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# Generalized Close to Convex Functions

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# Thesis Approval

## Generalized Close to Convex Functions

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## Dedication

To:

My Mother

A strong and gentle soul who taught me my first words and to trust in god,

believe in life and that so much could be done with little.

My Father

For earning an honest living for us and for supporting and encouraging me

to believe in myself and that anything is possible.

My Wife and My Children

For raising and supporting me to make everything is possible.

Sameer Omran

## Declaration

I certify that this submitted for the degree of master is the result of my own research, except where otherwise acknowledge. And that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: Sameer Omran

Sameer Hussein Mahmoud Omran

Date: 09/12/2017

# Acknowledgment

Praise is to God, the Cherisher and Sustainer of the words. Peace and blessing be upon my first teacher and educator, Prophet Mohammad, who has taught the world.

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# Abstract

In this thesis we considered and analyzed the generalized close to convex functions and its properties. The set of close to convex functions is a subclass of family of analytic, univalent, normalized functions in the open unit disk and the real part of quotient between derivative of $f$ and derivative of convex function is greater than zero.

In Chapter One, we presented theory of univalent functions, and we started with the definition of univalent functions, some theories and examples. Then we introduced the family of analytic and univalent functions normalized by $f(0) = 0$ and $f'(0) = 1$ which denoted by family S, and study some of its properties. Then, we briefly talked about the coefficients estimate of the function in family S. Furthermore, we presented the growth and distortion theorems. And we studied the convex and starlike functions as a special subclasses of family S.

Next, in Chapter Two, we introduced some notations in complex fractional calculus. Therefore, we started with a historical brief of fractional calculus, then we discussed two important special functions which called gamma and beta functions. After that we defined the fractional integral and introduced some theorems and examples. We also used Dirichlet's formula to prove that the composite of fractional integral is commutative.

In addition, the definition of fractional derivative, some theorems and examples were presented. Furthermore, we show that the composite of fractional derivative is not commutative. Then we used the fractional derivative to define a fractional operator and we studied some of its properties.

Finally, in Chapter Three, we deeply reviewed and studied the class of close to convex functions as a subclass of family S, after that we generalized the close to convex functions by the fractional operator, and we studied its properties. Then we talked about the bounds of these functions and estimated the coefficient of the generalized close to convex functions in the open unit disk.

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# Introduction

The notation of univalent functions defined in similar way of the injective functions, and its play a central role in the geometric theory which basis was put in the early twenty century by P. Koebe, T. H. Gronwall and L. Bieberbach.

Many properties of univalent function studied by many of mathematicians, Such as Noshiro and Warschawski whom introduced a sufficient condition for univalence when they said that an analytic function in a convex domain is univalent if $\mathrm{Re}(f'(z)) > 0$, see [6].

In addition to that many families of analytic and univalent functions are defined and studied such as the family P which is family of analytic functions with positive real part and satisfies $\varphi(0) = 1$. Also family S, family of analytic and univalent functions in the open unit disk which normalized by $f\ (0) = 0$ and $f'(0) = 1$.

In 1907, Paul Koebe provided a bound for $|f'(z)|$, and in 1914, Gronwall introduced growth theorems which provided a bound for $|f(z)|$ overall $f \in$ S. In 1916, Ludwig Bieberbach conjectured that the coefficients $a_n$ of the functions in family S satisfy the inequality $|a_n| \leq n$ which proved by L. de Branges in 1985. See [13].

Furthermore, many subclasses of family S was introduced such as the subclass of starlike functions, the subclass of convex functions, and in 1915 Alexander reveal surprisingly close analytical connection between the subclass of convex functions and the subclass of starlike functions. See [6].

In 1952, Wilfred Kaplan defined a class of close to convex functions which is a subclass of family S and contains the subclass of starlike function. In 1955, M. O. Reade proved that the Bieberbach conjecture is satisfied by a subclass of close to convex functions. After that many of mathematicians studied the definition and properties of the subclasses of close to convex functions, see [4], [6], [8], [9], [10], [17].

The subject of fractional calculus has gained considerable popularity and importance during the past three decades or so. The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716).

The notation of fractional integral of order $\lambda$ defined by G. Riemann and J. Liouville in 1832 and denoted by $I^{v} f(z)$, see [11]. Then they used the fractional integral to define the fractional derivative as follows

$$D^{u} f(z) = D^{n}[I^{v} f(z)],$$

see [12].

In 2013, M. Aydogan, Y. Kahramaner and Y. Polatoglu used the fractional derivative to define $\lambda$-fractional operator

$$O^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D^{\lambda} f(z)$$

and they used the $\lambda$-fractional operator in order to generalize the close to convex functions. The new class called $\lambda$-fractional close to convex functions and denoted by $K(\lambda)$. In this thesis, we shall study and analyze their scientific research paper, see [2].

# Chapter One

## Theory of Univalent Functions

By considering the set of all complex numbers $\mathbb{C} = \{z = a+ib : i = \sqrt{-1}, a,b \in \mathbb{R}\}$ and if the domain $D \subset \mathbb{C}$ is an open non-empty connected subset of the complex plane, we recall that the region $\Delta = \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk with center at the origin and radius one, and the function $f : D \to \mathbb{C}$ assigns for each element in the domain one value in the co-domain.

In addition to that, the function $f(z)$ is said to be analytic at a point $z_0$ in the complex domain $D$ if it is differentiable at every $z$ in some $\varepsilon$ - neighborhood of the point $z_0$. If a function $f(z)$ is analytic at every point in $D$ then $f(z)$ is called analytic in $D$.

That is, if $f(z)$ analytic in $\Delta = \{z : |z| < 1\}$, then the function $f(z)$ has the following Taylor series expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_o)^n \tag{1.1}$$

where $|z - z_0| < 1$. For more details, see [3]. An analytic function in the whole complex plane $\mathbb{C}$ is said to be entire function such as polynomials. see [5].

In this chapter, we shall introduce the notation of univalent functions, some theorems, examples and properties. And we shall study an important family of univalent functions in the open unit disk $\Delta = \{z : |z| < 1\}$ and its properties, this family denoted by family S. We also interested to estimate the coefficient of these functions, and at the end of this chapter we shall present the convex and starlike functions as a subclass of family S.

## 1.1 Univalent Functions

In this subsection we are interested in studying the family of complex functions such that every element in the range of the function corresponds to one and only one element in its domain. This family of functions called the univalent functions, and it plays a central role in geometric theory of analytic functions.

**Definition 1.1.1:**

A function $f: D \to \mathbb{C}$ is called univalent function on $D$ if $f(z_1) \neq f(z_2)$ for all $z_1 \neq z_2$ where $z_1, z_2 \in D$. Or equivalently, if $f(z_1) = f(z_2)$ for every $z_1, z_2 \in D$ then $z_1 = z_2$.

**Example 1.1.2:**

Let $f: \Delta \to \mathbb{C}$ such that $f(z) = z - \frac{1}{2}z^2$, where $\Delta$ the open unit disk. Show that $f(z)$ is univalent function in $\Delta$.

**Solution:**

Suppose $f(z_1) = f(z_2)$ where $z_1$ and $z_2$ are two distinct points in the open unit disk $\Delta$, then $z_1 - \frac{1}{2}z_1^2 = z_2 - \frac{1}{2}z_2^2$, which implies $z_1 - z_2 - \frac{1}{2}(z_1^2 - z_2^2) = 0$. Take $(z_1 - z_2)$ as a common factor, we get $(z_1 - z_2)(1 - \frac{1}{2}(z_1 + z_2)) = 0$.

That is, either $(z_1 - z_2) = 0$ or $(1 - \frac{1}{2}(z_1 + z_2)) = 0$. If $(z_1 - z_2) = 0$ then $z_1 = z_2$, and if $(1 - \frac{1}{2}(z_1 + z_2)) = 0$, then $z_1 + z_2 = 2$ which is impossible, since $z_1$ and $z_2$ in $\Delta$, thus $f(z_1) = f(z_2)$ implies $z_1 = z_2$. Therefore by Definition 1.1.1 $f$ is univalent function in $\Delta$.

**Definition 1.1.3:**

A function $f: D \to \mathbb{C}$ is called conformal mapping if $f$ is analytic and univalent on $D$.

**Example 1.1.4:**

Let $f:\Delta \to \mathbb{C}$ such that $f(z)=z-\frac{1}{2}z^2$, where $\Delta$ is the open unit disk. Show that $f(z)$ is conformal mapping on $\Delta$.

**Solution:**

Since $f(z)=z-\frac{1}{2}z^2$ is polynomial, thus it is analytic in $\Delta$, also in Example 1.1.2 we show that $f(z)=z-\frac{1}{2}z^2$ is univalent. Therefore by Definition 1.1.3 $f(z)=z-\frac{1}{2}z^2$ is conformal mapping on the open unit disk $\Delta$.

Recall that the domain $D$ is called convex if the line segment joining any two points of $D$ lies entirely in $D$. In the next theorem, Noshiro and Warschawski introduced a sufficient condition for a function $f(z)$ to be univalent in a convex domain $D$, see [6].

**Theorem 1.1.5:**

Let $f:D \to C$ be analytic in a convex domain $D$ and $Re\{f'(z)\}>0$, then $f(z)$ is univalent in $D$.

**Proof:**

Let $z_1$ and $z_2$ be two distinct points in $D$, then $f$ is defined on the line segment joining $z_1$ to $z_2$ and

$$f(z_2)-f(z_1)=\int_{z_1}^{z_2} f'(z).dz \ .$$

Since $D$ is convex domain, then any point $z$ lies on the segment joining $z_1$ and $z_2$ can be written as $z=(1-t)z_1+t\,z_2$ where $0\le t\le 1$, and $z_1$, $z_2$ in $D$, so we get

$$f(z_2)-f(z_1)=\int_0^1 (z_2-z_1)f'(t\,z_2+(1-t)z_1)dt,$$

$$f(z_2)-f(z_1)=(z_2-z_1)\int_0^1 f'\left(t\,z_2+(1-t)z_1\right)dt,$$

but $\operatorname{Re}\{f'(z)\}>0$, so

$$\operatorname{Re}\left\{\int_0^1 f'(z(t))dt\right\}>0\ .$$

Hence $f(z_2)-f(z_1)\neq 0$, i.e. $f(z_2)\neq f(z_1)$. Therefore $f(z)$ is univalent in $D$.

In the next theorem, we show that the composition of two univalent functions is univalent.

**Theorem 1.1.6**:

Let $g(z)$ and $\phi(z)$ be two univalent functions in $D\subseteq\mathbb{C}$ such that $\phi(D)\subseteq D$ then $f(z)=g(\phi(z))$ is also univalent in $D$, [17].

**Proof:**

Suppose $g(z)$ and $\phi(z)$ are univalent in $\Delta$ , and let $z_1$ and $z_2$ be two distinct points in $\Delta$ then

$$\phi(z_1)\neq\phi(z_2).$$

Assume $\phi(z_1)=\alpha_1$, $\phi(z_2)=\alpha_2$ we get

$$f(z_1)=g(\phi(z_1))=g(\alpha_1) \text{ and } f(z_2)=g(\phi(z_2))=g(\alpha_2).$$

But $g(\alpha_1)\neq g(\alpha_2)$ since $g(z)$ is univalent, so $f(z_1)\neq f(z_2)$. Therefore $f(z)$ is univalent function in $D$.

Next, we shall study an important family of univalent functions in the open unit disk, and we present some examples, properties, and theorems about it.

## 1.2 Family S

In this section, we define the family S and present some examples.

**Definition 1.2.1:**

Let S be the set of analytic and univalent functions on the open unit disk $\Delta = \{z : |z| < 1\}$ that normalized by the conditions $f(0) = 0$ and $f'(0) = 1$.

That is

$$f \in S \text{ iff } f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and univalent, } \quad \text{where } z \in \Delta . \tag{1.2}$$

**Example: 1.2.2**

Let $f : \Delta \to \mathbb{C}$ such that $f(z) = z - \frac{1}{2}z^2$, show that $f(z)$ belongs to the family S.

**Solution:**

Suppose $f : \Delta \to \mathbb{C}$ defined by $f(z) = z - \frac{1}{2}z^2$, as we show in Example 1.1.4 $f(z)$ is conformal mapping in the open unit disk. That is analytic and univalent function in $\Delta$. Furthermore $f'(z) = 1 - z$, $\forall z \in \Delta$, and clearly normalized by $f(0) = 0$ and $f'(0) = 1$. Hence $f \in S$.

**Example: 1.2.3**

Let $L : \Delta \to \mathbb{C}$ such that $L(z) = \frac{z}{1-z}$, show that $L(z)$ belongs to family S.

