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OSCILLATIONS AND NON-OSCLLATIONS OF SOLUTIONS OF SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

By

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Introduction

Differential equations with a deviating argument are differential equations in which the unknown function and its derivatives enter, generally speaking, under different values of the argument. For example,

$$y''(t)=f(t,y(t),y'(t),y(t-\tau(t)),y'(t-\tau(t)),y''(t-\tau(t))).$$

The first isolated equation of such type appeared in the literature in the second half of the eighteenth century (Kondorse, 1771)[2]. But a systematic study of equations with a deviating argument began only in the twentieth century so, in the late thirties and early forties, Myshkis [2], [10] in his study of ship stabilization and automatic steering, pointed out very clearly the importance of the consideration of the delay in the feedback mechanism.

Differential equations with a deviating argument have many applications in the theory of automatic control, in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar which is encountered by Rubanik [10], and also as the Euler equations in some variational problems [9],[10],[4], and in many other areas of science and technology. The abundance of applications is stimulating a rapid development of the theory of differential Equations with a deviating argument, and at present, the theory is one of the most rapidly developing branches of mathematical analysis.

So, while in many applications it is assumed that the system under consideration is governed by a principle of causality[10], that is the future state of the system is independent of the past and is determined solely by the present. One should keep in mind that this is only a first approximation to the true situation. A more realistic model must include some of the past history of the system.

Technological and engineering improvements require accounting for the phenomenon of aftereffect in the traditional domains of technology. For example, for modern high speed diesels, approximately a fifty centimeter intake pipe with respect to the time of suction turns out to be a rather lengthy line, and to describe the process of fuel injection it is necessary to use differential equations with a deviating argument (of neutral type)[2].

For differential equations with a deviating argument a classification method was proposed by G. A. Kamenskii [16],[2], he classified these equations as : (i) Equations of a retarded type. (ii) Equations of advanced type. (iii) Equations of neutral type.

For equations with a retarded argument, the basic initial value problem is posed in details in the studies of A. D. Myshkis[2].

For equations of neutral type, G. A. Kamenskii[2] obtained the first theorems of a general theory. The basic initial value problem for equations of advanced type one still insufficiently studied.

In applications, equations with retarded arguments one often encountered, equations of the neutral type less often encountered, and equations with advanced arguments almost never encountered.

In recent years there has been a growing interest in oscillation theory of functional differential equations of neutral type, see for example [20],[10],[15] and the references cited therein, for discussions of the existence and uniqueness of solutions and some application of these equations, in addition to how the behavior of their solutions differs from the behavior of solutions of delay equations, we can refer to [2], [10], [11-16]. Most of the literature, however, is focused on first order linear equations with constant coefficients and constant deviations. The first step to word a systematic investigation of the second order was taken by Ruan[11] who studied the existence of non oscillatory solutions of second order equations.

In general, the theory of neutral delay differential equations of second order or higher orders presents complications, and results which are true for non-neutral equations may not be true for neutral equations[9],[7].

With such clear indications of the importance of equations of neutral type in the applications and also with the number of interesting mathematical problems involved, it is not surprising that the subject has undergone a rapid development in the last twenty five years. New applications also continue to arise and require modifications of even the definition of the basic equations.

We will focus our study and discussion on oscillation and asymptotic behavior of solution of second order neutral delay differential equations. The first chapter is a presentation of some preliminary results, which will be used throughout the thesis, the second chapter deals with oscillations and asymptotic behavior of solutions of second order neutral equations with constant coefficients.

The third chapter discusses the oscillations and asymptotic behavior of solutions of second order neutral equations with variable coefficients.

The fourth chapter contains the necessary conditions for nonlinear neutral equations of second order to be oscillatory, and studies the asymptotic behavior of non-oscillatory solutions of such equations, with some examples to illustrate the theorems.

CHAPTER ONE

Preliminaries

Introduction: -

This chapter is essentially introductory in nature. Its main purpose is to introduce some basic concepts from the theory of differential equations with deviating arguments and to give some important results from the theory of oscillation of ordinary differential equations.

So, in section one, we will classify equations with deviating arguments, while in section two we will provide definitions of oscillation of solutions with or without deviating arguments. And in section three, we will give necessary important results in the oscillation theory of ordinary differential equations. In section four, we will give some examples to illustrate new oscillation phenomena caused by deviating arguments. Finally in section five, we will introduce the existence and uniqueness theorems of neutral functional differential equations.

1.1 Classification of Equations with Deviating Arguments

A natural classification of equations with a deviating argument is the proposal of G. A. Kamenskii.

Definition (1.1.1): -

A differential equation with a retarded argument is a differential equation with deviating argument in which the highest – order derivative of the unknown function appears for just one value of the argument and this argument is not less than all arguments of the unknown function and its derivatives appearing in the equation.

Example (1.1.1):

(1) $y''(t)=f(t,y(t),y'(t),y(t-\tau(t)),y'(t-\tau(t))\ )\quad , \qquad \tau(t)\geq 0$

(2) $y''(t)=f(t,y(\frac{t}{2}),y'(\frac{t}{2}),y(t),y'(t))\quad , \qquad t\geq 0$

Definition (1.1.2):

A differential equation with advanced argument is a differential equation with a deviating argument in which the highest order derivative of the unknown function appears for just one value of the argument, and this argument is not larger than the remaining arguments of the unknown function and its derivatives appearing in the equation.

