_1 z + p_2 z^2 + ...})$$

$w(0) = \frac{p_1}{2}$, $p_1 = 2w(0)$ which implies that $|p_1| = |2w(0)| \leq 2$ is proved for $n = 1$ the

only functions with $|p_1| = 2$ are given by $p(z) = (1 + xz)/(1 - xz)$ where $|x| = 1$.

Now suppose that $n \geq 2$ and $y = e^{2\pi i/n}$, the function q defined by

$q(z) = \frac{1}{n}\left[p(z) + p(yz) + p(y^2 z) + ... + p(y^{n-1} z)\right]$ belongs to $\mathcal{P}$

$q(0) = \frac{1}{n}[1 + 1 + 1 + ...] = \frac{n}{n} = 1$ and $\operatorname{Re} q(z) > 0$

Also, q has the form $q(z) = s(z^n)$ where s is analytic in Δ, since

$$q(yz) = \frac{1}{n}\left[p(yz) + p(y^2 z) + p(y^3 z) + ...p(y^n z)\right] = q(z)$$

if we write $s(z) = 1 + s_1 z + s_2 z^2 + ...$ then

$s_1 = p_n$ since $s \in \mathcal{P}$ the inequality $|s_1| \leq 2 \rightarrow |p_n| \leq 2$

the only function with $|p_n| = 2$ are given by $p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n$.

Theorem 2.3.5:

Let $f \in S^*$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $(|z| < 1)$ then $|a_n| \leq n$ for $n = 2,3,......$

Proof:

if $f \in S^*$ then $p \in \mathcal{P}$, where $p(z) = \dfrac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (|z| < 1)$

By equating coefficients of the power series in the relation $zf'(z) = f(z)p(z)$,

where

$zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n = z + 2a_2 z^2 + 3a_3 z^3 + \ldots\ldots$ and

$f(z)p(z) = (1 + p_1 z + p_2 z^2 + \ldots\ldots)(z + a_2 z^2 + a_3 z^3 + \ldots\ldots)$

we conclude that

$2a_2 = a_2 + p_1$, so $a_2 = p_1$ and , in general, for $n \geq 2$,

$(n-1)a_n = a_{n-1}p_1 + a_{n-2}p_2 + \ldots + a_2 p_{n-2} + p_{n-1}$.

With $n = 1$, Theorem(2.3.4)implies that $|a_2| \leq 2$.

Theorem (2.3.4) and an inductive argument complete the proof, since

$(n-1)|a_n| \leq (n-1)2 + (n-2)2 + \ldots.. + 2.2 + 2 = (n-1)n$

Hence $|a_n| \leq n$.

We also note that if $|a_n| = n$ for a given n the argument shows that $|a_2| = 2$ from

which we conclude that $f(z) = z/(1-xz)^2 \quad (|x| = 1)$.

2.4 Convex Functions

Definition 2.4.1(convex functions K):

Let K denote the subset of S consisting of functions f for which $f(\Delta)$ is a convex

set.

Since each convex set is starlike with respect to the origin, we see that $K \subset S^*$.

Lemma 2.4.1:

If $f \in K$ and 0<r<1 then $f(\{z : |z| < r\})$ is a convex set.

Proof: see[20, page 14].

The following theorem showes the relation between the families S and K.

Theorem 2.4.1:

$f \in$ K if and only if $f \in$ S and

$$\mathrm{Re}(\frac{zf''(z)}{f'(z)}+1)>0 \quad (|z|<1\ ). \tag{2.4.1}$$

proof:

Suppose that $f \in$ K and $0<r<1$ .By lemma 2.4.1, $f(\{z:|z|<r\})$ is convex and thus the angle of the tangent to the curve $w=f(re^{i\theta})$, $0 \le \theta \le 2\pi$, is a non decreasing function of θ. This is the same as

$\frac{\partial}{\partial\theta}\{\arg f(z)\} \ge 0 \rightarrow \frac{\partial}{\partial\theta}\{\arg izf'(z)\} \ge 0$ Therefore

$\log izf'(z) = \ln izf'(z) + i \arg izf'(z)$ and so

$\mathrm{Im}\{\log izf'(z) = \arg izf'(z)$

$$\frac{\partial}{\partial\theta}\arg izf'(z) = \frac{\partial}{\partial\theta}\mathrm{Im}\log[izf'(z)$$

$$= \mathrm{Im}\left[\frac{i[zf''(z)+f'(z)]iz}{izf'(z)}\right]$$

$$= \mathrm{Im}\, i\{\frac{zf''(z)}{f'(z)}+1\} = \mathrm{Re}(\frac{zf''(z)}{f'(z)}+1) \ge 0$$

The converse result is a consequence of $\mathrm{K} \subset \mathrm{S}$ and the fact (2.4.1) is the assertion that the curve $w=f(re^{i\theta}), 0 \le \theta \le 2\pi,$ is locally convex. Since $f(\{z:|z|<r\})$ is convex for every r $(0<r<1)$ we conclude that $f(\Delta)$ is convex.

To make the family more clear, we give the following example.

Example 2.4.1:

The function $f(z) = \dfrac{z}{1-z}$ is in K.

proof:

$f(z)$ belongs to S (example 2.1.3).To show that $f(z)$ is in K ,as in theorem (2.4.1),

it is enough to show that $\text{Re}(\dfrac{zf''(z)}{f'(z)}+1) > 0$. But

$$f'(z) = \frac{1}{(1-z)^2} \qquad \text{and}$$

$$f''(z) = \frac{2}{(1-z)^3}, \text{ so}$$

$\dfrac{zf''(z)}{f'(z)} + 1 = \dfrac{1+z}{1-z}$, in example (2.2.1) we show that $\text{Re}(\dfrac{1+z}{1-z}) > 0$.

Hence , $f(z)$ is in K .

Example 2.4.2:

The function $f(z) = \dfrac{1}{2}\log[k(z)/z]$ is in K, where $k(z) = \dfrac{z}{(1-z)^2}$.

Proof:

By using theorem(2.4.1) we must show that $f(z)$ is in S and $\text{Re}\left[\dfrac{zf''(z)}{f'(z)}+1\right] > 0$.

$f(z) = \dfrac{1}{2}\log[k(z)/z] = -\log(1-z)$, but it is clear that $f(z)$ is analytic in Δ.

Also $f(0) = 0$. Now $f'(z) = \dfrac{1}{1-z}$, therefore $f'(0) = 1$.

It remains to show that $f(z)$ is univalent and $\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]>0$, but $f'(z)=\frac{1}{1-z}$

and $f''(z)=\frac{1}{(1-z)^2}$, so $\frac{zf''(z)}{f'(z)}+1=\frac{1}{1-z}$.

$\frac{1}{1-z}\times\frac{1-\bar{z}}{1-\bar{z}}=\frac{1-\bar{z}}{1+(\bar{z}+z)+|z|^2}$, let $z=re^{i\theta}$ which implies that

$\frac{1-r\cos\theta+ir\sin\theta}{1+r^2-2r\cos\theta}=\frac{1}{1-z}$.

$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]=\frac{1-r\cos\theta}{1+r^2-2r\cos\theta}>0.$

Now, we prove that $f(z)$ is univalent, by using theorem 2.3.3, $f'(z)=\frac{1}{1-z}$ is in $\mathcal{P}$,

$f'(0)=1$ and $\operatorname{Re} f'(z)>0$, then $f(z)$ is univalent.

Hence, $f(z)$ is in K.

The following theorem shows the relation between the families K and S^*.

Theorem 2.4.2:

$f\in$K if and only if $g\in S^*$ where $g(z)=zf'(z)$.

Proof:

Suppose that $f\in$K ,to show that $g(z)$ is in S^*,where $g(z)=zf'(z)$, we must prove that $g(z)$ is in S (by theorem 2.2.1), but it is clear that $g(z)$ is analytic in Δ.

Also , $g(0)=0$

And $g'(z)=zf''(z)+f'(z)$, therefore $g'(0)=1$

It remains to show that $g(z)$ is univalent and Re $\left(\frac{zg'(z)}{g(z)}\right)>0$

$\frac{zg'(z)}{g(z)}=\frac{z^2 f''(z)+zf'(z)}{zf'(z)}=\frac{zf''(z)}{f'(z)}+1$.Since

$f \in$ K ,therefore, by theorem (2.4.1) Re $(\frac{zf''(z)}{f'(z)}+1)>0$,so Re$(\frac{zg'(z)}{g(z)}>0$

But Re$(\frac{zg'(z)}{g(z)})=$ Re$(\frac{g'(z)}{f'(z)})>0$, by lemma(1.2.1), $g(z)$ is univalent, since

$f(z)$ is univalent .

Hence $g(z) \in S^*$.

Conversely,

Suppose that $g(z) \in S^*$,where $g(z)=zf'(z)$,to show that $f(z)$ is in K, we must

prove that $f(z)$ is in S, by theorem(2.4.1),but it is clear that $f(z)$ is analytic in

Δ.

Also , $f'(z)=\frac{g(z)}{z}=1+\sum_{n=2}^{\infty} a_n z^{n-1}$,therefore $f'(0)=1$.

And $f(z)=z+\sum_{n=2}^{\infty} a_n \frac{z^n}{n}$

Therefore $f(0)=0$.

It remains to show that $f(z)$ is univalent and Re$[\frac{zf''(z)}{f'(z)}+1]>0$

$\frac{zf''(z)}{f'(z)}+1=\frac{zg'(z)}{g(z)}$ since $g(z) \in S^*$,therefore by theorem (2.2.1) Re$(\frac{zg'(z)}{g(z)}>0$

so, Re$[\frac{zf''(z)}{f'(z)}+1]>0$.

But Re$(\frac{zg'(z)}{g(z)})=$ Re$(\frac{g'(z)}{f'(z)})$, $f(z)$ is univalent.

Theorem 2.4.3:

Let $f \in \mathrm{K}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|z| < 1)$,then $|a_n| \le 1$ for $n = 2,3,.......$

Proof:

By theorem(2.4.2), $g \in \mathrm{S}^*$ where

$$g(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n \qquad (|z| < 1).$$

Theorem (2.3.5) implies that $|na_n| \le n$ and thus $|a_n| \le 1$, the only function with $|a_n| = 1$ are given $f(z) = z/(1 - xz)$ where $|x| = 1$.

Theorem 2.4.4:

The range of every function of class K contains the disk $\{w : |w| < \frac{1}{2}\}$.

Proof: see[20, page 16].

Theorem 2.4.5:

If $f \in \mathrm{K}$ and $|z| < 1$, then

$$\frac{|z|}{1+|z|} \le |f(z)| \le \frac{|z|}{1-|z|}$$

and

$$\frac{1}{(1+|z|)^2} \le |f'(z)| \le \frac{1}{(1-|z|)^2}.$$

Proof: see[20, pages 16,17].

2.5 Close to convex Functions

Definition 2.5.1(close to convex function CL):

Let CL denote the set of functions f that are analytic in Δ, satisfy $f(0)=0$ and $f'(0)=1$ and have the property that given f there exist a real number α and a function g in K so that $\mathrm{Re}(\frac{f'(z)}{e^{i\alpha}g'(z)})>0$. $(|z|<1)$ and $|\alpha|<\frac{\pi}{2}$.

Example 2.5.1:

The function $f(z)=\dfrac{z-\frac{1}{2}(x+y)z^2}{(1-yz)^2}$ is in CL,

where $|x|=1$ and $|y|=1$ and $g(z)=\dfrac{z}{1-yz}$.

Proof:

$f'(z)=(1-xz)/(1-yz)^3$ and if $g(z)=\dfrac{z}{1-yz}$ then $\dfrac{f'(z)}{g'(z)}=\dfrac{1-xz}{1-yz}$.

If $y\neq x$ then $w=(1-xz)/(1-yz)$ maps Δ onto a half plane containing $w=0$ on its boundary and thus there is a real number α so that $\mathrm{Re}(e^{-i\alpha}w)>0$ when $|z|<1$.

Therefore $\mathrm{Re}[f'(z)/e^{i\alpha}\,g'(z)>0$ $(|z|<1)$ and since $g\in$K this shows that $f\in$CL.

In the case $y=x$, $f(z)=z/(1-xz)$ and so $f\in$CL as $K\subset$CL. Thus , in general, the functions $f(z)$ belong to CL (when $|x|=|y|=1$).

