

Deanship of Graduate Studies

Al-Quds University

Extreme Points of The Set of Univalent Functions

By

Ranya Mohammad Saleem Al-Najajreh

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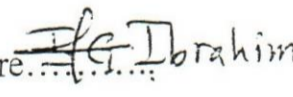
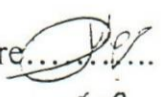

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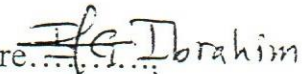


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Dedication

To my parents . To my Family . To my husband and to my daughter

Gaida .

Acknowledgment

I would like to express my thanks to those helped me to prepare and complete this work.

I owe the success of this work to Dr. Ibrahim Al -Grouz for his great help in giving me the appropriate references and suggestions for this work.

I am also very grateful , to Dr . Mohammed Khalil and Dr. Ibrahim Al-Masri for their efforts in preparing , writing , analysis and putting the final touches

My thanks to the principle of my school , Mr. Ashraf Shakarneh , for giving me the time to go and to leave the school to complete this work .

Abstract

Let Δ be the unit disk and \mathcal{A} be the set of all analytic functions in Δ . Let S be the set of all normalized univalent functions in the open unit disk

i.e. $\{ f \in \mathcal{A} : f \text{ is univalent, } f(0)=0 \text{ and } f'(0)=1 \}$.

Our main goal in the thesis is to study the extreme points and the convex hulls of several subfamilies of S such as the set of all starlike functions in S (S^*), the positive real functions (\mathcal{P}), the family of closed- to - convex functions (CL), the family of convex function (K) and the set of all function with real coefficients in S (T), and give a description about these extreme points by the integration .

Also , we prove some facts about the extreme point of the family S . An example of a function in S which is not extreme point of S was given.

In general , its proved that if $f \in S$ omits two values of equal modulus, then f is not an extreme point of S .

الخلاصة

افرض أن Δ هو قرص الوحدة وأن المجموعة \mathcal{R} هي مجموعة كل الاقترانات التحليلية في Δ .
افرض أن S مجموعة الاقترانات المركبة الأحادية المعرفة داخل دائرة الوحدة.
الهدف الأساسي من هذه الرسالة هو دراسة الاقترانات القصوى والغطاء لعدة مجموعات جزئية من S
مثل الاقترانات الشبه نجمية (S^*)، الاقترانات المقعرة (K)، الاقترانات القريبة من التقعر (CL)
وكذلك (T) و (ρ) . وقدمنا وصفا مفصلا حول الاقترانات القصوى باستخدام التكامل .
أيضا ، لقد أثبتنا بعض الحقائق حول الاقترانات القصوى للمجموعة S و أعطينا مثلا على اقتران من
المجموعة S كونه لا ينتمي الى مجموعة الاقترانات القصوى ، و أثبتنا بعض الصفات الضرورية
ليكون الاقتران من الاقترانات القصوى من S أو ليس من الاقترانات القصوى في S .

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Introduction

Let Δ be the open unit disk, $\Delta = \{z : |z| < 1\}$. A function f is said to be univalent on Δ if it is one – to – one in Δ .

The theory of univalent functions is an old subject, born around the beginning of this Century with the paper by P. Koebe [30] yet it remains an active field of current research.

A number of books on the subject have been written and an extensive introduction is provided by P. L. Duren [15], G. M. Golusin [16], W. K. Hayman [23] and Chr. Pommerenke [43]. Some additional books about univalent functions are by J. A. Hummel [29], I. M. Milin [38], P. Monteleone [39], and A. C. Schaeffer [52]. And D. C. Spencer [52].

Several books on complex analysis also contain information about univalent functions.

A number of survey articles have been written about the general theory of univalent functions and more specific developments and we mention [3], [10], [13], [17], [18], [24], [51], [36], [46] and [54]

The book [5] edited by D. A. Brannan and J. G. Clunie also contains survey articles, introductory lectures and current research work on univalent functions and other fields in complex analysis.

Let S be the set of all analytic and univalent functions f in the unit disk Δ with the normalization $f(0) = 0$ and $f'(0) = 1$.

Most books of complex analysis are concerned with this family S .

Also, P. Duren, T. Macgr. is a good reference for the set S .

One of the major problems of the field is the Bieberbach conjecture, dating from the year 1916, which asserts that the Taylor coefficients of each function of class S satisfy the inequality $|a_n| \leq n$. For many years this famous problem has stood as a challenge and has inspired the development of ingenious methods until the proof of de Brange [13] in 1985.

Most of the material in chapter two in the thesis deals with univalent functions. Also, a generalization of the subfamilies S^* and K were introduced by M.S. Roberstone in [45]. The geometric characterization mentioned for close-to-convex functions is due to Z. Lewandowski [32,33]. The class T of typically real functions was introduced and studied by W. Rogonsinski [48]. We give a detailed proof of some theorems which we need to develop the theory of the extreme point of S and the subfamilies of S .

The last two chapters give a brief introduction to the subordination which goes back to E. Lindelof [34]. Subordination was more formally introduced and studied by J. E Littlewood [35] and the later by Rogosinski [49].

we say that f is an extreme point of the determination of convex hull and extreme point of special families of univalent functions.

The determination of the extreme points of the family S is partially proved, until now it is not known the general characterization of the extreme points of S . Some results can be found in Brickman [7], G Springer [55].

Chapter One

Univalent Functions

In this chapter we deal with analytic functions on (a simply connected) domain that maps conformally this domain to the unit disk Δ .

With some normalization. Also, gives some elementary properties of these functions.

To do this, we want to define what we mean by the univalent functions.

1.1 The Set of Analytic Functions on Δ

Definition 1.1.1

Let Δ be the open unit disk , in notation $\{ z: |z| < 1 \}$. Its boundary is the unit circle , is denoted by $\partial\Delta$.

Definition 1.1.2

A complex function f is continuously differentiable if f' is continuous on Δ .

Definition 1.1.3

The function f is analytic on Δ if f is continuously differentiable on Δ . In notation \mathcal{A} denotes the set of all analytic functions.

In particular f is analytic at a point z_0 if it is analytic in a neighborhood of z_0 .

It is easy to show that if f and g are two analytic functions on Δ then

$f \pm g$ and $f \cdot g$ are also analytic on Δ .

Also if $g(z) \neq 0$, for all $z \in \Delta$ then $\frac{f}{g}$ is analytic on Δ , as we see in the following

example.

Example 1.1.1.

$f(z) = \frac{z}{1-z}$ is analytic function on Δ since $f'(z) = \frac{(1-z) + z}{(1-z)^2} = \frac{1}{(1-z)^2}$. Which

is continuous on Δ since $1-z \neq 0$. Hence f is analytic.

Example 1.1.2

Let $f(z) = \frac{z}{(1-z)^2}$ be a function on Δ . To show that f is analytic, want to find f'

Then $f'(z) = \frac{1+z}{(1-z)^3}$ since $1-z \neq 0$, then f' is continuous on Δ . Hence f is analytic

on Δ .

1.2 The Univalent Function

Definition 1.2.1

Let f be a complex function defined on Δ . We say f is univalent in domain Δ if

whenever $f(z_1) = f(z_2)$ implies $z_1 = z_2$, or when $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$ for all

$z_1, z_2 \in \Delta$.

Example 1.2.1

Let $f(z) = \frac{z}{1-z}$ be a function on Δ .

To show that f is univalent, suppose $f(z_1) = f(z_2)$ for some $z_1, z_2 \in \Delta$. Then

$$f(z_1) = \frac{z_1}{1-z_1} = \frac{z_2}{1-z_2} = f(z_2)$$

since $1-z_1 \neq 0$, $1-z_2 \neq 0$ then multiplying both sides by $(1-z_1)(1-z_2)$, we get

$$z_1(1-z_2) = z_2(1-z_1), \text{ therefore } \rightarrow z_1 - z_1z_2 = z_2 - z_1z_2 \text{ and hence } z_1 = z_2.$$

So f is univalent in Δ .

Example 1.2.2

$$\text{Let } f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \dots + nz^n + \dots$$

(Koebe function) be a function on Δ .

To show that f is univalent, suppose $f(z_1) = f(z_2)$ for some $z_1, z_2 \in \Delta$ then

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2}, \text{ since } 1-z_1 \neq 0, 1-z_2 \neq 0, \text{ then multiplying both sides by}$$

$(1-z_1)^2 \cdot (1-z_2)^2$, we get

$$z_1(1-z_2)^2 = z_2(1-z_1)^2$$

Therefore

$$z_1(1-2z_2+z_2^2) = z_2(1-2z_1+z_1^2)$$

$$z_1 - 2z_1z_2 + z_1z_2^2 = z_2 - 2z_1z_2 + z_2z_1^2$$

$$z_1 + z_1z_2^2 = z_2 + z_2z_1^2$$

$$z_1 + z_1z_2^2 - z_2 - z_2z_1^2 = 0$$

$$z_1 - z_2 + z_1z_2^2 - z_2z_1^2 = 0$$

$$(z_1 - z_2) + z_1z_2(z_2 - z_1) = 0$$

$$(z_1 - z_2)(1 - z_1z_2) = 0$$

Since $1 - z_1 z_2 \neq 0$ ($z_1, z_2 \in \Delta$), so $z_1 = z_2$. Hence f is a univalent in Δ .

Example 1.2.3

Let $f(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right)$ be a function on Δ .

To show that f is univalent, suppose $f(z_1) = f(z_2)$ for some $z_1, z_2 \in \Delta$ then

$\frac{1}{2} \log \frac{1+z_1}{1-z_1} = \frac{1}{2} \log \frac{1+z_2}{1-z_2}$ it is clearly that $\log \frac{1+z_1}{1-z_1} = \log \frac{1+z_2}{1-z_2}$, we get

$\frac{1+z_1}{1-z_1} = \frac{1+z_2}{1-z_2}$, since $1-z_1 \neq 0, 1-z_2 \neq 0$ then multiplying both sides by

$(1-z_1)(1-z_2)$, therefore

$$(1+z_1)(1-z_2) = (1+z_2)(1-z_1)$$

$$1 - z_2 + z_1 - z_1 z_2 = 1 - z_1 + z_2 - z_1 z_2$$

It is clearly that $-z_2 - z_2 + z_1 + z_1 = 0$, and so $2z_1 - 2z_2 = 0$. So clearly $z_1 = z_2$

Hence f is univalent on Δ .

To know how the univalent function behave in a neighborhood of some point z_0 , let

f be analytic at z_0 and $f'(z_0) \neq 0$, then there is a neighborhood $N(z_0)$ of z_0 such

that f is univalent in $N(z_0)$.

Theorem 1.2.1

Let $f'(z_0) = 0$ then $f(z)$ is not univalent in any neighborhood of z_0 .

Proof:

$$f'(z_0) = \lim_{z \rightarrow z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) = 0$$

There exist z such that $f(z) - f(z_0) = 0$ where $z \neq z_0$, implies that $f(z) = f(z_0)$.

Hence $f(z)$ is not univalent in any neighborhood of z_0 .

Theorem 1.2.2

Let f be univalent and g is analytic and univalent function on the range of f , with $g(0) = 0$, and $g'(0) = 1$ then $h = g \circ f$ is univalent.

Proof:

To show that $h(z)$ is univalent, let $h(z_1) = h(z_2)$, so $g(f(z_1)) = g(f(z_2))$, therefore $f(z_1) = f(z_2)$ since g is univalent, so $z_1 = z_2$, hence $h(z)$ is univalent.

Definition 1.2.2

A function $f: \Delta \rightarrow C$ is a conformal mapping if f is analytic and univalent in Δ .

Example 1.2.4

As in example (1.2.1), we show that $f(z) = \frac{z}{1-z}$ is analytic and univalent in Δ , so f is a conformal mapping. In fact, f maps Δ onto the half-plane $\text{Re}\{w\} > \frac{1}{2}$.

Example 1.2.5

Let θ be any real number and α is a complex number with $|\alpha| < 1$. The

$$f(z) = e^{i\theta} \cdot \frac{z - \alpha}{1 - \alpha z}$$
 is conformal mapping and maps Δ onto Δ .

The following theorem is well-known theorem and can be found in [1].

Theorem 1.2.3 (Riemman Mapping Theorem)

Every simply connected domain is conformal mapped to the unit disk Δ .

From this theorem, we see the importance of the unit disk.

To prove or to find properties of any simply connected domain, we just prove or find it in the unit disk and then conformally translated to it. In fact, let f be a univalent and analytic function on Δ , then

$$g(z) = \frac{f(z) - f(0)}{f'(0)}$$

is also analytic and univalent in Δ and satisfies the normalization $g(0) = 0$ and

$g'(0) = 1$. Where $g'(z) = \frac{f'(z)}{f'(0)}$, to show that g is univalent, let

$g(z_1) = g(z_2)$, so $\frac{f(z_1) - f(0)}{f'(0)} = \frac{f(z_2) - f(0)}{f'(0)}$ which implies that $f(z_1) = f(z_2)$,

since $f(z)$ is univalent, so $z_1 = z_2$. Hence $g(z)$ is univalent.