Example (1.1.2):

(1) $y''(t)=f(t,y(t),y'(t),y(t-\tau(t)),y'(t-\tau(t)), \qquad \tau(t)\leq 0$

(2) $y''(t)=f(t,y(\frac{t}{2}),y'(\frac{t}{2}),y(t),y'(t)), \qquad t\leq 0$

Definition (1.1.3): -

A neutral delay differential equation is a differential equation in which the highest order derivative of the unknown function appears both with and without delays.

Example (1.1.3): -

(1) $y''(t)=f(t,y(t), y'(t), y(t-1), y'(t-1), y''(t-1))$

(2) $y''(t)=f(t,y(t),y'(t),y(t-\tau),y'(t-\tau(t)),y''(t-\tau)),\qquad \tau>0$

Definition (1.1.4):

A neutral delay differential equation is said to have a regular solution $y(t)$ on some ray $[T_y,\infty)$ if

$Sup\ \{|y(t)|:\ t\geq T\}>0$ for every $T\geq T_y$. In other words, $|y(t)|\neq 0$ on any infinite interval $[T,\infty)$.

1.2 Definition of Oscillation: -

The definition of oscillation of regular solutions can have two different forms.

Definition (1.2.1): -

A nontrivial solution $y(t)$ is said to be oscillatory if it changes sign on (T,∞), where T is any number.

Definition (1.2.2): -

A nontrivial solution $y(t)$ is said to be oscillatory if and only if it has arbitrarily large zeros for $t \geq t_0$, that is, there exist a sequence of zeros $\{t_n\}$ $(y(t_n)=0)$ of $y(t)$ such that

$$\lim_{n\to\infty} t_n = +\infty,$$

Otherwise, $y(t)$ is said to be non oscillatory.

For non oscillatory solutions there exist a t_1 such that $y(t) \neq 0$, for all $t \geq t_1$

Example (1.2.1)

The ordinary differential equation $y''(t)+ y(t) = 0$ has periodic oscillatory solutions.

$$y(t)= \cos t,\ y(t)= \sin t \tag{1.2.1}$$

Example (1.2.2):- Consider the equation

$$y''(t) - (1/t)y'(t)+ 4t^2 y(t)=0 \tag{1.2.2}$$

Whose solution is $y(t)= \sin t^2$

This solution is not periodic but has an oscillatory property.

Example (1.2.3):- Consider the delay differential equation

$$y''(t)+\frac{1}{2}y(t)-\frac{1}{2}y(t-\pi)=0,\quad t\geq 0 \tag{1.2.3}$$

Whose solution is $y(t)= 1- \sin t$ has an infinite sequence of multiple zeros.

This solution also has an oscillatory property.

Example (1.2.4):- Consider

$$y''(t) - y(-t)= 0 \tag{1.2.4}$$

Which has an oscillatory solution $y_1(t)= \sin t$ and a non oscillatory solution

$$y_2(t) = e^t + e^{-t}$$

Let us now restrict our discussion to those solutions $y(t)$ of the equation

$$y''(t) + a(t)y(t-\tau(t))= 0 \qquad (1.2.5)$$

Which exist on some ray $[T_y, \infty)$ and satisfy sup $\{|y(t)| : t \geq T\}>0$ for every $T \geq T_y$.

We usually assume that $a(t) \geq 0$ or $a(t) \leq 0$ in Equation (1.2.5), and in doing so we mean to imply that $a(t) \neq 0$ on any infinite interval $[T, \infty)$.

When $\tau(t) \equiv 0$ and $a(t)$ is continuous in Equation (1.2.5), the two definitions given above are equivalent. This is because of the fact that the uniqueness of the solution makes multiple zeros impossible. However, as example (1.2.3) suggests, a differential equation with deviating arguments can have solutions with multiple zeros. These two definitions are different, especially for higher order ordinary differential equations, which may have solutions with multiple zeros.

Definition (1.2.2) is more general than definition (1.2.1), the solution $y(t)=1-\sin t$ of equation(1.2.3) is oscillatory according to definition (1.2.2) and is non oscillatory according to definition (1.2.1).

In example (1.2.3) the possibility of multiple zeros of nontrivial solutions is a consequence of the retardation, since if $\tau(t) \equiv 0$, the corresponding equation has no solutions with multiple zeros.

1.3 Review of the Oscillation Theory of Ordinary Differential Equations (ODE): -

Since Sturm (1836) introduced the concept of oscillation when he studied the problem of Heat transmission [15], oscillation theory has been an important area of research in the qualitative theory of ODE. Oscillation theory of differential Equations with deviating argument (DEWDA) is a natural extension of ODE generated from oscillation theory of (ODE), while certain known results in oscillation theory for (ODE) carry over to (ODEWDA) some what.

Therefore some background in oscillation theory for ODE is essential for understanding oscillation theory of (ODEWDA).

We shall recall only some facts concerning oscillation theory of ODE that are useful for our discussion. The following result which is known as the Sturm comparison theorem is a very important result in oscillation theory.