**Solution:**

Suppose $L : \Delta \to \mathbb{C}$ is a function defined by $L(z) = \frac{z}{1-z}$, since $L(z)$ is a quotient of two polynomials which are analytic in $\Delta$, so $L(z)$ analytic function in $\Delta$ when the denominator of $L(z)$ does not equal zero in $\Delta$, but $1 - z \neq 0$ for all $z \in \Delta$. Thus, $L(z)$ analytic in $\Delta$. Also

$$L'(z)=\frac{((1-z)\times 1)-(z)(-1))}{(1-z)^2}=\frac{1-z+z}{(1-z)^2}=\frac{1}{(1-z)^2}.$$

To show that $L(z)$ is univalent, we use Definition 1.1.1, hence suppose $L(z_1)=L(z_2)$ we get,

$$\frac{z_1}{1-z_1}=\frac{z_2}{1-z_2}.$$

By cross multiplication we have

$$z_1-z_1z_2=z_2-z_1z_2,$$

which implies $z_1=z_2$, Thus by Definition 1.1.1 $L(z)$ is univalent. Also, clearly $L(z)$ normalized by $L(0)=0$ and $L'(0)=1$. Therefore $L(z)$ belongs to family S.

**Example: 1.2.4**

Let $g:\Delta\to\mathbb{C}$ such that $g(z)=\frac{z}{1+iz}$, show that $g(z)$ belongs to the family S.

**Solution:**

Suppose $g:\Delta\to\mathbb{C}$ is a function defined by $g(z)=\frac{z}{1+iz}$, since $g(z)$ is a quotient of two polynomials which are analytic in $\Delta$, so $g(z)$ analytic function in $\Delta$ when the denominator of $g(z)$ does not equal zero in $\Delta$ because $1+iz\neq 0$ for all $z\in\Delta$. Thus $g(z)$ analytic function in $\Delta$.

We compute

$$g'(z)=\frac{(1+iz)(1)-(z)(i)}{(1+iz)^2}=\frac{1+iz-iz}{(1+iz)^2}=\frac{1}{(1+iz)^2}.$$

To show that $g(z)$ is univalent we use Definition 1.1.1, hence suppose $g(z_1)=g(z_2)$ we get,

$$\frac{z_1}{1+iz_1}=\frac{z_2}{1+iz_2}.$$

By cross multiplication we have,

$$z_1 + iz_1z_2 = z_2 + iz_1z_2,$$

which implies $z_1 = z_2$. Hence by Definition 1.1.1 $g(z)$ is univalent.

Thus $g(z)$ is analytic and univalent and clearly normalized by $g(0) = 0$ and $g'(0) = 1$. Therefore $g(z)$ belongs to the family S.

**Example: 1.2.5**

The function $f(z) = z^2 + z$ is analytic in $\Delta$ being a polynomial, further $f'(z) = 2z + 1$, and clearly $f(z)$ normalized by $f(0) = 0$ and $f'(0) = 1$. But $\frac{-1}{4}$ and $\frac{-3}{4}$ in $\Delta$ and $f(\frac{-1}{4}) = f(\frac{-3}{4}) = \frac{-3}{16}$. Hence by Definition 1.1.1 $f(z)$ is not univalent in $\Delta$. Therefore $f(z)$ does not belong to family S.

## 1.3 Properties of Family S

After we introduced family S and proposed some examples, we are going to study some of its properties, for more details, see [13].

**Remark 1.3.1:**

The family S is not closed under addition. Since if $f(z)$ and $g(z)$ are in family S, then $f + g$ is not belonging to S, because $f'(0) + g'(0) = 2$. As the following example shows.

**Example 1.3.2:**

In Examples 1.2.3 and 1.2.4, we show that $L(z) = \frac{z}{1-z}$ and $g(z) = \frac{z}{1+iz}$ are in family S. But the function $h(z) = L(z) + g(z) = \frac{z}{1-z} + \frac{z}{1+iz}$ not belongs to family S, since $h(0) = L'(0) + g'(0) = 2$, i.e. $h(z)$ is not normalized by $h'(0) = 1$.

**Theorem 1.3.3: (Rotation)**

The family S is closed under the Rotation: If $f \in S$, $\theta \in \mathbb{R}$ and $g(z) = e^{-i\theta} f(e^{i\theta} z)$ then $g \in S$.

**Proof:**

Suppose $f(z) \in S$ and let $R : \mathbb{C} \to \mathbb{C}$ defined by $R(z) = e^{i\theta} z$, and $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = e^{-i\theta} z$. It is not difficult to show that that $R(z)$ and $T(z)$ are univalent in $\Delta$.

Now, let $g(z) = e^{-i\theta} f(e^{i\theta} z) = (T \circ f \circ R)(z)$, $g(z)$ is a composition of univalent functions, hence by theorem 1.1.6 $g(z)$ is univalent function also $g(z)$ is analytic function since it is a composition of analytic functions. Furthermore, $g'(z) = e^{-i\theta}(f'(e^{i\theta} z) e^{i\theta}) = f'(e^{i\theta} z)$. So $g(0) = e^{-i\theta} f(0) = 0$ and $g'(0) = f'(0) = 1$, therefore $g(z)$ belongs to the family S.

**Theorem 1.3.4: (Dilation)**

The family S is closed under the Dilation: If $f \in S$, $0 < r < 1$ and $g(z) = \frac{1}{r} f(r\, z)$ then $g \in S$.

**Proof:**

Suppose $f \in S$, and let $R : \mathbb{C} \to \mathbb{C}$ be defined by $R(z) = r\, z$, also $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = r^{-1} z$ where $0 < r < 1$. It's easy to show that $R(z)$ and $T(z)$ are univalent functions.

Now, let $g(z) = r^{-1} f(r\, z) = (T \circ f \circ R)(z)$. Then $g(z)$ is a composition of two univalent functions. So by Theorem 1.1.6 $g(z)$ is univalent and it is analytic since it is a composition of analytic functions. Furthermore,

$$g'(z) = r^{-1}(f'(r\, z) r) = f'(r\, z).$$

Clearly $g(z)$ is normalized by $g(0) = 0$ and $g'(0) = 1$. Therefore $g(z)$ belongs to the family S.

**Theorem 1.3.5: (Conjugation)**

The family S is closed under the Conjugation: If $f \in S$ , and $g(z) = \overline{f(\overline{z})}$ then $g \in S$.

**Proof:**

Suppose $f \in S$ , then by Definition 1.2.1 $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ . So

$$g(z) = \overline{f(\overline{z})} = \overline{\overline{z} + \sum_{n=2}^{\infty} a_n \overline{z}^n} = z + \sum_{n=2}^{\infty} \overline{a}_n z^n .$$

Thus $g(z)$ is analytic in $\Delta$ and $g'(z) = 1 + \sum_{n=2}^{\infty} \overline{a}_n \cdot n \cdot z^{n-1}$ .

To show that $g(z)$ univalent in $\Delta$ , let $g(z) = g(w)$ then $\overline{f(\overline{z})} = \overline{f(\overline{w})}$ and if we take the conjugate for both side then $f(\overline{z}) = f(\overline{w})$ , since $f(z)$ is univalent we get $\overline{z} = \overline{w}$ and so $z = w$ . Hence $g(z)$ univalent in $\Delta$ .

Since $g(z)$ satisfies the relation (1.2), that $g(z)$ is analytic, univalent and clearly normalized by $g(0) = 0$ and $g'(0) = 1$ . Therefore $g(z)$ belongs to the family S.

**Theorem 1.3.6: (Disk Automorphisim)**

The family S is closed under the disk automorphism: If $f \in S$ , then for any $z_0 \in \Delta$

$$g(z) = \frac{f\left(\frac{z + z_0}{1 + \overline{z}_0 z}\right) - f(z_0)}{(1 - |z_0|^2) f'(z_0)} \in S.$$

**Proof:**

Suppose $f \in$ S and let $w(z) = \frac{z + z_0}{1 + \overline{z}_0 z}$, and define

$$g(z) = \frac{f(w(z)) - f(z_0)}{(1 - |z_0|^2) f'(z_0)} .$$

To show that $w(z)=\dfrac{z+z_0}{1+\overline{z}_0 z}$ is univalent, we use Definition 1.1.1. So let $w(z_1)=w(z_2)$, hence

$$\frac{z_1+z_0}{1+\overline{z}_0 z_1}=\frac{z_2+z_0}{1+\overline{z}_0 z_2}.$$

By cross multiplication we get

$$(z_1+z_0)(1+\overline{z}_0 z_2)=(z_2+z_0)(1+\overline{z}_0 z_1).$$

That is

$$z_1+z_1\overline{z}_0 z_2+z_0+z_0\overline{z}_0 z_2=z_2+z_2\overline{z}_0 z_1+z_0+z_0\overline{z}_0 z_1.$$

So $z_1+|z_0|^2 z_2=z_2+|z_0|^2 z_1$. Thus we have $z_1+|z_0|^2 z_2-z_2-|z_0|^2 z_1=0$ which implies $(z_1-z_2)(1-|z_0|^2)=0$. But $1-|z_0|^2>0$ since $|z|<1$ for every $z\in\Delta$, therefore $z_1-z_2=0$, that is $z_1=z_2$, hence $w(z)$ is univalent in $\Delta$.

Also, $w(z)$ is a quotient of two polynomials which are analytic in $\Delta$, so $w(z)$ analytic function in $\Delta$ when the denominator of $w(z)$ does not equal zero in $\Delta$. But $1+\overline{z}_0 z=0$ implies $z=\dfrac{-1}{\overline{z}_0}$. By taking the modulus for each side we get $|z|=\dfrac{1}{|z_0|}>1$, since $|z|<1$, so $w(z)$ is analytic in the open unit disk. Furthermore,

$$w'(z)=\frac{(1+\overline{z}_0 z)(1)-(z+z_0)(\overline{z}_0)}{(1+\overline{z}_0 z)^2}=\frac{1-|z_0|^2}{(1+\overline{z}_0 z)^2}.$$

Now, to show that $g(z)$ is univalent, we use Definition 1.1.1. So let $g(z_1)=g(z_2)$, hence

$$\frac{f(w(z_1))-f(z_0)}{(1-|z_0|^2)f'(z_0)}=\frac{f(w(z_2)-f(z_0)}{(1-|z_0|^2)f'(z_0)}$$

which implies $f(w(z_1))=f(w(z_2))$, but $w(z)$ and $f(z)$ are univalent so by Theorem 1.1.6 $f(w(z))$ is univalent. i.e. $f(w(z_1))=f(w(z_2))$ implies $z_1=z_2$. That is $g(z)$ is univalent.

Also, $g(z)$ is analytic since $f(z)$ and $w(z)$ are analytic in $\Delta$, and $g'(z)=\dfrac{f'(w(z))w'(z)}{(1-|z_0|^2)f'(z_0)}$.

Clearly $g(0)=0$ and $g'(0)=\dfrac{f'(w(0))w'(0)}{(1-|z_0|^2)f'(z_0)}=\dfrac{f'(z_0)w'(0)}{(1-|z_0|^2)f'(z_0)}=1$. Therefore $g(z)$ belongs to family S.

## 1.4 Coefficients Estimate

In General, many coefficients bound for classes of family of univalent functions are proved, for example if $g(z)$ is analytic in the open unit disk, $g(0)=1, \mathrm{Re}(g(z))>0$ such that $g(z)=\sum_{n=1}^{\infty} c_n z^n$ then $|c_n| \le 2$. Another example, if $\phi(z)$ is normalized, convex function on the unit disk and $\phi(z)=z+\sum_{n=2}^{\infty} b_n z^n$ then $|b_n| \le 1$. For details, see [17].

By considering the family S, Bieberbach conjectured that the coefficient of the power series, see the form (1.2), satisfies $|a_n| \le n$ in 1916. Then he proved that $|a_2| \le 2$.

In 1921 Nevalinna proved that the Bieberbach conjecture is true for functions that map the unit disk onto star shaped domain while Lowner proved in 1923 that $|a_3| \le 3$. Also in 1925 J. E. Littlewood proved that $|a_n| \le en$.

Finally, the difficult open problem which known as Bieberbach conjecture proved by Branges in 1985. Hence the following theorem named by their names.

**Theorem 1.4.1: (Bieberbach conjecture- de Branges Theorem)**

If $f(z) \in \mathrm{S}$, then the coefficients $a_n$ of the function $f(z)$ satisfies the inequality

$$|a_n| \le n, \qquad \text{for all } n \ge 2.$$

For the proof and more details, see [6].

## 1.5 The Growth and Distortion Theorems

In this section we establish two fundamental theorems about univalent functions, which provide bounds on $|f'(z)|$ and $|f(z)|$ respectively, over all $f \in \mathrm{S}$.

The first theorem called distortion theorem which gives us an upper and lower bound for $|f'(z)|$ over all $f \in S$.

**Theorem 1.5.1: (Distortion Theorem)**

If $f \in S$, then

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}.$$

For the proof refer to [17].