1.3 Normalized Univalent Functions

As we see the importance of the analytic and univalent functions, we will study the properties of these functions with some normalization, as in the following definition.

Definition 1.3.1

Let S be the set of all functions f that are analytic, univalent in Δ and satisfying the normalization conditions $f(0) = 0$ and $f'(0) = 1$, S is called the class of normalized univalent functions. Now if $f \in S$, then using Taylor's Expansion, we get

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n \\ &= z + a_2 z^2 + a_3 z^3 + \dots \\ &= z + \sum_{n=2}^{\infty} a_n z^n \end{aligned} \tag{1.1}$$

Example 1.3.1 (Koebe Function)

Let $K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots + nz^n + \dots$ be a function on Δ . To show that

f is in S .

First, it is clearly that $K(z)$ is analytic in Δ . As in example (1.2.2), we show that

$K(z)$ is univalent in Δ . Also $K(0)=0$ and

$$K'(z) = \frac{1+z}{(1-z)^3}, \text{ so } K'(0)=1$$

Hence $K(z)$ is in S .

Theorem 1.3.1

If $f(z)$ is a function in S , then each of the following functions are in S :

1) $e^{-i\alpha} f(e^{i\alpha} z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\alpha} z^n, \alpha$ is real

2) $\overline{f(\overline{z})} = z + \sum_{n=2}^{\infty} \overline{a_n} z^n$

3) $\sqrt{f(z^2)}$.

For proofs of (1) , (3) see [13]

To prove (2) , let $g(z) = \overline{f(\overline{z})}$ where $f \in S$. To show that g is in S .

First , it is clearly that $g(z)$ is analytic in Δ , and to show g is univalent , to do this let

$g(z_1) = g(z_2)$, then $\overline{f(\overline{z_1})} = \overline{f(\overline{z_2})}$, by taking the conjugant to both sides we have

$f(\overline{z_1}) = f(\overline{z_2})$ since f is univalent then $\overline{z_1} = \overline{z_2}$, therefore $z_1 = z_2$. Hence g is

univalent . finally $g(0) = \overline{f(0)} = 0$, and also $g'(0) = \overline{f'(0)} = 1$. Hence $g(z)$ is in S .

Theorem 1.3.2

If $f \in S$ and f has the power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ then $|a_2| \leq 2$. The only

function in S with $|a_2| = 2$ is given by $f(z) = \frac{z}{(1-xz)^2}$ where $|x| = 1$.

Proof: see [22]

Theorem 1.3.3 (Koebe covering Theorem)

If $f \in S$ then $f(\Delta) \supset \left\{ w : |w| < \frac{1}{4} \right\}$

Proof:

Suppose that $f(z) \neq c$ for $|z| < 1$, the function $g = \frac{cf}{c-f}$ clearly belongs to S .

Since g is analytic and univalent, $g(0) = 0$ and $g'(0) = 1$ and has a power series

beginning $g(z) = z + \left(a_2 + \frac{1}{c} \right) z^2 + \dots$

By using theorem (1.3.2) $\left| a_2 + \frac{1}{c} \right| \leq 2$ and therefore

$$\left| \frac{1}{c} \right| = \left| \frac{1}{c} - a_2 + a_2 \right| \leq \left| a_2 + \frac{1}{c} \right| + |a_2| \leq 4$$

Hence $|c| \geq \frac{1}{4}$ i.e. $\left\{ w : |w| \geq \frac{1}{4} \right\} \supseteq C - f(\Delta)$ and by taking the complement, we get

$$\left\{ w : |w| < \frac{1}{4} \right\} \subset f(\Delta).$$

Note:

The topology in $\mathcal{A} = \{ f : f \text{ is analytic in } \Delta \}$ is defined in the following way .

Let $\{r_n\}$ be a sequence of real numbers so that $0 < r_n < 1$ and $r_n \rightarrow 1$. If $f \in \mathcal{A}$ then

$\|f\|_n = \max \{f(z) : |z| = r_n\}$. we define ρ by

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}$$

whenever f and g belong to \mathcal{A} . Then ρ is a metric space.

Chapter Two

Special Families of Univalent Function

In this chapter we introduce several subsets of S , which consists of the set of starlike functions, convex functions and close to convex functions, we also introduce the class functions with a positive real part or typically real functions.

2.1 Starlike Function

Definition 2.1.1

A set $E \subseteq C$ is said to be starlike with respect to $w_0 \in E$ provided that

$$w_0 + t(w - w_0) \in E \text{ whenever } w \in E \text{ and } 0 \leq t \leq 1.$$

Definition 2.1.2

A function f belongs to S is called a starlike if the image of f is starlike with respect to 0.

Definition 2.1.3

Let S^* be the set of all starlike functions in S . In other words,

$$S^* = \{f \in S: f(\Delta) \text{ is starlike with respect to } 0\}.$$

Example 2.1.1

The identity function $f(z)=z$ is a function in S^* .

Example 2.1.2

The Koebe function $K(z)=\frac{z}{(1-z)^2}$ is another example of the functions in S^* .

Theorem 2.1.1 " Schwarz's Lemma "

If f is analytic on Δ with $f(0)=0$ and $|f(z)|<1$ for all $z\in\Delta$ then $|f'(0)|\leq 1$ and $|f(z)|\leq|z|$ in Δ , also if $|f'(0)|=1$ and $|f(z)|=|z|$ then $f(z)=e^{i\theta}z$ for some θ .

Proof: [1]

Theorem 2.1.2

If $f\in S^*$, $0<r<1$ then $f(\Delta_r)$ is starlike with respect to the origin where $\Delta_r=\{z:|z|<r\}$.

Proof:

Suppose that $0<t<1$ and let ϕ be defined by

$$\phi(z)=f^{-1}(tf(z)) \text{ where } |z|<1.$$

ϕ is well defined since $f\in S^*$ and also it is clear that ϕ is analytic in Δ and ϕ is univalent in Δ . Also $\phi(0)=0$ and $|\phi(z)|<1$ for $|z|<1$, by Schwarz Lemma $|\phi(z)|<|z|$ for $|z|<1$, since $\phi(z)=f^{-1}(tf(z))$, then $f(\phi(z))=tf(z)$, now

if $z_0 \in \Delta$, then $|\phi(z_0)| \leq |z_0| < r$ and $f(\phi(z_0)) \in f(\{w: |w| < r\})$, so

$f(z_0) \in f(\{w: |w| < r\})$. Therefore $f(\Delta_r)$ is starlike with respect to 0.

Theorem 2.1.3

1. $f \in S^*$ if and only if $f \in S$ and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{where} \quad |z| < 1 \quad (2.1)$$

2. If $f \in S^*$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1, \quad (2.2)$$

then $|a_n| \leq n$ where $n = 2, 3, \dots$.

3. S^* is compact.

Proof: see [12].

2.2 Convex Functions

Definition 2.2.1

Let A be a set. A is called convex set if the line segment joining any two points of

A is contained in A that is to say if $x_1, x_2 \in A$ then $tx_1 + (1-t)x_2 \in A, 0 \leq t \leq 1$.

Definition 2.2.2

A function f is convex on Δ if and only if $f(\Delta)$ is convex.

Definition 2.2.3(convex functions K)

Let K be the set of all functions which are convex on Δ , in other words,

$$K = \{ f \in S : f(\Delta) \text{ is convex} \}$$

It is clear that every convex function is starlike function, and so $K \subset S^*$.

Example 2.2.1

The function $f(z) = \frac{z}{1-z}$ is a convex function which maps Δ onto

$$\left\{ w : \operatorname{Re} w > -\frac{1}{2} \right\}.$$

Theorem 2.2.1

If $f \in K$ and $0 < r < 1$ then $f(\{z : |z| < r\})$ is convex.

Proof:

Suppose that $z_1 \neq z_2$, $|z_1| \leq |z_2| < r$ and $0 < t < 1$. Now let ϕ be defined by

$$\phi(z) = f^{-1} \left(t \cdot f \left(\frac{z_1}{z_2} \cdot z \right) + (1-t) f(z) \right).$$

Since $f(0) = 0$, then $\phi(0) = 0$, since $f(\Delta)$ is convex, ϕ is well defined, analytic in

Δ and $|\phi(z)| < 1$, then Schwarz's Lemma implies that $|\phi(z)| \leq |z|$ and so

$|\phi(z_2)| < r$. Now letting $z = z_2$, from the definition of ϕ , we show that

$$t \cdot f(z_1) + (1-t) f(z_2) = f(\phi(z_2)) \in f(\{z : |z| < r\}).$$

Hence, $f(\{z : |z| < r\})$ is convex.

Theorem 2.2.2

1. $f \in K$ if and only if $f \in S$ and

$$\operatorname{Re} \left(\frac{z f''(z)}{f'(z)} + 1 \right) > 0 \quad \text{where } |z| < 1 \quad (2.3)$$

2. $f \in K$ iff $g \in S^*$ where $g = z f'(z)$.

3. K is compact.

4. If $f \in K$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ($n = 2, 3, \dots$), then $|a_n| \leq 1$ for $n = 2, 3, \dots$

Proof: See [22]

Theorem 2.2.3

If $f \in K$ then $f(\Delta) \supset \left\{ w : |w| < \frac{1}{2} \right\}$.

Proof:

Suppose that $f(z) \neq c$ for $|z| < 1$, let the function $g(z) = (f - c)^2$

To show that g is univalent, let $g(z_1) = g(z_2)$ therefore

$$f(z_1)^2 - 2cf(z_1) + c^2 = f(z_2)^2 - 2cf(z_2) + c^2$$

then

$$f(z_1)^2 - f(z_2)^2 - 2cf(z_1) + 2cf(z_2) = 0$$

therefore

$$(f(z_1) - f(z_2)) [f(z_1) + f(z_2) - 2c] = 0$$

hence, either $f(z_1) = f(z_2)$ or $f(z_1) + f(z_2) = 2c$.

which is cannot hold since $f(\Delta)$ is convex and $c \notin f(\Delta)$. Therefore $f(z_1) = f(z_2)$.

Hence $z_1 = z_2$, hence g is univalent in Δ .

The function $h = \frac{g - c^2}{-2c}$ clearly belongs to S since h is analytic and univalent,

$h(0) = 0$ and $h'(0) = 1$ by using theorem (1.3.3) we conclude

$$\left| \frac{c}{2} \right| \geq \frac{1}{4} \quad \text{hence} \quad |c| \geq \frac{1}{2}$$

i.e $\left\{ w : |w| \geq \frac{1}{2} \right\} \supseteq C - f(\Delta)$ and by taking the complement, we get

$$f(\Delta) \supset \left\{ w : |w| < \frac{1}{2} \right\}.$$

Theorem 2.2.4

If $f \in K$ and $|z| < 1$ then

$$1. \frac{|z|}{1+|z|} \leq f(z) \leq \frac{|z|}{1-|z|}.$$

$$2. \frac{1}{(1+|z|)^2} \leq f'(z) \leq \frac{1}{(1-|z|)^2}.$$

For proof, see [22]

Lemma 2.2.1

If h is analytic in Δ and satisfies $\operatorname{Re} h'(z) > 0$ where $z \in \Delta$ then h is univalent in Δ .

Proof :

Suppose that $z_1, z_2 \in \Delta$ and $z_1 \neq z_2$, let γ be the closed line segment from z_1 to z_2 parameterized by

$$z_t = z_1 + t(z_2 - z_1) \quad \text{where} \quad 0 \leq t \leq 1 \quad 2.4$$

Since Δ is convex, we may write

$$\begin{aligned} h(z_2) - h(z_1) &= \int_{\gamma} h'(z_t) dz_t \\ &= \int_0^1 h'(z_t)(z_2 - z_1) dt \end{aligned}$$

by using 2.4

$$\begin{aligned} |h(z_2) - h(z_1)| &= \left| (z_2 - z_1) \int_0^1 h'(z) \cdot dt \right| = |z_2 - z_1| \left| \int_0^1 h'(z) dt \right| \\ &\geq |z_2 - z_1| \operatorname{Re} \int_0^1 h'(z) dt \\ &= |z_2 - z_1| \int_0^1 \operatorname{Re} h'(z) dt \\ &> 0 \end{aligned}$$

Hence $h(z_1) \neq h(z_2)$ and h is univalent in Δ .

Lemma 2.2.2

Suppose that f is analytic in Δ and satisfies $\operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > 0 \dots |z| < 1$, where g is

analytic and univalent in Δ and $g(\Delta)$ is convex then f is univalent in Δ .

Proof:

Let $h = f \circ g^{-1}$ and $D = g(\Delta)$, then h is analytic in the convex domain D .

$$h(w) = f(g^{-1}(w)), w \in D.$$

$$\text{Then } h'(w) = \frac{f'(g^{-1}(w))}{g'(g^{-1}(w))} = \frac{f'(z)}{g'(z)} \text{ where } w = g(z)$$

So $\text{Re } h'(w) > 0$, then lemma 2.2.1 implies that h is univalent in Δ . since

$$f = h \circ g \text{ and } g \text{ is univalent in } \Delta$$

This shows f is univalent in Δ .