[·]

Sturm Comparison Theorem:- let y(t) and Z(t) be nontrivial solutions of

$$y''(t) + q(t)\,y(t) = 0$$

And

$$Z''(t) + r(t)\,Z(t) = 0$$

Where $q(t)$ and $r(t)$ are positive functions such that $q(t) > r(t)$, then $y(t)$ vanishes at least once between any two successive zeros of $Z(t)$.

We consider a second order linear ODE

$$y''(t) + a(t)y(t)=0 \qquad (1.3.1)$$

By using Sturm comparison theorem for equation (1.3.1), it is easy to see the following conclusions: -

a) For the linear differential equation (1.3.1), solutions are either all oscillatory or all non-oscillatory, equation (1.3.1) is said to be oscillatory if every solution of equation (1.3.1) is oscillatory and it is said to be non-oscillatory otherwise.

b) We consider another second order linear ODE

$$y''(t) + b(t)y(t)=0 \qquad (1.3.2)$$

If $a(t) \leq b(t)$ for all $t \geq t_0$ and Equation (1.3.1) is oscillatory , then so is Equation(1.3.2).

The comparison method is one of the important methods in oscillation theory of second order linear ODE. There is much literature dealing with extensions of the comparison method to nonlinear ODE and higher order ODE.

Consider a second order non linear ODE

$$y''(t) + q(t)\, f(y(t))=0 \qquad (1.3.3)$$

Interested in non-linear oscillation problems for equations of this type began with the publication of the pioneering work by Atkinson [15]. We would like to point out the fact that the non-linearity in equation (1.3.3) may generate both oscillatory and non-oscillatory solution.

1.4 Some Oscillatory and Non Oscillatory Phenomena Caused by Deviating Arguments

We shall present some examples to show the effect of the delay on the behavior of the solution of the differential equation.

Example (1.4.1): - Consider the second order equation with a delay: -

$$y''(t) + y(\pi - t) = 0 \qquad (1.4.1)$$

It has both an oscillatory solution $y_1 = \sin t$, and a non-oscillatory solution $y_2 = e^t - e^{\pi - t}$

As we mentioned in section (1.3), for second order linear ODE either all solutions oscillate or all solutions are non-oscillatory. Thus we see that second order equations with delay create some new problems in oscillation theory.

For example, consider

$$y''(t) + P(t)y''(t-\tau) + Q(t)y(t-\sigma) = 0 \qquad (1.4.2)$$

the study of (ODEWDA) establishes various sets of conditions under which either: -

(a) All solutions are oscillatory.

(b) All solutions are non-oscillatory.

(c) The equation has a non-oscillatory solution.

(d) The equation has an oscillatory solution

(e) The equation has both oscillatory and non-oscillatory solutions.

(f) The asymptotic behavior of the non-oscillatory solution and its derivative.

Example (1.4.2): - The equation with a delay

$$y''(t) + 4y''(t-\pi) - 3y(t-2\pi) = 0, \tag{1.4.3}$$

has the oscillatory solutions $y = \sin t$, and $y = \cos t$.

but

$$y''(t) + 4y''(t) - 3y(t) = 0,$$

or

$5y''(t) - 3y(t) = 0$ has no oscillatory solution.

This example suggests that we need to find conditions for oscillatory solutions of equation (1.4.2).

Therefor we need to study the effect of deviating arguments on the oscillation of solutions.

Deviating argument can occur in many complex forms. For example, consider equations where the deviating argument depends on the solution itself,

$$y''(t) + P(t)\, y(t-\tau(y(t))) = 0 \tag{1.4.4}$$

Since the oscillation theory of (ODEWDA) presents some new problems that one not relevant for the corresponding ODE, a study of the oscillation and non-oscillation caused by deviating arguments is most interesting.

1.5 Existence and Uniqueness Theorems for Solutions of Neutral Delay Differential Equations

Applying the method of steps to equations of neutral type reduces differential equations with a deviating argument to an equation without a deviating argument, to which we may apply known existence and uniqueness theorems for the solution of the basic initial value problem.

For example, for the equation

$$X''(t)=f(t,X(t),\ X(t-\tau(t)),\ X'(t-\tau(t)),\ X''(t-\tau(t))) \qquad (1.5.1)$$

$$\tau(t) \geq 0,\ X(t) = \phi(t),\ t \in E_{t0} \qquad (1.5.2)$$

Where $E_{t0} = [t_0-\tau,\ t_0]$

Applying the method of steps, we obtain

$$X''(t)=f(t,X(t),\ \phi(t-\tau(t)),\ \phi'(t-\tau(t)),\ \phi''(t-\tau(t))) \qquad (1.5.3)$$

$$X(t_0)=\phi(t_0) \qquad (1.5.4)$$

Equation (1.5.3) with initial condition (1.5.4), and consequently, Equation (1.5.1) with initial condition (1.5.2) **has a solution if f and ϕ are continuous and this solution is unique if the function $f(t,x,y)$ satisfied a lipschitz condition in its second and third arguments for t near t_0, for y near $\phi(t_0 - \tau(t_0))$, and X near X_0,** (for this it suffices for

$$\left|\frac{\partial f}{\partial x}\right|,\ \left|\frac{\partial f(t,x,y)}{\partial y}\right|,\ |\phi'(t)| \text{ and } |\phi''(t)|$$

to be bounded in a neighborhood of the initial values).