The second theorem called distortion theorem which gives us an upper and lower bound for $|f(z)|$ overall $f \in S$.

**Theorem 1.5.2: (Growth theorem)**

If $f \in S$, then

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

For the proof see [17].

Next result is a corollary of growth theorem and it gives an upper and a lower bound for $\left|\frac{zf'(z)}{f(z)}\right|$, over all $f \in S$. See [6].

**Corollary 1.5.3:**

If $f \in S$, then

$$\frac{1-|z|}{1+|z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|z|}.$$

**Proof:**

Suppose $f \in S$ and let $\zeta \in \Delta$ be fixed and let $T_\zeta = \frac{z+\zeta}{1+\bar{\zeta} z}$. Note that $T_\zeta(0) = \zeta$ and $T'_\zeta(0) = 1-|\zeta|^2$.

Define $F_\zeta(z)$ to be

$$F_\zeta(z)=\frac{f(T_\zeta(z)-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}\ .$$

Since $T_\zeta(-\zeta)=f(0)=0$ we get

$$F_\zeta(-\varsigma)=\frac{f(T_\varsigma(-\varsigma)-f(\varsigma)}{(1-|-\varsigma|^2)f'(\varsigma)}=\frac{f(0)-f(\varsigma)}{(1-|-\varsigma|^2)f'(\varsigma)}\ .$$

That is,

$$F_\zeta(-\zeta)=\frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}. \qquad (1)$$

Also, $F_\zeta(z)\in S$ by Theorem 1.3.6, therefore by Theorem 1.5.2 we get

$$\frac{|\zeta|}{(1+|\zeta|)^2}\le\left|F_\zeta(-\zeta)\right|\le\frac{|\zeta|}{(1-|\zeta|)^2}. \qquad (2)$$

Substitute (1) in (2) we have

$$\frac{|\zeta|}{(1+|\zeta|)^2}\le\frac{|f(\zeta)|}{|(1-|\zeta|^2)||f'(\zeta)|}\le\frac{|\zeta|}{(1-|\zeta|)^2}.$$

If we take the reciprocal of this inequality we get,

$$\frac{(1-|\zeta|)^2}{|\zeta|}\le\frac{(1-|\zeta|^2)|f'(\zeta)|}{|f(\zeta)|}\le\frac{(1+|\zeta|)^2}{|\zeta|},$$

and if we divide each side by $(1-|\zeta|^2)$ and multiply each side by $|\zeta|$, obtain

$$\frac{1-|\zeta|}{1+|\zeta|}\le\frac{|\zeta f'(\zeta)|}{|f(\zeta)|}\le\frac{1+|\zeta|}{1-|\zeta|}.$$

But $\zeta$ is arbitrary, so we can replace $\zeta$ by $Z$ to get the required result

$$\frac{1-|z|}{1+|z|}\le\left|\frac{zf'(z)}{f(z)}\right|\le\frac{1+|z|}{1-|z|}.$$

## 1.6 Convex and Starlike Functions

Recall that if $D \subset \mathbb{C}$ is an open connected set, then the set $D$ is said to be starlike domain with respect to a point $w_0 \in D$, if the line segment joining $w_0$ to every other point $w \in D$ lies entirely in $D$. If the set is starlike domain with respect to each of its points, then the set $D$ is said to be convex domain.

In more picturesque language, the set $D$ is called starlike with respect to $w_0$ if that every point of $D$ be visible from $w_0$, whereas the set $D$ is called convex set, if the line segment joining any two points of $D$ lies entirely in $D$. Hence every convex domain is starlike domain, but the converse is not true, see [6].

In this section, we introduce a special class of analytic functions in order to help us to discuss two special subclasses of the family S and their properties, these subclasses are starlike functions and convex functions.

**Definition 1.6.1**

The class $P$ is the class of analytic functions such that every function $\varphi$ in $P$ have a positive real part in $\Delta$, with $\varphi(0) = 1$. That is

$$P = \{\varphi(z) : \phi(z) \text{ is analytic}, \mathrm{Re}(\phi(z) > 0) \text{ and } \phi(0) = 1 \}$$

**Example 1.6.2**

Let $\varphi : \Delta \to \mathbb{C}$ such that $\varphi(z) = \dfrac{1+z}{1-z}$. Show that the function $\varphi(z)$ belongs to the class $P$ .

**Solution:**

Suppose $\varphi(z) = \dfrac{1+z}{1-z}$ is defined from $\Delta$ into the complex plane $\mathbb{C}$ , then $\varphi(z)$ is quotient of two polynomials which they are analytic in $\Delta$ . So $\varphi(z)$ is analytic in $\Delta$ when the denominator does not equal zero. But $1-z = 0$ only when $z = 1 \notin \Delta$ . Thus $\varphi(z)$ is analytic in $\Delta$ , and

$$\varphi'(z) = \frac{(1-z)\times(1)-(1+z)\times(-1))}{(1-z)^2} = \frac{1-z+1+z}{(1-z)^2} = \frac{2}{(1-z)^2}.$$

Now, let $z = x+iy$ where $x, y \in \mathbb{R}$, then $\varphi(z) = \frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy}$. Multiply by denominator conjugate we get

$$\varphi(z) = \frac{1+x+iy}{1-x-iy} \cdot \frac{1-x+iy}{1-x+iy} = \frac{1-x^2+i(1-x)y}{(1-x)^2+y^2}.$$

Thus $\mathrm{Re}(\varphi(z)) = \frac{1-x^2}{(1-x)^2+y^2}$, since $|x|<1$ then $1-x^2>0$, also we have $(1-x)^2+y^2>0$ so we get,

$$\mathrm{Re}(\varphi(z)) = \frac{1-x^2}{(1-x)^2+y^2} > 0,$$

Finally $\varphi(0)=1$ .Therefore $\varphi(z) \in P$ .

Now, we present the class that introduced by J. Alexander in 1915 which denoted by Starlike Functions, see [6].

**Definition 1.6.3:**

Let $f : \Delta \to \mathbb{C}$ be an analytic function with $f(0) = 0$ . We say that $f$ is starlike in $\Delta$ with respect to zero (or, starlike in brief) if $f(\Delta)$ is starlike domain with respect to zero. In other word, $f(z)$ is starlike function in $\Delta$ , if the line segment between the origin and $f(z)$ lies in $f(\Delta)$ for every $z \in \Delta$ .The subclass of the family S that contains starlike functions with respect to the origin denoted by $S^*$ .

An analytical description of starlike functions in $\Delta$ , introduced in the next theorem which we present without proof.

**Theorem 1.6.4:**

Let $f$ be analytic in $\Delta$ , with $f(0)=0$ and $f'(0)=1$, then $f \in S^*$ if and only if $[zf'(z)/f(z)] \in P$. See [6].

**Example: 1.6.5**

Let $K:\Delta\rightarrow\mathbb{C}$ be such that $K(z)=\dfrac{z}{(1-z)^2}$ , Show that Keobe function $K(z)\in S^*$.

**Solution:**

Suppose $K(z)=\dfrac{z}{(1-z)^2}$ is defined from $\Delta$ into the complex plane $\mathbb{C}$ , then $K(z)$ is quotient of two polynomials which they are analytic in $\Delta$ . So $K(z)$ is analytic in $\Delta$ when the denominator does not equal zero. But $1-z=0$ only when $z=1\notin\Delta$ . Thus $K(z)$ is analytic in $\Delta$ . Also,

$$K'(z)=\frac{((1-z)^2\times 1)-(z\times 2(1-z)(-1))}{(1-z)^4}=\frac{1-2z+z^2+2Z-2z^2}{(1-z)^4}=\frac{1+z}{(1-z)^3}.$$

To show that $K(z)$ is univalent, we use Definition 1.1.1. Hence suppose $K(z_1)=K(z_2)$, then $\dfrac{z_1}{(1-z_1)^2}=\dfrac{z_2}{(1-z_2)^2}$. By cross multiplication we get $z_1(1-z_2)^2=z_2(1-z_1)^2$. So we have $z_1-2z_1z_2+z_1z_2^{\ 2}=z_2-2z_2z_1+z_2z_1^{\ 2}$, and so

$$z_1-z_2+z_1z_2^{\ 2}-z_2z_1^{\ 2}=(z_1-z_2)-(z_1z_2(z_1-z_2))=0.$$

Take $(z_1-z_2)$ as a common factor we get $(z_1-z_2)(1-z_1z_2)=0$. But $1-z_1z_2\neq 0$ since $|z|<1$ , so we get $z_1=z_2$. Therefore by Definition 1.1.1 $K(z)$ is univalent in $\Delta$ .

Finally, to show that $K(z)$ is starlike function, we use Theorem 1.6.4, thus

$$\frac{z\,K'(z)}{K(z)}=z\cdot\frac{1+z}{(1-z)^3}\cdot\left(\frac{z}{(1-z)^2}\right)^{-1}=z\cdot\frac{1+z}{(1-z)^3}\cdot\frac{(1-z)^2}{z}=\frac{1+z}{1-z}.$$

Also in Example 1.6.2, we show that $\varphi(z)=\dfrac{1+z}{1-z}=\dfrac{zK'(z)}{K(z)}$ belongs to the class $P$. Clearly $K(z)$ normalized by $K(0)=0$ and $K'(0)=1$. Therefore by Theorem 1.6.4, we get $K(z)$ belongs to the class $S^*$.

Now, we present a subclass of the family S introduced by E. Study in 1913 and known by the class of convex functions, see [17].

**Definition 1.6.6:**

Let $f:\Delta\to\mathbb{C}$ be an analytic function. We say that $f$ is convex function in $\Delta$ (or, in brief, convex) if $f(\Delta)$ is convex domain.

That means the line segment between $f(z_1)$ and $f(z_2)$ lies in $f(\Delta)$ for every $z_1, z_2 \in \Delta$. The subclass of the family S that contains all convex functions in $\Delta$ denoted by $C$.

The following theorem is presented without proof describe the convex function in a similar way that used to describe the starlike functions. For proof of this theorem we refer the reader to reference [6].

**Theorem 1.6.7:**

Let $f$ be analytic in $\Delta$, with $f(0)=0$ and $f'(0)=1$, then $f\in C$ if and only if $[1+zf''(z)/f'(z)]\in P$.

**Example 1.6.8**

Let $g:\Delta\to\mathbb{C}$ be such that $g(z)=\dfrac{z}{1+iz}$. Show that $g(z)$ belongs to the class $C$.

**Solution:**

In Example 1.2.4 we show that $g(z)$ belongs to the family S. That $g(z)$ is analytic, univalent and normalized by $g(0)=0$ and $g'(0)=1$. Furthermore,

$$g'(z)=\frac{(1+iz)\times(1)-(z)(i)}{(1+iz)^2}=\frac{1+iz-iz}{(1+iz)^2}=\frac{1}{(1+iz)^2},$$

and

$$g''(z)=\frac{-1(2)(1+iz)(i)}{(1+iz)^4}=\frac{-2i}{(1+iz)^3}.$$

Now, we apply Theorem 1.6.7 to show that $g(z)$ belongs to the class $C$ . So let

$$\varphi(z)=1+\frac{zf''(z)}{f'(z)}=1+\frac{-2iz}{(1+iz)^3}\cdot\frac{(1+iz)^2}{1}=1-\frac{2iz}{1+iz}=\frac{1-iz}{1+iz}.$$

Since $\varphi(z)$ is a quotient of two polynomials which are analytic in $\Delta$ , so $\varphi(z)$ is analytic function in $\Delta$ because the denominator of $\varphi(z)$, $1+iz$ does not equal zero in $\Delta$ . Also, if $z=x+iy$ where $x,y\in\mathbb{R}$ then

$$\operatorname{Re}(\varphi(z))=\operatorname{Re}\frac{1-iz}{1+iz}=\operatorname{Re}(\frac{1-i(x+iy)}{1+i(x+iy)})=\operatorname{Re}\frac{1+y-ix}{1-y+ix},$$

and so

$$\frac{1-iz}{1+iz}=\frac{1+y-ix}{1-y+ix}\cdot\frac{1-y-ix}{1-y-ix}=\frac{1-y^2-i(1+y)x-i(1-y)x-x^2}{(1-y)^2+x^2}.$$

Since $|z|<1$ then $1-x^2-y^2>0$ so we get,

$$\operatorname{Re}(\varphi(z))=\frac{1-x^2-y^2}{(1-y)^2+x^2}>0,$$

and clearly $\varphi(0)=1$, hence $\varphi(z)\in P$ . Therefore by Theorem 1.6.7 $g\in C$.

Next theorem is observed by Alexander in 1915 who shows that the two preceding theorems reveal surprisingly close analytic connection between convex and starlike mappings.