Theorem 2.5.1:

Let $f\in$ CL and $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ $(|z|<1)$ then $|a_n|\leq n$ for $n=2,3,.......$

Proof:

If $f \in \mathrm{CL}$ then there is a function g in K and a real number α such that

$\mathrm{Re}[f'(z)/e^{i\alpha}g'(z)] > 0 \quad (|z| < 1)$. If

$$q(z) = \frac{f'(z)}{e^{i\alpha}g'(z)} = \sum_{n=0}^{\infty} q_n z^n$$

then the function

$$p(z) = \frac{q(z) + i\sin\alpha}{\cos\alpha} = 1 + \sum_{n=1}^{\infty} p_n z^n$$

belongs to $\mathcal{P}$,since

$$p(0) = \frac{e^{-i\alpha} + i\sin\alpha}{\cos\alpha} = \frac{\cos\alpha - i\sin\alpha + i\sin\alpha}{\cos\alpha} = 1$$

and $\mathrm{Re}\, p(z) = \dfrac{q(z)}{\cos\alpha} > 0$,

By equating coefficient of the power series in the relation

$p(z)\cos\alpha = q(z) + i\sin\alpha$ where

$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $q(z) = \sum_{n=0}^{\infty} q_n z^n$

we conclude that

$q_n z^n = \cos\alpha p_n z^n$

$q_n = (\cos\alpha) p_n \quad (n = 1,2,.......)$.Theorem(2.3.4) implies that $|p_n| \le 2$, so

$$|q_n| \le |p_n| \le 2 \qquad (n = 1,2,......) \tag{2.5.1}$$

if $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (|z| < 1)$ then theorem (2.4.3) asserts that

$$|b_n| \le 1 \qquad (n = 2,3,.....) \tag{2.5.2}$$

As $q_0 = e^{-i\alpha}$ the relation $f'(z) = e^{i\alpha} g'(z) q(z)$ where

$f'(z) = 1 + \sum_{n=2}^{\infty} na_n z^{n-1}$, $g'(z) = 1 + \sum_{n=2}^{\infty} nb_n z^{n-1}$ and

$q(z) = \sum_{n=0}^{\infty} q_n z^n$ implies that

$$na_n = e^{i\alpha}[nb_n e^{-i\alpha} + (n-1)b_{n-1}q_1 + (n-2)b_{n-2}q_2 + \dots + 2b_2 q_{n-2} + q_{n-1}]$$

applying (2.5.1) and (2.5.2) to this equality we find that

$$|na_n| \le n|b_n| + (n-1)|b_{n-1}||q_1| + (n-2)|b_{n-2}||q_2| + \dots + 2|b_2||q_{n-2}| + |q_{n-1}|$$

$$\le n + (n-1)2 + (n-2)2 + \dots + 2.2 + 2 = n^2$$

$$|na_n| \le n^2$$

Therefore $|a_n| \le n$ for $n = 2,3,\dots\dots$

Theorem 2.5.2:

Every close to convex function is univalent.

Proof :

If f is close to convex, then $\text{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0$, for some convex function g .Let

D be the range of g and the consider the function $h(w) = f(g^{-1}(w))$, $w \in D$.

Then

$$h'(w) = \frac{f'(g^{-1}(w)}{g'(g^{-1}(w)} = \frac{f'(z)}{g'(z)} \quad \text{,so}$$

$\text{Re}(h'(w)) > 0$ in D .Thus h is univalent , by Theorem(1.2.8), and so f is univalent.

2.6 Typically real part functions

Definition 2.6.1:

Let S_R denote the subset of functions f in the set S such that $f(z)$ is real when z is real.

Definition 2.6.2:

Let S^*_R denote the subset of S^* consisting of functions with real coefficients .

Definition 2.6.3:

Let $\mathcal{P}_R$ denote the subset of $\mathcal{P}$ of functions p such that $p(z)$ is real when z is real. $(|z|<1)$.

Definition 2.6.4(Typically real part) :

Let T denote the set of functions that are analytic in Δ, normalized by $f(0)=0$ and $f'(0)=1$ and satisfy $f(z)$ is real if and only if z is real $(|z|<1)$.

Facts:

- $f(z)\in T$ is real when z is real.
- If $f(z)\in T$, then $f(\Delta)$ is symmetric with respect to the real axis.
- $S_R \subset T$.

The following lemma showes the relation between the families Typically real part and positive real part.

Lemma 2.6.1:

$f \in T$ if and only if there is a function p in $\mathcal{P}_R$ so that

$$f(z)=\frac{z}{1-z^2}p(z) \qquad (|z|<1).$$

Proof: see[20, page 20].

As a fact of lemma we introduce the fact , If $f(z) \in \mathrm{T}$, then $(\operatorname{Im} z)[\operatorname{Im} f(z)] \geq 0$.

$(|z|<1)$.

Example 2.6.1:

The function $f(z) = z + z^3$ is in T

Proof:

To show that $f(z)$ is in T , as in lemma(2.6.1) ,we must prove that p is in $\mathcal{P}_{\mathbb{R}}$, where

$p(z) = f(z)\dfrac{1-z^2}{z} = 1 - z^4$,that is, we must show that $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ and

$p(z)$ is real when z is real, clearly from the definition of $p(z)$, $p(0) = 1$.

To show that $\operatorname{Re} p(z) > 0$,let $z = re^{i\theta}$ so $p(z) = 1 - r^4 e^{i4\theta}$ and therefore,

$\operatorname{Re} p(z) = 1 - r^4 \cos 4\theta > 0$, (since $r < 1$).

It remains to show that $p(z)$ is real when z is real, since $z = \bar{z}$, so $p(z) = p(\bar{z})$

$p(\bar{z}) = 1 - r^4 \cos 4\theta + ir^4 \sin 4\theta = \overline{p(z)}$,that is , $p(z) = \overline{p(z)}$ which proves that

$p(z)$ is real when z is real. Hence $f(z) \in \mathrm{T}$.

Example 2.6.2:

The function $f(z) = \dfrac{z}{(1-xz)(1-\bar{x}z)}$ is in T ,where $p(z) = \dfrac{1-z^2}{(1-xz)(1-\bar{x}z)}$, $|x| = 1$

Theorem 2. 6.1 :

Let $f \in \mathrm{T}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $(|z|<1)$ then $|a_n| \leq n$ for $n = 2,3,\ldots..$

Proof:

Let $p(z)=\frac{1-z^2}{z}f(z)=\sum_{n=0}^{\infty}p_n z^n$, then by lemma(2.6.1) , $p\in \mathcal{P}_R$. In particular,

$p\in\mathcal{P}$ and so theorem(2.3.4) implies that $|p_n|\le 2$ $\quad (n=1,2,.....)$

By equating coefficient of the power series in the relation

$$p(z)=(1-z^2)\frac{f(z)}{z} \quad \text{where} \quad p(z)=\sum_{n=0}^{\infty}p_n z^n \quad \text{and} \quad f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

we conclude that

$$p(z)=(1-z^2)(1+\sum_{n=2}^{\infty}a_n z^{n-1})$$

$$p(z)=1+a_2 z+(a_3-1)z^2+(a_4-a_2)z^3+............+(a_n-a_{n-2})z^{n-1}+........ \quad ,\text{so}$$

$$|p_{n-1}|=|a_n-a_{n-2}|\le 2 \quad .$$

To show that $|a_n|\le n$, we use mathematical induction. Since $a_1=1$ then $|a_1|\le 1$.

Now suppose $|a_k|\le k$, then

$$|a_{k+1}|=|a_{k+1}-a_{k-1}+a_{k-1}|\le|a_{k+1}-a_{k-1}|+|a_{k-1}|\le 2+k-1=k+1$$

Hence , $|a_n|\le n$, $\forall n$.

Chapter Three

Subordination and Linear topological spaces.

In the first part of this chapter we introduce the concept of subordination between pairs of analytic functions and prove some initial facts about subordination.

In the second part of this chapter we discuss the linear topological structure of the set of analytic functions.

Finally, we identify the closed convex hulls and the extreme points of the sets S^*,K,C and T, in the remaining of this chapter.

3.1 Subordination

Definition 3.1.1(Subordination):

Suppose that the functions f and g are analytic in Δ. We say that f is subordinate to g in Δ if there exists a function ϕ analytic in Δ such that

$\phi(0)=0$,

$|\phi(z)|<1$, ($|z|<1$) and $f(z)=g(\phi(z))$, ($|z|<1$).

The subordination relation shall be denoted by $f \prec g$.

Note:

Let B denote the set of functions ω that are analytic in Δ and satisfy

$|\omega(z)|\leq 1$, ($|z|<1$).

Let B_0 denote the subset of B of functions ϕ which additionally satisfy

$\phi(0)=0$,that is , B_0 consists of functions satisfying Schwarz's lemma.

Note that $\omega \in B$ if and only if $\phi \in B_0$ where $\phi(z)=z\omega(z)$.

Example 3.1.1:

Let $f(z)=z^n$ and $g(z)=z$. Then $f \prec g$.

Let $\phi(z)=z^n$ .clearly, $f(z)$ and $g(z)$ are analytic in Δ, and

$f(z)=g(\phi(z))=\phi(z)=z^n$,but $\phi(z)$ is analytic in Δ and $\phi(0)=0$,so $f \prec g$.

Example 3.1.2:

Let $f(z)=z^{2n}$ and $g(z)=z^2$ .then $f \prec g$.

Let $\phi(z)=z^n$ .clearly, $f(z)$ and $g(z)$ are analytic in Δ,and

$f(z)=g(\phi(z))=(\phi(z))^2=z^{2n}$, but $\phi(z)$ is analytic in Δ and $\phi(0)=0$,so

$f \prec g$

Theorem 3.1.1:

suppose that f and g are analytic in Δ and g is univalent. If $f(0)=g(0)$ and

$f(\Delta) \subset g(\Delta)$ then f is subordinate to g in Δ.

Proof:

Define ϕ by $\phi=g^{-1}\circ f$. Since g is univalent and $f(\Delta)\subset g(\Delta)$, ϕ is well

defined and analytic in Δ. Also, $|\phi(z)|<1$ when $|z|<1$ and

$\phi(0)=g^{-1}(f(0))=g^{-1}(g(0))=0$. Therefore f is subordinate to g is in Δ.

Theorem 3.1.2:

Let $f \prec g$ in Δ and $0<r<1$, then

$|f'(0)| \le |g'(0)|$ and (3.1.1)

$f(\{z:|z|<r\}) \subset g(\{z:|z|<r\})$. (3.1.2)

Proof:

$f(z) = g(\phi(z))$ implies that $f'(0) = g'(0)\phi'(0)$ .Since $|\phi(z)| < 1$ when $|z| < 1$ we conclude that $|\phi'(0)| \leq 1$.

Hence $|f'(0)| \leq |g'(0)|$.

Suppose that $|z| < r$ and $\zeta = \phi(z)$. Then by Schwarz's lemma,

$|\zeta| = |\phi(z)| \leq |z| < r$. since $f(z) = g(\zeta)$ this show that (3.1.2) is valid.

Theorem 3.1.3:

Let $f \prec g$ in Δ, $0 < r < 1$ and $\lambda > 0$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^{\lambda}\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^{\lambda}\, d\theta.$$

The special case of Theorem (3.1.3) with $\lambda = 2$ asserts that

$$\sum_{n=0}^{\infty}|a_n|^2 r^{2n} \leq \sum_{n=0}^{\infty}|b_n|^2 r^{2n}.$$

Where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (|z| < 1)$

proof: see[20, pages 25-27].

Theorem 3.1.4:

Let $f \prec g$,$f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $(|z| < 1)$, then

$$\sum_{n=0}^{N}|a_n|^2 \leq \sum_{n=0}^{N}|b_n|^2 \quad (N=0,1,2,\ldots\ldots.).$$

proof:

If $f = g \circ \phi$ where $\phi \in B_0$,then

$$f(z)=\sum_{n=0}^{\infty}a_nz^n=g(\phi(z))=\sum_{n=0}^{\infty}b_n[\phi(z)]^n=\sum_{n=0}^{N}b_n[\phi(z)]^n+\sum_{n=N+1}^{\infty}c_nz^n$$

for suitable c_n .Since $\phi(0)=0$. Therefore

$\sum_{n=0}^{N}a_nz^n+\sum_{n=N+1}^{\infty}d_nz^n=\sum_{n=0}^{N}b_n[\phi(z)]^n$ for suitable d_n. Since the function on the right

hand side of this equality is subordinate to $\sum_{n=0}^{N}b_nz^n$ an application of theorem

(3.1.3) in the case $\lambda=2$ shows that if $0<r<1$ then

$\sum_{n=0}^{N}|a_n|^2r^{2n}+\sum_{n=N+1}^{\infty}|d_n|^2r^{2n}\le\sum_{n=0}^{N}|b_n|^2r^{2n}$. Therefore

$\sum_{n=0}^{N}|a_n|^2r^{2n}\le\sum_{n=0}^{N}|b_n|^2r^{2n}$ for every r $(0<r<1)$.

Letting $r\to1$ we obtain the desired result.

Theorem 3.1.5:

Let $f\in S^*$ and g , h are defined by $g(z)=\frac{f(z)}{z}$ and $h(z)=1/(1-z)^2$, then

$g\prec h$.

Proof: see[20, page 37].

As we know in theorem (2.4.2) then we can prove the following corollary.

Corollary 3.1.1:

Let $f\in K$ and $h(z)=1/(1-z)^2$, then $f'\prec h$..

3.2 The linear topological structure of the set of analytic functions.

We begin by recalling certain simple facts about $\mathcal{A}$={ $f : f$ is analytic in Δ }, $\mathcal{A}$ is a vector space over the set of complex numbers C,with the usual definitions of addition and scalar multiplication for functions .

The topology on $\mathcal{A}$ is defined in the following way. Let $\{r_n\}$ be a sequence of real numbers so that $0 < r_n < 1$ and $r_n \to 1$. If $f \in \mathcal{A}$ then define

$\|f\|_n = \max\{|f(z)| : |z| = r_n\}$. Now let P be defined by

$$P(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n}$$

whenever f and g belong to $\mathcal{A}$,

then P is a metric on $\mathcal{A}$. Moreover, a sequence $\{f_n\}$in $\mathcal{A}$ converges to f in the metric P if and only if $\{f_n\}$ converges to f uniformly on each compact subset of Δ, which we denote by $f_n \to f$.