Chapter Two

Oscillations and Asymptotic Behavior of Solutions of Second Order Linear Neutral Delay Differential Equations with Constant Coefficients

Introduction: -

In this chapter, which consists of two sections, we will study the following second order linear neutral delay differential equation (LNDDE) of the form:

$$\frac{d^2}{dt^2}[y(t) + p\,y(t-\tau)] + Q\,y(t-\sigma) = 0 \qquad (2.1.1)$$

Where p, Q are real numbers and the deviating arguments τ, σ are non-negative real numbers.

In section (2.1), we will study the asymptotic behavior of non-oscillatory solutions of the (LNDDE) (2.1.1).

In section (2.2), we will study and discuss the necessary and sufficient conditions, under which all solutions of the (LNDDE) (2.1.1) oscillate,

2.1 Asymptotic Behavior of Non-Oscillatory Solutions

In this section, we will study the asymptotic behavior of the non-oscillatory solutions of equation (2.1.1) under the hypothesis:

(H): Q is a positive constant, p is a real number, and τ and σ are nonnegative constants

Theorem (2.1.1): consider the NDDE (2.1.1) and assume that the hypothesis (H) is satisfied. If $p>-1$, Then every non-oscillatory solution of equation (2.1.1) tends to zero as $t \to \infty$.

Proof:

Let $y(t)$ be an eventually positive solution of equation (2.1.1).

Set $Z(t) = y(t) + py(t-\tau)$ $\quad (2.1.2)$

Then from equation (2.1.1)

$$Z''(t) = -Qy(t-\sigma) < 0 \quad (2.1.3)$$

And so $Z'(t)$ is strictly decreasing.

From (2.1.3) either

$$\lim_{t\to\infty} Z'(t) = -\infty \quad (2.1.4)$$

or,

$$\lim_{t\to\infty} Z'(t) = L \text{ is finite} \quad (2.1.5)$$

We claim that, in this case (2.1.4) is impossible, otherwise, we shall also have that

$$\lim_{t\to\infty} Z(t) = -\infty \quad (2.1.6)$$

And $Z(t) < 0$, which implies that

$$0 < y(t) < -py(t-\tau) \quad (2.1.7)$$

and by iteration

$$0 < y(t+k\tau) < (-p)^k y(t) \quad (2.1.8)$$

We have $p>-1$ and so (2.1.7) is impossible for $p \geq 0$, while (2.1.8) implies that

$$\lim_{t\to\infty} y(t) = 0 \quad for \quad -1 < p < 0$$

But in the later case (2.1.6) is impossible.

We have established the validity of (2.1.5) for any $p>-1$.

Now integrating (2.1.3) from t_1 to t with t_1 sufficiently large and letting $t\to\infty$ we find

$$L - Z'(t_1) = -Q\int_{t_1}^{\infty} y(s-\sigma)ds,$$

which implies that $y \in L^1[t_1,\infty)$.

Thus, from (2.1.2) $Z \in L^1[t_1,\infty)$, and since Z is monotonic, it follows that

$$\lim_{t\to\infty} Z(t) = 0, \qquad (2.1.9)$$

and so $L=0$

As the function $Z'(t)$ decreases to zero, it follows that

$$Z'(t) > 0. \qquad (2.1.10)$$

And (2.1.10) implies that $Z'(t)Z''(t) < 0$. Next, we shall prove that

$$\lim_{t\to\infty} y(t) = 0 \quad \text{for} \quad p \geq o$$

This follows from the observation that

$$0 < y(t) \leq y(t) + p\,y(t-\tau) = Z(t) \to 0 \quad \text{as} \quad t\to\infty$$

Hence

$$\lim_{t\to\infty} y(t) = 0 \quad \text{for} \quad p \geq 0$$

For $-1 < p < 0$ we shall apply the following lemma for Ladas and Sficas[14].

If $f(t)= g(t) + p\,g(t-\tau)$ where $-1 < p < 0$ and if g is bounded and if $\lim_{t\to\infty} f(t)$ exists then, the $\lim_{t\to\infty} g(t)$ also exists.

So, in order to apply this lemma we have to prove first that y(t) is bounded for $-1 < p < 0$.

Hence, from (2.1.10) and the fact that $Z''(t) < 0$ we find that $Z(t)<0$ and $y(t) < -p\,y(t-\tau) < y(t-\tau)$ which establishes the boundedness of $y(t)$.

So, by the previous lemma we have:

If $y(t)$ is bounded and

$$\lim_{t\to\infty} Z(t) = 0$$

Then

$$\lim_{t\to\infty} y(t) = 0$$

The proof of the theorem is complete.

Remark: If $p<-1$, it is not true that all non-oscillatory solutions of equation (2.1.1) tend to 0 as $t\to\infty$. This follows from the following example:

$$\frac{d^2}{dt^2}[y(t) - 4y(t - Log2)] + e\,y(t-1) = 0$$

Which has the non-oscillatory solutions $\pm e^t$, and this solution tends to $\pm\infty$ as $t\to\infty$.

Corollary (2.1.1)[14]: If $p = -1$, then every non-oscillatory solution of equation (2.1.1) is bounded, and tends to zero as $t\to\infty$.