Definition 2.2.4(Positive real part functions)

Let \wp the set of all functions \wp that are analytic in Δ , $\wp(0) = 1$ and $\text{Re}(\wp(z)) > 0$.

In other words

$$\wp = \{f \in \mathcal{A} : \wp(0) = 1 \text{ and } \text{Re}(\wp(z)) > 0\}.$$

The following theorems give some properties of the class \wp and more details can be found in [22], also we are here not interested in the proofs.

Theorem 2.2.5

$$f \in S^* \text{ if and only if } f \in S \text{ and } \frac{zf'(z)}{f(z)} \in \wp$$

Theorem 2.2.6

If $p \in \wp$ and $p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n$ ($|z| < 1$)

Then $|p_n| \leq 2$ ($n = 2, 3, \dots$)

Theorem 2.2.7

Let $f'(z)$ be in \wp then $f(z)$ is univalent in Δ .

Theorem 2.2.8

\wp is compact.

Proof : see [12]

Example 2.2.2

The function $p(z) = \frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n$ is in \wp .

Proof:

To show that $p(z) \in \wp$, since $p(z) = \frac{1+z}{1-z}$ then $p(0) = 1$. Also

$$p(z) = \frac{1+z}{1-z} \cdot \frac{1-\bar{z}}{1-\bar{z}} = \frac{1-\bar{z}+z-|z|^2}{1-(\bar{z}+z)+|z|^2}, \text{ let } z = r e^{i\theta} \text{ which implies that}$$

$$p(z) = \frac{2ir \sin \theta - r^2 + 1}{1+r^2 - 2r \cos \theta}, \text{ Re}(p(z)) > 0, \text{ Hence } p(z) \in \wp.$$

2.3 Close-to-Convex Function

Definition 2.3.1

A function $f \in H(\Delta)$ is said to be close to convex if there exist a convex function

$$g \text{ such that } \operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > 0.$$

Definition 2.3.2

Let CL be the set of all functions f such that $f \in H(\Delta)$, $f(0) = 0$, $f'(0) = 1$, and

$$\operatorname{Re} \left(\frac{f'(z)}{e^{i\alpha} g'(z)} \right) > 0 \text{ for some } \alpha \in R, g \in K. \text{ In other words,}$$

$$CL = \left\{ f \in H(\Delta) : f(0) = 0, f'(0) = 1, \text{ and } \operatorname{Re} \left(\frac{f'(z)}{e^{i\alpha} g'(z)} \right) > 0 \right\}.$$

Notes

1. If $f \in CL$, then by the definition $\exists g \in K$ such that $\operatorname{Re} \left(\frac{f'(z)}{e^{i\alpha} g'(z)} \right) > 0$, but

$g \in K$ implies g is univalent and so $(e^{i\alpha} g)$ is also univalent and convex.

Hence, f is univalent.

2. The upper bound of such $|\alpha|$ satisfying the definition of CL is $\frac{\pi}{2}$, $|\alpha| < \frac{\pi}{2}$.
3. $K \subset S^* \subset CL \subset S$.

Example 2.3.1

Let x, y be complex numbers such that $|x| = 1 = |y|$ define

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}.$$

Clearly f is analytic and $f(0) = 0$ and since

$$f'(z) = \frac{1-xz}{(1-yz)^3}.$$

Hence $f'(0) = 1$.

Now, let

$$g'(z) = \frac{(1-yz) - z(-y)}{(1-yz)^2},$$

therefore

$$g'(z) = \frac{1}{(1-yz)^2}.$$

Hence,

$$\frac{f'(z)}{g'(z)} = \frac{1-xz}{1-yz}$$

For some $\alpha \in \mathbb{R}$ then

$$\operatorname{Re} \left(\frac{f'(z)}{e^{i\alpha} g'(z)} \right) > 0$$

then f is a function in CL .

Theorem 2.3.1

If $f \in CL$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ($|z| < 1$), then $|a_n| \leq n$ for $(n = 1, 2, 3, \dots)$

Proof:

If $f \in CL$ then there is a function g in K and a real number α such that

$$\operatorname{Re} \left(\frac{f'(z)}{e^{i\alpha} g'(z)} \right) > 0 \quad (|z| < 1)$$

If $q(z) = \frac{f'(z)}{e^{i\alpha} g'(z)} = \sum_{n=0}^{\infty} q_n z^n$ then, the

function $p(z) = \frac{q(z) + i \sin \alpha}{\cos \alpha} = 1 + \sum_{n=1}^{\infty} p_n z^n$ is belong to \wp . Since

$$p(0) = \frac{e^{-i\alpha} + i \sin \alpha}{\cos \alpha} = \frac{\cos \alpha - i \sin \alpha + i \sin \alpha}{\cos \alpha} = 1 \quad \text{and} \quad \operatorname{Re} p(z) = \frac{q(z)}{\cos \alpha} > 0$$

By equating coefficient of the power series in the relation

$$p(z) \cos \alpha = q(z) + i \sin \alpha \quad \text{where} \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad \text{and} \quad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

We conclude that $q_n z^n = \cos \alpha p_n z^n$ then $q_n = p_n \cos \alpha \quad (n = 1, 2, \dots)$

Theorem (2.2.6) implies that $|p_n| \leq 2$ so

$$|q_n| \leq |p_n| \leq 2 \quad (n = 1, 2, \dots) \quad 2.5$$

If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (n = 1, 2, \dots)$ then theorem 2.2.2(4) asserts that

$$|b_n| \leq 1 \quad (n = 1, 2, \dots) \quad 2.6$$

as $q_0 = e^{-i\alpha}$ the relation $f'(z) = e^{i\alpha} g'(z) q(z)$

$$\text{where } f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1}, \quad g'(z) = 1 + \sum_{n=2}^{\infty} n b_n z^{n-1} \quad \text{and} \quad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

Which implies that

$$n a_n = e^{i\alpha} (n b_n e^{-i\alpha} + (n-1) b_{n-1} q_1 + (n-2) b_{n-2} q_2 + \dots + 2 b_2 q_{n-2} + q_{n-1})$$

Applying (2.5) and (2.6) to this equality we find that

$$\begin{aligned}
|na_n| &\leq n|b_n| + (n-1)|b_{n-1}|q_1 + (n-2)|b_{n-2}|q_2 + \dots + 2|b_2|q_{n-2} + |q_{n-1}| \\
&\leq n + (n-1)2 + (n-2)2 + \dots + 2 \cdot 2 + 2 \\
&\leq n^2
\end{aligned}$$

Therefore $|a_n| \leq n$ for $n = 2, 3, \dots$

Theorem 2.3.2

CL is compact.

Proof : see [1].

Theorem 2.3.3

Every close to convex function is univalent .

Definition 2.3.3

Let S_R denote the subset of functions f in the set S such that $f(z)$ is real when z is real .

Definition 2.3.4

Let p_R denote the subset of p of function p such that $p(z)$ is real z is real, $|z| < 1$.

Definition 2.3.5

Let T be a subset of S of all functions with real coefficient. In other words a_n is real for all n .

Example 2.3.2

Let $f(z) = z + 2z^2 + 3z^3 + \dots + nz^n + \dots$. Then, clearly, $f \in T$.

Facts

1. $f \in T \Rightarrow f(\Delta)$ is symmetric with respect to x-axis.
2. $f \in T$ is real if z is real.
3. $f \in T$, then $(\operatorname{Im} z)(\operatorname{Im} f(z)) \geq 0$.

The following theorem shows that the relation between the families typically real part and positive real part.

Theorem 2.3.3

$f \in T$ if and only if there is a function p in \mathcal{P}_R so that

$$f(z) = \frac{z}{1-z^2} \cdot p(z) \quad \dots\dots(2.1) .$$

Proof:

Suppose that $f \in T$ and $p(z) = \frac{f(z)}{z} \cdot (1-z^2)$ ($|z| < 1$). Therefore,

$$\begin{aligned} p(z) &= \left(\frac{z + \sum_{n=2}^{\infty} a_n z^n}{z} \right) (1-z^2) \\ &= \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right) (1-z^2) \end{aligned}$$

Hence $p(0) = 1$, and it is clear p is analytic in Δ , and $p(z)$ is real when z is real.

Suppose that $0 < r < 1$ and $q(z) = \frac{f(rz)}{z}(1 - z^2)$ ($|z| < 1$).

If θ is real and $z = e^{i\theta}$ then

$$\begin{aligned} q(e^{i\theta}) &= \frac{f(re^{i\theta})}{e^{i\theta}}(1 - e^{i\theta}) \\ &= f(re^{i\theta})(e^{-i\theta} - e^{i\theta}) \\ &= f(re^{i\theta})(\cos \theta - i \sin \theta - \cos \theta - i \sin \theta) \\ &= f(re^{i\theta})(-2i \sin \theta) \end{aligned}$$

therefore

$$\begin{aligned} \operatorname{Re} q(e^{i\theta}) &= \operatorname{Re}[-2i \sin \theta \cdot f(re^{i\theta})] \\ &= 2 \sin \theta \cdot \operatorname{Im} f(re^{i\theta}) \end{aligned}$$

we conclude that $\operatorname{Re} q(e^{i\theta}) \geq 0$. Therefore $\operatorname{Re} q(z) \geq 0$ when $|z| < 1$.

By letting $r \rightarrow 1$ in the inequality then $\operatorname{Re}\left(\frac{f(rz)}{z} \cdot (1 - z^2)\right) \geq 0$, we conclude that

$\operatorname{Re}\left(\frac{f(z)}{z} \cdot (1 - z^2)\right) \geq 0$ when $|z| < 1$. Equality cannot occur because of the minimum principle for harmonic functions and the normalizations $f(0) = 0$ and $f'(0) = 1$.

This proves $p \in \mathcal{P}_R$.

Conversely, suppose that $p \in \mathcal{P}_R$ and f is defined by (2.1). Then f is analytic in Δ

$f(0) = 0$, $f'(0) = 1$ and f is real when z is real. Suppose that $r \in (0, 1)$ and

$g(z) = \left[\frac{z}{(1 - z^2)} \right] \cdot p(rz)$ ($|z| < 1$), Then g is analytic in $\bar{\Delta} = \{z : |z| \leq 1\}$ except for the

simple poles at $z = \pm 1$ and $g(z)$ is real when z is real.

As

$$\begin{aligned}
g(e^{i\theta}) &= \frac{e^{i\theta}}{1 - e^{2i\theta}} \cdot p(re^{i\theta}) \\
&= \frac{e^{i\theta}(1 + e^{-2i\theta})}{(1 - e^{2i\theta})(1 + e^{-2i\theta})} \cdot p(re^{i\theta}) \\
&= \frac{e^{i\theta} + e^{-i\theta}}{1 - 1 - e^{2i\theta} + e^{-2i\theta}} \cdot p(re^{i\theta}) \\
&= \frac{2 \cos \theta}{-2i \sin 2\theta} \cdot p(re^{i\theta}) \\
&= \frac{\cos \theta}{\sin 2\theta} \cdot i \cdot p(re^{i\theta})
\end{aligned}$$

Therefore $\operatorname{Im} g(e^{i\theta}) = \frac{1}{2 \sin \theta} \operatorname{Re} p(re^{i\theta}) \quad (\theta \neq 0, \pi)$

We conclude that $\operatorname{Im} g(e^{i\theta}) > 0$ when $0 < \theta < \pi$ and $\operatorname{Im} g(e^{i\theta}) < 0$ when $\pi < \theta < 2\pi$. Suppose $\gamma = \{z : |z| \leq 1, \operatorname{Im} z \geq 0, |z - 1| = \varepsilon\}$ where $\varepsilon > 0$,

Since $p(r) > 0$ the local behaviour of g at $z = 1$ implies that if ε is sufficiently small then $g(\gamma)$ is a curve

with one endpoint on the positive real axis and with the other endpoint having an argument close to $\frac{\pi}{2}$. As z varies clockwise, $g(z)$ varies with an increasing argument

In particular, this implies that if ε is sufficiently small then $\operatorname{Im} g(z) \geq 0$ when z

is on γ . A similar argument may be applied at $z = -1$ to show that $\operatorname{Im} g(z) \geq 0$

If $z \in \gamma' = \{z : |z| \leq 1, \operatorname{Im} z \geq 0, |z + 1| = \varepsilon\}$. By applying the minimum principle for

harmonic functions in the regions formed by deleting open disks centered at

$z = \pm$ from the closed upper half - disk , we conclude that $\text{Im } g(z) \geq 0$ when $|z| < 1$ and $\text{Im } z \geq 0$. A similar argument shows that $\text{Im } g(z) \leq 0$ when $|z| < 1$ and $\text{Im } z \leq 0$.