Definition 3.2.1(Normal families):

A family $\mathcal{F}$ of functions analytic in a domain D ,contained in $\mathcal{A}$, is called a normal family if every sequence of functions f_n belong to $\mathcal{F}$ has a subsequence which converges uniformly on each compact subset of D .

Definition 3.2.2:

A family $\mathcal{F}$ of functions analytic in a domain D ,is said to be uniformly bounded in D if there is an $M > 0$ such that for every f in $\mathcal{F}$ and every z in D

$|f(z)| \leq M$.

Also, the family $\mathcal{F}$ is said to be locally uniformly bounded in D if each point z_0 in D has a neighborhood N in which $\mathcal{F}$ is uniformly bounded on it.

Example 3.2.1:

If D is the unit disk and $\mathcal{F}$ is the set of functions

$f_\alpha(z) = 1/(z - e^{i\alpha})$, α real, then $\mathcal{F}$ is not uniformly bounded, but $\mathcal{F}$ is locally uniformly bounded.

Definition 3.2.3:

A family of functions $\mathcal{F}$ is said to be compact if every convergent sequence of functions from $\mathcal{F}$ converges to a function that is also in $\mathcal{F}$.

Theorem 3.2.1 (Montel' s theorem):

Every locally bounded family of analytic functions is normal.

Proof : see[13, pages 7-8].

Note the converse of Montel' s theorem is also true.

Theorem 3.2.2:

Every normal family is locally bounded.

Proof : see[13, pages8-9].

The following theorems can be found in [19] and give some properties of the family $\mathcal{A}$.

Theorem 3.2.3 (Hurwitz theorem):

Let f_n be analytic and univalent in a domain D, and suppose $f_n(z) \to f(z)$ as $n \to \infty$,uniformly on each compact subset of D , then f is either univalent or constant in D .

Theorem 3.2.4:

If $\{f_n(z)\}$ is sequence of functions from S that converges in any disk $|z| \le r$, where $0 < r < 1$, then the limit function is also in S .

Theorem 3.2.5:

A family $\mathcal{F}$ contained in $\mathcal{A}$ is compact if and only if is closed and locally uniformly bounded.

Theorem 3.2.6:

Let f_n be analytic in a domain D, and suppose $f_n(z) \to f(z)$ as $n \to \infty$, uniformly on compact subsets of D, then f is analytic in a domain D.

Theorem 3.2.7:

The families S,S^*,K,CL,$\mathcal{P}$ and T are compact subsets of $\mathcal{A}$.

Proof:

To prove that any of these set compact, we must show that set is locally uniformly bounded and closed, by theorem(3.2.5).

Here , we prove S is compact, if $f \in$ S then (2.1.3) implies that S is locally uniformly bounded , suppose that $f_n \in$ S and $f_n \to f$ we want to prove f belongs in S .By theorem(3.2.3) either f is univalent or f is constant , f cannot be constant since $\lim_{n\to\infty} f_n'(0) = f'(0) = 1$, hence f is univalent, and so $\lim_{n\to\infty} f(0) = f(0) = 0$ and so f is analytic , by theorem (3.2.6).Hence f belongs to S .

We next consider the family $\mathcal{P}$, if $p \in \mathcal{P}$ then (theorem 2.3.2) implies that $\mathcal{P}$ is locally uniformly bounded, suppose that $p_n \in \mathcal{P}$ and $p_n \to p$ .It clearly follows that $\operatorname{Re} p(z) \ge 0$ $(|z| < 1)$. If there is a number z_0 so that $\operatorname{Re} p(z_0) = 0$ and $|z_0| < 1$ then the

Finally, If $f \in \mathrm{T}$ then by lemma 2.6.1 $f(z) = [z/(1-z^2)]p(z)$ where $p \in \mathcal{P}_R$. In particular, $p \in \mathcal{P}$ and so $|p(z)| \leq (1+|z|)/(1-|z|)$ from which we obtain $|f(z)| \leq |z|/(1-|z|)^2$. The family T is locally uniformly bounded.

Suppose that $f_n \in \mathrm{T}$ and $f_n \to f$ .Since $(\operatorname{Im} z)[\operatorname{Im} f_n(z)] \geq 0$ when $|z| < 1$ $(n = 1,2,......)$, we see that $(\operatorname{Im} z)[\operatorname{Im} f(z)] \geq 0$ $(|z| < 1)$. Also, the normalizations $f_n(0) = 0$ and $f_n'(0) = 1$ imply that $f(0) = 0$ and $f'(0) = 1$ and so $f \in \mathrm{T}$.

Definition 3.2.4:

A functional defined on a family of functions $\mathcal{A}$ is a function whose domain is $\mathcal{A}$ and whose range is some subset of the set of all complex numbers. The functional is usually denoted by J and its value at f is $J(f)$. If the range of J is a subset of the real number, then J is called a real-valued functional.

Definition 3.2.5:

A functional J is linear if for every complex numbers λ, μ and every f, g in $\mathcal{A}$

$$J[\lambda f + \mu g] = \lambda J[f] + \mu J[g]$$

Definition 3.2.6:

A real-valued functional is convex if for every t in $[0,1]$ and every f, g in $\mathcal{A}$

$$J[tf + (1-t)g] \leq tJ[f] + (1-t)J[g]$$

Definition 3.2.7:

A functional J is said to be a continuous functional on $\mathcal{A}$ if for every sequence $\{f_n\}$ of functions in $\mathcal{A}$ that converges to an f in $\mathcal{A}$, we have

$\lim_{n\to\infty} J[f_n] = J[f]$ so $J[f_n] \to J[f]$.

Example 3.2.2:

Suppose that $f \in \mathcal{A}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(|z| < 1)$, for each fixed value of n the functional $J[f] = a_n$ is continuous on $\mathcal{A}$.

suppose that $f_k \to f$, $f_k(z) = \sum_{n=0}^{\infty} a_n{}^k z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we need to show that

$a_n{}^k \to a_n$ as $k \to \infty$, by Cauchy's formula on $|z| = r$

$$a_n{}^k = \frac{1}{2\pi i} \int_{|z|=r} \frac{f_k(z)}{z^{n+1}} dz, \quad a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

$$a_n{}^k - a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f_k(z) - f(z)}{z^{n+1}} \qquad n = 0,1,2,\ldots\ldots\ldots$$

and thus

$$\left|a_n{}^k - a_n\right| \leq \frac{1}{r^n} \max\left|f_k(z) - f(z)\right|$$

since $f_k \to f$ uniformly on $\{z : |z| \leq r\}$, $\max|f_k(z) - f(z)| \to 0$ as $k \to \infty$

thus $a_n{}^k \to a_n$ as $k \to \infty$. Hence $J(f) = a_n$ is continuous.

Example 3.2.3:

Let $f \in \mathcal{A}$, define $J(f) = f^{(n)}(z_0)$

where z_0 and n are fixed ($|z_0| < 1$ and n is a non negative integer).clearly, J is a continuous functional.

Example 3.2.4:

Define $J(f)=\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\lambda}d\theta$

where $f\in\mathcal{A}, 0<r<1$ and $\lambda>0$, clearly, J is continuous functional.

Theorem 3.2.8:

Let $\mathcal{F}$ be a compact subset of $\mathcal{A}$ and let J be a complex-valued continuous functional on $\mathcal{A}$,there exists a function F in $\mathcal{F}$ so that

$\operatorname{Re} J(f)\le \operatorname{Re} J(F)$ for all f in $\mathcal{F}$. (3.2.1)

Also there is a function G in $\mathcal{F}$ so that

$|J(f)|\le|J(G)|$ for all f in $\mathcal{F}$. (3.2.2)

proof:

Suppose $M=Max\{\operatorname{Re} J(f): f\in\mathcal{F}\}$and so $M\le\infty$.There exists a sequence $\{f_n\}$so that $f_n\in\mathcal{F}$ and $\operatorname{Re} J(f_n)\to M$.By Montel's theorem, the local uniform boundedness of $\mathcal{F}$ implies that there is a convergent subsequence $\{f_{nk}\}$ of $\{f_n\}$. If $F=\lim_{k\to\infty} f_{nk}$ then $F\in\mathcal{F}$ because $\mathcal{F}$ is closed .The continuity of J implies that $J(f_{nk})\to J(F)$ as $k\to\infty$.Therefore $\operatorname{Re} J(f_{nk})\to \operatorname{Re} J(F)=M$. This prove that $M<\infty$ and that (3.2.1) holds.

The proof of (3.2.2) is similar, ending with the fact that $J(f_{nk})\to J(G)$ implies that $|J(f_{nk})|\to|J(G)|$.

Theorem 3.2.9:

Let J be a complex- valued continuous linear functional on $\mathcal{A}$. Then

(1) there is a sequence $\{b_n\}$ of complex numbers satisfying

$\overline{\lim}(|b_n|)^{\frac{1}{n}} < 1$ and such that

$$J(f) = \sum_{n=0}^{\infty} b_n a_n \quad \text{where } f \in \mathcal{A} \text{ and}$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1).$$

(2) there is a complex valued regular Borel measure λ supported on a compact subset of Δ such that $J(f) = \int_{\Delta} f(z) d\lambda\,(z)$ for $f \in \mathcal{A}$.

Conversely any such sequence or any such measure defines a continuous linear functional on $\mathcal{A}$ according to the formula in (1) and (2).

Proof: see[20, pages 42,43].

3.3 Extreme points of some class of univalent functions.

Definition 3.3.1:

The line segment $L[f, g]$ joining f and g in a vector space is the set of all points h of the form $h = tf + (1-t)g$ for which $0 \le t \le 1$. The points f and g are the end points of $L[f, g]$.The point h is an interior point of $L[f, g]$ if $0 < t < 1$.

Definition 3.3.2:

Let X be a linear topological space and suppose that a set U in X is convex if for every pair points f, g in U, the set $L[f, g]$ is also in U.

Definition 3.3.3:

The closed convex hull (or cover) of a set U is the smallest closed convex set that contains U, we denote this set by HU ,the closed hull of U.

The convex hull of U consists of all elements of the form $\sum_{k=1}^{n} t_k u_k$ where

$u_k \in U$, $t_k \geq 0$ and $\sum_{k=1}^{n} t_k = 1$, with variable n.

Definition 3.3.4:

A point f in a convex set U is called an extreme point of U if it has no representation of the form

$f = tf_1 + (1-t)f_2$ with $0 < t < 1$

as a proper convex combination of two distinct points f_1 and f_2 in U. The set of all extreme points of U denoted by EU.

Theorem 3.3.1(Krein-Milman theorem): .

Let U be a compact convex set in a locally convex topological vector space X. Then U is the closed convex hull of its extreme points. In symbols $U = HEU$.

Theorem 3.3.2:

Let $\mathcal{A}$ be a locally convex linear topological space and let $\mathcal{F}$ be a compact subset of $\mathcal{A}$.

- If $\mathcal{F}$ is non –empty then E$\mathcal{F}$ is non-empty.
- HE$\mathcal{F}$ = H$\mathcal{F}$
- If H$\mathcal{F}$ is compact then EH$\mathcal{F} \subset \mathcal{F}$.

Proof: see[20, page 44].

Theorem 3.3.3:

Let $\mathcal{F}$ be a compact subset of $\mathcal{A}$ and let J be a complex- valued continuous functional on $\mathcal{A}$.

then $\max\{\operatorname{Re} J(f) : f \in H\mathcal{F}\} = \max\{\operatorname{Re} J(f) : f \in \mathcal{F}\} = \max\{\operatorname{Re} J(f) : f \in EH\mathcal{F}\}$

proof: see[20, pages 44,45].

Theorem 3.3.4:

Let $\mathcal{F}$ be a compact subset of $\mathcal{A}$ and let J be a real-valued , continuous, convex functional on $H\mathcal{F}$. Then ,

$$\max\{J(f): f \in \mathrm{H}\mathcal{F}\} = \max\{J(f): f \in \mathcal{F}\} = \max\{J(f): f \in E\mathrm{H}\mathcal{F}\}.$$

Proof : see[20, page 45].

The following theorems can be found in [31] and will be given without proof as an examples of an extreme points of some class of univalent functions.

Theorem 3.3.5:

The set of extreme points of $\mathcal{P}$ consists of the functions

$$p(z) = \frac{1+xz}{1-xz} \qquad \text{where } |x| = 1.$$

Theorem 3.3.6:

The set of extreme points of T consists of the functions

$$f(z) = \frac{z}{(1-xz)(1-\bar{x}z)} \qquad \text{where } |x| = 1 \quad \text{and } \operatorname{Im} x \geq 0.$$

Corollary 3.3.1:

$HS_R = HS^*_R = T$ and $EHS_R = EHS^*_R = ET$.

Theorem 3.3.7:

HS^* consists of all functions represented by the formula

$$f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2} d\mu(x)$$

where $\mu \in \Lambda$. Also, EHS* consists of the Koebe functions

$$f(z) = \frac{z}{(1-xz)^2} \text{ where } |x| = 1.$$

Theorem 3.3.8:

HK consists of all functions represented by the formula

$$f(z) = \int_{|x|=1} \frac{z}{1-xz} d\mu(x)$$

where $\mu \in \Lambda$. Also, EHK consists of the functions $f(z) = \dfrac{z}{1-xz}$ where $|x| = 1$

Theorem 3.3.9:

HCL consists of the functions represented by

$$f(z) = \int_R \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x,y)$$

where μ varies over the probability measures on R = $\partial\Delta \times \partial\Delta$. EHCL consists

of the functions defined by

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$$

where $|x| = 1$, $|y| = 1$ and x $\neq$ y.