Corollary (2.1.2)[14]: If $p <-1$, $y(t)$ is eventually positive solution of equation (2.1.1) then the following statements are satisfied:

Either

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} Z'(t) = -\infty \qquad (2.1.11)$$

or

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} Z'(t) = 0, \quad Z(t) < 0, \quad Z'(t) > 0 \tag{2.1.12}$$

Corollary (2.1.3): a) If $p < -1$ and (2.1.11) holds then every non-oscillatory solution of equation (2.1.1) tends to ∞ as $t \to \infty$.

b) If $p < -1$ and (2.1.12) holds then every non-oscillatory solution of equation (2.1.1) tends to zero as $t \to \infty$.

Proof:

a) $p\, y(t-\tau) < Z(t) \to -\infty$ as $t \to \infty$

but since $p<-1$ it follows that,

$$\lim_{t\to\infty} y(t) = \infty$$

b) Let $y(t)$ be an eventually positive solution of the equation (2.1.1) then

$$p\, y(t-\tau) < y(t) + p\, y(t-\tau) = Z(t) \to 0 \text{ as } t \to \infty$$

Hence

$$\lim y(t) = 0 \text{ as } t \to \infty.$$

2.2 Oscillation

We will discuss the (LNDDE) of the following form:-

$$\frac{d^2}{dt^2}[y(t) + p\, y(t-\tau)] + Qy(t-\sigma) = 0, \tag{2.2.1}$$

where $p, Q \in R$ and τ, σ are non-negative constants (2.2.2)

For such an equation, its characteristic equation is

$$\lambda^2 + \lambda^2 p\, e^{-\lambda\tau} + Q\, e^{-\lambda\sigma} = 0 \tag{2.2.3}$$

The function $y(t)$ is a solution of equation (2.2.1) if $y \in C\,[t_0 - m, \infty)$ where $m = \max\{\tau, \sigma\}$ such that $y(t) + p\,y(t-\tau) \in C^2[t_0 - m, t_0]$ and Equation (2.2.1) is satisfied for $t \geq t_0$. And $y(t)$ is a unique solution of equation (2.2.1) if given an initial function $\phi \in C[t_0 - m,\ t_0]$ and a number $A \in R$, and

$$y(t) = \phi(t), \qquad t_0 - m \leq t \leq t_0 \tag{2.2.4}$$

and

$$y'(t_0) = A \tag{2.2.5}$$

Equation (2.2.1) is called oscillatory if for every $t_0 \in R$, every function $\phi \in C\ [t_{0-m,}\ t_0]$ and every $A \in R$, the unique solution of equation (2.2.1) which satisfies (2.2.4) and (2.2.5) oscillate.

Theorem (2.2.1): - Assume that (2.2.2) holds, then the following statements are equivalent.

(a) Every solution of equation (2.2.1) oscillates.

(b) Equation (2.2.3) has no real roots.

Proof:

If every solution of equation (2.2.1) oscillates then equation (2.2.3) cannot have any real root. Since if λ_0 were a real root of equation (2.2.3) then

$$y(t) = e^{\lambda_0 t}$$

Would be a non-oscillatory solution of equation (2.2.1). this proofs (a) → (b).

The proof of (b) → (a) will be accomplished by a series of lemmas From 1 to 6 and we will give the proof of Lemma (2.2.1), for the proof of the remaining lemmas consult [11], [12]. We will assume that equation (2.2.3) has no real root, and for the sake of contradiction we assume that equation (2.2.1) has eventually positive solution.

When $p\tau = 0$, $p \neq -1$ and $Q > 0$ the NDDE reduces to a delay equation of the form

$$y''(t) + q\, y(t-\sigma) = 0 \tag{2.2.6}$$

And equation (2.2.3) reduces to

$$G(\lambda) = \lambda^2 + q\, e^{-\lambda\sigma} = 0 \tag{2.2.7}$$

The hypothesis that equation (2.2.7) has no real root and the fact that $G(\infty) = \infty$ imply that

$G(\lambda) > 0$ for $\lambda \in R$

In particular $G(0) = q > 0$

And so, every solution of equation (2.2.6) oscillates[11].

Lemma (2.2.1):-

Assume that (b) holds $p\tau \neq 0$ then the following statements are true.

(i) $Q>0$

(ii) If $p<0$ then $\sigma > \tau$.

(iii) There exist a positive constant m such that

$$F(\lambda) = \lambda^2 + \lambda^2 p\, e^{-\lambda\tau} + Q\, e^{-\lambda\sigma} \geq m \text{ for all } \lambda \in R \qquad (2.2.8)$$

Proof:

(i) Indeed, $F(\infty)=\infty$ and since $F(\lambda)=0$ has no real zeros.

$$F(\lambda) > 0 \text{ for all } \lambda \in R \qquad (2.2.9)$$

In particular

$$F(0) = Q > 0$$

Which proves (i)

(ii) If $\sigma \leq \tau$ then

$$F(-\infty) = -\infty \text{ and } F(\infty) = \infty$$

So $F(\lambda)=0$ would have a real root.

$$\therefore \sigma > \tau$$

(iii) Since $F(-\infty) = F(\infty) = \infty$ which together with $F(\lambda)>0$ for all $\lambda \in R$ implies