**Theorem 1.6.9: (Alexander's Theorem).**

Let $f$ be analytic in $\Delta$, with $f(0)=0$ and $f'(0)=1$, then $f\in C$ if and only if $zf'(z)\in S^*$.

**Proof:**

($\Rightarrow$) Suppose $f$ is a convex function in $\Delta$, with $f(0)=0$ and $f'(0)=1$, let $g(z)=z\,f'(z)$ then

$$\frac{zg'(z)}{g(z)}=\frac{z((1)f'(z)+z\,f''(z))}{z\,f'(z)}=\frac{z\,f'(z)}{z\,f'(z)}+\frac{z^2\,f''(z)}{z\,f'(z)},$$

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

Since $f$ is a convex function, then $1 + \frac{zf''(z)}{f'(z)} \in P$ and so $\frac{zg'(z)}{g(z)} \in P$.

Also since $f(z)$ is analytic $g(z) = z\,f'(z)$ is analytic function. And clearly $g(0)=0$ and $g'(0)=1$ is analytic. Therefore by Theorem 1.6.4 $g(z)=z\,f'(z)\in S^*$.

($\Leftarrow$) Suppose $g(z) = z\,f'(z)$ belongs to starlike class in $\Delta$, and let $f$ be analytic function in with $f(0)=0$ and $f'(0)=1$. Since $g(z)=z\,f'(z)\in S^*$ then

$$\frac{zg'(z)}{g(z)} \in P \text{ implies } 1 + \frac{zf''(z)}{f'(z)} \in P.$$

Therefore, by Theorem 1.6.7 $f \in C$.

Also by definition of starlike and convex function we notice that the subclass of the convex functions is proper subset of the sub class of the starlike functions since Keobe function belongs to $S^*$ while does not belongs to $C$. Therefore,

$$C \subset S^* \subset S.$$

# Chapter Two

## Complex Fractional Calculus and Fractional Operator

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering.

Fractional calculus does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

## 2.1 History of Complex Fractional Calculus

The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in N_0 = \{0, 1, 2, ....\}$ when $n = \frac{1}{2}$.

In his reply, dated 30 September 1695, Leibniz wrote to L'Hopital "This is an apparent paradox from which, one day, useful consequences will be drawn". Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917.

In fact, in his 700-pagetextbook, entitled "Traite du Calcul Differentiel et du Calcul Integral" (Second edition; Courcier, Paris, 1819), S. F. Lacroix devoted two pages (pp. 409-410) to fractional calculus, showing eventually that

$$\frac{d^{\frac{1}{2}}v}{dx^{\frac{1}{2}}} = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

In addition to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present day applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on. See [11].

As all branches in mathematics, since we cannot start from zero and before we discuss the notation of fractional calculus and its properties, we present important functions that help us to understand the discussion about the main concept of fractional calculus.

## 2.2 The Gamma Function

The gamma function was first introduced by Swiss mathematician Leon hard Euler (1707-1855), in his goal to generalize the factorial to non-integer value. Later because of its great importance, it was studied by other eminent mathematicians like Adrien- Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), As well as many others.

The gamma function belongs to the category of special transcendental functions. And we will see that some famous mathematical constant is occurring in its study.

During the years 1729 and 1730, Euler introduced a function which has the property to interpolate the factorial whenever the argument of the function is an integer. In a letter he proposed the following definition. See [16].

**Definition: 2.2.1 (Gamma Function)**

Let $Z$ be complex number such that $\mathrm{Re}(z) > 0$, then

$$\Gamma(z) = \int_0^1 (-\log(t))^{z-1} dt . \qquad (2.1)$$

By elementary change of variable this historical definition takes the more usual forms.

**Theorem: 2.2.2**

Let $Z$ be complex number such that $\mathrm{Re}(z) > 0$, then

$$\Gamma(z) = \int_0^\infty (t)^{z-1} e^{-t} dt . \qquad (2.2)$$

Or sometime

$$\Gamma(z) = 2\int_0^\infty (t)^{2z-1} e^{-t^2} dt . \qquad (2.3)$$

**Proof:**

Let $u = -\log(t)$ so $-tdu = dt$, also when $t \to 0$ then $u \to \infty$ and when $t \to 1$ then $u \to 0$.

Thus by substitution $u = -\log(t)$ in (2.1) we get,

$$\Gamma(z) = -\int_\infty^0 (u)^{z-1} t du .$$

But $u = -\log(t)$ implies $t = e^{-u}$, therefore

$$\Gamma(z) = \int_0^\infty (u)^{z-1} e^{-u} du .$$

That is the equation (2.2).

To get equation (2.3) set $u^2 = -\log(t)$ so $-t(2u)du = dt$, clearly when $t \to 0$ then $u \to \infty$ and when $t \to 1$ then $u \to 0$. By substituting $u^2 = -\log(t)$ in (2.1) we have

$$\Gamma(z) = -2\int_\infty^0 (u^2)^{z-1} t(u) du .$$

Since $u^2 = -\log(t)$ then $t = e^{-u^2}$. Therefore

$$\Gamma(z) = \int_0^1 (-\log(t))^{z-1} dt = 2\int_0^\infty (u^2)^{z-1} e^{-u^2} (u) du$$

$$= 2\int_0^\infty (u)^{2z-1} e^{-u^2} du$$

That is the equation (2.3).

From this theorem, we see that the gamma function $\Gamma(z)$ or the Eulerian integral of second kind is well defined and analytic function for a complex number $Z$ with positive real part.

Next we derive and justify some of important function equations, in order to help us later in some calculations.

**Corollary 2.2.3:**

Let $Z$ be complex number such that $\operatorname{Re}(z) > 0$, then

1- $\Gamma(1) = 1$.

2- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

3- $\Gamma(z+1) = z\,\Gamma(z)$. (2.4)

**Proof:**

Suppose $Z$ be complex number such that $\operatorname{Re}(z) > 0$, to prove (1) substitute $z = 1$ in the relation (2.2), we get

$$\Gamma(1) = \int_0^\infty (t)^{1-1} e^{-t} dt = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 1$$

.

Now, to prove (2) we substitute $z = \frac{1}{2}$ in the relation (2.3) and so we get,

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty (t)^{2\frac{1}{2}-1} e^{-t^2} dt = 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}.$$

Finally to show that $\Gamma(z+1) = z\,\Gamma(z)$. We use relation (2.2) and replace $Z$ by $z+1$ to get

$$\Gamma(z+1) = \int_0^\infty (t)^z e^{-t} dt \,.$$

Now, we use integration by parts, so let $u = t^z, dv = e^{-t}dt$ then, $du = zt^{z-1}dt$ and $v = -e^{-t}$.

Thus we have

$$\Gamma(z+1) = \int_0^\infty (t)^z e^{-t} dt = t^z(-e^{-t})\Big|_0^\infty - \int_0^\infty (-e^{-t}) z t^{z-1} dt$$

$$= \int_0^\infty (e^{-t}) z t^{z-1} dt = z\int_0^\infty (e^{-t}) t^{z-1} dt$$

$$\Gamma(z+1) = z\int_0^\infty t^{z-1} e^{-t} dt = z\,\Gamma(z).$$

Let us now consider the useful and related function to the gamma function which occurs in the computation of many definite integrals.

## 2.3 The Beta function

The name of beta function was introduced for the first time by Jacques Binet (1786-1856) and he made various contributions on the subject. Beta function which also called Eulerian integral of first kind introduced by Euler in 1730. See [16].

**Definition: 2.3.1**

Let $u$ and $v$ be two complex numbers such that $\mathrm{Re}(u) > 0$ and $\mathrm{Re}(v) > 0$ then the Beta function defined as

$$B(u,v) = \int_0^1 (t)^{u-1}(1-t)^{v-1} dt. \qquad (2.5)$$

Using the substitution $s = 1-t$ we get

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt = \int_0^1 s^{v-1}(1-s)^{v-1} ds = B(v,u) \,.$$

Hence $B(u,v)$ is symmetric function.

The following theorem gives the trigonometric form of the beta function.

**Theorem: 2.3.2**

Let $u$ and $v$are two complex numbers such that $\mathrm{Re}(u) > 0$ $\mathrm{Re}(v) > 0$ . Then

$$B(u,v) = 2\int_0^{\pi/2} \sin(t)^{2u-1}\cos(t)^{2v-1}dt.$$

**Proof:**

Let $t = \sin^2(s)$ so $dt = 2\sin(s)\cos(s)ds$ .Then as $t \to 0, s \to 0$ and as $t \to 1, s \to \pi/2$.
Thus the equation (2.5)

$$B(u,v) = \int_0^1 (t)^{u-1}(1-t)^{v-1}dt$$

$$= \int_0^{\pi/2} (\sin^2(s))^{u-1}(1-\sin^2(s))^{v-1}.2\sin(s)\cos(s)ds$$

$$= 2\int_0^{\pi/2} (\sin(s))^{2u-1}(\cos(s))^{2v-1}ds.$$

Among the most important relations of the beta function is the one that relates it to the gamma function as given in the following theorem.

**Theorem 2.3.3:**

Let $\mathrm{Re}(u) > 0, \mathrm{Re}(v) > 0$ , then

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{2.6}$$

**Proof:**

Using Theorem 2.2.2 we form the following product

$$\Gamma(u)\Gamma(v) = 4\int_0^{\infty} t^{2u-1}e^{-t^2}dt\int_0^{\infty} s^{2v-1}e^{-s^2}ds,$$

and so

$$\Gamma(u)\Gamma(v) = 4\int_0^\infty\int_0^\infty t^{2u-1}s^{2v-1}e^{-(t^2+s^2)}dtds.$$

Introduce polar variables $t = r\cos\theta$ and $s = r\sin\theta$, we have

$$\Gamma(u)\Gamma(u) = 4\int_0^\infty\int_0^{\pi/2}(r\cos\theta)^{2u-1}(r\sin\theta)^{2v-1}e^{-r^2}rdrd\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}r^{2(u+v)-1}e^{-r^2}drd\theta$$

$$= (2\int_0^\infty r^{2(u+v)-1}e^{-r^2}dr)(2\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}d\theta)$$

$$= \Gamma(u+v)\mathrm{B}(u,v).$$

Therefore

$$\mathrm{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

**Corollary: 2.3.4**

Let $\lambda$ , $\mu$ , $a$ and $b$ be complex numbers having positive real parts, then

$$(b-a)^{\mu+\lambda-1}B(\mu,\lambda) = \int_a^b (b-u)^{\lambda-1}(u-a)^{\mu-1}du.$$

**Proof:**

Suppose $\lambda$ , $\mu$ , $a$ and $b$ are complex numbers having positive real parts, and let $t = \dfrac{u-a}{b-a}$ then, $u = b$ when $t = 1$ and $u = a$ when $t = 0$, also $du = (b-a)dt$ . Hence

$$B(\mu,\lambda) = \int_0^1 (t)^{\mu-1}(1-t)^{\lambda-1}dt$$

$$= \int_a^b (\frac{u-a}{b-a})^{\mu-1}(1-\frac{u-a}{b-a})^{\lambda-1}\frac{du}{b-a}$$

$$= \int_a^b \frac{(u-a)^{\mu-1}}{(b-a)^{\mu-1}}\frac{(b-a-u+a)^{\lambda-1}}{(b-a)^{\lambda-1}}\cdot\frac{du}{b-a}$$

$$B(\mu,\lambda)=\frac{1}{(b-a)^{\mu+\lambda-1}}\int_a^b (b-u)^{\lambda-1}(u-a)^{\mu-1}du.$$

Therefore, we get

$$(b-a)^{\mu+\lambda-1}B(\mu,\lambda)=\int_a^b (b-u)^{\lambda-1}(u-a)^{\mu-1}du\,. \tag{2.7}$$

## 2.4 Fractional Integral

In this section we shall introduce the idea of fractional integral which called Riemann-Liouville fractional integral of order $\lambda$. We also consider some basic properties of the fractional integral. See [11].