Chapter Four

Support points of several classes

In this chapter we identify the support points of $\mathcal{P}$, S^*,K,CL and T .

In the first part of this chapter, we will prove that, for each of the families K,S^*,S_R and CL, a support point is necessarily an extreme point of the families closed convex hull.

In the second part of this chapter we consider the classes of functions analytic in Δ and subordinate to a fixed analytic function F.

Later in the chapter we describe some facts related to supp S.

4.1 Introduction

Let $\Delta = \{z \in C : |z| < 1\}$ and let $\mathcal{A}$ denote the set of functions analytic in Δ ,as we showed in chapter three, then $\mathcal{A}$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ .By a continuous linear functional on $\mathcal{A}$ we mean a complex- valued functional defined on $\mathcal{A}$ that is linear and continuous . Also, if $f_n \in \mathcal{A}$ $(n = 1,2,........)$ and $f_n \to f$,uniformly on compact subset, then $J(f_n) \to J(f)$, for each continuous linear functional J on $\mathcal{A}$. Theorem (3.2.9) shows that J is given by a sequence $\{b_n\}$ $(n = 0,1,.....)$ which satisfies $\overline{\lim}_{n\to\infty} \sqrt[n]{|b_n|} < 1$ and

$$J(f) = \sum_{n=0}^{\infty} a_n b_n \qquad \text{where } f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1).$$

Definition 4.1.1:

A function f is called a support point of a compact subset $\mathcal{F}$ of $\mathcal{A}$ if $f \in \mathcal{F}$ and if there is a continuous, linear functional J on $\mathcal{A}$ so that Re J is non constant on $\mathcal{F}$ and $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in \mathcal{F}\}$. or

$$\operatorname{Re} J(g) \le \operatorname{Re} J(f) \quad \text{for all } g \in \mathcal{F}.$$

Example 4.1.1:

Let $f(z) = \dfrac{z}{(1-z)^2}$. Clearly $f(z) \in S^*$.

As in theorem (3.2.7), S^* is compact subset of $\mathcal{A}$.

Define J on S^* by $J(g) = a_n$ where $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Clearly , J is continuous linear functional on S^* but $f(z) = \sum_{n=1}^{\infty} nz^n$,so $\operatorname{Re} J(f) = n$ but $\operatorname{Re}(J(g)) = \operatorname{Re} a_n \le |a_n|$,

By theorem (2.3.5), $|a_n| \le n = \operatorname{Re} J(f)$.

Hence $\operatorname{Re} J(g) \le \operatorname{Re} J(f)$,for all $g \in S^*$.

Therefore, f is a support of S^*.

We begin by pointing out some general relationships between support points of $\mathcal{F}$ and extreme points of closed convex hull of $\mathcal{F}$ whenever $\mathcal{F}$ is a compact subset of $\mathcal{A}$.

Theorem 4.1.1:

If $\mathcal{F}$ is compact then:

1. $\mathcal{F} \subset \overline{H \text{ supp } \mathcal{F}}$
2. $H\mathcal{F} = \overline{H \text{ supp } \mathcal{F}}$
3. $EH\mathcal{F} \subset \overline{\text{supp } \mathcal{F}}$

4. $H\mathcal{F} = H(\text{supp}\,\mathcal{F} \cap EH\mathcal{F})$

Proof:

1. We start by proving that $\mathcal{F} \subset H\,\text{supp}\,\mathcal{F}$.

On the contrary, suppose that there is an element f of $\mathcal{F}$ such that $f \notin H\,\text{supp}\,\mathcal{F}$, since H supp $\mathcal{F}$ is the intersection of all closed convex sets containing supp $\mathcal{F}$, there exists a closed convex set B such that supp $\mathcal{F} \subset B$ and $f \notin B$. A basic separation theorem applied to the two sets B and $\{f\}$ implies that there exists a continuous linear functional J such that $\text{Re}\,J(g) \leq \text{Re}\,J(f) - \varepsilon$ $(\varepsilon > 0)$ whenever $g \in B$.

In particular , this implies that $\max_{g \in \mathcal{F}} \text{Re}\,J(g) = \text{Re}\,J(h)$ for some h in supp $\mathcal{F}$ Which contradicts $\text{Re}\,J(h) < \text{Re}\,J(f)$ for some h in B ,since $f \in H\,\text{supp}\,\mathcal{F}$.

Therefore , $\mathcal{F} \subset H\,\text{supp}\,\mathcal{F}$.

2. we want to prove $H\mathcal{F} = H\,\overline{\text{supp}\,\mathcal{F}}$

since $\mathcal{F} \subset H\,\text{supp}\,\mathcal{F}$, it follows that $H\,\mathcal{F} \subset H\,\text{supp}\,\mathcal{F} = H\,\overline{\text{supp}}\,\mathcal{F}$

Also supp $\mathcal{F} \subset \mathcal{F}$ and so $\overline{\text{supp}}\,\mathcal{F} \subset \overline{\mathcal{F}}$ and then H supp $\mathcal{F} \subset H\mathcal{F}$.

Therefore ,we conclude that $H\mathcal{F} = H\,\overline{\text{supp}\,\mathcal{F}}$.

3. we want to prove $EH\mathcal{F} \subset \overline{\text{supp}\,\mathcal{F}}$

Suppose that $f \in EH\mathcal{F}$. As $H\mathcal{F} = H\,\overline{\text{supp}}\,\mathcal{F}$ this implies that $f \in EH\,\overline{\text{supp}}\,\mathcal{F}$, since supp $\mathcal{F}$ is compact, so by theorem (3.3.2) if Hsupp $\mathcal{F}$ is compact implies that $EH\,\overline{\text{supp}}\,\mathcal{F} \subset \overline{\text{supp}}\,\mathcal{F}$ and so $f \in \overline{\text{supp}}\,\mathcal{F}$. Therefore,

$EH\mathcal{F} \subset \overline{\text{supp}\,\mathcal{F}}$.

4. In order to prove that $H\mathcal{F} = H(\text{supp}\,\mathcal{F} \cap EH\mathcal{F})$,

we first prove that $\mathcal{F} \subset$ H(supp$\mathcal{F} \cap$ EH$\mathcal{F}$)

On the contrary, suppose that there is an element f of $\mathcal{F}$ so that

$f \notin$ H(supp$\mathcal{F} \cap$ EH$\mathcal{F}$). Since H(supp$\mathcal{F} \cap$ EH$\mathcal{F}$) is the intersection of all closed, convex sets containing (supp$\mathcal{F} \cap$ EH$\mathcal{F}$),there exists a closed convex set B such that (supp$\mathcal{F} \cap$ EH$\mathcal{F}$) $\subset B$ and $f \notin B$.A basic separation theorem applied to the two sets B and $\{f\}$ such that $\operatorname{Re} J(g) \leq \operatorname{Re} J(f) - \varepsilon$ $(\varepsilon > 0)$ whenever $g \in B$. In particular, this implies that $\max_{g \in \mathcal{F}} \operatorname{Re} J(g) = \max_{g \in \mathrm{EH}\mathcal{F}} \operatorname{Re} J(g) = \operatorname{Re} J(h)$ for some h in supp$\mathcal{F} \cap$ EH$\mathcal{F}$. This contradicts $\operatorname{Re} J(h) \leq \operatorname{Re} J(f)$ and so that we conclude that $\mathcal{F} \subset$ H(supp$\mathcal{F} \cap$ EH$\mathcal{F}$).

Since H $\mathcal{F} \subset$ H(supp$\mathcal{F} \cap$ EH$\mathcal{F}$).

Conversely, since supp $\mathcal{F} \subset \mathcal{F}$ and EH$\mathcal{F} \subset \mathcal{F}$ we have

Supp $\mathcal{F} \cap$EH$\mathcal{F} \subset \mathcal{F}$ and so H (supp $\mathcal{F} \cap$EH$\mathcal{F}$) $\subset$H$\mathcal{F}$.

Consequently, H$\mathcal{F}$= H (Supp $\mathcal{F} \cap$EH$\mathcal{F}$).

The next two lemmas have a number of applications and, in particular, will be used to describe the support points of several families.

Lemma 4.1.1:

Let f be analytic functions in $\overline{\Delta}$. If $f(e^{i\theta})$ lies on a line for infinitely many values of θ in $[0,2\pi)$ then f is constant.

Proof:

We may assume that the given line is the imaginary axis. If $g(z) = \frac{1}{2}[f(z) + \overline{f(1/\overline{z})}]$.

then g is analytic in a neighbourhood of the unit circle and $g(e^{i\theta}) = \operatorname{Re} f(e^{i\theta})$ for θ in $[0,2\pi)$. Hence g vanishes on an infinite set with a limit point in its domain of analyticity .Therefore g vanishes identically, and in particular $\operatorname{Re} f(e^{i\theta}) = 0$ for all θ in $[0,2\pi)$. But then $\operatorname{Re} f(z) = 0$ when $|z| \leq 1$ by the maximum and minimum principle for harmonic functions. Hence f is constant.

Lemma 4.1.2:

Let B any finite set of points on the unit circle. There is a function F analytic on $\overline{\Delta}$ such that $\operatorname{Re} F(z) \geq 0$ when $z \in \overline{\Delta}$ and $\operatorname{Re} F(z) = 0$ only for z in B.

Proof:

Let $x_1, x_2, ... x_m$ be m distinct points on the unit circle. Define the complex numbers c_j by

$$\prod_{k=1}^{m} (e^{i\theta} - x_k)(e^{-i\theta} - \overline{x}_k) = \sum_{j=-m}^{m} c_j e^{ij\theta} .$$

Since

$$\sum_{j=-m}^{m} c_j e^{ij\theta} = \overline{\sum_{j=-m}^{m} c_j e^{ij\theta}} = \sum_{j=-m}^{m} \overline{c}_j e^{-ij\theta}$$

we get $c_{-j} = \overline{c}_j$ and c_0 is real . If F(z)= $c_0 + 2\sum_{j=1}^{m} c_j z^j$ then

$$\operatorname{Re} F(e^{i\theta}) = \frac{1}{2} \left[F(e^{i\theta}) + \overline{F(e^{i\theta})}\right]$$

$$= \frac{1}{2}\left[c_0 + 2\sum_{j=1}^{m} c_j e^{ij\theta} + c_0 + 2\sum_{j=1}^{m} \overline{c}_j e^{-ij\theta} \right]$$

$$= c_0 + \sum_{j=1}^{m} c_j e^{ij\theta} + \sum_{j=1}^{m} \overline{c}_j e^{-ij\theta}$$

$$=c_0+\sum_{j=1}^{m}c_je^{ij\theta}+\sum_{j=-m}^{-1}c_je^{ij\theta}$$

$$=\sum_{j=-m}^{m}c_je^{ij\theta}=\prod_{k=1}^{m}(e^{i\theta}-x_k)(e^{-i\theta}-\overline{x}_k)$$

$$=\prod_{k=1}^{m}\left|e^{i\theta}-x_k\right|^2$$

This shows that $\operatorname{Re}F(e^{i\theta})\geq 0$ and thus $\operatorname{Re}F(z)\geq 0$ for $|z|\leq 1$.Also ,$\operatorname{Re}F(z)$ can be zero only on $|z|=1$ and therefore only at $e^{i\theta}=x_n$ $\quad (n=1,2,......m)$.

Lemma 4.1.3:

Let G be defined by $G=\{f\in F:\operatorname{Re}(J(f))=\operatorname{Re}(J(f_0))\}$,where $f_0(z)$ is a support point of F , then G is convex, $E\,G\subset EF$,and

$$G=\{f\in F:f(z)=\sum_{k=1}^{\infty}\lambda_kf_k(z),\lambda_k\geq 0,\sum_{k=1}^{\infty}\lambda_k=1,\ f_k(z)\in EG\}.$$

Proof: see[10, pages 103-111].

Lemma 4.1.4:

Let J be a continuous ,linear functional on $\mathcal{A}$. $\operatorname{Re}J$ is constant on $\mathcal{P}$ if and only if has the form

$$J(f)=\alpha f(0) \qquad (4.1.1)$$

where $f\in\mathcal{A}$ and α is a complex constant .

Proof:

If J is given by (4.1.1) and $p\in\mathcal{P}$, then $J(p)=\alpha\, p(0)=\alpha$ and so $\operatorname{Re}J$ is constant on $\mathcal{P}$.

Conversely, suppose that $\operatorname{Re} J$ is constant on $\mathcal{P}$, if $p(z) = 1 + xz^n$ where $|x| = 1$ and $n = 1,2,.......$, then $p \in \mathcal{P}$, since $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. Let J be a given by the sequence $\{b_n\}$ .as in theorem (3.2.9).

Since $\operatorname{Re} J$ is constant on $\mathcal{P}$, and if

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n .$$

then $J(p) = \sum_{n=0}^{\infty} p_n b_n$,where

$p_0 = p(0) = 1$.

$p_1 = p_2 = ...p_{n-1} = 0$ and $p_n = x$,

since $J(p) = b_0 + xb_n$.