$m = \min_{\lambda \in R} F(\lambda)$ is a positive number which satisfies (2.2.8)

Lemma (2.2.2):

Set

$$V(t) = -[y(t) + p\, y(t-\tau)] \qquad (2.2.10)$$

(i) $V(t)$ is a twice continuously differentiable solution of equation (2.1.1), that is

$$V''(t) + p\, V''(t-\tau) + Q\, V(t-\sigma) = 0 \qquad (2.2.11)$$

(ii) One of the following holds:-

Either

$$\left.\begin{aligned} & V(t) > 0,\ V'(t) < 0,\ V''(t) > 0 \\ \text{and} & \\ & \mathrm{Lim}_{t\to\infty} V(t) = \mathrm{Lim}_{t\to\infty} V'(t) = 0 \end{aligned}\right\} \qquad (2.2.12)$$

or

$$\left.\begin{aligned} & V(t) > 0,\ V'(t) > 0,\ V''(t) > 0 \\ \text{and} & \\ & \mathrm{Lim}_{t\to\infty} V(t) = \mathrm{Lim}_{t\to\infty} V'(t) = \infty \end{aligned}\right\} \qquad (2.2.13)$$

(iii) $p < 0$ and $\sigma > \tau$, and if (2.2.13) holds then $p < -1$.

Next we will define two sets corresponding to whether (2.2.12) or (2.2.13) is satisfied.

Definition (2.2.1):

Let W^- be the set of all functions of the form

$$w(t) = -[V(t) + p\,V(t-\tau)],$$

which satisfies (2.2.12) that is for every function $w \in W^-$.

$$\left.\begin{aligned} & w(t) > 0,\ w'(t) < 0,\ w''(t) > 0, \\ \text{and} & \\ & \mathop{\mathrm{Lim}}_{t\to\infty} w(t) = \mathop{\mathrm{Lim}}_{t\to\infty} w'(t) = 0 \end{aligned}\right\} \qquad (2.2.14)$$

Definition (2.2.2):

Let W^+ be the set of all functions of the form

$$w(t) = -[V(t) + p\,V(t-\tau)],$$

which satisfies (2.2.13) that is for every function $w \in W^{+}$.

$$\left.\begin{array}{l} w(t) > 0,\ w'(t) > 0,\ w''(t) > 0, \\ \text{and} \\ \lim\limits_{t\to\infty} w(t) = \lim\limits_{t\to\infty} w'(t) = \infty \end{array}\right\} \qquad (2.2.15)$$

In view of lemma (2.2.2) either W^{-} *or* W^{+} is non-empty.

Also, an argument similar to that of lemma (2.2.2) (i) shows that each function $w \in (W^{-} \cup W^{+})$ is a four times continuously differentiable solution of equation (2.2.1), that is $w \in C^{4}$ and

$$w''(t) + p\,w''(t-\tau) + Q\,w(t-\sigma) = 0 \qquad (2.2.16)$$

Also, there is a solution V $\in C^{2}$ of equation (2.2.1) which satisfies (2.2.12) if $w \in W^{-}$, or (2.2.13) if $w \in W^{+}$ such that

$$w(t) = -[v(t) + pv(t-\tau)] + QV(t-\sigma) = 0$$

$$[v(t) + pv(t-\tau)]'' + QV(t-\sigma) = 0$$

$$[-w(t)]'' + QV(t-\sigma) = 0$$

$$-w''(t) = -QV(t-\sigma)$$

$$w''(t) = QV(t-\sigma) \qquad (2.2.17)$$

Further more,

$$w \in W^{-} \rightarrow -[w(t) + p\,w(t-\tau)] \in W^{-}$$

and

$$w \in W^{+} \to -[w(t) + p\, w(t-\tau)] \in W^{+}$$

If w_1 and $w_2 \in W^{-}$ (respectively in W^{+}) and $a, b > 0$ then

$$aw_1 + bw_2 \in W^{-} \text{ (respectively in } W^{+}\text{)}.$$

With each function $w \in (W^{-} \cup W^{+})$ define the set

$$\Gamma(w)=\{\lambda \geq 0 : w''(t) - \lambda^2 w(t) \geq 0\}$$

$0 \in \Gamma(w)$ since $w''(t) - 0 > 0$ i.e. $w''(t) > 0$ and if $\lambda \in \Gamma(w)$ then $[0,\lambda] \subseteq \Gamma(w)$, that is $\Gamma(w)$ is a non-empty subinterval of R^{+}.

For the proof of the theorem, we will assume that $W^{-} \neq \phi$ and we will show that leads to a contradiction.

Lemma 2.2.3:

(i) Let $w \in W^{-}$. Then $[Q / -p]^{1/2} \in \Gamma(w)$.

(ii) There is a constant μ (independent of w) such that $\Gamma(w)$ is bounded above by μ for all $w \in W^{-}$

(iii) Let $w \in W^{-}$ and $\lambda \in \Gamma(w)$. then $w'(t) + \lambda w(t) \leq 0$.