**Definition 2.4.1**:

Let $f(z)$ be analytic function in $D$, then the fractional integral of order $\lambda$ is given by:

$$I^{\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(w)}{(z-w)^{1-\lambda}}dw,\quad \lambda>0\,. \tag{2.8}$$

**Theorem: 2.4.2**

Let $\lambda>0, \mu>-1, \mathrm{Re}(z)>0$ then,

$$I^{\lambda}z^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\lambda+1)}z^{\lambda+\mu}\,. \tag{2.9}$$

**Proof:**

By applying Equation (2.8) with $f(z)=z^{\mu}$ we get

$$I^{\lambda}z^{\mu}=\frac{1}{\Gamma(\lambda)}\int_0^z (z-t)^{\lambda-1}z^{\mu}dt,$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^z (1-\frac{t}{z})^{\lambda-1}z^{\lambda-1}z^{\lambda}dt.$$

Let $u=\frac{t}{z}$, then $dt=zdu$ and $u=1$ when $t=z$, $u=0$ when $t=0$. Therefore

$$I^{\lambda} z^{\mu} = \frac{1}{\Gamma(\lambda)} \int_0^1 (1-u)^{\lambda-1} z^{\lambda-1} (zu)^{\mu} z du$$

$$= \frac{1}{\Gamma(\lambda)} z^{\lambda+\mu} \int_0^1 (1-u)^{\lambda-1} (u)^{\mu} du .$$

Thus by Definition 2.3.1 we get

$$I^{\lambda} z^{\mu} = \frac{1}{\Gamma(\lambda)} z^{\lambda+\mu} \mathrm{B}(\mu+1, \lambda) .$$

Using Theorem 2.3.3 we obtain

$$I^{\lambda} z^{\mu} = \frac{1}{\Gamma(\lambda)} z^{\lambda+\mu} \frac{\Gamma(\mu+1)\Gamma(\lambda)}{\Gamma(\mu+1+\lambda)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\lambda+1)} z^{\lambda+\mu} .$$

**Example: 2.4.3**

In Equation (2.9), if $f(z) = k$ is constant then $\mu = 0$ and so,

$$I^{\lambda} k = \frac{\Gamma(0+1)}{\Gamma(0+\lambda+1)} z^{\lambda+0} = \frac{k}{\Gamma(\lambda+1)} z^{\lambda} .$$

In particular, if $\lambda = \frac{1}{2}$ then

For $\mu = 0$ we have,

$$I^{1/2} z^{0} = \frac{\Gamma(0+1)}{\Gamma(0+\frac{1}{2}+1)} z^{\frac{1}{2}+0} = \frac{1}{\Gamma(\frac{3}{2})} z^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} z^{\frac{1}{2}} = 2\sqrt{\frac{z}{\pi}} .$$

**Example: 2.4.4**

For $\mu = 1$ and if $\lambda = \frac{1}{2}$ then,

$$I^{1/2} z^{1} = \frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2}+1)} z^{\frac{1}{2}+1} = \frac{1}{\Gamma(\frac{5}{2})} z^{\frac{3}{2}} = \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} z^{\frac{3}{2}} = \frac{4}{3}\sqrt{\frac{z^3}{\pi}} .$$

For $\mu = 2$ and if $\lambda = \frac{1}{2}$ then,

$$I^{1/2} z^{2} = \frac{\Gamma(2+1)}{\Gamma(2+\frac{1}{2}+1)} z^{\frac{1}{2}+2} = \frac{\Gamma(3)}{\Gamma(\frac{7}{2})} z^{\frac{5}{2}} = \frac{2!}{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} z^{\frac{5}{2}} = \frac{16}{15}\sqrt{\frac{z^5}{\pi}}$$

In the next theorem we show that the fractional integral operator possesses the linear property.

**Theorem 2.4.5**

Let $f(z)$ and $g(z)$ be two analytic functions in $D$ and $c$ be a constant, then

1- $I^{\lambda}(cf(z)) = cI^{\lambda}(f(z))$.
2- $I^{\lambda}(f(z)+g(z)) = I^{\lambda}f(z)+I^{\lambda}g(z)$.

**Proof:**

To prove (1), we apply Definition 2.4.1 on the function $cf(z)$, so we get

$$I^{\lambda}cf(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{cf(w)}{(z-w)^{1-\lambda}}dw, \quad \lambda>0 .$$

Or we write

$$I^{\lambda}cf(z) = c\frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(w)}{(z-w)^{1-\lambda}}dw, \ \lambda>0$$

$$= cI^{\lambda}(f(z)).$$

Also, to prove (2), we apply Definition 2.4.1 on the function $f(z)+g(z)$ to get

$$I^{\lambda}(f(z)+g(z)) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(w)+g(w)}{(z-w)^{1-\lambda}}dw, \quad \lambda>0$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(w)}{(z-w)^{1-\lambda}}dw + \frac{1}{\Gamma(\lambda)}\int_0^z \frac{g(w)}{(z-w)^{1-\lambda}}dw$$

$$= I^{\lambda}f(z)+I^{\lambda}g(z).$$

Hence the fractional integral operator has the linear property.

**Theorem 2.4.6 (Dirichlet's Formula)**

Let $h(t,s)$ be jointly continuous and let $\mu$ and $\lambda$ be positive numbers. Then

$$\int_0^z (z-t)^{\mu-1}dt\int_0^t (t-s)^{\lambda-1}h(t,s)ds = \int_0^z ds\int_s^z (z-t)^{\mu-1}(t-s)^{\lambda-1}h(t,s)dt .$$

For more details, see [14].

The next corollary of Dirichlet's formula is of particular interest.

**Corollary 2.4.7:**

Let $h(t,s)$ be jointly continuous such that $h(t,s) = g(t)f(s), g(t) \equiv 1$ and let $\mu$ and $\lambda$ be positive numbers, then

$$\int_0^z (z-t)^{\mu-1} dt \int_0^t (t-s)^{\lambda-1} f(s)ds = B(\mu,\lambda)\int_0^z (z-s)^{\mu+\lambda-1} f(s)ds .$$

**Proof:**

Suppose $h(t,s)$ is jointly continuous and $h(t,s) = g(t)f(s), g(t) \equiv 1$ , and let $\mu$ and $\lambda$ be positive numbers, then

$$\int_0^z (z-t)^{\mu-1} dt \int_0^t (t-s)^{\lambda-1} h(t,s)ds = \int_0^z (z-t)^{\mu-1} dt \int_0^t (t-s)^{\lambda-1} g(t)f(s)ds$$
$$= \int_0^z (z-t)^{\mu-1} dt \int_0^t (t-s)^{\lambda-1} f(s)ds.$$

By Theorem 2.4.6, the last integral is equal to

$$\int_0^z ds \int_s^z (z-t)^{\mu-1}(t-s)^{\lambda-1} f(s)dt. \qquad (1)$$

Applying Equation (2.6) with $b = z$ and $a = s$ we get

$$(z-s)^{\mu+\lambda-1} B(\mu,\lambda) = \int_s^z (z-t)^{\lambda-1}(t-s)^{\mu-1} dt . \qquad (2)$$

By substituting (2) in (1) we have

$$\int_0^z ds \int_s^z (z-t)^{\mu-1}(t-s)^{\lambda-1} f(s)dt = \int_0^z (z-s)^{\mu+\lambda-1} B(\mu,\lambda) f(s)ds .$$

Therefore,

$$\int_0^z (z-t)^{\mu-1} dt \int_0^t (t-s)^{\lambda-1} f(s)ds = B(\mu,\lambda)\int_0^z (z-s)^{\mu+\lambda-1} f(s)ds .$$

Now, in the next theorem we shall use the previous corollary in order to help us to prove the law of exponents for fractional integrals.

**Theorem 2.4.8:**

Let $f(z)$ be analytic in $D$ with $\operatorname{Re}(z) > 0$ and let $\mu, \lambda > 0$. then

$$I^{\mu}[I^{\lambda} f(z)] = I^{(\mu+\lambda)} f(z) = I^{\lambda}[I^{\mu} f(z)]. \tag{2.10}$$

**Proof:**

By the definition of fractional integral we have

$$I^{\mu}[I^{\lambda} f(z)] = \frac{1}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} I^{\lambda} f(t) dt$$

$$= \frac{1}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} \left( \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s) ds \right) dt$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\lambda)} \int_0^z (z-t)^{\mu-1} dt \left( \int_0^t (t-s)^{\lambda-1} f(s) ds \right).$$

By Corollary 2.4.6 we get

$$I^{\mu}[I^{\lambda} f(z)] = \frac{1}{\Gamma(\mu)\Gamma(\lambda)} B(\mu,\lambda) \int_0^z (z-s)^{\mu+\lambda-1} f(s) ds$$

$$= \frac{1}{\Gamma(\mu+\lambda)} \int_0^z (z-s)^{\mu+\lambda-1} f(s) ds.$$

Also we have

$$I^{(\mu+\lambda)} f(z) = \frac{1}{\Gamma(\mu+\lambda)} \int_0^z (z-s)^{\mu+\lambda-1} f(s) ds .$$

Therefore, the following result holds

$$I^{\mu}[I^{\lambda} f(z)] = I^{(\mu+\lambda)} f(z) = I^{\lambda}[I^{\mu} f(z)].$$

Next we introduce the Leibniz's rule which helps us to determine if the derivative of fractional integral equal or not equal the fractional integral of the derivative.

**Theorem 2.4.9: (Leibniz's Rule)**

The derivative of integral of the form

$$\int_0^{b(z)} f(z,t)dt$$

expressed as

$$\frac{d}{dz}(\int_0^{b(z)} f(z,t)dt) = f(z,b(z))b'(z) + \int_0^{b(z)} \frac{\partial}{\partial z} f(z,t)dt.$$

For more details, see [14].

The following theorem generally, shows that the derivative of fractional integral is not equal the fractional integral of the derivative.

**Theorem 2.4.10:**

Let $f(z)$ and $D(f(z))$ be analytic in $D$ with $\mathrm{Re}(z) > 0$, then

$$D[I^v f(z)] = I^v[Df(z)] + \frac{f(0)}{\Gamma(v)} z^{v-1}, \quad v > 0. \tag{2.11}$$

**Proof:**

By Definition (2.4.1) of fractional integral let us make the substitution $t = z - \zeta^\lambda$, and $\lambda = 1/v$, then $dt = -\lambda\zeta^{\lambda-1}d\zeta$, when $t = 0$ then $\zeta = (z-t)^v$ implies $\zeta = z^v$, and when $t = z$ then $\zeta = 0$, so the equation (2.8) becomes

$$I^v f(z) = \frac{1}{\Gamma(v)} \int_{z^v}^{0} (\zeta^\lambda)^{v-1} f(z-\zeta^\lambda)(-\lambda\zeta^{\lambda-1})d\zeta .$$

By Applying properties of integral, we get

$$I^v f(z) = \frac{\lambda}{\Gamma(v)} \int_0^{z^v} \zeta^{\lambda v-\lambda} f(z-\zeta^\lambda)\zeta^{\lambda-1}d\zeta,$$

but $\lambda = 1/v$, so we have $I^v f(z) = \frac{1}{\Gamma(v+1)} \int_0^{z^v} f(z-\zeta^\lambda)d\zeta$ .Applying Theorem 2.4.9 we get

$$\frac{d}{dz}(I^v f(z)) = \frac{1}{\Gamma(v+1)} \frac{d}{dz} \int_0^{z^v} f(z-\zeta^\lambda)d\zeta$$

$$\frac{d}{dz}(I^{v}f(z)) = \frac{1}{\Gamma(v+1)}[f(z-(z^{v})^{\lambda})\frac{d}{dz}z^{v} + \int_{0}^{z^{v}} \frac{\partial}{\partial z} f(z-\zeta^{\lambda})d\zeta]$$

$$= \frac{1}{\Gamma(v+1)}[f(0)(vz^{v-1}) + \int_{0}^{z^{v}} \frac{\partial}{\partial z} f(z-\zeta^{\lambda})d\zeta].$$

Now, if we reverse our substitution $z-\zeta^{\lambda}=t$ we obtain

$$D[I^{v}f(z)] = \frac{1}{v\Gamma(v)} f(0)vt^{v-1} + \frac{1}{v\Gamma(v)} \int_{z}^{o} \frac{\partial}{\partial t} f(t)(-\frac{1}{\lambda}\zeta^{1-\lambda})dt.$$

Since $\zeta = (z-t)^{1/v}$, where $\lambda = 1/v$, the preceding equation simplifies to

$$D[I^{v}f(z)] = \frac{1}{\Gamma(v)} f(0)z^{v-1} + \frac{1}{\Gamma(v)} \int_{0}^{z} (z-t)^{v-1} \frac{\partial}{\partial t} f(t)dt.$$

So the result $D[I^{v}f(z)] = I^{v}[Df(z)] + \frac{1}{\Gamma(v)} f(0)z^{v-1}$ holds.

Therefore, generally we

$$D[I^{v}f(z)] = I^{v}[Df(z)] + \frac{1}{\Gamma(v)} f(0)z^{v-1}.$$

## 2.5 Fractional Derivative

In the previous section, we introduced the notation of the fractional integral $I^{v}f(z)$. In this section we use the notation of the fractional integral $I^{v}f(z)$ in order to introduce the notation of the fractional derivative of a function $f(z)$ of an arbitrary order which is denoted $D^{\lambda}f(z)$.

The following definition is called Riemann-Liouville Fractional Derivative which is defined using Definition (2.4.1) of the fractional integral, see [12].

**Definition 2.5.1**

Suppose that $n=u+v$, where $n$ is the smallest integer greater than $u$, then the fractional derivative of analytic function $f(z)$ of order $u$ is

$$D^{u}f(z) = D^{n}[I^{v}f(z)]. \quad (2.12)$$

In the Next theorem we define the fraction derivative in another way.