$\operatorname{Re} J(p) = \operatorname{Re} b_0 + \operatorname{Re}(xb_n)$ is constant , with n fixed, this implies that $\operatorname{Re}(xb_n)$ is constant for $|x| = 1$ and so $b_n = 0$ for $n = 1,2,......$ Thus, $J(f) = b_0 a_0$ whenever

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$ Hence J has the form (4.1.1).

4.2 Support points of subclasses of Normalized univalent functions

Theorem 4.2.1:

The set supp $\mathcal{P}$ consists of all functions which may be written

$$p(z) = \sum_{k=1}^{m} \lambda_k \frac{1 + x_k z}{1 - x_k z} \tag{4.2.1}$$

where $\lambda_k \geq 0$, $\sum_{k=1}^{m} \lambda_K = 1$, and $|x_k| = 1$ $(m = 1,2,.......)$

Proof:

Let p_0 belong to supp $\mathcal{P}$. There is a continuous linear functional $J:\mathcal{A}\rightarrow C$ such that

$\operatorname{Re}J(p_0)=\max\{\operatorname{Re}J(p): p\in \mathcal{P}\}$, and $\operatorname{Re}J$ is not constant on $\mathcal{P}$.Let J be given

by the sequence $\{ b_n \}$as in theorem (3.2.9) and suppose $p(z)=\dfrac{(1+z)}{(1-z)}$.

$p(0)=0$ and $\operatorname{Re}p(z)>0$ since $p(z)\in \mathcal{P}$. Also , we can easily shows,

$p(z)=\dfrac{(1+z)}{(1-z)}=1+\sum\limits_{n=1}^{\infty}2z^n$ .Apply J on p,

$J(p)=\sum\limits_{n=0}^{\infty}2b_n$ and since

$p(xz)=1+\sum\limits_{n=1}^{\infty}2x^n z^n$

then

$J(p(xz))=b_0+\sum\limits_{n=1}^{\infty}2b_n x^n=G(x)$ defines a function which is analytic in $\overline{\Delta}$,since

$\overline{\lim}\,(|b_n|)^{\frac{1}{n}}<1$ by theorem (3.2.9).

By theorem (3.3.5) the set of extreme points of $\mathcal{P}$ consists of the functions

$p(z)=\dfrac{(1+xz)}{(1-xz)}$ where $|x|=1$

E $\mathcal{P}=\{p(xz):|x|=1\}$.

Since $\operatorname{Re}J$ is not constant on $\mathcal{P}$,then $\operatorname{Re}G$ is not constant when $|x|=1$

(alternatively, if $\operatorname{Re}G$ is constant then $\operatorname{Re}J$ is constant on E $\mathcal{P}$ and thus on $\mathcal{P}$).

Therefore, there are a finite number of distinct values of x ,say $x_1,x_2,x_3,\ldots\ldots x_m$,so

that $\operatorname{Re}G(x)=\max\{\operatorname{Re}G(y):|y|=1\}$.

If $\nu = \{q \in \mathcal{P} : \operatorname{Re} J(q) = \max_{p \in \mathcal{P}} \operatorname{Re} J(p)\}$ then we must show that ν is compact and convex, and so non-empty by theorem(3.3.2).To show that ν is compact, it suffusion to prove ν is bounded and closed , but $\nu \subset \mathcal{P}$ and since $\mathcal{P}$ is locally uniformly bounded, then ν is bounded.

To show that ν is closed ,suppose that $q_n \in \nu$ and $q_n \to q$ .since $q_n \in \nu$ then

$\operatorname{Re} J(q_n) = \max_{p \in \mathcal{P}} \operatorname{Re} J(p)$.

$q_n \subseteq \mathcal{P}$ and $\mathcal{P}$ is compact, so $q \in \mathcal{P}$ and $J(q_n) \to J(q)$ .so $\operatorname{Re} J(q_n) \to \operatorname{Re} J(q)$

but $\operatorname{Re} J(q_n) = \max_{p \in \mathcal{P}} \operatorname{Re} J(p)$,so $\operatorname{Re} J(q_n) \to \max_{p \in \mathcal{P}} \operatorname{Re} J(p)$,hence

$\operatorname{Re} J(q) = \max_{p \in \mathcal{P}} \operatorname{Re} J(p)$.

Therefore, $q \in \nu$, ν is closed .

To show that ν is convex ,let f and $g \in \nu$, then

$\operatorname{Re} J(g) = M$ where $M = \max\{\operatorname{Re} J(p) : p \in \mathcal{P}\}$ and

$\operatorname{Re} J(f) = M$ where $M = \max\{\operatorname{Re} J(p) : p \in \mathcal{P}\}$.

let $h = tf + (1-t)g$ then $\operatorname{Re} J(h) = t \operatorname{Re} J(f) + (1-t)\operatorname{Re} J(g) = tM + (1-t)M = M$

Hence , $h \in \nu$.

Also, we can show that ν is an extremal subset of $\mathcal{P}$ and so $E\nu \subset E\mathcal{P}$. To do this, Suppose $M = \max\{\operatorname{Re} J(q) : q \in \mathcal{P}\}$ and $\nu = \{q : q \in \mathcal{P}$ and $\operatorname{Re} J(q) = M\}\}$.

We claim that ν is an extremal subset of $\mathcal{P}$, suppose that $h \in \nu$ and

$h = tf + (1-t)g$ where $f, g \in \mathcal{P}$ and $0 < t < 1$, then

$M = \operatorname{Re} J(h) = t \operatorname{Re} J(f) + (1-t)\operatorname{Re} J(g) \le tM + (1-t)M = M$ and so we must have

$\operatorname{Re} J(f) = \operatorname{Re} J(g) = M$ .Hence, f and g belongs to ν .

By lemma (4.1.1),the equations $\operatorname{Re} J(p(xz)) = \operatorname{Re} G(x) = \operatorname{Re} J(p_0)$ has only a finite number of solutions which we denote by $x_1, x_2, \ldots\ldots. x_m$ and where $|x_k| = 1$ $(k = 1,2,\ldots\ldots.m)$.

Therefore , by theorem(3.3.5), Eν ={ $(1 + x_k z)/(1 - x_k z)$: $k = 1,2,\ldots\ldots.m$ } and so ν consists of the functions given by (4.1.2) In particular , p_0 must have the form.

Conversely, we now show that each function of the form(4.1.2) belongs to supp $\mathcal{P}$

Suppose p_0 has the form(4.1.2) where the x_k are distinct. By lemma (4.1.2) there is a function F analytic on $\overline{\Delta}$ such that $\operatorname{Re} F(z) \geq 0$ when $z \in \overline{\Delta}$ and $\operatorname{Re} F(z) = 0$ if and only if $z = x_k$ $(k = 1,2,\ldots\ldots., m)$. If $G(z) = -F(z) = \sum_{n=0}^{\infty} d_n z^n$ then $\operatorname{Re} G(z) \leq 0$ when $z \in \overline{\Delta}$ and $\operatorname{Re} G(z) = 0$ if and only if $z = x_k$ $(k = 1,2,\ldots\ldots., m)$.

Suppose $b_0 = d_0$ and $b_n = d_n/2$ when $n = 1,2,\ldots\ldots.$ Let the continuous linear functional J be defined on $\mathcal{A}$ by $J(f) = \sum_{n=0}^{\infty} a_n b_n$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Note that $J((1 + x_k z)/(1 - x_k z)) = G(x)$ and so

$\max\{\operatorname{Re} J(q) : q \in \mathcal{P}\} = \max\{\operatorname{Re} J(q) : \in \mathrm{E}\, \mathcal{P}\} = 0$ and

$\operatorname{Re} J((1 + x_k z)/(1 - x_k z)) = \operatorname{Re} G(x_k) = 0$ when $k = 1,2,\ldots\ldots, m$ Hence $\operatorname{Re} J(p_0) = 0$, and $p_0 \in$ supp $\mathcal{P}$. note that $\operatorname{Re} J$ is not constant on $\mathcal{P}$.

Theorem 4.2.2:

supp K= EHK= $\{z/(1 - xz) : |x| = 1\}$.

Proof:

The inclusion EHK $\subset$ supp K , was proved by theorem (4.1.1).

To prove supp K$\subset$EHK, we first show that supp HK consist of all functions of the form $\sum_{k=1}^{m}\lambda_k(z/(1-x_k z)$, where $\lambda_k \ge 0, \sum_{k=1}^{m}\lambda_k = 1$,and $|x_k| = 1$ $(m = 1,2,......)$ This follows from theorem(4.2.1) and the fact that $(\ell f)(z) = \frac{1}{2}z[f(z)+1]$defines a linear homeomorphism between $\mathcal{P}$ and HK. This is easily since

$\ell(((1+xz)/(1-xz)) = z/(1-xz)$ if $|x| = 1$. since K is compact, it is clear that suppK$\subset$suppHK. So suppose $f \in$ suppK and hence $f(z) = \sum_{k=1}^{m}\lambda_k(z/(1-x_k z),$ where λ_k and x_k satisfy the condition noted above. by theorem(2.4.2),

$g(z) = zf'(z) = \sum_{k=1}^{m}\lambda_k(z/(1-x_k z)^2 \in S^*$ and so is univalent. If $\lambda_k \neq 0$ then g has a pole of order 2 at $z = x_k$ .Therefore, if $\lambda_k \neq 0$ for at least two values of k, then g has at least two poles on the unit circle, each of order 2 and such a function is not univalent on Δ. Hence, $g(z) = z/(1-xz)^2$ for some x with $|x| = 1$ and so

$f(z) = z/(1-xz)$ .This proves that supp K $\subset$EHK.

Theorem 4.2.3:

Let $f_1, f_2, f_3, ...f_m$ be m distinct and $f_k(z) = \dfrac{z}{1-x_k z}$ where

$|x_k| = 1, (k = 1,2,3......., m)$.

There is a continuous linear functional J on $\mathcal{A}$ such that the functions in K which satisfy $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in K\}$ are precisely the functions $f_1, f_2, f_3, ...f_m$.

Proof :

By lemma(4.1.2) there is a function F analytic on $\overline{\Delta}$ such that $\operatorname{Re} F(z) \geq 0$ when

$z \in \overline{\Delta}$ and $\operatorname{Re} F(z) = 0$ if and only if $z = x_k$ $(k = 1,2,......., m)$ .If $G(z) = -F(z) =$

$\sum_{n=0}^{\infty} d_n z^n$,then $\operatorname{Re} G(z) \leq 0$ when $z \in \overline{\Delta}$ and $\operatorname{Re} G(z) = 0$ if and only if $z = x_k$

$(k = 1,2,......, m)$.

Suppose $b_0 = 0$ and $b_n = d_{n-1}$ when $n = 1,2,.......$Let the continuous linear functional

J be defined on $\mathcal{A}$ by $J(f) = \sum_{n=0}^{\infty} x^{n-1} b_n$ where $f(z) = \dfrac{z}{(1-xz)} = \sum_{n=1}^{\infty} x^{n-1} z^n$.

Note that $J(z/(1-xz)) = d_0 + d_1 x + d_2 x^2 + ... = G(x)$ and so

$$\max\{\operatorname{Re} J(f) : f \in \mathrm{K}\} = \max\{\operatorname{Re} J(f) : f \in \mathrm{HK}\}$$

$$= \max\{\operatorname{Re} J(f) : f \in \mathrm{EHK}\}$$

$$= \max\{\operatorname{Re} G(x) : |x| = 1\}.$$

Hence

$\max\{\operatorname{Re} J(f) : f \in \mathrm{K}\} = 0$

and $\operatorname{Re} J(z/(1-x_k z) = \operatorname{Re} G(x_k) = 0$ when $k = 1,2,......, m$ .It is clear that

$\operatorname{Re} J(f) < 0$ when $f \in \mathrm{K}$ and $f \neq f_k$ $(k = 1,2,......., m)$.

Theorem 4.2.4:

Let J be a continuous linear functional on $\mathcal{A}$ not of the form

$J(f) = af(0) + bf'(0)$. The only functions f in K satisfying

$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in \mathrm{K}\}$ are the functions $f(z) = z/(1-xz)$, $|x| = 1$, and

there are only finitely many of them.

Proof:

According to theorem(3.2.9) J is given by a suitable sequence $\{b_n\}$.If

$f(z) = z/(1-xz) = \sum_{n=1}^{\infty} x^{n-1} z^n, |x| = 1$, then its n[th] Taylor coefficient is $a_n = x^{n-1}$,and

therefore $J(f) = \sum_{n=1}^{\infty} x^{n-1} b_n$.This defines an analytic function $F(x)$ for $|x| \le 1$ since

$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} < 1$,the image of the circle $|x| = 1$ under F can intersect a line for a finite number of values of x,unless F is constant .But F is not constant because J does has not the form $J(f) = af(0) + bf'(0)$. Therefore, there are only finitely many numbers x such that $|x| = 1$ and $\operatorname{Re} F(x) = \max\{\operatorname{Re} F(y) : |y| = 1\}$.

Equivalently, if K= $\{f : f(z) = z/(1-xz), |x| = 1\}$ then there are only finitely many functions f in K such that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in \mathrm{K}\}$.