For any function $w \in W^{-}$, by integrating (2.2.16) from t to t_1 twice and by letting $t_1 \to \infty$ and using (2.2.14) we find that

$$-[w(t) + pw(t-\tau)] = Q\int_t^{\infty}\int_s^{\infty} w(\xi - \sigma)\, d\xi\, ds, \qquad (2.2.18)$$

In particular, (2.2.18) shows that the right-hand side of (2.2.18) is an element of W^-.

Also for any $w \in W^-$ and any $\lambda \in R$, the function

$$Z(t) = -[w(t) + pw(t-\tau)] + \lambda^2 \int_{t-\sigma}^{\infty} \int_{s}^{\infty} w(\xi)\, d\xi\, ds \tag{2.2.19}$$

is an element of W^-

Lemma 2.2.4: -

Let $w \in W^-$ and let $\lambda \in \Gamma(w)$

Set

$$k = \frac{m}{-pe^{\mu\tau} + e^{\mu\sigma}},$$

Where m is the constant which was defined in lemma (2.2.1) (iii) and μ is the constant which was defined in lemma (2.2.3)(ii).

Then

$$(\lambda^2 + k)^{1/2} \in \Gamma(Z),$$

Where Z is the function defined by (2.2.19).

Now, consider the sequence of the functions

$$Z_n(t) = -[Z_{n-1}(t) + pZ_{n-1}(t-\tau)] + \lambda_n^2 \int_{t-\sigma}^{\infty} \int_{s}^{\infty} Z_{n-1}(\xi)\, d\xi\, ds$$

For $n=1,2,\ldots$ where $Z_0(t)$ is the function $Z(t)$ defined in (2.2.19), λ_0 is the number in lemma (2.2.3)(i)

$$\lambda_0 = \left(\frac{Q}{-p}\right)^{1/2}, \lambda_n = [\lambda_{n-1}^2 + k]^{1/2}$$

And a repeated application of lemma (2.2.4) shows that

$\lambda_n \in \Gamma(Z_{n-1})$, For $n = 1,2,.....$

Clearly,

$$\lim_{n\to\infty} \lambda_n = \infty,$$

which contradicts the fact in lemma (2.2.3)(ii) that

$\lambda_n \leq \mu$ for all $n=1,2,...$

The proof of the theorem is complete when $W^- \neq \phi$.

Next, we will assume that $W^- = \phi$. Then $W^+ \neq \phi$.

Lemma (2.2.5):

(i) Let $w \in W^+$ and $\lambda \in \Gamma(w)$. then

$w'(t) - \lambda w(t) \geq 0$

(ii) Let k be a positive integer such that $k\tau > \sigma - \tau$. Then

$[\frac{Q}{(-p)^{k+1}}]^{1/2} \in \Gamma(w)$, For every $w \in W^+$

(iii)

$(\frac{1}{\tau})\ln(-p) \notin \Gamma(w)$, For any $w \in W^+$

Now by integrating both sides of (2.2.16) from $t_0+\sigma$ to $t-\tau$ we find

$$[w'(t-\tau)+pw'(t-2\tau)]-[w'(t_0+\sigma)+pw'(t_0+\sigma-\tau)]+Q\int_{t_0+\sigma}^{t-\tau} w(s-\sigma)ds = 0$$

or

$$-[w'(t+\sigma-\tau)+pw'(t+\sigma-2\tau)] = C + Q\int_{t_0}^{t-\tau} w(s)ds \qquad (2.2.20)$$

Where

$$C = -[w'(t_0 + \sigma) + pw'(t_0 + \sigma - \tau)] \tag{2.2.21}$$

As $w'(t)$ is a solution of equation (2.2.1), it follows from (2.2.20) that if $w \in W^+$ then

$$C + Q\int_{t_0}^{t-\tau} w(s)ds \in W^+$$

Where C is a constant given by (2.2.21)

Lemma (2.2.6)

Let $w \in W^+$ and let $\lambda \in \Gamma(w)$,

$$\lambda \geq \lambda_0 = [\frac{Q}{(-p)^{k+1}}]^{1/2}$$

Set

$$N = \frac{m}{2(-p + \frac{Q}{\lambda_0^2})}$$

Where m is the constant which was defined in lemma (2.2.1)(iii), then

$(\lambda^2 + N)^{1/2} \in \Gamma(Z)$, where

$$Z(t) = -[w(t) + pw(t-\tau)] + \frac{Q}{\lambda}\int_{t_0}^{t-\tau} w(s)ds + \frac{C}{\lambda}$$

and C is the constant given by (2.2.21)

Finally, consider the sequence of functions

$$Z_n(t) = -[Z_{n-1}(t) + pZ_{n-1}(t-\tau)] + \frac{Q}{\lambda_n}\int_{t_0}^{t-\tau} Z_{n-1}(s)ds + \frac{C_n}{\lambda_n},$$

for n=1,2,…where $Z_0(t)$ is the function $Z(t)$ of lemma (2.2.6), λ_0 is as in lemma (2.2.6).

$$C_n = -[Z_n'(t_0+\sigma) + pZ_n'(t_0+\sigma-\tau)]$$

and

$$\lambda_n = (\lambda_{n-1}^2 + N)^{1/2}$$

where N is a positive constant defined in lemma (2.2.6).