**Theorem 2.5.2**

Let $f(z)$ be analytic function in $D$, the fractional derivative of $f(z)$ given by

$$D^{\lambda} f(z)=\frac{d}{dz} I^{1-\lambda} f(z)=\frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(w)}{(z-w)^{\lambda}} dw, \quad 0 \leq \lambda<1.$$

**Proof:**

In order to use Definition 2.5.1, let $n=1$ then we get

$$D^{\lambda} f(z)=D^{1}[I^{(1-\lambda)} f(z)].$$

But by Definition 2.4.1

$$I^{(1-\lambda)} f(z)=\frac{1}{\Gamma(1-\lambda)} \int_{0}^{z} \frac{f(w)}{(z-w)^{\lambda}} dw, \quad \lambda>0.$$

Thus,

$$D^{\lambda} f(z)=D^{1}[\frac{1}{\Gamma(1-\lambda)} \int_{0}^{z} \frac{f(w)}{(z-w)^{\lambda}} dw]$$

$$=\frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(w)}{(z-w)^{\lambda}} dw, \quad 0 \leq \lambda<1.$$

**Theorem 2.5.3**:

The fractional derivative of $f(z)=z^{\mu}$ of order $\lambda$, where $\mu \geq 0$ is given by

$$D^{\lambda} z^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)} z^{\mu-\lambda}.$$

**Proof:**

In Definition 2.5.1, let $n=1$ and $f(z)=z^{\mu}$ we get

$$D^{\lambda} z^{\mu}=D^{1}[I^{(1-\lambda)} z^{\mu}],$$

and by Theorem 2.4.2

$$D^{\lambda} z^{\mu} = D^{1}[\frac{\Gamma(\mu+1)}{\Gamma((\mu-\lambda+1)+1)} z^{\mu-\lambda+1}]$$

$$= (\mu-\lambda+1)\frac{\Gamma(\mu+1)}{(\mu-\lambda+1)\Gamma(\mu-\lambda+1)} z^{\mu-\lambda}$$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)} z^{\mu-\lambda}. \qquad (2.13)$$

**Example 2.5.4:**

Use Theorem (2.5.3) to find the $(\frac{1}{2})^{th}$ order derivative of $f(z)=z^{\mu}$ for $\mu = 0,1,2$.

**Solution:**

1- $D^{1/2} z^{0} = \frac{\Gamma(0+1)}{\Gamma(0-(\frac{1}{2})+1)} z^{0-\frac{1}{2}} = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} z^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi z}}.$

2- $D^{1/2} z^{1} = \frac{\Gamma(1+1)}{\Gamma(1-(\frac{1}{2})+1)} z^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} z^{\frac{1}{2}} = 2\frac{\sqrt{z}}{\sqrt{\pi}}.$

3- $D^{1/2} z^{2} = \frac{\Gamma(2+1)}{\Gamma(2-(\frac{1}{2})+1)} z^{2-\frac{1}{2}} = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})} z^{\frac{3}{2}} = \frac{8\sqrt{z^{3}}}{3\sqrt{\pi}}.$

Under the hypothesis of fractional derivative, the fractional derivative of order $(n+\lambda)$ for analytic function $f(z)$ defined by:

$$D^{n+\lambda} f(z) = \frac{d^{n}}{dz^{n}} D^{\lambda} f(z), \quad 0 \le \lambda < 1, \ n \in \{0,1,2,.....\} \qquad (2.14)$$

and

$$D^{n+\lambda} z^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-n-\lambda}, \quad 0 \le \lambda < 1. \qquad (2.15)$$

Where $n \in \{0,1,2,.....\}$, $k-n \neq -1,-2,-3,......$

Therefore, for any positive real number $\lambda$,

$$D^{\lambda} z^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-1}, \quad \lambda>0. \qquad (2.16)$$

As a fraction integral, next theorem shows that the fractional derivative satisfies the linear properties.

**Theorem 2.5.5:**

Let $f(z)$ and $g(z)$ be two analytic functions in $D$ and let $c$ be a constant, then

1- $D^{\lambda}(cf(z)) = cD^{\lambda}(f(z))$.

2- $D^{\lambda}(f(z)+g(z)) = D^{\lambda}f(z)+D^{\lambda}g(z)$.

**Proof:**

To prove (1) we use Definition 2.5.1 and Theorem (2.5.2), thus

$$D^{\lambda}(cf(z)) = \frac{d}{dz}I^{1-\lambda}(cf(z)) = \frac{1}{\Gamma(1-\lambda}\frac{d}{dz}\int_0^z \frac{cf(w)}{(z-w)^{\lambda}}dw, \quad 0 \le \lambda < 1.$$

By Theorem 2.4.5 part (1) we get

$$= c\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(w)}{(z-w)^{\lambda}}dw$$

$$= c\frac{d}{dz}I^{1-\lambda}(f(z)) = cD^{\lambda}(f(z))..$$

Also, to prove (2) we use Definition 2.5.1 and Theorem (2.5.2), thus

$$D^{\lambda}(f(z)+g(z)) = \frac{d}{dz}I^{\lambda}(f(z)+g(z))$$

$$= \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{(f(w)+g(w))}{(z-w)^{\lambda}}dw, 0 \le \lambda \le 1.$$

Now, by Theorem 2.4.5 part (2) we get

$$= \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(w)}{(z-w)^{\lambda}}dw + \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{g(w)}{(z-w)^{\lambda}}dw.$$

That is,

$$= \frac{d}{dz}I^{\lambda}f(z)+\frac{d}{dz}I^{\lambda}g(z) = D^{\lambda}f(z)+D^{\lambda}g(z).$$

And so the fractional derivative operator has the linear property.

## 2.6 $\lambda$-Fractional Operator

Fractional operators play important role in generalization some notation. In 2013, M. Aydogan, Y. Kahramaner and Y. Polatoglu defined $\lambda$-fractional operator, in order to generalize a special subclass of family S, see [2].

**Definition 2.6.1**:

Let $f(z)$ be analytic function and normalized by $f(0)=0$ and $f'(0)=1$, then $\lambda$-fractional operator for analytic function $f(z)$ in $D$ defined by

$$O^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D^{\lambda} f(z).$$

By using the relation (2.16) of fractional derivative and by Theorem 2.5.5 of linearity of fractional derivative, we express $\lambda$-fractional operator in terms of power series as in the following theorem.

**Theorem 2.6.2**:

Let $f(z)$ be analytic function in $D$ and normalized by $f(0)=0$ and $f'(0)=1$, then

$$O^{\lambda} f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n .$$

**Proof:**

Suppose $f(z)$ is analytic function in $D$ and normalized by $f(0)=0$ and $f'(0)=1$, then $f(z)$ from (1.2) has a Taylor series expansion, thus $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

By linearity of fractional derivatives, we have

$$D^{\lambda}(f(z)) = D^{\lambda}(z + \sum_{n=2}^{\infty} a_n z^n)$$

$$= D^{\lambda} z + D^{\lambda} \sum_{n=2}^{\infty} a_n z^n$$

$$= D^{\lambda} z + \sum_{n=2}^{\infty} a_n D^{\lambda} z^n ,$$

then by (2.16) we have

$$D^{\lambda} f(z) = \frac{\Gamma(1+1)}{\Gamma(1+1-\lambda)} z^{1-\lambda} + \frac{\Gamma(2+1)}{\Gamma(2+1-\lambda)} a_2 z^{2-\lambda} + ......+ \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} + .....$$

$$D^{\lambda} f(z) = \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + \frac{\Gamma(3)}{\Gamma(3-\lambda)} a_2 z^{2-\lambda} + ..........+ \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} + ........$$

Multiply both sides by $\Gamma(2-\lambda)\, z^{\lambda}$ we have,

$$\Gamma(2-\lambda) z^{\lambda} D^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + \Gamma(2-\lambda) z^{\lambda} \frac{\Gamma(3)}{\Gamma(3-\lambda)} a_2 z^{2-\lambda} + ...$$
$$+ \Gamma(2-\lambda) z^{\lambda} \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} + ...$$

By Definition 2.6.1 and since $\Gamma(2) = 1\,\Gamma(1)$ we get,

$$O^{\lambda} f(z) = z + \Gamma(2-\lambda) \frac{\Gamma(3)}{\Gamma(3-\lambda)} a_2 z^2 + ... + \Gamma(2-\lambda) \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n + ...$$

Thus

$$O^{\lambda} f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n .$$

By applying Definition 2.6.1 and theorem 2.6.2, we drive some properties of $\lambda$ -fractional operator as stated in the following theorem.

**Theorem 2.6.3**

Let $f(z)$ be analytic function and normalized by $f(0) = 0$ and $f'(0) = 1$, Then:

1- $O f(z) = \lim_{\lambda \to 1} O^{\lambda} f(z) = z f'(z)$.

2- $O^{\lambda}(O^{\delta} f(z)) = O^{\delta}(O^{\lambda} f(z)) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} z^n$.

3- $O(O^{\delta} f(z)) = z\,(O^{\delta} f(z))'$.

4- $O(O^{\lambda} f(z)) = \Gamma(2-\lambda) z^{\lambda} (\lambda D^{\lambda} f(z) + z D^{\lambda+1} f(z))$.

5- If $\lambda = 0$ then

$$\frac{O(O^{\lambda} f(z))}{O^{\lambda} f(z)} = \frac{z f'(z)}{f(z)},$$

and if $\lambda = 1$ then

$$\frac{O(O^{\lambda} f(z))}{O^{\lambda} f(z)} = 1 + \frac{z f''(z)}{f'(z)} .$$

**Proof**:

Suppose $f(z)$ is analytic function and normalized by $f(0)=0$ and $f'(0)=1$, to prove (1) we use Definition 2.6.1 of $\lambda$-fractional operator $O^{\lambda} f(z)=\Gamma(2-\lambda)z^{\lambda}D^{\lambda}f(z)$, thus

$$\lim_{\lambda\to 1} O^{\lambda} f(z)=\Gamma(2-1)z^{1}D^{1}f(z)$$

$$=\Gamma(1)z f'(z),$$

therefore, we get

$$O f(z)=\lim_{\lambda\to 1} O^{\lambda} f(z)=z f'(z).$$

To prove (2) we substitute $\lambda=\delta$ in Theorem 2.6.2 we get

$$O^{\delta} f(z)=z+\sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)} z^n.$$

Now let $a'_n=a_n\dfrac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}$ and $b'_n=a_n\dfrac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)}$, thus by Theorem 2.6.2 we have

$$O^{\delta}(O^{\lambda} f(z))=z+\sum_{n=2}^{\infty} a'_n \frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)} z^n$$

$$=z+\sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)} z^n.$$

Hence

$$O^{\lambda}(O^{\delta} f(z))=z+\sum_{n=2}^{\infty} b'_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n$$

$$=z+\sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n.$$

Therefore

$$O^{\lambda}(O^{\delta} f(z))=O^{\delta}(O^{\lambda} f(z))=z+\sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} z^n.$$

To prove (3), we apply the first part of Theorem 2.6.3 on Theorem 2.6.2 so we have

$$O(O^{\delta} f(z))=z\,(O^{\delta} f(z))'=z(1+\sum_{n=2}^{\infty} n a_n \frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n-1}),$$

therefore

$$O(O^{\delta} f(z)) = z\ (O^{\delta} f(z))' = z + \sum_{n=2}^{\infty} na_n \frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)} z^n .$$

To prove (4), we apply third part of Theorem 2.6.3 on the Definition 2.6.1 to get,

$$O(O^{\lambda} f(z)) = z(O^{\lambda} f(z))' = z(\Gamma(2-\lambda)z^{\lambda} D^{\lambda} f(z))',$$

$$= z(\Gamma(2-\lambda)(z^{\lambda} D(D^{\lambda} f(z)) + \lambda z^{\lambda-1} D^{\lambda} f(z)).$$

Now, by relation (2.14) we get

$$= z^{\lambda}\Gamma(2-\lambda)(zD^{\lambda+1} f(z) + \lambda D^{\lambda} f(z)),$$

therefore

$$O(O^{\lambda} f(z)) = \Gamma(2-\lambda)z^{\lambda}(\lambda D^{\lambda} f(z) + zD^{\lambda+1} f(z)).$$

Finally, To prove (5), we substitute $\lambda = 0$ and apply first part of Theorem 2.6.3, thus

$$\frac{O(O^{\lambda} f(z))}{O^{\lambda} f(z)} = \frac{O(O^{0} f(z))}{O^{0} f(z)} = \frac{O(f(z))}{f(z)} = \frac{z f'(z)}{f(z)} .$$

we substitute $\lambda = 1$ then

$$\frac{O(O^{\lambda} f(z))}{O^{\lambda} f(z)} = \frac{O(O^{1} f(z))}{O^{1} f(z)} = \frac{O(O\ f(z))}{O\ f(z)} .$$

Now by first part of Theorem 2.6.3 we have

$$\frac{O(z f'(z))}{z f'(z)} = \frac{z(z f'(z))'}{z f'(z)} = \frac{z(f'(z) + z f''(z))}{z f'(z)}$$

$$= \frac{z f'(z) + z^2 f''(z)}{z f'(z)} = 1 + \frac{z\ f''(z)}{f'(z)} .$$

We showed that the fractional integral and fractional derivative satisfy the linear property in Theorems 2.4.5 and 2.5.5 respectively. Now we show also the fractional operator satisfies the linear property.