Letting $r \rightarrow 1$ in the relation $g(z) = \frac{z}{1-z^2} \cdot p(rz)$ we conclude that

$$[\text{Im } f(z)](\text{Im } z) \geq 0 \quad (|z| < 1) \text{ and so } f \in T$$

Example 2.3.3

Let $p(z) = \frac{1+z}{1-z}$ be a function in T since ρ in \mathcal{P} and therefore

$$\begin{aligned} \frac{z}{1-z^2} \cdot p(z) &= \frac{z}{1-z^2} \cdot \frac{1+z}{1-z} \\ &= \frac{z}{(1-z)(1+z)} \cdot \frac{1+z}{1-z} = \frac{z}{(1-z)^2} = K(z) \end{aligned}$$

be a function in T .

Example 2.3.4

Let $p(z) = \frac{1-z^2}{(1-xz)(1-\bar{x}z)}$, where $|x| = 1$ is clearly in \mathcal{P} , therefore

$$\frac{z}{1-z^2} \cdot p(z) = \frac{z}{(1-xz)(1-\bar{x}z)}$$

belong to T .

Chapter Three

Convex and Convex Hulls of Some Classical Families of Univalent Functions

We shall be concerned with the closed convex hulls of various families of functions that are analytic and univalent in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For each family considered we obtain integral representations for the closed convex hull.

Let us establish some notation, we shall let \mathcal{A} denote the set of all analytic functions in Δ . With the natural topology of uniform convergence on compact subsets of Δ , \mathcal{A} is locally convex linear topological space which means at each point in \mathcal{A} there is a convex neighborhood of this point.

Recall that S is the subset of \mathcal{A} consisting of functions f that are univalent in Δ and satisfy $f(0)=0$, $f'(0)=1$. It is well known that S is compact in \mathcal{A} . We shall be particularly interested in the following subfamilies of S

$$K = \{ f \in S : f(\Delta) \text{ is convex set} \}$$

$$S^* = \{ f \in S : f(\Delta) \text{ is star-like with respect to } O \}$$

$$CL = \{ f \in S : f \text{ is close-to-convex function} \}.$$

3.1 Basic Definition

Definition 3.1.1

Suppose $f, g \in \mathcal{A}$. Let $L[f, g]$ be the set of all points in \mathcal{A} lying on the line segment between f and g .

A set M in \mathcal{A} is convex if for every pair of points f, g in M , the set $L[f, g]$ is also in M .

Example 3.1.1

The sets \emptyset and T are clearly convex subsets of \mathcal{A} . The sets S^* , K and C are not convex subsets of \mathcal{A} .

Definition 3.1.2

The intersection of all convex sets containing M is called a convex hull of M .

Definition 3.1.3

Let Λ denote the set of probability measures on $\partial\Delta$.

Example 3.1.2

Λ is convex and convex hull of Λ is Λ .

Definition 3.1.4

The closed convex hull of a set M is the smallest closed convex set that contains M . We denote this set by $H(M)$, the closed hull of M .

Example 3.1.3

Let $f(z) = \frac{z}{(1-xz)^2}$ and $g(z) = \frac{z}{(1-yz)^2}$ where $|x|=|y|=1$, $0 < t < 1$

Let $h = tf + (1-t)g$, it is clear that $h \in H(S)$, but h is not in S .

Proof:

To prove that want to prove $h(z)$ is not univalent. Let $x = i$, $y = -i$ then

$$h(z) = t \frac{z}{(1-iz)^2} + (1-t) \frac{z}{(1+iz)^2}$$

$$\begin{aligned} \text{Therefore } h'(z) &= \frac{t(1+iz)^2 + (1-t)(1-iz)^2}{(1-iz)^2(1+iz)^2} \\ &= \frac{t(1+2iz-iz^2) + (1-t)(1-2iz-iz^2)}{(1-iz)^2(1+iz)^2} \\ &= \frac{t+2itz-it^2+1-2iz-t+2itz+itz^2}{(1-iz)^2(1+iz)^2} \\ &= \frac{1-2iz-iz^2}{(1-iz)^2(1+iz)^2} = \frac{(1-iz)^2}{(1-iz)^2(1+iz)^2} \end{aligned}$$

want to prove $h(z)$ is not univalent, let $h'(z) = 0$, then

$$(1-iz)^2 = 0 \text{ which is satisfied at } z = -i \text{ then by theorem (1.2.1) } h(z) \text{ is}$$

not univalent ,hence $h \notin S$.

Note :-

Let \mathcal{F} be a subfamily of \mathcal{A} . Then $H\mathcal{F} \supseteq \mathcal{F}$.

As we showed in example (3.1.3) that the function $h \in HS$ but $h \notin S$, so $S \subsetneq HS$.

Definition 3.1.5

A family B of functions analytic in Δ is said to be locally bounded if the functions are uniformly bounded on each closed disk . $B \subset \mathcal{A}$;that is , if $|f(z)| \leq M$ for all $z \in B$ and for every $f \in B$, where the bound M depends only on B .

Definition 3.1.6

A family B of functions analytic in a domain Δ is called a normal family if every sequence of functions $B_n \in B$ has a subsequence which converges uniformly on each compact subset of Δ .

Theorem 3.1.1 “Montel’s theorem”

Every locally bounded family of analytic functions is normal.

Theorem 3.1.2

Let \mathcal{A} be a locally convex linear topological space and let \mathcal{F} be a compact subset of \mathcal{A} then $H\mathcal{A}$ is compact.

Proof:

Let \mathcal{F} be a compact subset of \mathcal{A} . Want to show that $H(\mathcal{A})$ is compact, let (f_n)

be a sequence in $H(\mathcal{A})$ and $f_n \rightarrow f$. Since $f_n = \sum_{k=1}^m c_k f_{n_k}$, then $(f_{n_k})_{k=1}^{\infty}$ is a

sequence in \mathcal{F} , but \mathcal{F} is compact, So $f_{n_k} \rightarrow f_k$ as $n \rightarrow \infty$ and $f_k \in \mathcal{F}$.

Thus, $f = \sum c_k f_k$ and $|f_k(z)| \leq M, \forall M$, so $|f(z)| \leq M$. Therefore,

$$f \in H(\mathcal{A})$$

3.2 Convex Hull of Some Families**Definition 3.2.1**

Let f and g be analytic functions in Δ , then f is subordinate to g in Δ if there

exists a function ϕ analytic in Δ such that $\phi(0)=0, |\phi(z)| < 1$ ($|z| < 1$) and

$f(z) = g(\phi(z))$ ($|z| < 1$). It is denoted by $f \prec g$.

Example 3.2.1

Let $f = z^n$ and let $g = z$, if ϕ defined by $\phi(z) = z^n$

Since $f(z) = g(\phi(z)) = g(z^n) = z^n$. Then $f \prec g$

Note:

We use the notation $s(F) = \{ f : f \prec F \text{ when } F \in \mathcal{A} \}$.

Lemma 3.2.1

Suppose that c is a complex number satisfying $|c| \leq 1$ and $c \neq -1$ and

$$F_\alpha(z) = \left(\frac{1+cz}{1-z} \right)^\alpha.$$

If $f \in s(F_\alpha)$ and $g \in s(F_\beta)$ ($\alpha > 0$ and $\beta > 0$) then $fg \in s(F_{\alpha+\beta})$.

Proof:

Since $f \in s(F_\alpha)$ then $f \prec F_\alpha$ this means that $\exists \phi$ such that

$$f = F_\alpha \circ \phi = \left(\frac{1+c\phi(z)}{1-\phi(z)} \right)^\alpha$$

$$\rightarrow f^{1/\alpha} = \frac{1+c\phi(z)}{1-\phi(z)} \quad \rightarrow f^{1/\alpha}(\phi(z)) = \frac{1+c\phi(z)}{1-\phi(z)}$$

Hence, $f^{1/\alpha} \prec F_1$ then $\frac{1}{\alpha} \log f \prec \log F_1$

And also since $g \in s(F_\beta)$ then $g \prec F_\beta$, hence $g^{1/\beta} \prec F_1$ then $\frac{1}{\beta} \log g \prec \log F_1$.

Let $G = \log F_1$. Then G is convex and univalent in Δ , because $G' = \frac{F_1'}{F_1}$, where

$$F_1(z) = \frac{1+cz}{1-z} \quad \text{therefore} \quad F_1' = \frac{(1-z)c+1+cz}{(1-z)^2}$$

Therefore $F_1'(0) = c+1$ hence $G'(0) = \frac{c+1}{1} = c+1 \neq 0$

Where $G'(z) = \frac{c+1}{(1-z)^2} \cdot \frac{1-z}{1+cz} = \frac{c+1}{(1-z)(1+cz)}$, therefore

$$G''(z) = \frac{-c^2 + (2c^2 + 2c) \cdot z + 1}{(1-z)^2(1+cz)^2} \quad . \text{ And}$$

$$\begin{aligned} z \cdot \frac{G''(z)}{G'(z)} + 1 &= \frac{z(-c^2 + (2c^2 + 2c)z + 1)}{(1-z)^2(1+cz)^2} \cdot \frac{(1-z)(1+cz)}{c+1} + 1 \\ &= \frac{c^2 z^2 + cz^2 - c^2 z + c + 1}{(1-z)^2(1+cz)^2(c+1)} \end{aligned}$$

By partial fraction we have

$$z \cdot \frac{G''(z)}{G'(z)} + 1 = \frac{z}{1-z} + \frac{-cz}{1+cz} + 1 \quad \text{Therefore}$$

$$\begin{aligned} \operatorname{Re} \left(z \frac{G''(0)}{G'(0)} + 1 \right) &= \operatorname{Re} \left(\frac{-cz}{1+cz} \right) + \operatorname{Re} \left(\frac{z}{1-z} \right) + \operatorname{Re}(1) \\ &\geq -\frac{1}{2} - \frac{1}{2} + 1 = 0 \end{aligned}$$

Therefore $\frac{1}{\alpha} \log f \prec \log F_1$ and $\frac{1}{\beta} \log g \prec \log F_1$. Since the function G is convex then

$$t \cdot \frac{1}{\alpha} \log f + (1-t) \cdot \frac{1}{\beta} \log g \prec \log F_1 \quad \text{where } 0 < t < 1.$$

If $t = \frac{\alpha}{\alpha + \beta}$ then

$$\frac{\alpha}{\alpha + \beta} \cdot \frac{1}{\alpha} \log f + \left(1 - \frac{\alpha}{\alpha + \beta} \right) \cdot \frac{1}{\beta} \log g \prec \log F_1$$

therefore

$$\frac{1}{\alpha + \beta} [\log f + \log g] \prec \log F_1$$

$$\log(f \cdot g) \prec (\alpha + \beta) \log F_1 \Rightarrow \log(f \cdot g) \prec \log F_1^{\alpha + \beta}$$

Hence $f \cdot g \prec F_1^{\alpha + \beta} = F_{\alpha + \beta}$.

Theorem 3.2.1

Suppose that F_α is defined as in lemma (3.2.1), $|c| \leq 1$ and $c \neq -1$. If $\alpha \geq 1$ then the convex hull of F_α consists of all functions in \mathcal{A} represented by

$$f(z) = \int_{|x|=1} \left(\frac{1+cxz}{1-xz} \right)^\alpha d\mu(x) \quad \text{where} \quad \mu \in \Lambda.$$

Proof : see [22]

Theorem 3.2.2

The convex hull of S^* consists of all functions represented by the formula

$$f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2} d\mu(x)$$

where $\mu \in \Lambda$.

Proof :

If $f \in S^*$, then by using example (3.2.1), $g \prec G$ where

$$g(z) = \frac{f(z)}{z} \quad \text{and} \quad G(z) = \frac{1}{(1-z)^2}$$

i.e. $g \in s(G)$. But $G = F_2$ with $c = 0$. So using theorem (3.2.1) with $\alpha = 2$ and $c = 0$, the equation

$$g(z) = \int_{|x|=1} \left(\frac{1+cxz}{1-xz} \right)^\alpha d\mu(x) \quad \text{implies} \quad g(z) = \int_{|x|=1} \frac{1}{(1-xz)^2} d\mu(x)$$

whenever $f \in S^*$.

Let G be the set of the functions represented by $f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2} d\mu(x)$. By

the compactness of Λ , G is closed convex set and so $HS^* \subset G$.

Definition 3.2.3

Let S_R denotes the subset of S consisting of functions with real coefficients, and also let S_R^* denote the subset of S^* consisting of functions with real coefficients.

Theorem 3.2.3

HK consists of all functions represented by the formula

$$f(z) = \int_{|x|=1} \frac{z}{1-xz} d\mu(x)$$

where $\mu \in \Lambda$.

Proof:

Theorem 2.2.2 part (2) asserts that $f \in K$ if and only if $g \in S^*$ where

$$g(z) = z f'(z)$$

Which provides a linear homeomorphism of K onto S^* . This map also gives a linear homeomorphism of HK onto HS^* .