A repeated application of lemma (2.2.6) shows that

$\lambda_n \in \Gamma(Z_{n-1})$ *for* $n=1,2,\dots\dots$

Clearly

$$\lim_{n\to\infty} \lambda_n = \infty,$$

which contradicts the fact in lemma (2.2.5)(iii), that

$$\frac{1}{\tau}\ln(-p),$$

is an upper bound of $\Gamma(w)$ for $w \in W^+$.

The proof of the theorem is complete.

Example (2.2.1): -Consider the NDDE

$$\frac{d^2}{dt^2}[y(t) + y(t-2\pi)] + 2y(t-2\pi) = 0$$

$$p = 1, \qquad Q = 2, \qquad \tau = \sigma = 2\pi$$

The characteristic equation of the NDDE is

$$\lambda^2 + \lambda^2 e^{-2\lambda\pi} + 2e^{-2\lambda\pi} = 0$$

$$\lambda^2(1 + e^{-2\lambda\pi}) = -2e^{-2\lambda\pi}$$

$$\lambda^2 = \frac{-2}{1+e^{2\lambda\pi}}$$

$$\lambda = \left(\frac{-2}{1+e^{2\lambda\pi}}\right)^{1/2} \notin R$$

Since the characteristic equation has no real root then the NDDE has oscillatory solution $y(t) = C_1 \cos t + C_2 \sin t$.

The following two lemmas are essential in the proofs of the next oscillation theorems.

Lemma (2.2.7): - [14]

If $y(t)$ is an eventually positive solution of equation (2.2.1) and if $p \geq -1$ then $Z(t)$ is bounded negative solution of equation (2.2.1)

Lemma (2.2.8):- [14]

Assume that r and μ are positive constants and suppose that:

$$\sqrt{r}\left(\frac{\mu}{2}\right) > \left(\frac{1}{e}\right), \qquad (2.2.22)$$

Then the inequality

$$X''(t) - r\, X(t-\mu) \leq 0,$$

has no eventually negative bounded solution.

Remark:- the following theorems are proved under the hypothesis:

(H) Q is a positive constant, p is a real number, and τ and σ are nonnegative constants.

Theorem (2.2.2): Assume that (H) is satisfied, suppose

$$p \geq 0 \qquad (2.2.23)$$

Then every solution of equation (2.2.1) oscillates.

Proof: -

Assume, for the sake of contradiction, that equation (2.2.1) has an eventually positive solution $y(t)$, then from (2.2.23).

$$Z(t) = y(t) + p\,y(t - \tau) > 0,$$

which contradicts lemma (2.2.7).

The proof is complete.

Theorem (2.2.3):

Assume that (H) is satisfied, suppose that

$$-1 \leq p < 0 \tag{2.2.24}$$

and

$$\left[\left(\frac{-Q}{p}\right)^{1/2}\left(\frac{\sigma-\tau}{2}\right)\right] > \frac{1}{e} \tag{2.2.25}$$

Then every solution of equation (2.2.1) oscillates.

Proof:

Assume for the sake of contradiction, that $y(t)$ is an eventually positive solution of equation (2.2.1). Then,

$$Z(t) = y(t) + p\,y(t-\tau) > p\,y(t-\tau),$$

and so,

$$y(t-\tau) > \frac{1}{p} Z(t)$$

Now replace t by $t - \sigma + \tau$

We get:

$$y(t-\sigma) > \frac{1}{p} Z(t-(\sigma-\tau))$$

It follows that

$$Z''(t) = -Q\,y(t-\sigma) < -\frac{Q}{p} Z(t-(\sigma-\tau))$$

or

Also from lemma (2.2.7) we have

$$Z''(t)+\frac{Q}{p}Z(t-(\sigma-\tau))<0 \tag{2.2.26}$$

$Z(t) < 0$ and $Z(t)$ is bounded (2.2.27)

Now, let

$$r=\frac{-Q}{p} \text{ and } \mu=\sigma-\tau$$

then, in view of (2.2.25) and lemma (2.2.8) it follows that inequality (2.2.26) cannot have an eventually negative bounded solution.

This contradicts (2.2.27) and completes the proof of the theorem.

Theorem (2.2.4):

Assume that (H) and condition (2.2.25) are satisfied.

Suppose that

$$p < -1$$

Then every bounded solution of equation (2.2.1) oscillates.

Proof:

Assume, for the sake of contradiction, that equation (2.2.1) has an eventually positive and bounded solution $y(t)$. Then as in the proof of Theorem (2.2.3), (2.2.26) holds.

Also, in view of the fact that $y(t)$ is bounded, (2.1.12) and in particular (2.2.27) is satisfied, this leads to a contradiction.

The proof is complete.

Example (2.2.2):

Consider the NDDE:

$y''(t) + -3y''(t-\pi) + 4y(t-2\pi) = 0$

$p= -3 \;\; < -1$

$Q= 4 \;\; > 0$

This equation satisfies condition (2.2.25) which is

$[(-\frac{Q}{p})^{1/2}\frac{(\sigma-\tau)}{2}] > \frac{1}{e}$, that is $[\frac{-4}{-3}]^{1/2}\pi > \frac{1}{e}$

and all conditions of theorem (2.2.4).

So every bounded solution of equation (2.2.1) oscillates.

For example: $y(t) = \sin t$ is a bounded oscillatory solution.