**Theorem 2.6.4**

Let $f(z)$ and $g(z)$ be two analytic functions in $D$ and let $a$ and $b$ be two constants, then

$$O^{\lambda}(af(z) + bg(z)) = aO^{\lambda} f(z) + bO^{\lambda} g(z) .$$

**Proof:**

Suppose $f(z)$ and $g(z)$ are two analytic functions in $D$ and let $a$ and $b$ be two constants, so applying Definition (2.6.1) we get

$$O^{\lambda}(a\,f(z)+b\,g(z))=\Gamma(2-\lambda)z^{\lambda}D^{\lambda}(a\,f(z)+b\,g(z)).$$

Now, by linearity of fractional derivative we have

$$\begin{aligned} O^{\lambda}(a\,f(z)+b\,g(z)) &= \Gamma(2-\lambda)z^{\lambda}(a\,D^{\lambda}f(z)+b\,D^{\lambda}g(z)) \\ &= a\,\Gamma(2-\lambda)z^{\lambda}\,D^{\lambda}f(z)+b\,\Gamma(2-\lambda)z^{\lambda}\,D^{\lambda}g(z)) \\ &= aO^{\lambda}f(z)+bO^{\lambda}g(z), \end{aligned}$$

and so the fractional operator has the linear property.

# Chapter Three

## Generalized Close to Convex Functions

A class of analytic close to convex functions in the open unit disk which are normalized by $f(0)=0$ and $f'(0)=1$ is a subclass of the family S defined in Chapter one. In this chapter we introduce some definitions and properties of the class of close to convex functions. Then we present $\lambda$ - fractional close to convex function as a generalization to the close to convex function using the $\lambda-$fractional operator that is defined in chapter two.

### 3.1 Close to Convex Functions

In 1952, Wilfred Kaplan introduced the class of close to convex univalent functions, when he said:

If $g(z)$ is a convex univalent function for $|z|<\mathrm{R}$, and $f(z)$ is analytic function for $|z|<\mathrm{R}$ such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right)>0, \qquad |z|<\mathrm{R}$$

Then $f(z)$ is also univalent for $|z|<\mathrm{R}$. See [10].

After that he defined the subclass of close to convex function by the following definition.

**Definition 3.1.1:**

Let $f(z)$ be analytic for $|z|<\mathrm{R}$, then $f(z)$ is close to convex for $|z|<\mathrm{R}$, if there exists a function $g(z)$ convex and univalent for $|z|<\mathrm{R}$, such that $f'(z)/g'(z)$ has positive real part for $|z| < R$. Kaplan denote this classes of functions by close to convex functions, Since the vectors $f'$ and $g'$ never differ in direction by more than $90^o$, see [10].

When $R = 1$, it will be convenient to omit reference to the circular domain of definition. Therefore, a close to convex function will mean a function which is close to convex in the open unit disk $\Delta$.

In 1977 A.W. Goodman and E.B. Staff, explain that one can begin with more general expression, but any additive constants disappear on differentiation and hence may be drooped at the beginning.

Furthermore, there is no less of generality in assuming that $f'(0) = 1$ and $g'(0) = e^{i\alpha}$. Here it seems natural to set $g'(0) = 1$, but there are several places in the literature where this second normalization is expressly forbidden. The reason for maintaining $e^{i\alpha}$ is never explicitly given. Therefore the class of close to convex is naturally divided into subclasses in according with the value of $\alpha$ denoted by:

$$\operatorname{Re}\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0, \qquad \text{for } |z| < 1.$$

Where now, the subclasses of close to convex exhaustive, but they are certainly not mutually exclusive. Thus if $f(z)$ is itself convex then we may take $g(z) = f(z)$. Hence a convex function is close to convex function for every $\alpha \in (-\pi/2, \pi/2)$.

In fact, it is easy to show using a normal families argument that the intersection of all close to convex with respect to $\alpha$, is precisely the collection of all normalized convex function. see [9].

Since our interest is to study the analytic and univalent functions normalized by $f(0) = 0$ and $f'(0) = 1$, that is functions in the family S, therefore we define the close to convex function as follows.

**Definition 3.1.2:**

An analytic function $f(z)$ in the unit disk normalized by $f(0) = 0$ and $f'(0) = 1$ is said to be close to convex if there is a convex function $g$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)>0 \qquad \text{for all } z\in\Delta \tag{3.1}$$

The class of all close to convex functions in the open unit disk is a subclass of family $S$ and denoted by $K$, see [6]. That is

$$K=\{f(z): f(z) \text{ close to convex on } \Delta\}.$$

**Example 3.1.3:**

Let $f:\Delta\to\mathbb{C}$ such that $f(z)=z-\frac{1}{2}z^2$, show that $f(z)$ is close to convex in $\Delta$ .

**Solution:**

In Example 1.2.2 we show that $f(z)$ belongs to family S. That is, it is analytic, univalent function and normalized by $f(0)=0$ and $f'(0)=1$, in order to show that $f(z)$ is close to convex in $\Delta$ , we apply Definition 3.1.2, so we choose $g(z)=z$ which is a convex function, thus if $z=x+iy$ in $\Delta$ , then

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)=\operatorname{Re}\left(\frac{1-z}{1}\right)=\operatorname{Re}(1-x-iy)=1-x..$$

Since $|z|<1$, we get

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)=1-x>0 .$$

Therefore, $f(z)$ satisfies Definition 3.1.2, hence $f(z)=z-\frac{1}{2}z^2$ is close to convex in $\Delta$ .

In Chapter One, we show that every convex function is starlike function. Thus someone can ask, could we define the close to convex functions through starlike function? The answer is the following definition.

**Definition 3.1.4:**

A function $f(z)$ of the form $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is said to be close to convex in the open unit disk $\Delta$ , if there is a starlike function $g(z)$ such that

$$\operatorname{Re}\left(\frac{z f'(z)}{g(z)}\right) > 0 \quad \text{for all} \quad z \in \Delta . \qquad (3.2)$$

For more details, see [8].

**Example 3.1.5:**

Let $K : \Delta \to \mathbb{C}$ such that $K(z) = \dfrac{z}{(1-z)^2}$ , show that Keobe function $K(z)$ belongs to the class $K$ .

**Solution:**

To show that $K(z)$ is close to convex in $\Delta$ , we use Definition 3.1.4. In order to apply Definition 3.1.4, we want a starlike function to complete our solution. In Example 1.6.5 we show that Keobe function itself is a starlike function. That it is analytic, univalent, normalized by $K(0) = 0, K'(0) = 1$ and satisfy Theorem 1.6.4.

To end our problem, we only need to satisfy Equation (3.2), so take $g(z) = K(z)$, then $\operatorname{Re}\left(\dfrac{z K'(z)}{g(z)}\right) = \operatorname{Re}\left(\dfrac{1+z}{1-z}\right) > 0$. But in Example 1.6.2 we show that $\varphi(z) = \dfrac{1+z}{1-z}$ belongs to family $P$ . That is , $\operatorname{Re}\varphi(z) = \operatorname{Re}\dfrac{1+z}{1-z} > 0$ so $\operatorname{Re}\left(\dfrac{z K'(z)}{g(z)}\right) = \operatorname{Re}\left(\dfrac{1+z}{1-z}\right) > 0$. Hence, Koebe function is close to convex in $\Delta$ .

**Theorem 3.1.6:**

let $f(z) \in K$ then, there is a convex function $g(z)$ such that $\operatorname{Re}\left(\dfrac{f'(z)}{g'(z)}\right) > 0$ if and only if there is a starlike function $h(z)$ such that $\operatorname{Re}\left(\dfrac{z f'(z)}{h(z)}\right) > 0$, for all $z \in \Delta$ .

**Proof:**

$(\Rightarrow)$ Suppose $f(z) \in K$ and there is a convex function $g(z)$ in $C$ such that $\operatorname{Re}\left(\dfrac{f'(z)}{g'(z)}\right) > 0$ for all $z \in \Delta$ . But by Theorem (1.6.9) $g(z) \in C \Leftrightarrow zg'(z) \in S^*$ , let $h(z) = zg'(z)$ then $g'(z) = z^{-1}h(z)$ so

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)=\operatorname{Re}\left(\frac{f'(z)}{z^{-1}h(z)}\right)=\operatorname{Re}\left(\frac{z\,f'(z)}{h(z)}\right)>0.$$

Hence we find a starlike function $h(z)$ such that $\operatorname{Re}\left(\frac{z\,f'(z)}{h(z)}\right)>0$, for all $z\in\Delta$.

($\Leftarrow$) Suppose $f(z)\in K$ and there is a starlike function $h(z)$ in $S^*$ such that $\operatorname{Re}\left(\frac{z\,f'(z)}{h(z)}\right)>0$ for all $z\in\Delta$. Define the function $g$ by

$$g(z)=\int_0^z\frac{h(t)}{t}\,dt.$$

Since $h(z)\in S^*$ starlike function normalized by $h(0)=0$ and $h'(0)=1$, then

$$\lim_{t\to 0}\frac{h(t)}{t}=\lim_{t\to 0}\frac{h(t)-h(0)}{t-0}=h'(0)=1,$$

hence

$$g(z)=\int_0^z\frac{h(t)}{t}\,dt \text{ is defined for } z\in\Delta.$$

Also $g'(z)=\frac{h(z)}{z}$, and so $h(z)=z\,g'(z)$, thus by Theorem 1.6.9, $g(z)\in C$. Thus

$$\operatorname{Re}\left(\frac{z\,f'(z)}{h(z)}\right)=\operatorname{Re}\left(\frac{z\,f'(z)}{z\,g'(z)}\right)=\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)>0.$$

Therefore, there is a convex function $g(z)$ such that $\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)>0$ for all $z\in\Delta$.

**Theorem 3.1.7:**

Every close to convex function in $\Delta$ is univalent function in the open unit disk $\Delta$.

**Proof:**

Suppose $f(z)$ is close to convex in $\Delta$, then by Definition 3.1.2 for some convex function $g(z)$ we have $\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)>0$, for all $z\in\Delta$.

Now, let $\Omega$ be the range of $g(z)$ and consider the function

$$h(w) = f(g^{-1}(w)), \qquad w \in \Omega.$$

Then

$$h'(w) = \frac{f'(g^{-1}(w))}{g'(g^{-1}(w))} = \frac{f'(z)}{g'(z)},$$

hence

$$\mathrm{Re}\{h'(w)\} = \mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0.$$

Therefore by Theorem 1.1.5 $h(w)$ is univalent. Hence $f(z)$ is univalent, see [6].

**Theorem 3.1.8:**

If $f(z)$ is a convex function in $\Delta$, then $f(z)$ is close to convex function.

**Proof:**

Suppose $f(z)$ is any convex function in the open unit disk $\Delta$. To show that $f(z)$ is close to convex function in $\Delta$, we use Definition 3.1.2, by choosing $\phi(z) = f(z)$ thus

$$\phi'(z) = f'(z) \text{ and } \mathrm{Re}\left(\frac{f'(z)}{\phi'(z)}\right) = \mathrm{Re}\left(\frac{f'(z)}{f'(z)}\right) = 1 > 0.$$

Since $f(z)$ is convex on the open unit disk $\Delta$, then $f(z)$ is analytic, univalent, normalized by $f(0) = 0$ and $f'(0) = 1$, and satisfies Definition 3.1.2, hence $f(z)$ is close to convex function. See [17].

**Theorem 3.1.9:**

If $f(z)$ is a starlike function in $\Delta$, then $f(z)$ is close to convex function.

**Proof:**

Suppose $f(z)$ is any starlike function. Choose $\phi(z) = \int_0^z \frac{f(t)}{t} .dt$ .Since $f(z) \in S^*$ starlike function normalized by $f(0) = 0$ and $f'(0) = 1$, then

$$\lim_{t\to 0}\frac{f(t)}{t}=\lim_{t\to 0} f'(t)=f'(0)\neq 0 .$$

Hence,

$$\phi(z)=\int_0^z \frac{f(t)}{t}.dt \text{ exists for } z\in\Delta .$$

Also $\phi'(z)=\dfrac{f(z)}{z}$, thus by Theorem 1.6.9 $f(z)=z\phi'(z)$ implies that $\phi(z)\in C$ .