Let $G = \{f : f \in \mathrm{HK}, \operatorname{Re} J(f) = \max \operatorname{Re} J(g) : g \in \mathrm{HK}\}$. G is compact, convex, and nonempty and therefore has extreme points . G also is an extremely subset of HK, that is , if $tf + (1-t)g \in G,\quad 0 < t < 1, f \in \mathrm{HK}$, and $g \in \mathrm{HK}$, then f and g are in G. Thus the extreme points of G are extreme points of HK ,that is, they are $f(z) = z/(1-xz)$.Because of what was first proved there can be only a finite number of such functions , say $f_1, f_2, f_3, \ldots\ldots, f_m$, where

$f_k(z) = z/(1-x_k z), |x_k| = 1 \ (k = 1,2,\ldots\ldots,m), x_j \ne x_k$ for $j \ne k.$

Since $\mathrm{E}(G) = \{f_1, f_2, f_3, \ldots\ldots, f_m\}$ it follows that

$$G = \left\{ f : f = \sum_{k=1}^{m} \lambda_k f_k, \lambda_k \ge 0, \sum_{k=1}^{m} \lambda_k = 1 \right\}.$$

By Theorem(2.4.2), $g(z) = zf'(z) = \sum_{k=1}^{m} \lambda_k (z/(1-x_k z)^2) \in S^*$ and so is univalent ,the function g has a pole of order two at $z = \bar{x}_k$, and therefore if $\lambda_k \neq 0$ for at least two values of k, the function g has two poles on the unit circle each of order two. Such that a function is not univalent in Δ since its local behavior at the poles shows that it takes on most large values at least twice in Δ. Hence the only functions in G which are in K are the functions $f_1, f_2, f_3, \ldots\ldots, f_m$. Because

$\max\{\mathrm{Re}\, J(g) : g \in \mathrm{K}\} = \max\{\mathrm{Re}\, J(g) : g \in \mathrm{HK}\}$.

This completes the proof that $f_1, f_2, f_3, \ldots\ldots, f_m$ are the only functions in K satisfying $\mathrm{Re}\, J(f) = \max\{\mathrm{Re}\, J(g) : g \in \mathrm{K}\}$.

Theorem 4.2.5:

$$\text{supp } S^* = \text{EHS}^* = \{ \frac{z}{(1-x\,z)^2} : |x| = 1 \}.$$

Proof:

The inclusion $\text{EHS}^* \subset \text{supp } S^*$ was proved by theorem (4.1.1).

To prove $\text{supp } S^* \subset \text{EHS}^*$,we first show that supp HS^* consist of all functions of the form $\sum_{k=1}^{m} \lambda_k (z/(1-x_k z)^2)$,where $\lambda_k \geq 0, \sum_{k=1}^{m} \lambda_k = 1,$ and $|x_k| = 1 \quad (m = 1,2,\ldots..)$.This follows from theorem(4.2.2) and the fact that $(\ell f)(z) = zf'(z)$ defines a linear homeomorphism between HK and HS^*.

This is easily since $\ell(z/(1-xz)) = z/(1-xz)^2$ if $|x| = 1$. Since S^* is compact, it is clear that $\text{supp } S^* \subset \text{supp } \text{HS}^*$. So suppose $f \in \text{supp } S^*$and hence

$f(z) = \sum_{k=1}^{m} \lambda_k (z/(1-x_k z)^2)$,where λ_k and x_k satisfy the conditions noted above .If

$\lambda_k \neq 0$ for at least two values of k, then f has at least two poles on the unit circle, each of order 2 and such a function is not univalent on Δ. Hence, $f(z) = z/(1-xz)^2$ for some x with $|x| = 1$.

This proves that supp $S^* \subset EHS^*$.

Theorem 4.2.6:

Let $f_1, f_2, f_3, \ldots\ldots, f_m$ be m distinct koebe functions. There is a continuous linear functional J on $\mathcal{A}$ such that the functions f in S^* which satisfy

$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in S^*\}$are precisely the functions $f_1, f_2, f_3, \ldots\ldots, f_m$.

Proof:

Let f_k be given by $f_k(z) = \dfrac{z}{(1-x_k z)^2}$ $(k = 1,2,\ldots\ldots, m)$, $|x_k| = 1$,where the numbers $x_1, x_2, \ldots\ldots, x_m$ are distinct. Let F be the polynomial defined in the proof of lemma(4.1.2)and defined G by $G(z) = -F(z) = \sum_{n=0}^{\infty} d_n z^n$ .then $\operatorname{Re} G(z) \leq 0$ when $z \in \overline{\Delta}$ and $\operatorname{Re} G(z) = 0$ if and only if $z = x_k$ $(k = 1,2,\ldots\ldots, m)$.

Suppose $b_0 = 0$ and $b_n = \dfrac{d_{n-1}}{n}$ when $n = 1,2,\ldots\ldots$

Let the continuous linear functional J be defined on $\mathcal{A}$ by $J(f) = \sum_{n=1}^{\infty} n x^{n-1} b_n$,where

$$f(z) = \frac{z}{(1-xz)^2} = \sum_{n=1}^{\infty} n x^{n-1} z^n \ .$$

Note that $J(z/(1-xz)^2) = d_0 + d_1 x + d_2 x^2 + \ldots. = G(x)$ and so

$$\max\{\operatorname{Re} J(f) : f \in S^*\} = \max\{\operatorname{Re} J(f) : f \in HS^*\}$$

$$= \max\{\operatorname{Re} J(f) : f \in EHS^*\}$$

$= \max\{\operatorname{Re} G(x) : |x| = 1\}$.

Hence

$\max\{\operatorname{Re} J(f) : f \in S^*\}=0$ and $\operatorname{Re} J(z/(1-x_k z)^2) = \operatorname{Re} G(x_k) = 0$ when

$k = 1,2,\ldots\ldots,m$. It is clear that $\operatorname{Re} J(f) < 0$ when $f \in K$ and $f \neq f_k\ (k = 1,2,\ldots\ldots,m)$.

Theorem 4.2.7:

Let J be a continuous linear functional on $\mathcal{A}$ not of the form

$J(f) = af(0) + bf'(0)$. The only functions f in S^* that satisfy

$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in S^*\}$ are the koebe functions $z/(1-xz)^2$, $|x| = 1$, and

there are only finitely many of them.

Proof:

According to theorem (3.2.9) J is given by a suitable sequence $\{b_n\}$ .If f is a koebe

function $f(z) = z/(1-xz)^2 = \sum_{n=1}^{\infty} nx^{n-1}z^n, |x| = 1$, then its nth Taylor coefficient is

$a_n = nx^{n-1}$, and therefore $J(f) = \sum_{n=1}^{\infty} nb_n x^{n-1}$. This defines an analytic function

$F(x)$ for $|x| \leq 1$ since $\limsup_{n\to\infty} |nb_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |b_n|^{\frac{1}{n}} < 1$. The image of the circle

$|x| = 1$ under F can intersect a line for only a finite number of values of x ,unless

F is constant .But F is not a constant because J does has not the form

$J(f) = af(0) + bf'(0)$. Therefore there are only finitely many numbers x such that

$|x| = 1$ and $\operatorname{Re} F(x) = \max\{\operatorname{Re} F(y) : |y| = 1\}$.

Equivalently ,if $S^* = \{f : f(z) = z/(1-xz)^2 , |x| = 1\}$ then there are only finitely many

functions f in S^* such that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in S^*\}$.

Let $G=\{f: f\in \mathrm{HS}^*, \operatorname{Re}J(f)=\max\operatorname{Re}J(g), g\in \mathrm{HS}^*\}$. G is compact ,convex, and nonempty and therefore has extreme points. G also is an extremely subset of HS^*, that is, if $tf+(1-t)g\in G,\quad 0<t<1, f\in \mathrm{HS}^*$,and $g\in \mathrm{HS}^*$, then f and g are in G .Thus the extreme points of G are extreme points of HS^*, that is ,they are koebe functions. Because of what was first proved there can be only a finite number of such functions, say $f_1, f_2, f_3, ...f_m$, where

$f_k(z)=z/(1-x_k z)^2,\ |x_k|=1\quad (k=1,..........,m), x_j\neq x_k$ for $j\neq k$.

Since $\mathrm{E}G=\{f_1, f_2, f_3, ..., f_m\}$ it follows that

$$G=\left\{f: f=\sum_{k=1}^{m}\lambda_k f_k, \lambda_k\geq 0, \sum_{k=1}^{m}\lambda_k=1\right\}.$$

the function f_k has a pole of order two at $z=\bar{x}_k$, and therefore if $\lambda_k\neq 0$ for at least two values of k, the function $\sum_{k=1}^{m}\lambda_k f_k$ has two poles on the unit circle each of order two. Such a function is not univalent in Δ since its local behavior at the poles shows that it takes on most large values at least twice in Δ .Hence the only functions in G which are in S^* are the functions $f_1, f_2, f_3, ..., f_m$ .because

$\max\{\operatorname{Re}J(g): g\in \mathrm{S}^*\}=\max\{\operatorname{Re}J(g): g\in \mathrm{HS}^*\}$.

This completes the proof that $f_1, f_2, f_3, ..., f_m$ are the only functions in S^* satisfying

$\operatorname{Re}J(f)=\max\{\operatorname{Re}J(g): g\in \mathrm{S}^*\}$.

Theorem 4.2.8:

The set supp T consists of all functions which may be written

$$f(z)=\sum_{k=1}^{m}\lambda_k\frac{z}{(1-x_k z)(1-\bar{x}_k z)}\qquad (4.2.3)$$

where $\lambda_k \geq 0, \sum_{k=1}^{m} \lambda_k = 1, |x_k| = 1$ and $\operatorname{Im} x_k \geq 0$ $(m = 1,2,........)$.

proof:

we first prove that each support point of T has the form(4.2.3).Suppose that $|x| = 1$

and $F(z,x) = \dfrac{z}{(1 - xz)(1 - \bar{x}z)} = z + \sum_{n=2}^{\infty} a_n(x) z^n$

Where $a_n(x) = x^{n-1} + x^{n-2}\bar{x} + ... + x\bar{x}^{n-2} + \bar{x}^{n-1}$ and so we can show that

$|a_n(x)| \leq nr^{n-1}$ if 1/r< $|x| < r$ and r>1.To do this,

$|x| < r \Rightarrow |x|^{n-1} < r^{n-1}$ and $|\bar{x}| = |x| < r \Rightarrow |\bar{x}|^{n-1} < r^{n-1}$ and so

$|a_n(x)| \leq r^{n-1} + r^{n-2}r + ...rr^{n-2} + r^{n-1} = r^{n-1} + r^{n-1} + ...r^{n-1} = r^{n-1}[1 + 1 + ... + 1] = r^{n-1}n$

Hence, $|a_n(x)| \leq nr^{n-1}$.

A functional J associated with a support point is given by a suitable sequence $\{b_n\}$

according to theorem (3.2.9) if $G(x) = J(F(z,x)) = \sum_{n=0}^{\infty} b_n a_n(x) = b_1 + \sum_{n=2}^{\infty} b_n a_n(x)$

and $\overline{\lim}_{n\to\infty} (|b_n|)^{\frac{1}{n}} < 1$ imply that this series is dominated by a convergent geometric

series for r sufficiently close to 1 .Indeed , $(|b_n|)^{\frac{1}{n}} \leq \beta$ when $n > N$ for some N and

some β satisfying $0 < \beta < 1$.

It follows that $|b_n a_n(x)| \leq \beta^{\,n} nr^{n-1} = (n/r)(\beta r)^n$ .Choosing r so that $\beta r < 1$,we find

that the series for $G(x)$ is convergent if 1/r< $|x| < r$ .Hence G is analytic on

$\{x:1/r<|x|<r\}$ and so H defined by $H(x)=\frac{1}{2}\left[G(x)+\overline{G(1/\overline{x})}\right]$ also has this

property . Note that if $|x|=1$, then $H(x)=\operatorname{Re}G(x)=\operatorname{Re}J(F(z,x))$.

If $\operatorname{Re}J(F(z,x))=\max\{\operatorname{Re}J(F(z,y)):|y|=1,\operatorname{Im}y\geq 0\}=\max\{\operatorname{Re}J(f):f\in \mathrm{T}\}$ has an infinite number of solutions, then letting $M=\max\{\operatorname{Re}J(f):f\in \mathrm{T}\}$ we have $H(x)=M$ for an infinite number of values of x satisfying $|x|=1$ .Since H is analytic on $\{x:1/r<|x|<r\}$ we conclude that $H(x)=M$ when $|x|=1$ .Hence $\operatorname{Re}J(F(z,x))=M$ for all x . Since $\{F(z,x):|x|=1,\operatorname{Im}x\geq 0\}=\mathrm{ET}$, we have $\operatorname{Re}J$ constant on T ,which is impossible. This proves that there is only a finite number of functions $f_1,f_2,f_3,\ldots\ldots,f_m$ in ET that maximize Re J over T.

If $v=\{g:g\in\mathrm{T},\operatorname{Re}J(g)=\max\{\operatorname{Re}J(f):f\in\mathrm{T}\}\}$ then v is convex and compact and non-empty and since v is an extremal subset of T we conclude that E$v\subset$ET`.

As we showed above, there are only a finite number of members of ET in v .Hence there exist distinct functions f_k of the form

$$f_k(z)=\frac{z}{(1-x_k z)(1-\overline{x}_k z)}$$ where $|x_k|=1$ and $\operatorname{Im}x_k\geq 0$,such that

$$v=\{f:f=\sum_{k=1}^{m}\lambda_k f_k,,\ \lambda_k\geq 0\ ,\ \sum_{k=1}^{m}\lambda_k=1\}.$$

Hence each support point of T has the form (4.2.3).