Corollary 3.2.1

If $f \in K$ then

$$\operatorname{Re} \left(\frac{f(z)}{z} \right) > \frac{1}{2} \quad (|z| < 1)$$

Proof:

If $f \in K$ then theorem (3.2.3) implies that

$$f(z) = \int_{|x|=1} \frac{z}{1-xz} d\mu(x) \quad \text{where } \mu \in \Lambda$$

If $|xz| < 1$ then $\operatorname{Re}\left[\frac{1}{1-xz}\right] > \frac{1}{2}$ and therefore

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) = \int_{|x|=1} \operatorname{Re}\left(\frac{1}{1-xz}\right) d\mu(x) > \int_{|x|=1} \frac{1}{2} d\mu(x) = \frac{1}{2}$$

Theorem 3.2.4

Suppose that $\alpha > 0$ and G_α denotes the set of functions represented by

$$f(z) = \int_{|x|=1} \frac{1}{(1-xz)^\alpha} d\mu(x)$$

where $\mu \in \Lambda$.

If $f \in G_\alpha$, $g \in G_\beta$, $\alpha > 0$ and $\beta > 0$ then $f \cdot g \in G_{\alpha+\beta}$.

Proof :

Assume $\alpha + \beta \geq 1$, and suppose that $f \in G_\alpha$ then let

$$f(z) = \int_{|x|=1} \frac{1}{(1-xz)^\alpha} d\mu(x)$$

and let $g \in G_\beta$ then let

$$g(z) = \int \frac{1}{(1-yz)^\beta} d\nu(y)$$

where $\mu, \nu \in \Lambda$.

Then

$$f(z) \cdot g(z) = \int_{|x|=1} \int_{|y|=1} \frac{1}{(1-xz)^\alpha (1-yz)^\beta} d\mu(x) d\nu(y).$$

This integral over $\partial\Delta \times \partial\Delta$ want to show that integral is in $G_{\alpha+\beta}$ for every value of x and y on $\partial\Delta$, by using lemma (3.2.1)

$$f(z) \cdot g(z) \prec \frac{1}{(1-z)^{\alpha+\beta}}$$

Since $\alpha + \beta \geq 1$, we may apply Theorem 3.2.1 to conclude that the integrand belong to $G_{\alpha+\beta}$ whenever $|x| = |y| = 1$.

Now consider $\alpha + \beta < 1$, we show that the integrand in the representation of $f \cdot g$ Belongs to $G_{\alpha+\beta}$, suppose that

$$h(z) = \frac{z}{(1-xz)^\alpha (1-yz)^\beta}, \quad |x| = |y| = 1$$

We will find $\frac{1}{\alpha + \beta} \left(h'(z) + (\alpha + \beta - 1) \frac{h(z)}{z} \right)$, where

$$\begin{aligned} h'(z) &= \frac{(1-xz)^\alpha (1-yz)^\beta - z \left[-(1-xz)^\alpha \cdot \beta y (1-yz)^{\beta-1} + -(1-yz)^\beta \cdot \alpha x (1-xz)^{\alpha-1} \right]}{(1-xz)^{2\alpha} (1-yz)^{2\beta}} \\ &= \frac{(1-xz)^\alpha (1-yz)^\beta + \beta yz (1-xz)^\alpha (1-yz)^{\beta-1} + \alpha xz (1-xz)^{\alpha-1} (1-yz)^\beta}{(1-xz)^{2\alpha} (1-yz)^{2\beta}} \\ &= \frac{1}{(1-xz)^\alpha (1-yz)^\beta} + \frac{\beta yz (1-xz)^\alpha (1-yz)^{\beta-1}}{(1-xz)^{2\alpha} (1-yz)^{2\beta}} + \frac{\alpha xz (1-xz)^{\alpha-1} (1-yz)^\beta}{(1-xz)^{2\alpha} (1-yz)^{2\beta}} \\ &= \frac{1}{(1-xz)^\alpha (1-yz)^\beta} + \frac{\beta yz}{(1-xz)^\alpha (1-yz)^{\beta+1}} + \frac{\alpha xz}{(1-xz)^{\alpha+1} (1-yz)^\beta} \dots\dots\dots(1) \end{aligned}$$

If we multiply $h(z)$ by $\frac{(\alpha + \beta - 1)}{z}$, then

$$(\alpha + \beta - 1) \frac{h(z)}{z} = \frac{\alpha + \beta - 1}{z} \cdot \frac{1}{(1-xz)^\alpha (1-yz)^\beta} = \frac{\alpha}{(1-xz)^\alpha (1-yz)^\beta} + \frac{\beta}{(1-xz)^\alpha (1-yz)^\beta} - \frac{1}{(1-xz)^\beta (1-yz)^\beta} \dots\dots(2)$$

by adding (1),(2)

$$\begin{aligned} & \frac{\beta yz}{(1-xz)^\alpha(1-yz)^{\beta+1}} + \frac{\beta}{(1-xz)^\alpha(1-yz)^\beta} + \frac{\alpha xz}{(1-xz)^{\alpha+1}(1-yz)^\beta} + \frac{\alpha}{(1-xz)^\alpha(1-yz)^\beta} \\ &= h'(z) + (\alpha + \beta - 1) \cdot \frac{h(z)}{z} \\ &= \frac{\beta}{(1-xz)^\alpha(1-yz)^\beta} \left[\frac{yz}{1-yz} + 1 \right] + \frac{\alpha}{(1-xz)^\alpha(1-yz)^\beta} \left[\frac{xz}{1-xz} + 1 \right] \\ &= \frac{\beta}{(1-xz)^\alpha(1-yz)^\beta} \left[\frac{1}{1-yz} \right] + \frac{\alpha}{(1-xz)^\alpha(1-yz)^\beta} \left[\frac{1}{1-xz} \right] \\ h'(z) + (\alpha + \beta - 1) \cdot \frac{h(z)}{z} &= \frac{\beta}{(1-xz)^\alpha(1-yz)^{\beta+1}} + \frac{\alpha}{(1-xz)^{\alpha+1}(1-yz)^\beta} \end{aligned}$$

then

$$\frac{1}{\alpha + \beta} \left[h'(z) + (\alpha + \beta - 1) \cdot \frac{h(z)}{z} \right] = \frac{\alpha}{\alpha + \beta} \cdot \frac{1}{(1-yz)^\beta(1-xz)^{\alpha+1}} + \frac{\beta}{\alpha + \beta} \cdot \frac{1}{(1-xz)^\alpha(1-yz)^{\beta+1}} \dots 3$$

where $\alpha + \beta + 1 \geq 1$

so we show that the function defined by the right hand sides $\in G_{\alpha+\beta+1}$.

Let H denote the function defined by the left hand side of (3) then we may write

$$H(z) = \int \frac{1}{(1-uz)^{\alpha+\beta+1}} d\lambda(u) \dots \dots \dots (4) \text{ where } \lambda \in \Lambda$$

The mapping $\xi : A \rightarrow A$ defined by $\xi(h) = H$ where

$$H(z) = \frac{1}{\alpha + \beta} \left(h'(z) + (\alpha + \beta - 1) \frac{h(z)}{z} \right)$$

is an invertible linear map of the subset of A satisfying the condition

$$h(0) = 0 \text{ whenever } \alpha + \beta > 0. \text{ To see this, suppose that } h(z) = \sum_{n=1}^{\infty} c_n z^n \quad (|z| < 1).$$

Therefore

$$h'(z) = \sum_1^{\infty} n c_n z^{n-1} \dots\dots (5) \text{ and } \frac{h(z)}{z} = \sum_1^{\infty} c_n z^{n-1}$$

If we multiply $\frac{h(z)}{z}$ by $(\alpha + \beta - 1)$ then

$$(\alpha + \beta - 1) \frac{h(z)}{z} = \sum_1^{\infty} (\alpha + \beta - 1) c_n z^{n-1} \dots\dots (6)$$

Adding (5) and (6) we have

$$h'(z) + (\alpha + \beta - 1) \frac{h(z)}{z} = \sum_1^{\infty} (n + (\alpha + \beta - 1)) c_n z^{n-1} \dots\dots (7)$$

Now multiplying (7) by $\frac{1}{\alpha + \beta}$ then,

$$\begin{aligned} \frac{1}{\alpha + \beta} \left(h'(z) + (\alpha + \beta - 1) \frac{h(z)}{z} \right) &= \sum_{n=1}^{\infty} \frac{\alpha + \beta - 1 + n}{\alpha + \beta} \cdot c_n z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{\beta + \alpha + n}{\alpha + \beta} \cdot c_{n+1} z^n \end{aligned}$$

In general if we write $\sum_{n=0}^{\infty} \frac{\alpha + \beta + n}{\alpha + \beta} \cdot c_{n+1} z^n = \sum_{n=0}^{\infty} d_n z^n$

This determines c_n by the formula

$$c_n = \frac{\alpha + \beta}{\alpha + \beta - 1 + n} d_{n-1} \quad (n = 1, 2, \dots\dots)$$

Moreover, $\sum_{n=0}^{\infty} d_n z^n$ and $\sum_{n=1}^{\infty} c_n z^n$ have the same radius of converges as

$$\lim_{n \rightarrow \infty} \left(\frac{\alpha + \beta}{\alpha + \beta + n - 1} \right)^{1/n} = 1$$

If we apply ξ^{-1} to both sides of (4) we obtain

$$h(z) = \int_{|u|=1} \frac{z}{(1-uz)^{\alpha+\beta}} d\lambda(u)$$

since $\xi \left(\frac{z}{(1-uz)^{\alpha+\beta}} \right) = \frac{1}{(1-uz)^{\alpha+\beta+1}}$. This proves the function $w = \frac{1}{(1-xz)^\alpha (1-yz)^\beta}$

belongs to $G_{\alpha+\beta}$ for all complex number x, y with $|x| = |y| = 1$, and so the proof of theorem is finished.

Theorem 3.2.5

HCL consists of the functions represented by

$$f(z) = \int \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x, y) \quad \dots\dots(3.1)$$

Where μ varies over the probability on $\partial\Delta \times \partial\Delta$.

Proof:

Suppose $f \in CL$, then by notes of family C there exist g belong to K and a real

number α such that $h = \frac{f'}{e^{i\alpha} g'}$ which satisfies that $\text{Re}(h(z)) > 0, |z| < 1$.

Let $p(z) = \frac{h(z) + i \sin \alpha}{\cos \alpha}$ by theorem (2.3.1)

$$p(z) = \frac{h(z) - i \operatorname{Im} h(0)}{\operatorname{Re} h(0)} \quad (3.2)$$

Where $h(0) = \cos \alpha - i \sin \alpha$, such that $\operatorname{Re} h(0) = \cos \alpha$, $\operatorname{Im} h(0) = -\sin \alpha$

therefore $p \in \rho$. Since $p(0) = 1$ and $\operatorname{Re} p(z) > 1$

And also

$$p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\delta(x) \quad \text{where } \delta \in \Lambda. \quad (3.3)$$

From (3.2) and (3.3)

$$\frac{h(z) - i \operatorname{Im} h(0)}{\operatorname{Re} h(0)} = \int_{|x|=1} \frac{1+xz}{1-xz} d\delta(x)$$

$$h(z) = \operatorname{Re} h(0) \int_{|x|=1} \frac{1+xz}{1-xz} d\delta(x) + i \operatorname{Im} h(0) \quad \dots\dots\dots(3.4)$$

$$\begin{aligned} h(z) &= \int_{|x|=1} \cos \alpha \left(\frac{1+xz}{1-xz} \right) + i(-\sin \alpha) d\delta(x) \\ &= \int_{|x|=1} \frac{\cos \alpha(1+xz) - i \sin \alpha(1-xz)}{(1-xz)} d\delta(x) \\ &= \int_{|x|=1} \frac{\cos \alpha + \cos \alpha(xz) - i \sin \alpha + ixz \sin \alpha}{1-xz} d\delta(x) \\ &= \int_{|x|=1} \frac{e^{-i\alpha} + e^{i\alpha}xz}{1-xz} d\delta(x) \quad \dots\dots\dots(3.5) \end{aligned}$$

Since $g \in K$, and by using theorem (3.2.3) implies that

$$g'(z) = \int_{|y|=1} \frac{(1-yz) - z(-y)}{(1-yz)^2} d\nu(y)$$

$$g'(z) = \int_{|y|=1} \frac{1}{(1-yz)^2} d\nu(y) \quad \dots\dots\dots(3.6)$$

where $\nu \in \Lambda$. Since $h = \frac{f'}{e^{i\alpha} g'}$, then $f'(z) = h(z) \cdot g'(z) \cdot e^{i\alpha}$

Combining (3.5) and (3.6) then

$$f'(z) = \int_{|x|=1} \int_{|y|=1} \frac{1 + e^{2i\alpha} xz}{(1-xz)(1-yz)^2} d\delta(x) d\nu(y) \quad \dots\dots\dots(3.7)$$

From theorem (3.2.4); If $|x|=|y|=1$, then

$$\frac{1}{(1-xz)(1-yz)^2} = \int_{|u|=1} \frac{1}{(1-uz)^3} d\lambda(u) \quad \lambda \in \Lambda \quad \dots\dots\dots(3.8)$$

Since $\frac{1}{(1-xz)(1-yz)^2} \in G_3$. Let $w = e^{2i\alpha} x$. Consequently (3.7) may be written

as

$$f'(z) = \int_R \frac{1+wz}{(1-uz)^3} d\mu(w,u) \quad \dots\dots\dots(3.9)$$

where μ is probability measure on R . Letting $u = y$ and $w = -x$ and integrating (3.9) we find that

$$f(z) = \int \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x,y) \quad \dots\dots\dots(3.10)$$

Since CL contained in the set of functions represented by (3.10), HCL is also a subset of this closed convex collection. The fact that the collection is closed follows from the weak - star compactness of the probability measures on R . The functions

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} \in CL, |x|=|y|=1$$

And thus the functions given by (3.1) belong to HCL .