Conversely, we now show that each function of the form(4.2.3) belongs to supp T.

By lemma (4.1.2) there exists a function F analytic on $\overline{\Delta}$ such that $\operatorname{Re}F(z)\geq 0$ when $|z|\leq 1$,and $\operatorname{Re}F(z)=0$ if and only if $z=x_k$ or $\overline{x}_k$ $(k=1,2,\ldots\ldots,m)$ and $F(z)$ is real when $-1<z<1$.If $F(z)=\sum_{n=0}^{\infty}b_n z^n$ then b_n is real $(n=0,1,2,.....)$ If $c_0=-b_0$ and

$c_n=-b_n/2 \ \ (n=1,2,.....)$, then $G(z)=\sum_{n=0}^{\infty}c_n z^n$ is analytic in $\overline{\Delta}$ and

$$J_G(\frac{1-z^2}{(1-xz)(1-\bar{x}z)})= J_G(\frac{1}{2}\frac{1+xz}{1-xz}+\frac{1}{2}\frac{1+\bar{x}z}{1-\bar{x}z})$$

$$=J_G(\frac{1}{2}\left[\frac{1}{1-xz}+\frac{xz}{1-xz}+\frac{1}{1-\bar{x}z}+\frac{\bar{x}z}{1-\bar{x}z}\right])$$

$$=J_G(\frac{1}{2}\left[\sum_{n=0}^{\infty}x^n z^n+\sum_{n=1}^{\infty}x^n z^n+\sum_{n=0}^{\infty}\bar{x}^n z^n+\sum_{n=1}^{\infty}\bar{x}^n z^n\right])$$

$$=J_G(\frac{1}{2}\left[1+\sum_{n=1}^{\infty}x^n z^n+\sum_{n=1}^{\infty}x^n z^n+1+\sum_{n=1}^{\infty}\bar{x}^n z^n+\sum_{n=1}^{\infty}\bar{x}^n z^n\right])$$

$$=J_G(\frac{1}{2}\left[2+2\sum_{n=1}^{\infty}x^n z^n+2\sum_{n=1}^{\infty}\bar{x}^n z^n\right])$$

$$=J_G(1+\sum_{n-1}^{\infty}(x^n+\bar{x}^n)z^n)=\sum_{n=0}^{\infty}c_n a_n \qquad \text{when } a_n=x^n+\bar{x}^n$$

$$J_G=c_0+\sum_{n=1}^{\infty}c_n(x^n+\bar{x}^n)=c_0+\sum_{n=1}^{\infty}c_n 2\operatorname{Re}(x^n) \quad ,\operatorname{Re}(x^n)=\frac{x^n+\bar{x}^n}{2} \ .$$

If $c_0=-b_0$ and $c_n=-b_n/2\,(n=1,2,.....)$ and b_n is real, then

$$J_G=\operatorname{Re}(-b_0-\sum_{n=1}^{\infty}b_n x^n)=\operatorname{Re}[-F(x)].$$

It follows that $\operatorname{Re}J_G(f)\le 0$ whenever $f\in \mathrm{T}$ with equality exactly for the functions

f in T of the form (4.2.3). Therefore each function given by (4.2.3) is in supp T.

Theorem 4.2.9:

Supp S_R= EHS_R=ET=$\{z/(1-xz)(1-\bar{x}z):|x|=1, \qquad \operatorname{Im}x\ge 0\}$.

Proof :

It follows from the argument given in the second part of the theorem (4.2.8) that

ET $\subset$ supp T and, since HS_R=T ,corollary (3.3.1) implies that ET $\subset$ supp S_R.

To prove supp $S_R \subset EHS_R$ we first note that supp HS_R consists of all functions of the form (4.2.3) .Since S_R is compact it is clear that supp $S_R \subset$ supp HS_R and so

supp $S_R \subset$ suppT.

Now assume that $f \in$ supp S_R and so

$$f(z)=\sum_{k=1}^{m}\lambda_k\frac{z}{(1-x_k z)(1-\bar{x}_k z)} \text{ where } \lambda_k \geq 0,\ \sum_{k=1}^{m}\lambda_k=1,\ |x_k|=1, \text{ and}$$

$\operatorname{Im} x \geq 0$. Assume that $\lambda_k \neq 0$ for at least two values of k .Then f has poles on the unit circle with a total order of at least 4 and such a function is not univalent in Δ.

This implies that $f(z)=\dfrac{z}{(1-xz)(1-\bar{x}z)}$ $(|x|=1,\ \operatorname{Im} x \geq 0)$ and so $f \in EHS_R$.

Theorem 4.2.10:

Let J be a continuous linear functional on $\mathcal{A}$ such that Re $J(f)$ is non constant on S_R. The only functions f in S_R satisfying

$\operatorname{Re} J(f)=\max\{\operatorname{Re} J(g): g \in S_R\}$ are the functions

$f(z)=z/(1-2az+z^2), -1 \leq a \leq 1,$ and there are only a finite number of them.

Proof: see[4 ,pages 102-103].

Theorem 4.2.11:

$$\text{supp CL} \subset \text{EHCL}=\left\{\frac{z-\frac{1}{2}(x+y)z^2}{(1-yz)^2}:|x|=|y|=1, x \neq y\right\}.$$

proof :

Let J be a continuous linear functional on $\mathcal{A}$ such that $\operatorname{Re} J$ is non constant on CL. By Theorem (3.2.9), J is given by a sequence $\{b_n\}$ satisfying

$\overline{\lim}_{n\to\infty}(|b_n|)^{\frac{1}{n}} < 1.$

Suppose that $f \in \text{CL}$ and $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in HCL\} = M$. By theorem(3.3.9),

f has the representation $$f(z) = \int_R \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu\,(x,y) \tag{4.2.3}$$

where μ is a probability measure on $R = \partial\Delta \times \partial\Delta$. If

$$f(z,x,y) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2},$$

Theorem(3.3.9) asserts that $EHCL = \{f(z,x,y) : |x| = |y| = 1, \quad x \neq y\}$. From (4.2.3) it follows that there is a set W contained in R such that $\mu\,(W) = 1$ and

$\operatorname{Re} J(f(z,x,y)) = M$ whenever $(x,y) \in W$. If

$A(y) = J((z - \frac{1}{2}yz^2)/(1-yz)^2)$ and $B(y) = J(z^2/2(1-yz)^2)$ then it is easy to verify

that $A(y) = \sum_{n=1}^{\infty} nb_n y^{n-1} - \sum_{n=1}^{\infty} nb_{n+1} y^n$

and

$$B(y) = \frac{1}{2}\sum_{n=1}^{\infty} nb_{n+1} y^{n-1}$$

Since $\overline{\lim}_{n\to\infty}(|b_n|)^{\frac{1}{n}} < 1$ it is clear that A and B are analytic in $\overline{\Delta}$. Also, $(x,y) \in W$ is equivalent to $\operatorname{Re} A(y) - \operatorname{Re} xB(y) = M$. If $(x_0, y_0) \in W$ then $B(y_0) \neq 0$. To show this, assume that $B(y_0) = 0$. Then $\operatorname{Re} A(y_0) = M$, and $\operatorname{Re} A(y_0) - \operatorname{Re} xB(y_0) = M$

whenever $|x|=1$ .In particular , $f(z,y_0,y_0)=(z-y_0z^2)/(1-y_0z)^2 = z/(1-y_0 z)$ is

then a support point of C L.This is a contradiction of theorem (4.2.5) since $S^* \subset$ CL,

and therefore $B(y)\neq 0$ when $(x,y)\in W$ .If $\operatorname{Re} A(y)-\operatorname{Re} xB(y)=M$, then

$x=-|B(y)|/B(y)$ and $\operatorname{Re} A(y)+|B(y)|=M$. In particular, x is uniquely determined

by y .We show that W is finite .Suppose W is an infinite set; then it follows from

$\operatorname{Re} A(y)+|B(y)|=M$ and $\overline{A(y)}=\overline{A(1/\overline{y})}$ that

$$[\frac{1}{2}(A(y)+\overline{A(1/\overline{y})})-M]^2 = B(y)\overline{B(1/\overline{y})} \tag{4.2.4}$$

holds that infinitely many y with $|y|=1$ .Since the expressions in (4.2.4) are

analytic in a neighbourhood of the unit circle $|y|=1$,(4.2.4) must be an identity on

the unit circle ; that is , $\operatorname{Re} A(y)\pm|B(y)|=M$ when $|y|=1$. Since

$\operatorname{Re} A(y)+|B(y)|=M$ for y in W and $B(y)\neq 0$ for such y, we see that

$\operatorname{Re} A(y)+|B(y)|=M$ when $|y|=1$. If $F(y)=yB(y)$ when $y\leq 1$, then as $F(0)=0$ the

winding number of $\{F(x):|x|=1\}$

with respect to the origin is non zero. Hence, there exists y_1 $(|y_1|=1)$ such that

$F(y_1)<0$, that is , $y_1B(y_1)=-|B(y_1)|$. This implies that $(y_1,y_1)\in W$ .Hence ,the

corresponding function $z/(1-y_1 z)\in$ supp C L which, as we noted earlier, is

impossible. Therefore W is a finite set, consisting of n elements.

It follows that $f(z)=\sum_{k=1}^{m}\lambda_k f_k(z)$ where

$\lambda_k\geq 0$, $\sum_{k=1}^{m}\lambda_k=1$, $f_k(z)=f(z,x_k,y_k)$, $|x_k|=|y_k|=1$, $x_k\neq y_k$ and

$x_k = -|B(y_k)|/B(y_k)$ .Since f is univalent only one second order pole on the unit circle is possible. Hence $f(z) = \dfrac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$ where $|x| = |y| = 1$ and $x \neq y$ and so $f \in EHCL$ .This completes the proof.

The inclusion $EHCL \subset$ supp CL is also true. This implies that

supp $HCL =$ supp $CL = EHCL$.

The next theorem describes the minimal set for supp $s(F)$ when F is a non constant analytic function in Δ.

Theorem 4.2.12 :

Let F be analytic and non – constant in Δ.Then the set $\{f : f(z) = F(xz), |x| = 1\}$ is contained in supp $s(F)$, where $s(F) = \{f : f \prec F\}$.

Proof:

We may assume that F is non constant. The power series for F begins $F(z) = A_0 + A_n z^n + ...$ where $n \geq 1$ and $A_n \neq 0$.If $f \in s(F)$ then $f = F \circ \varphi$ where $\phi \in B_0$ .Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\phi(z) = \sum_{n=1}^{\infty} c_n z^n$ .The form of F implies that $a_0 = A_0$, $a_1 = 0$, $a_2 = 0$,......, $a_{n-1} = 0$ and $a_n = A_n c_1{}^n$. Let J be a continuous linear functional on $\mathcal{A}$ defined by $J(h) = b_n$ whenever $h \in \mathcal{A}$ and $h(z) = \sum_{n=0}^{\infty} b_n z^n$.

Because $|c_1| \leq 1$ and equality holds if and only if $\phi(z) = xz$ with $|x| = 1$ it follows that $\operatorname{Re} J(f) = \operatorname{Re} a_n \leq |a_n| = |A_n c_1{}^n| \leq |A_n|$, therefore

$\max\{\operatorname{Re} J(f) : f \in s(F)\} = |A_n|$.Moreover , the only functions in $s(F)$ for which $\operatorname{Re} J(f) = |A_n|$ have the form $f(z) = F(xz)$ for suitable x with $|x| = 1$.

Hence $f(z)$ is contained in supp $s(F)$.

Lemma 4.2.1:

Suppose that $\alpha > 1, |c| \leq 1, c \neq -1$ and $F_\alpha(z) = \left(\frac{1+cz}{1-z}\right)^\alpha$

If

$f(z) = \sum_{k=1}^{n} \lambda_k (\frac{1+cx_k z}{1-x_k z})^\alpha$ where $\lambda_k > 0$, $\sum_{k=1}^{n} \lambda_k = 1, |x_k| = 1$ and $n \geq 2$, then f is not

subordination to F_α in Δ.

Theorem 4.2.13:

Suppose that F_α is defined as in lemma 4.2.1, $|c| \leq 1$ and $c \neq -1$ .If $\alpha \geq 1$ then

$Hs(F_\alpha)$ consists of all functions in $\mathcal{A}$ represented by

$f(z) = \int_{|x|=1} \left(\frac{1+cxz}{1-xz}\right)^\alpha d\mu(x)$ where $\mu \in \Lambda$ .Moreover, $EHs(F_\alpha)$ consists of the

functions given by $f(z) = \left(\frac{1+cxz}{1-xz}\right)^\alpha$ where $|x| = 1$.

Theorem 4.2.14:

Suppose that $\alpha > 1, |c| \leq 1, c \neq -1$ and $F_\alpha(z) = \left(\frac{1+cz}{1-z}\right)^\alpha$.

Then supp $s(F_\alpha) = EHs(F_\alpha) = \{F_\alpha(xz) : |x| = 1\}$.

Proof :

The second equality is theorem (4.2.13) and by theorem (4.2.12)

$EHs(F_\alpha) \subset \operatorname{supp} s(F_\alpha)$.

We must prove that supp $s(F_\alpha) \subset EHs(F_\alpha)$.

Suppose that J is a continuous linear functional on $\mathcal{A}$ such that $\operatorname{Re} J$ is non constant on $s(F_\alpha)$ and J is given by the sequence $\{b_n\}$ as indicted by theorem (3.2.9). If $F_\alpha(z) = 1 + \sum_{n=1}^{\infty} A_n z^n \quad (|z| < 1)$ then $J(F_\alpha(xz)) = b_0 + \sum_{n=1}^{\infty} A_n b_n x^n = G(x)$. Since

$\overline{\lim}_{n\to\infty} (|A_n||b_n|)^{\frac{1}{n}} = \overline{\lim}_{n\to\infty} (|b_n|)^{\frac{1}{n}} < 1$, G is analytic in $\overline{\Delta}$. G is non constant, since otherwise $\operatorname{Re} J$ is constant on $EHs(F_\alpha)$ and so is constant on $s(F_\alpha)$, contrary to our assumption. Hence, by lemma(4.1.1) if $M = \max\{\operatorname{Re} J(f) : f \in s(F_\alpha)\}$ then $\operatorname{Re} G(x) = M$ for only a finite number of values of x satisfying $|x| = 1$, say $x_1, x_2, \ldots\ldots, x_n$. If $\chi = \{f : f \in Hs(F_\alpha)$ and $\operatorname{Re} J(f) = M$, then by familiar arguments we conclude that $\chi = \{f : f(z) = \sum_{k=1}^{n} \lambda_k F_\alpha(x_k z), \lambda_k \geq 0$ and $\sum_{k=1}^{n} \lambda_k = 1\}$.

Assume that $f \in s(F_\alpha)$ and $\operatorname{Re} J(f) = M$. Then $f \in \chi$ but by lemma (4.2.1) the only functions in χ which are in $s(F_\alpha)$ are functions of the form $F_\alpha(xz)$ $(|x| = 1)$. Hence $f(z) = F_\alpha(xz)$ where $|x| = 1$ and so $f \in EHs(F_\alpha)$. This proves that $\operatorname{supp} s(F_\alpha) \subset EHs(F_\alpha)$.

4.3 Support points of Normalized univalent functions

Let J be a continuous linear functional on a locally convex space X, and let E be a compact subset of X. Let

$$\sigma_J(E) = \{x \in E : \operatorname{Re}\{J(x)\} = \max_{y \in E} \operatorname{Re}\{J(y)\}\}$$

be the associated set of support points. Clearly, $\sigma_J(E)$ is a nonempty compact subset of E. By the Krein Milman theorem, $\sigma_J(E)$ contains an extreme point. This point

$x \in \sigma_J(E)$ is not a proper convex combination of two distinct points in $\sigma_J(E)$ .If it had a more general representation $x = ty + (1-t)z$ for $y, z \in E$,then it would follow at once from the linearity of J that $y, z \in \sigma_J(E)$ .Thus x is actually an extreme point of E .

Although a support point need not be an extreme point , it seems intuitively obvious that every extreme point must be a support point. This is false ,however, even in a Hilbert space. The following example.

Example 4.3.1:

If X is the complex plane and E is the closed interior of a sequre as in Figure 4.3.1, every boundary point is a support point but only the vertices are extreme points.

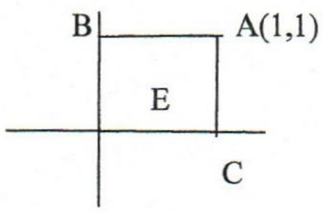

Figure 4.3.1

Vertices are not interior point of the line segement contained in E, therefore the vertices are extreme points, but every boundary point is not extreme points.

We want to prove every boundary point is a support point.

A point $z \in$E is called a support point of E if there is a continuous linear functional J , not constant on E , such that $\operatorname{Re} J\{(z)\} \geq \operatorname{Re} J\{(z_0)\}$ for all $z_0 \in$E.

Let w be a point on the line segement AB.

Define $J : \text{E} \rightarrow C \quad J(z) = -iz = -ix + y$, where $z = x + iy$.

Clearly, $\operatorname{Re} J(w) = \operatorname{Re} J(x+i) = 1 > \operatorname{Re} J(z_0) = y_0$, where $z_0 = x_0 + y_0$.

Hence w is a support point.

Example 4.3.2:

Consider the space ℓ^2 of all complex sequence $x=(x_1,x_2,....)$ such that

$$\|x\|^2=\sum_{n=1}^{\infty}|x_n|^2<\infty.$$

Let $T:\ell^2\to\ell^2$ be the compact (or completely continuous) operator defined by $y=Tx,$ where $y_n=2^{-n}x_n$. Let $B=\{x\in\ell^2:\|x\|\le 1\}$ be the closed unit ball in ℓ^2 ,and let $E=TB$ be its image under T. Then E is a compact set, because it is the image under a compact operator of the closed bounded set B (which is weakly compact).Clearly, E is convex. The extreme points of B are simply the points x with $\|x\|=1$. Since T is linear and one to one. It follows that the extreme points of E are the points $y=Tx$ with $\|x\|=1$ .Now let $y=Tx$ be a support point of E .Then by the Riesz representation theorem there is a point $z\ne 0$ in ℓ^2 such that $\mathrm{Re}\{(Tx,z)\}\ge \mathrm{Re}\{(Tw,z)\}=\mathrm{Re}\{(w,Tz)\}$ for all $w\in B$ .Thus $\|Tz\|\le \mathrm{Re}\{(Tx,z)\}$, which gives $\|x\|\,\|Tz\|\le \mathrm{Re}\{(Tx,z)\}\le |(Tx,z)|=|(x,Tz)|$.

Since $|(x,Tz)|\le\|x\|\,\|Tz\|$, it follows that $(x,Tz)=\|x\|\,\|Tz\|$, which implies $Tz=\lambda x$ for some constant $\lambda>0$ .In particular, $x\in T(\ell^2)$ for every support point Tx.

It now remains only to choose a point $x\in\ell^2$ with $\|x\|=1$ and $x\notin T(\ell^2)$.

For example, let $x_n=2^{-n/2}, n=1,2,.....$ Then $\|x\|=1$ but $T^{-1}x=w\notin\ell^2$, since $w_n=2^{n/2}, n=1,2,....$ Thus Tx is an extreme point of E but not a support point.

Definition 4.3.1(monotonic modulus property):

A function f has the monotonic modulus property if and only if f maps the unit disk onto the complement of a continuous arc which extends to ∞ with increasing modulus.

Theorem 4.3.1 :

If a function $f \in S$ omits two values of equal modulus, then f has the form

$f = tf_1 + (1-t)f_2,\quad 0 < t < 1,$ where f_1 and f_2 are distinct functions in S which omit open sets.

Proof:

Let Δ be the range of f. If f omits α and $\beta, \alpha \neq \beta$, then some branch of the

function $$\psi(w) = \{(w-\alpha)(w-\beta)\}^{\frac{1}{2}}$$

is analytic and single valued in D. We claim that the two functions $w \pm \psi(w)$ are univalent and have disjoint ranges. To prove the univalence, suppose

$$w_1 \pm \psi(w_1) = w_2 \pm \psi(w_2), \qquad \text{or}$$

$\psi(w_1) - \psi(w_2) = \pm(w_2 - w_1)$. Squaring both sides, we have

$2\psi(w_1)\psi(w_2) = (w_1-\alpha)(w_1-\beta) + (w_2-\alpha)(w_2-\beta) - (w_2-w_1)^2$. Squaring again, we obtain after some labor $(\alpha-\beta)^2(w_1-w_2)^2 = 0,$ which is impossible unless $w_1 = w_2$. Thus both of the functions $w \pm \psi(w)$ are univalent in D. A similar argument shows they have disjoint ranges. Indeed, if $w_1 + \psi(w_1) = w_2 - \psi(w_2)$, essentially the same calculation shows that $w_1 = w_2$, which implies $\psi(w_1) = \psi(w_2) = 0$. This is clearly impossible.

In particular, we have shown that the functions $w \pm \psi(w)$ are univalent and omit open sets. Both properties are preserved under the normalizations

$\psi_1(w) = \dfrac{w + \psi(w) - \psi(0)}{1 + \psi'(0)}$, $\psi_2(w) = \dfrac{w + \psi(w) + \psi(0)}{1 - \psi'(0)}$. These functions ψ_1 and ψ_2 are analytic and univalent in D ,omit open sets, and satisfy $\psi_j(0) = 0$ and $\psi'_j(0) = 1$, $j = 1,2$. Therefore, the compositions $f_1 = \psi_1 \circ f$ and $f_2 = \psi_2 \circ f$ are distinct functions in S which omit open sets. Furthermore, since $[1 + \psi'(0)]\psi_1(w) + [1 - \psi'(0)]\psi_2(w) = 2w$, the function f can be expressed by $f(z) = tf_1(z) + (1-t)f_2(z)$, where $t = \frac{1}{2}[1 + \psi'(0)]$. It remains to show that $0 < t < 1$ under the additional assumption that $|\alpha| = |\beta|$. Equivalently, it is to be shown that $-1 < \psi'(0) < 1$ if $\alpha = re^{i\theta}$ and $\beta = re^{i\phi}$, where $0 < \theta - \phi < 2\pi$ .But an easy calculation gives $\psi'(0) = -(\alpha + \beta)/2\psi(0) = \pm\cos\frac{1}{2}(\theta - \phi)$, which proves $-1 < \psi'(0) < 1$.

Definition 4.3.2:

Let Γ be a closed simply connected set in the extended complex plane, consisting of more than a single point. Thus the complement of Γ is a simply connected domain Δ..

A variety of techniques and variational methods have been exploited to identify geometric analytic properties of these support points. For example, if $f \in$ S and $f(\Delta)$ omits an open set, it is easy to show, for w in an open subset omitted by f, that $f_\varepsilon(z) = f(z) + \varepsilon \dfrac{f^2(z)}{w - f(z)} \in$ S for all sufficiently small complex numbers ε. This exhibits an elementary variation of f which constructs functions in S close to f and is the initial step in the proof we present of the following theorem.

Theorem 4.3.2:

If f is a support point of S then $f(\Delta)$ is dense in C.

Proof :

Let f yield the maximum of $\operatorname{Re} J$ over S for the continuous linear functional J ,whose real part is non constant on S .Suppose $f(\Delta)$ is not dense in C and choose w in an open subset of $C \setminus f(\Delta)$. Then, with f_ε as given above, $\operatorname{Re} J(f_\varepsilon) \le \operatorname{Re} J(f)$ so that $\operatorname{Re} J(\varepsilon f^2 /(w-f)) \le 0$ for all sufficiently small complex numbers ε. It follows that $J(f^2/(w-f)) = 0$. According to Theorem (3.2.9) there is a compact set $Q \subset D$ and a measure λ supported on Q with $J(h) = \int_Q h(z) d\lambda\,(z)$ for $h \in \mathcal{A}$. The univalence of f assures that any $w \notin F(\Delta)$ is connected to ∞ so that w lies in an open, connected neighborhood of ∞ disjoint from $f(Q)$. The function

$H(w) = \int_Q \frac{f^2(z)}{w - f(z)} d\lambda\,(z)$ is analytic on this neighborhood and vanishes on an open subset by the above argument, hence is identically zero. Expanding $H(w)$ in a series at ∞ yields $J(f^n) = \int_Q f^n(z) d\lambda\,(z) = 0$ for every integer $n \ge 2$.

By applying Runge's Theorem to functions analytic on the range of f and transferring this information back to D we obtain $J(h) = J(1)h(0) + J(f)h'(0)$ for every $h \in \mathcal{A}$. Thus J is constant on S, contrary to hypothesis.

Theorem 4.3.3:

If f is a support point of S then f maps the unit disk onto the complement of a continuous arc tending to ∞ with increasing modulus.

Proof:

Let f be a support point of S , and let J be the corresponding linear functional. Thus f maximizes $\operatorname{Re}\{J\}$ over S. If f does not have the monotonic modulus property, it must have the form $f = tf_1 + (1-t)f_2, \quad 0 < t < 1,$ where f_1 and f_2 are functions in S which omit open sets. In particular, neither f_1 nor f_2 is a support point of S, since the support points have dense range. Thus $\operatorname{Re}\{J(f_j)\} < \operatorname{Re}\{J(f)\}, \quad j = 1,2.$

By the linearity of J, this implies

$\operatorname{Re}\{(J(f)\} = t\operatorname{Re}\{J(f_1)\} + (1-t)\operatorname{Re}\{J(f_2)\} < \operatorname{Re}\{J(f)\}.$

This contradiction.

Lemma 4.3.1:

Let f be a support point of S then $J(f^2/(f-w))$ is an analytic function of w in the complement of a compact subset of $f(\Delta)$.

Proof : see[5, page 525].

Lemma 4.3.2:

Let w_0 be the end point of Γ and let $w \in \Gamma \setminus \{w_0\}$. Then $\operatorname{Re} J(f^2/(f-w)) > 0.$

Proof :

Let $f_w = wf/w - f$. Then $f_w \in$ S and $C \setminus f_w(\Delta)$ consists of two disjoint unbounded arcs. By Theorem (4.3.3) (applied to f_w) f_w is not a support point of S .Therefore $\operatorname{Re} J(f_w) < \operatorname{Re} J(f)$. But $f - f_w = f^2/(f-w)$, so the desired inequality follows.

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