Definition 4.1.1

If $U \subset V$ then U is called an extremal subset of V provided that whenever

$$u = tx + (1-t)y$$

where $u \in U, x, y \in V$, and $0 < t < 1$ then x and y both belong to U .

We shall use the notation FU to denote the set of all functions $f \in HCL$ such that $f \in FU$.

Chapter Four

Extreme Points of Special Families of Univalent Functions

In this chapter we introduce the extreme points of \wp, S^*, K, CL, T and several classes by subordinations. And also we obtain information about extreme point of the family S .

4.1 Extreme Points of Several Classes

Definition 4.1.1

If $U \subset V$ then U is called an extremal subset of V provided that whenever

$$u = tx + (1-t)y$$

where $u \in U, x, y \in V$, and $0 < t < 1$ then x and y both belong to U .

We shall use the notation EU to denote the set of extreme points of U .

Theorem 4.1.1 “ Krein – Mil’man Theorem “

Let M be a compact convex set in a locally convex topological vector space X .

Then M is the closed convex hull of its extreme points. In symbols $M = H(E(M))$.

Proof : [see Dunfared and Schwarz] .

Theorem 4.1.2

Let \mathcal{A} be a locally convex linear topological space and let \mathcal{F} be a compact subset of \mathcal{A} .

$$1) \text{ If } \mathcal{A} \text{ is non empty then } E(\mathcal{F}) \text{ is non empty} \quad (4.1)..$$

$$2) HE(\mathcal{F}) = H(\mathcal{F}) \quad (4.2).$$

Proof: see [22]

Theorem 4.1.3

Let \mathcal{F} be a compact subset of \mathcal{A} and let J be a complex valued continuous functional on \mathcal{A} , then

$$\max\{\operatorname{Re}J(f): f \in \mathcal{F}\} = \max\{\operatorname{Re}J(f): f \in H\mathcal{F}\} = \max\{\operatorname{Re}J(f) = f \in EHF\}.$$

Proof:

Since $H\mathcal{F}$ is compact the second maximum exists . As \mathcal{F} is compact the first maximum exists .

Suppose $M = \max\{\operatorname{Re}J(f): f \in H\mathcal{F}\}$ and $G = \{f: f \in H\mathcal{F} \text{ and } \operatorname{Re}J(f) = M\}$ then

G is non empty . We claim that that is an extremal subset of $H(\mathcal{F})$. Namely ,

suppose that $h \in G$ and $h = tf + (1-t)g$ where $f, g \in H\mathcal{F}$ and $0 < t < 1$ then

$$\begin{aligned} M = \operatorname{Re}J(h) &= t\operatorname{Re}J(f) + (1-t)\operatorname{Re}J(g) \\ &\leq tM + (1-t)M \\ &= M \end{aligned}$$

and so we must have $\operatorname{Re} J(f) = \operatorname{Re} J(g) = M$. Finally, we note that G is compact, this follows from the continuity of J and the fact that HG is locally uniformly bounded.

Because G is a non-empty compact subset of A , (4.1) implies that G has an extreme point f_0 . As G is an extremal subset of HG , f_0 is an extreme point of HG . Note that (4.2) implies that $f_0 \in F$. Since f_0 belongs to all of the sets G , HF , F and EHF it is clear that the third maximum in the statement of the theorem exists and all three maximums are equal.

Lemma 4.1.1

The set of extreme points of Λ consists of the point masses.

Proof:

Suppose $\mu = \delta_x$, the point mass of x , and $\mu = t\nu + (1-t)\lambda$, where $\nu, \lambda \in \Lambda$, $0 < t < 1$. And since Λ is convex then

$$\begin{aligned} 1 = \mu(\{x\}) &= t(\nu\{x\}) + (1-t)(\lambda\{x\}) \\ &\leq t \cdot 1 + (1-t) \cdot 1 \\ &= t + 1 - t = 1 \end{aligned}$$

so $\mu = \lambda = \nu$, then μ is an extreme point of Λ .

Now suppose μ is an extreme point of Λ , and μ is not a point mass.

Claim, \exists a measurable set A , $A \subset \partial\Delta$, so that $0 < \mu(A) < 1$

Let $B = \partial\Delta \setminus A$ so that $0 < \mu(B) < 1$. Let ν, λ be measures and defined by

$$\nu(E) = \frac{\mu(E \cap A)}{\mu(A)} \quad \text{and} \quad \lambda(E) = \frac{\mu(E \cap B)}{\mu(B)}.$$

For each Borel set E contained in $\partial\Delta$ then $\nu, \lambda \in \Lambda$ and $\mu = t\nu + (1-t)\lambda$ where $t = \mu(A)$ then μ is not an extreme point of Λ , a contradiction, so μ is a point mass

Theorem 4.1.4

A function f is analytic in Δ and satisfies $\operatorname{Re} f(z) \geq 0$ ($|z| < 1$) if and only if there exists a non-decreasing function μ on $[0, 2\pi]$ such that

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) + ib$$

and b is a real number. Moreover, if μ is normalized $\left(\mu(t) = \frac{1}{2}\mu(t^{-1}) + \frac{1}{2}\mu(t^{+1})\right)$

then μ is unique.

Proof : see [22]

Theorem 4.1.5

$p \in \wp$ if and only if there is a probability measure μ on $\partial\Delta$ such that

$$p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x) \quad , |z| < 1 \quad (4.3)$$

the correspondence between \wp and the set of probability measure $\{\mu\}$ on $\partial\Delta$ given by equation (4.3) is one-to-one.

Proof :

Let $p \in \wp$. Since $\operatorname{Re} p(z) \geq 0$ ($|z| < 1$), then by theorem 4.1.4 there is a non-decreasing function μ on $[0, 2\pi]$ such that

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

we may assume μ is a probability measure on $\partial\Delta$ such that $\mu(\partial\Delta) = 1$

i.e $f(z) = \int_{\partial\Delta} \frac{1+xz}{1-xz} d\mu(x)$ where $x = e^{-it}$

to show that there is a one - to - one correspondence between the set μ and the set f

write $f(z) = \int_{\partial\Delta} \frac{\bar{x} + z}{x - z} \mu(\bar{x})$ and so $f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$

Hence , the map is one - to one .

Theorem 4.1.6 :

The set of extreme points of \wp consists of the functions

$$p(z) = \frac{1+xz}{1-xz} \quad \text{where} \quad |x|=1 \quad (4.4)$$

Proof :

\Rightarrow) If $f(z) = 1 + p_1 z + \dots$ is in \wp , then $|p_n| \leq 2$ and we will see that $|p_n| = 2$ if

and only if it has the form (4.4). let $p(z)$ has the form (4.4), we want to show

$p(z)$ extreme point of \wp .

Suppose to the contrary that $p(z)$ is not in $E(\wp)$. Then we have two functions,

$f(z)$ and $g(z)$, in \wp such that

$$\begin{aligned} p(z) &= t f(z) + (1-t)g(z) \\ &= t(1+a_1 z + \dots) + (1-t)(1+b_1 z + \dots) \end{aligned}$$

so that

$$|P_n| = 2 \leq t |a_n| + (1-t) |b_n|, \quad 0 < t < 1 \quad (4.5)$$

Since $|a_n| \leq 2$ and $|b_n| \leq 2$, this is possible if and only if $|a_n| = |b_n| = 2$ and

$\text{Arg } a_n = \text{Arg } b_n$. But then both $f(z)$ and $g(z)$ have the form (4.4) and, in fact,

$f(z) = g(z)$. Hence $p(z)$ is an extreme point of \wp .

\Leftarrow) Now, if $f(z)$ does not have the form (4.4) and

$$f(z) = 1 + p_1 z + p_2 z^2 + \dots \quad \text{then} \quad |p_1| < 2.$$

We will show that in this case $f(z)$ is an interior point of the line joining two functions $g(z)$ and $h(z)$ in \wp . This will prove that $f(z)$ is not an extreme point.

First, we select c so that $\bar{c} p_1 = |p_1|$. Then $|c| = 1$. If $p_1 = 0$, then we can set $c = 1$.

We next construct the auxiliary function

$$F(z, c) = \frac{f(z)(1-cz)(z-\bar{c})}{z} + \frac{\bar{c}}{z} - cz$$

$$= -p_1 \bar{c} + 1 + |c|^2 + [-p_2 \bar{c} + p_1 + |c|^2 |p_1 - 2c|]z + \dots \quad (4.6)$$

$$= -p_1 \bar{c} + 2 + [-p_2 \bar{c} + 2p_1 - 2c]z + \dots \quad (4.7)$$

We observe that the constant term $2 - p_1 \bar{c} = 2 - |p_1| > 0$, because $|p_1| < 2$.

Now suppose that $f(z)$ is regular on ∂E . Then for $z = e^{i\theta}$

$$\text{Re } F(e^{i\theta}, c) = \text{Re} \left[f(e^{i\theta}) (2 - ce^{i\theta} - \bar{c} e^{-i\theta}) + \bar{c} e^{-i\theta} - ce^{i\theta} \right]$$

$$= \text{Re} \left[2 - ce^{i\theta} - \bar{c} e^{-i\theta} \right] \text{Re } f(e^{i\theta}) \geq 0.$$

Since $F(z, c)$ is regular in \bar{E} , this inequality holds throughout E . If $f(z)$ has singularities on ∂E , we apply the work just completed to $f(rz)$ and let

$r \rightarrow 1^-$. Hence if $f(z)$ is in p , then $\text{Re } F(z, c) > 0$ in E . Further, each step

from equation (4.6) and on is valid if we replace c by $-c$. We define $g(z)$ and

$h(z)$ by

$$g(z) = \frac{F(z, c)}{2 - |p_1|} \quad \text{and} \quad h(z) = \frac{F(z, -c)}{2 + |p_1|} \quad 4.8$$

Clearly $g(0) = h(0) = 1$, so that both $g(z)$ and $h(z)$ are in p . A brief computation shows that

$$f(z) = \frac{2 - |p_1|}{4} g(z) + \frac{2 + |p_1|}{4} h(z) \quad 4.9$$

Where $0 < (2 \pm |p_1|)/4 < 1$. Thus, from (4.9) it appears as though $f(z)$ is not in extreme point in p . There is one minor detail to check. It is possible that $g(z)$ and $h(z)$ are the same function, and if so the argument fails. Assume that $g(z) = h(z)$. Then a moderately long computation will give

$$f(z) = \frac{1 - c^2 z^2}{1 - |p_1| c z + c z^2} \quad 4.10$$

And if $|p_1| = 2 \cos \beta$, then $|\cos \beta| < 1$ and

$$f(z) = \frac{1}{2} \left(\frac{1 + e^{i\beta} z}{1 - e^{i\beta} z} + \frac{1 + e^{-i\beta} z}{1 - e^{-i\beta} z} \right). \quad 4.11$$

Again, $f(z)$ is not an extreme point in p .

Theorem 4.1.7:

$f \in T$ if and only if there is a probability measure on $X = \{x | |x| = 1, \text{Im } x \geq 0\}$ so that

$$f(z) = \int_{|x|=1} \frac{z}{(1 - xz)(1 + \bar{x}z)} d\mu(x) \quad (4.12).$$

The correspondence from the set of probability measures on X to T defined through equation (4.12) is one-to-one mapping onto T .

Proof :

T is characterized by the representation

$$f(z) = \frac{z}{1-z} \cdot p(z) \quad (|z| < 1)$$

where $p \in \wp_R$ since $p \in \wp$, theorem (4.4) implies that

$$p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\nu(x)$$

where ν is a probability measure on $\partial\Delta$.

Suppose $\bar{A} = \{x: \bar{x} \in A\}$ and $\mu(A) = \nu(A) + \nu(\bar{A})$,

where A is any measurable subset of X .

If p is also real then $p(\bar{z}) = \overline{p(z)}$, we find that

$$\begin{aligned} p(z) &= \frac{1}{2} \left[p(z) + \overline{p(\bar{z})} \right] \\ &= \frac{1}{2} \left[\int_{|x|=1} \frac{1+xz}{1-xz} d\nu(x) + \int_{|x|=1} \overline{\left(\frac{1+xz}{1-xz} \right)} d\nu(x) \right] \\ &= \frac{1}{2} \left[\int_{|x|=1} \frac{1+xz}{1-xz} d\nu(x) + \int_{|x|=1} \frac{1+\bar{x}z}{1-\bar{x}z} d\nu(x) \right] \\ &= \frac{1}{2} \left[\int_{|x|=1} \frac{1+xz - \bar{x}z - z^2 + 1 + \bar{x}z - xz - z^2}{(1-xz)(1-\bar{x}z)} d\nu(x) \right] \\ &= \frac{1}{2} \left[\int_{|x|=1} \frac{2-2z^2}{(1-xz)(1-\bar{x}z)} d\nu(x) \right] \end{aligned}$$

The function $\frac{1-z^2}{(1-xz)(1-\bar{x}z)}$ belong to \wp_R for each x ($|x|=1$) it follows that if μ

is a probability measure on X then $p(z)$ is belong to \wp_R .

Let μ is a probability measure on X then

$$p(z) = \int_{|x|=1} \frac{1-z^2}{(1-xz)(1-\bar{x}z)} d\mu(x) \in \mathcal{P}_R$$

let $f(z) = \frac{z}{1-z^2} \cdot p(z)$ yields equation (4.12).

The one-to-one correspondence between T and the set of probability measure on X follows from the uniqueness of measure ν by using theorem (4.1.5) and the fact of the measure on $\partial\Delta$ are is one-to-one correspondence with measures on X through the relation $\mu(A) = \nu(A) + \nu(\bar{A})$.

The technique given in proof of Theorem 4.1.6 may be used to prove the following result based on 4.1.5 and the fact that the extreme probability measures on $\{x : |x|=1, \text{Im } x > 0\}$ are the point masses.

Theorem 4.1.8

The set of extreme points of T consists of the functions

$$f(z) = \frac{z}{(1-xz)(1-\bar{x}z)}$$

where $|x|=1$ and $\text{Im } x \geq 0$.

Theorem 4.1.9

If $F \in \mathcal{A}$, then each function of the form $f(z) = F(xz)$ where $|x|=1$ belongs to $EHs(F)$.

Proof:

Assume F is not constant, the power series of F begins

$$F(z) = A_0 + A_n z^n + \dots \quad n \geq 1, \quad A_n \neq 0$$

If $f \in s(F)$ then

$$f = F \circ \phi$$

where ϕ is analytic in Δ , and $\phi(0) = 0$,

$$|\phi(z)| < 1 \quad |z| < 1$$

and $\phi \in B_0$, suppose that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad \phi(z) = \sum_{k=1}^{\infty} c_k z^k$$

$$\begin{aligned} f &= F \circ \phi \\ &= F(\phi(z)) \\ &= A_0 + A_n (\phi(z))^n + \dots \\ &= A_0 + A_n \left(\sum_{k=1}^{\infty} c_k z^k \right)^n \\ &= A_0 + A_n (c_1 z + c_2 z^2 + \dots + c_k z^k + \dots)^n + \dots \end{aligned}$$

$$a_0 + a_1 z + \dots + a_n z^n = A_0 + A_n c_1^n z + \dots + A_n c_n^n z^{2n}$$

$$a_0 = A_0$$

$$a_1 = a_2 = a_3 = \dots = a_{n-1} = 0$$

$$a_n = A_n c_1^n$$

Let J be the continuous linear functional on \mathcal{A} defined by $J(h) = b_n$, $h \in \mathcal{A}$ and

$$h(z) = \sum_{n=0}^{\infty} b_n z^n.$$

$$\begin{aligned} |\phi(z)| &= \left| \sum_{n=1}^{\infty} c_n z^n \right| \leq \sum_{n=1}^{\infty} |c_n| |z|^n \\ &= |c_1| |z| + |c_2| |z|^2 + \dots + |c_n| |z|^n + \dots \end{aligned}$$

Because $|c_1| < 1$ and equality holds if and only if $\phi(z) = xz$, $|x| = 1$.

$$\max \{ \operatorname{Re} J(f) : f \in s(F) \} = |A_n|$$

because

$$J(f) = \text{coefficient of } a_n$$

$$= |A_n c_1|^n$$

$$\leq |A_n| \cdot 1$$

$$= |A_n|$$

Moreover the only function in $s(F)$ for which $\operatorname{Re} J(f) = |A_n|$ have the form

$f(z) = F(xz)$ for suitable x with $|x| = 1$ by the theorem (4.1.6) the solution set

continuous a member $EHs(F)$, this proves that there is a complex number x_0 such

that $|x_0| = 1$ and $f_0(z) = F(x_0 z) \in EHs(F)$.

Now want to show that if $f(z) = F(xz)$ and $|x| = 1$ then $f \in EHs(F)$. Suppose

that $f = tg + (1-t)h$, $g, h \in Hs(F)$ and $0 < t < 1$ then

$$\begin{aligned} f_0(z) &= F(x_0 z) \\ &= F\left(x \cdot \frac{x_0}{x} z\right) \\ &= f\left(\frac{x_0}{x} z\right) \\ &= t g\left(\frac{x_0}{x} z\right) + (1-t) h\left(\frac{x_0}{x} z\right) \end{aligned}$$

Since $f_0 \in EHs(F)$ and $g\left(\frac{x_0}{x} z\right)$ and $h\left(\frac{x_0}{x} z\right) \in Hs(F)$ they must be equal.

Therefore $g = h$ thus $f \in EHs(F)$.

Theorem 4.1.10

$Hs(K)$ Consists of the functions represented by

$$f(z) = \int \frac{xz}{1-yz} d\mu(x, y)$$

Where μ varies over the probability measures on $R = \partial\Delta \times \partial\Delta$.

Proof:

Which follows from the Krein_Milman theorem and the weak_star compactness probability measures.

Theorem 4.1.11

$Hs(S^*)$ Consists of the functions represented by

$$f(z) = \int_R \frac{xz}{(1-yz)^2} d\mu(x, y)$$

Where μ varies over the probability measures on $\partial\Delta \times \partial\Delta$.

Proof:

Which follows from the Krein_Milman theorem and the weak_star compactness probability measures.

Theorem 4.1.12

$Hs(CL)$ Consists of the functions represented by

$$f(z) = \int_u \frac{wz - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x, y, w).$$

Where μ varies over the probability measures on $\partial\Delta \times \partial\Delta \times \partial\Delta$.

Proof:

Which follows from the Krein _Milman theorem and the weak _star compactness probability measures.

Theorem 4.1.13

EHS^* consists of the Koebe functions $f(z) = \frac{z}{(1-xz)^2}$ where $|x|=1$.

Proof :

The extreme points of G are the Koebe functions according to theorem (3.2.1)

where $G = \frac{1}{(1-z)^2}$. Since the Koebe functions belong to S^* we conclude that

$G \subset HS^*$ and thus $HS^* = G$ moreover, as $EG = EHS^*$ the set of keobe functions is the same as EHS^* .

Theorem 4.1.14

EHK consists of the functions $f(z) = \frac{z}{1-xz}$ where $|x|=1$.

Proof :

Since a map the set EHK is in one-to-one correspondence with EHS^* . The conclusion of this theorem, follows from the equivalent assertions for S^* given in

theorem (4.1.13). In particular, we note that if $g(z) = \frac{z}{(1-xz)^2}$ where

$$g(z) = z f'(z) \text{ then } f(z) = \frac{z}{1-xz} \text{ belongs to } EHK .$$

Theorem 4.1.15

EHCL consists of the functions defined by

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} \quad \text{where } |x|=1, |y|=1 \text{ and } x \neq y. \quad (4.13)$$

Proof :

To show this let G denote the set of functions having the representation $G = \frac{1}{(1-z)^2}$

and suppose that $f \in EG$. Consider the set of all probability measures on R which correspond to f .

This is a compact set and thus has an extreme point of μ , thus $f \in EG$ implies that μ is an extreme probability measure on $R = \partial\Delta \times \partial\Delta$. Therefore μ is a point mass and thus

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$$

for suitable (x, y) in R .

We now show that if

$$f(z) = \frac{z - \frac{1}{2}(x_0 + y_0)z^2}{(1 - y_0 z)^2} \quad \text{with } |x_0| = |y_0| = 1 \quad \text{and } x_0 \neq y_0$$

then $f \in EHCL$.

It suffices to show that if

$$\frac{z - \frac{1}{2}(x_0 + y_0)z^2}{(1 - y_0z)^2} = \int_R \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2} d\mu(x, y) \quad \dots 4.14$$

where μ is a probability measure on R , then μ is a point mass at (x_0, y_0) .

Equation (4.14) is equivalent to

$$\frac{1 - x_0z}{(1 - y_0z)^3} = \int_R \frac{1 - xz}{(1 - yz)^3} d\mu(x, y) \quad (4.15)$$

If we multiply (4.15) by $(1 - y_0z)^3$ we may write

$$1 - x_0z = \int_{\Gamma} (1 - xz) d\mu(x, y) + \int_{\frac{R}{\Gamma}} \left(\frac{1 - y_0z}{1 - yz} \right)^3 (1 - xz) d\mu(x, y) \quad (4.16)$$

where $\Gamma = \{(x, y): y = y_0\}$ if we let $z \rightarrow \overline{y_0}$ radially, because of the Lebesgue dominated convergence theorem, we find that the second integral in equation (4.16) has the limit zero.

Therefore

$$1 - x_0\overline{y_0} = \int_{\Gamma} (1 - x\overline{y_0}) d\mu(x, y) = \int_{\Gamma} d\mu(x, y) - \overline{y_0} \int_{\Gamma} x d\mu(x, y).$$

If $b = \int_{\Gamma} d\mu(x, y)$ then $0 \leq b \leq 1$ and as $\left| \int_{\Gamma} x d\mu(x, y) \right| \leq \int_{\Gamma} d\mu(x, y)$ we conclude

that $|1 - x_0\overline{y_0} - b| \leq b$. Since $|1 - x_0\overline{y_0} - b| \geq |x_0\overline{y_0}| - (1 - b) = b$ we must have

$x_0\overline{y_0} = 1$ or $b = 1$ since $x_0 \neq y_0$, $x_0\overline{y_0} \neq 1$ and so $b = 1$. Thus $x_0 = \int_{\Gamma} x d\mu(x, y)$ and

consequently μ is unit point mass at (x_0, y_0) .

To verify this last assertion, note that if

$$P(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x, y_0)$$

Then $p \in \wp$ and

$$P'(0) = \int_{|x|=1} 2x d\mu(x, y_0) = 2x_0.$$

Since $|P'(0)| = 2$ we conclude that $p(z) = \frac{1+x_0z}{1-x_0z}$. The one-to-one property implies that $\mu(x, y_0)$ is unit point mass at $\{x_0\}$ and so $\mu(x, y)$ is unit mass at (x_0, y_0) .

It remains by to verify that if $|x|=|y|=1$ and $x=y$ the functions defined by

(4.13), namely $f(z) = \frac{z}{1-xz}$ where $|x|=1$, are not members of EHC . Since such

functions belong to S^* and do not belong to EHS^* , according to theorem (4.9), they

cannot belong to $EHCL$ as $HS^* \subseteq HCL$.

Note

EHC is not closed, in contrast to the fact that $EH\mathfrak{I}$ is closed when \mathfrak{I} is any one of the families \wp, T, S^* or K .

Corollary 4.1.1

$$HS_R = HS_R^* = T$$

Proof:

The function described in theorem 4.1.8 belong to

S^* and so $ET \subset S_R^*$, then $HET \subseteq HS_R^*$. It follows from (4.2) and since

$S_R^* \subseteq S_R \subseteq T$ that

$T \subseteq HT \subseteq HS_R^* \subseteq HS_R \subseteq T$ Therefore

$$T = H S_R^* = H S_R$$

Theorem 4.1.16

$$EHs(\mathbb{K}) = \left\{ f : f(z) = \frac{xz}{1-y^2}, |x|=|y|=1 \right\}$$

Proof:

Since z and $-z$ are in $s(\mathbb{K})$ it is clear that $0 \notin EHs(\mathbb{K})$. If $f \in EHs(\mathbb{K})$ then from

theorem (4.1.10), we conclude that $f(z) = \frac{z}{1-xz}$ for some x where $|x|=1$. For a

given x with $|x|=1$, suppose

$$\mathbb{K}(x) = \left\{ f : f \prec g, g(z) = \frac{z}{1-xz} \right\}$$

since

$$\frac{z}{1-xz} = -\frac{1}{x} + \frac{1}{x} \frac{1}{1-xz}$$

it is clear that $f(z) = -\frac{1}{x} + \frac{1}{x}h(z)$ where $h \prec \mathbb{K}$ and $h(z) = \frac{1}{1-xz}$.

The set $\{h\}$ consists of functions subordinate to $w = \frac{1}{1-z}$ and so

$$EH(h) = \left\{ h : h(z) = \frac{1}{1-yz}, |y|=1 \right\}.$$

As $EHs(\mathbb{K}) \subset EHK(x)$ for each x ($|x|=1$) it follows that f belongs to

$EHK(x)$ and so

$$f(z) = -\frac{1}{x} + \frac{1}{x} \frac{1}{1-yz}$$

for some y ($|y|=1$). Consequently

$$f(z) = \frac{\overline{y}xz}{1-yz} = \frac{uz}{1-vz}$$

where $|u|=|v|=1$ and so

$$f \in E = \left\{ g: g(z) = \frac{xz}{1-yz}, |x|=|y|=1 \right\}.$$

We claim that $E \subset EHs(\mathbb{K})$. As $E \subset s(\mathbb{K})$ it suffices to show that each function in E uniquely maximizes a real-valued, continuous linear functional over E . Let

$$J(g) = \alpha g'(0) + \beta \frac{g''(0)}{2} \text{ whenever } g \in A \text{ and where } |\alpha|=|\beta|=1. \text{ As}$$

$$f(z) = \frac{xz}{1-yz} = xz + xyz^2 + \dots$$

we see that

$$\begin{aligned} \operatorname{Re} J(f) &= \operatorname{Re} \left(\alpha x + \frac{\beta}{2} \cdot 2xy \right) \\ &\leq |\alpha x + \beta xy| \\ &\leq |\alpha| + |\beta| = 2 \end{aligned}$$

Equality requires that $x = \frac{1}{\alpha}$ and $y = \frac{\alpha}{\beta}$.

By varying α and β we obtain all possible pairs (x, y) with $|x|=|y|=1$ such that

$\operatorname{Re} J(f) = 2$. Hence $EHs(\mathbb{K}) = E$ and the statement about $Hs(\mathbb{K})$ follows from

the Krein- Milman theorem and the weak-star compactness of the probability measures.

Theorem 4.1.17

$$EHs(S^*) = \left\{ f : f(z) = \frac{xz}{(1-yz)^2}, |x|=|y|=1 \right\}.$$

Proof :

It is clear that $0 \notin EHs(S^*)$. If $f \in EHs(S^*)$, then by theorem (4.1.9),

$f(z) \prec \frac{z}{(1-xz)^2}$ for some x ($|x|=1$). For a given x with $|x|=1$, let

$$M(x) = \left\{ f : f \prec g, g(z) = \frac{z}{(1-xz)^2} \right\}.$$

It is clear that

$$\frac{z}{(1-xz)^2} = \frac{1}{4x} \left[\left(\frac{1+xz}{1-xz} \right)^2 - 1 \right]$$

which shows the linear map $\frac{1}{4x}(g-1)$ is one-to-one correspondence between the

family $M(x)$ and the class of functions \mathfrak{S} subordinate to $F(z) = \left(\frac{1+z}{1-z} \right)^2$.

Therefore, $HM(x) = \frac{1}{4x}(H\mathfrak{S}-1)$ and $EHM(x) = \frac{1}{4x}(H\mathfrak{S}-1)$. By using theorem

(3.2.1) with $c=1$ and $\alpha=2$ yields

$$EH\mathfrak{S} = \left\{ h : h(z) = \left(\frac{1+yz}{1-yz} \right)^2, |y|=1 \right\}.$$

Since $EHs(S^*) \subset EHM(x)$ we conclude that

$$f(z) = \frac{1}{4x} \left[\left(\frac{1+yz}{1-yz} \right)^2 - 1 \right] = \frac{y\bar{x}z}{(1-yz)^2} = \frac{uz}{(1-vz)^2}$$

and $|u|=|v|=1$.

Hence

$$f \in E = \left\{ g: g(z) = \frac{xz}{(1-yz)^2}, |x|=|y|=1 \right\}.$$

We claim that $E \subset EHs(S^*)$. As $E \subset s(S^*)$ it suffices to show that each function in E uniquely maximizes a real-valued functional over E . Let

$$J(g) = \alpha g'(0) + \beta \left(\frac{g''(0)}{2} \right)$$

whenever $g \in A$ and where $|\alpha|=|\beta|=1$. As

$$f(z) = \frac{xz}{(1-yz)^2} = xz + 2xyz^2 + \dots$$

we conclude that

$$\begin{aligned} \operatorname{Re} J(f) &= \operatorname{Re} \left(\alpha x + \frac{\beta}{2} 4xy \right) \\ &\leq |\alpha x + 2\beta xy| = |\alpha + 2\beta y| \\ &\leq |\alpha| + 2|\beta| = 3 \end{aligned}$$

The equality requires that $x = \frac{1}{\alpha}$ and $y = \frac{\alpha}{\beta}$. By varying α and β with all possible

pairs (x, y) with $|x|=|y|=1$ such that $\operatorname{Re} J(f) = 3$. Hence $EHs(S^*) = E$.

Theorem 4.1.18

$$EHs(CL) = \left\{ f : f(z) = w \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, |x|=|y|=|w|=1, x \neq y \right\}.$$

Proof:

Clearly, $o \notin EHs(CL)$. Assume that $f \in EHs(CL)$ and

$$f(z) \prec \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$$

for some x, y where $x \neq y$ and $|x|=|y|=1$. For a given x and y with $x \neq y$, and

$|x|=|y|=1$, suppose

$$K(x, y) = \left\{ f : f \prec g, g(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} \right\}.$$

The identity

$$f = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} = \frac{1}{2(y-x)} \left[\left(\frac{1-xz}{1-yz} \right)^2 - 1 \right].$$

Shows that the linear map $f = \left(\frac{1}{2(y-x)} \right) (g-1)$ is a one-to-one correspondence

between the family $K(x, y)$ and the class of functions \mathfrak{S} subordinate to

$F(z) = \left(\frac{1-xz}{1-yz} \right)^2$. Since $\phi \in \beta_0$ if and only if $w = y\phi \in \beta_0$, \mathfrak{S} is the same as the set

of functions subordinate to $\left[\frac{1-uz}{1-z} \right]^2$ where $u = \frac{x}{y}$ ($u \neq 1, |u|=1$). By theorem

with $c = -u$ and $\alpha = 2$ we conclude that

$$EH\mathfrak{S} = \left\{ h : h(z) = \left(\frac{1-uvz}{1-vz} \right)^2, |v|=1 \right\}.$$

Since $EHs(C) \subset EHK(x, y)$ we find that

$$f(z) = \frac{1}{2(y-x)} \left[\left(\frac{1 - (\frac{x}{y})vz}{1-vz} \right)^2 - 1 \right] = \frac{v}{y} \left(\frac{z - (\frac{v}{2y})(x+y)z^2}{(1-vz)^2} \right).$$

Hence

$$f \in E = \left\{ f : f(z) = w \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, x \neq y, |x|=|y|=|w|=1 \right\}.$$

We now prove that $E \subset EHs(CL)$. It is sufficient to prove that if

$$w_0 \frac{z - \frac{1}{2}(x_0 + y_0)z^2}{(1-y_0z)^2} = \int_u w \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x, y, w) \quad (4.17)$$

Where μ is a probability measure on u , and $x_0 \neq y_0$, then μ is unit point mass at (x_0, y_0, w_0) . Equation (4.1) is equivalent to

$$w_0 \frac{z - x_0 z}{(1 - y_0 z)^3} = \int_u w \frac{z - xz}{(1 - yz)^3} d\mu(x, y, w). \quad (4.18)$$

By setting $z = 0$ in equation (4.18) we have

$$w_0 = \int_u w d\mu(x, y, w). \quad (4.19)$$

Let A be any measurable subset of $\partial\Delta$ and define a measure λ on $\partial\Delta$ by

$\lambda(A) = \mu(\partial\Delta \times \partial\Delta \times A)$. Then λ is a probability measure on $\partial\Delta$ and we may write

equation (4.19) as

$$w_0 = \int_{\partial\Delta} w d\lambda(w). \quad (4.20)$$

It now follows that λ is point mass at w_0 and equation (4.18) becomes

$$\frac{1 - x_0 z}{(1 - y_0 z)^3} = \int_{\partial\Delta \times \partial\Delta \times \{w_0\}} \frac{1 - xz}{(1 - yz)^3} d\mu(x, y, w_0) . \quad (4.21)$$

Since μ is a probability measure on $\partial\Delta \times \partial\Delta \times \{w_0\}$. The technique used above can be applied to prove that if

$$\frac{x_0 z}{(1 - y_0 z)^n} = \int_{\mathbb{R}} \frac{xz}{(1 - yz)^n} d\mu(x, y)$$

For a positive integer n then μ is point mass at (x_0, y_0) .

4.2 Extreme Point of \mathcal{S}

The describing of EHS and ES are open and difficult . We have shown that the Keobe functions belong to EHS and so to ES , since $EHS \subset ES$. It is known that there are other functions belonging to EHS .

Also, several necessary conditions for a function to belong to ES have been obtained

Theorem 4.2.1

If $f \in ES$ and $\Gamma = C \setminus f(\Delta)$ then Γ is an unbounded continuous curve having a strictly increasing modulus.

Proof : see [22] .

Corollary 4.2.1

Suppose that $f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-z)^2}$ and $x = e^{i\theta}$ ($0 \leq \theta < 2\pi$). If $|\theta| < \frac{\pi}{2}$ then

$f \notin ES$

Proof :

Since $C \setminus f(\Delta)$ consists of a ray having the tip $-\frac{1}{4} - \frac{i}{4} \cot\left(\frac{\theta}{2}\right)$, $\theta \neq 0$ and passing

through $-\frac{1}{2}$. It is a geometrically evident that if $|\cot\left(\frac{\theta}{2}\right)| > 1$ then the ray will not

have a strictly increasing modulus. Therefore, if $|\theta| < \frac{\pi}{2}$ then $C \setminus f(\Delta)$ does not

satisfy the conclusion of theorem 4.2.1 and thus $f \notin ES$.

Corollary 4.2.2

$S \not\subset HCL$.

Proof :

If $S \not\subset HCL$ then $HS \subset HCL$ and this implies $EHCL \subset EHS$ which is impossible as each function described there belongs to $EHCL$ (when $\varphi \neq 0$).

Theorem 4.2.2

If a function $f \in S$ omits two values of equal modulus, then f has the form

$$f = tf_1 + (1-t)f_2 \quad 0 < t < 1$$

where $f_1 \neq f_2$ and f_1 and f_2 are in S which omit open set.

Proof :

Let D be the range of f . If f omits α and β , $\alpha \neq \beta$ then some branch of the function

$$\psi(w) = \{(w - \alpha)(w - \beta)\}^{1/2}$$

is analytic and single-valued in D . We claim that the two functions $w \pm \psi(w)$ are univalent and have disjoint ranges. To prove the univalence, suppose

$$w_1 \pm \psi(w_1) = w_2 \pm \psi(w_2),$$

or

$$\psi(w_1) - \psi(w_2) = \pm(w_2 - w_1).$$

Squaring both sides, we have

$$2\psi(w_1)\psi(w_2) = (w_1 - \alpha)(w_1 - \beta) + (w_2 - \alpha)(w_2 - \beta) - (w_2 - w_1)^2.$$

Squaring again, we obtain after some labor

$$(\alpha - \beta)^2 (w_1 - w_2)^2 = 0,$$

Which is impossible unless $w_1 = w_2$. Thus both of the functions $w \pm \psi(w)$ are univalent in D . A similar argument shows they have disjoint ranges. Indeed, if

$$w_1 + \psi(w_1) = w_2 - \psi(w_2),$$

essentially the same calculation shows that $w_1 = w_2$, which implies

$$\psi(w_1) = \psi(w_2) = 0. \text{ This is clearly impossible.}$$

In particular, we have shown that the functions $w \pm \psi(w)$ are univalent and omit open sets. Both properties are preserved under the normalizations

$$\psi_1(w) = \frac{w + \psi(w) - \psi(0)}{1 + \psi'(0)}, \quad \psi_2(w) = \frac{w - \psi(w) + \psi(0)}{1 - \psi'(0)}.$$

These functions ψ_1 and ψ_2 are analytic and univalent in D , omit open sets, and satisfy $\psi_j(0) = 0$ and $\psi'_j(0) = 0$, $j = 1, 2$. Therefore, the compositions $f_1 = \psi_1 \circ f$ and $f_2 = \psi_2 \circ f$ are distinct functions in S which omit open sets. Furthermore, since

$$[1 + \psi'(0)]\psi_1(w) + [1 - \psi'(0)]\psi_2(w) = 2w,$$

the function f can be expressed by

$$f(z) = t f_1(z) + (1 - t) f_2(z),$$

where $t = \frac{1}{2}[1 + \psi'(0)]$.

It remains to show that $0 < t < 1$ under the additional assumption that $|\alpha| = |\beta|$.

Equivalently, it is to be shown that $-1 < \psi'(0) < 1$ if $\alpha = re^{i\theta}$ and $\beta = re^{i\varphi}$, where $0 < \theta - \varphi < 2\pi$. But an easy calculation gives

$$\psi'(0) = -(\alpha + \beta) / 2\psi(0) = \pm \cos \frac{1}{2}(\theta - \varphi),$$

which proves $-1 < \psi'(0) < 1$.

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