Now apply Definition 3.1.2 we get,

$$\operatorname{Re}\left(\frac{f'(z)}{\varphi'(z)}\right)=\operatorname{Re}\left(\frac{z\varphi''(z)+\varphi'(z)}{\varphi'(z)}\right)=\operatorname{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) \quad \text{for all } z\in\Delta$$

But by Theorem 1.6.7

$$\varphi(z)\in C \Leftrightarrow \left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)\in P$$

hence by Definition 1.6.1

$$\operatorname{Re}\left(\frac{f'(z)}{\varphi'(z)}\right)=\operatorname{Re}\left(\frac{z\varphi''(z)+\varphi'(z)}{\varphi'(z)}\right)=\operatorname{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)>0$$

Therefore so $f(z)$ is close to convex in $\Delta$.

By definitions of family of convex functions, family of starlike functions, family S and the last two theorems we conclude the following relation

$$C\subset S^{*}\subset K\subset S.$$

## 3.2 $\lambda$-Fractional Close to Convex Functions

In 2013, M. Aydogan, Y. Kahramaner and Y. Polatoglu defined $\lambda$-fractional operator to generalize the close to convex function and denoted the new close to convex function by $\lambda$-fractional close to convex function. See [2].

**Definition 3.2.1**:

If $f(z)=\sum_{n=1}^{\infty} a_n z^n, a_1=1$, is an element of $S$, then $f(z)$ is said to be $\lambda$-fractional close to convex function in $\Delta$, if there exist a function $g(z)$ of $S^{*}$ such that

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right) > 0, \qquad \text{for all } z \in \Delta .$$

The class $\lambda$ -fractional close to convex functions in $\Delta$ is denoted by $K(\lambda)$.

We notice that, Aydogan, Y. Kahramaner and Y. Polatoglu, define $\lambda$ -fractional close to convex in Definition 3.2.1 to generalize the close to convex function according to Definition 3.1.4.

But, we know that Definition 3.1.4 is equivalent to Definition 3.1.2, and by Alexander Theorem 1.6.9, we know that

$$g(z) \in C \Leftrightarrow z\, g'(z) \in S^*$$

Therefore we can redefine $\lambda$ -fractional close to convex function by the following theorem which is in fact equivalent to Definition 3.2.1.

**Theorem 3.2.2:**

If $f(z) = \sum_{n=1}^{\infty} a_n\, z^n, a_1 = 1$, be an element of $S$, and if there exist a function $h(z)$ in $C$ such that

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{O(h(z))}\right) > 0, \qquad \text{for all } z \in \Delta$$

Then $f(z)$ is $\lambda$ -fractional close to convex function in $\Delta$ .

**Proof:**

Let $f(z) = \sum_{n=1}^{\infty} a_n\, z^n, a_1 = 1$ be an element of $S$, and suppose there exist a function $h(z)$ in $C$ such that

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{O(h(z))}\right) > 0, \qquad \text{for all } z \in \Delta .$$

Thus by Theorem 2.6.3 part (1), we get

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{O(h(z))}\right) = \operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{z(h(z))'}\right) > 0.$$

Suppose $g(z) = z\,h'(z)$, by Theorem 1.6.9 $h(z) \in C \Leftrightarrow z\,h'(z) \in S^*$, so $g(z)$ is starlike function, hence there exist a function $g(z)$ in $S^*$ such that

$$\mathrm{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right) > 0, \text{ for all } z \in \Delta .$$

Therefore $f(z)$ satisfies Definition 3.2.1 and so it belongs to the family of $\lambda$-fractional close to convex function in $\Delta$.

**Example 3.2.3:**

Let $f : \Delta \to \mathbb{C}$ such that $f(z) = z - \frac{1}{8}z^2$, show that $f(z) \in K\left(\frac{1}{2}\right)$.

**Solution:**

To show that $f(z) \in K\left(\frac{1}{2}\right)$ we must show that $f(z)$ satisfies Definition 3.2.1. Since $f(z)$ is polynomial so it is analytic everywhere, and so it is analytic in $\Delta$, furthermore $f'(z) = 1 - \frac{1}{4}z$, then by Theorem 1.1.5 $f(z)$ is univalent since $\mathrm{Re}\{f(z)\} = 1 - \frac{1}{4}x > 0$, and clearly $f(z)$ normalized by $f(0) = 0$ and $f'(0) = 1$.

Now, by Definition 2.6.1 we get

$$O^{\frac{1}{2}} f(z) = O^{\frac{1}{2}}\left(z - \frac{1}{8}z^2\right) = \Gamma(2 - \frac{1}{2}) z^{\frac{1}{2}} D^{\frac{1}{2}}\left(z - \frac{1}{8}z^2\right)$$

$$= \Gamma(2 - \frac{1}{2})\, z^{\frac{1}{2}}\left(\frac{2\sqrt{z}}{\sqrt{\pi}} - \frac{1}{8}\cdot\frac{8\sqrt{z^3}}{3\sqrt{\pi}}\right)$$

$$= \frac{1}{2}\Gamma(\frac{1}{2})\left(\frac{2z}{\sqrt{\pi}} - \frac{z^2}{3\sqrt{\pi}}\right) = \left(z - \frac{z^2}{6}\right).$$

If we take $g(z) = z$, then we have

$$\frac{O(O^{\lambda} f(z))}{g(z)} = \frac{1}{z} O\left(z - \frac{z^2}{6}\right) = \frac{1}{z} z\left(1 - \frac{z}{3}\right) = 1 - \frac{z}{3}$$

And so we get, $\mathrm{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right) = \mathrm{Re}(1 - \frac{z}{3}) = 1 - \frac{x}{3}$.

Since $|z|<1$ we conclude that

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right)=1-\frac{x}{3}>0.$$

Therefore $f(z)$ satisfies Definition 3.2.1 when $\lambda=\frac{1}{2}$, thus $f(z)\in K\left(\frac{1}{2}\right)$.

## 3.3 Properties of $\lambda$-Fractional Close to Convex Functions

By using the definition 3.2.1 and properties of $\lambda$-fractional operator $O^{\lambda} f(z)$, we have the following properties:

**Theorem 3.3.1:**

If $f(z)\in K(0)$ then $f(z)\in K$.

**Proof:**

Suppose $f(z)\in K(0)$, that is $f(z)$ belongs to the family $0$-fractional close to convex functions hence satisfies Definition 3.2.1 when $\lambda=0$ thus there exist a function $g(z)$ in $S^{*}$ such that

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z)}{g(z)}\right)>0, \quad \text{for all } z\in\Delta.$$

Substitute $\lambda=0$ we get

$$\operatorname{Re}\left(\frac{O(O^{0} f(z)}{g(z)}\right)=\operatorname{Re}(\frac{O(f(z)}{g(z)})=\operatorname{Re}\left(\frac{z f'(z)}{g(z)}\right)>0.$$

Hence $f(z)$ satisfies Definition 3.1.4, and therefore $f(z)\in K$.

**Theorem 3.3.2:**

Let $\lambda=0$ then,

$$\operatorname{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right)>0 \text{ implies } \operatorname{Re}\left(\frac{z f'(z)}{g(z)}\right)>0.$$

**Proof:**

Let $\lambda = 0$ then $O(O^{\lambda} f(z)) = z(O^{0} f(z))' = z \cdot f'(z)$.

Therefore

$$\mathrm{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right) > 0 \text{ implies } \mathrm{Re}\left(\frac{z f'(z)}{g(z)}\right) > 0.$$

**Theorem 3.3.3**:

Let $\lambda = 1$ then

$$\mathrm{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right) > 0 \text{ implies } \mathrm{Re}\left(\frac{z \cdot f'(z)}{g(z)} \cdot (1 + \frac{z \cdot f''(z)}{g(z)})\right) > 0.$$

**Proof:**

Let $\lambda = 1$ then

$$O(O^{\lambda} f(z)) = z(O\ f(z))' = z(z \cdot f'(z))'$$

$$= z(z f''(z) + f'(z)) = z^2 f''(z) + z f'(z).$$

Thus

$$\mathrm{Re}\left(\frac{O(O^{1} f(z))}{g(z)}\right) = \mathrm{Re}\left(\frac{z^2 \cdot f''(z) + z \cdot f'(z)}{g(z)}\right) > 0.$$

That is

$$\mathrm{Re}\left(\frac{z f'(z)}{g(z)} \cdot (1 + \frac{z f''(z)}{f'(z)})\right) > 0.$$

**Theorem 3.3.4:**

Let $f\ (z\ )$ be an element of $k\ (\lambda)$ then,

$$\frac{r(r-1)}{(1+r)^3} \le \left|O(O^{\lambda} f(z))\right| \le \frac{r(r+1)}{(1-r)^3}, \qquad \text{where } |z| = r.$$

**Proof**:

Using the definition of the class of generalized close to convex function $k\ (\lambda)$ we can write $\dfrac{O(O^{\lambda} f(z))}{g(z)} = p(z) \Rightarrow O(O^{\lambda} f(z)) = p(z) \cdot g(z)$, where $p(z\ ) \in P$. (1)

On other hand if we apply Schwarz lemma, see[5], on $p(z)$ with suitable Mobius transformation, we get

$$\frac{1-r}{1+r} \le |p(z)| \le \frac{1+r}{1-r} \text{ for every } p(z) \in P . \qquad (2)$$

Also, recall that by growth Theorem 1.5.2 we have for every starlike function,

$$\frac{r}{(1+r)^2} \le |g(z)| \le \frac{r}{(1-r)^2} \text{ for every } g(z) \in S^* . \qquad (3)$$

So by considering Equations (1), (2) and (3) we get

$$\frac{r(1-r)}{(1+r)^3} \le |p(z)g(z)| \le \frac{r(1+r)}{(1-r)^3} .$$

When $\lambda = 0$ then

$$\frac{O(O^\lambda f(z))}{g(z)} = \frac{O(O^0 f(z))}{g(z)} = \frac{O f(z))}{g(z)} = \frac{z \cdot f'(z)}{g(z)} .$$

Thus

$$\frac{z\ f'(z)}{g(z)} = p(z) \Rightarrow p(z)g(z) = z\ f'(z) .$$

Therefore

$$\frac{r(1-r)}{(1+r)^3} \le |z\ f'(z)| \le \frac{r(1+r)}{(1-r)^3} .$$

Since $|z| = r > 0$, divide all side, we get

$$\frac{(1-r)}{1+r)^3} \le |f'(z)| \le \frac{(1+r)}{(1-r)^3} .$$

Also when $\lambda = 1$ then

$$O(O^\lambda f(z)) = z(O\ f(z))' = z(z \cdot f'(z))'$$

$$= z(z\ f'(z))' = z(z\ f''(z) + f'(z)) .$$

Hence

$$\frac{r(1-r)}{(1+r)^3} \le |z(z\ f''(z) + f'(z))| \le \frac{r(1+r)}{(1-r)^3} .$$

Since $|z| = r > 0$, divide all side, we get

$$\frac{(1-r)}{(1+r)^3} \le |\ (f'(z)) + z\ f''(z)| \le \frac{(1+r)}{(1-r)^3} .$$

**Theorem 3.3.5:**

$k(\lambda)$ is not closed under addition.

**Proof:**

Suppose $f(z)$ and $g(z)$ belong to $k(\lambda)$ then, by definition 3.2.1 $f(z)$ are normalized by $f(0)=0$ and $f'(0)=1$, also $g(z)$ normalized by $g(0)=0$ and $g'(0)=1$, thus $f'(0)+g'(0)=2$.

Therefore, $f(z)+g(z)$ does not normalize by $f'(0)+g'(0)=1$ so $f(z)+g(z)$ does not belong to $k(\lambda)$, hence $k(\lambda)$ is not closed under addition.

## 3.4 Coefficient Estimates for Generalized Close to Convex Functions

In section 1.4, we discuss the coefficient estimate for the functions of family S, and since we show that the close to convex functions is subclass of family S, then its satisfy the coefficient bounds of family S. Furthermore, in 1955, M. O. Reade proved that the Bieberbach conjecture is satisfied by a class of close to convex functions which defined by Kaplan. See [15].

**Theorem 3.4.1**

If $f(z)$ is analytic normalized univalent close to convex function in $\Delta$, then the coefficients satisfy the inequality $|a_n| \le n$ for $n = 1,2,3,\ldots$.

**Theorem 3.4.2:**

Let $f(z)$ be an element of $k(\lambda)$ then,

$$|a_n| \le \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} n$$

**Proof:**

Suppose $f(z) \in k(\lambda)$ then by definition of the class $k(\lambda)$ there exist a starlike function $g(z)$ such that

$$\mathrm{Re}\left(\frac{O(O^{\lambda} f(z))}{g(z)}\right) > 0.$$

That is

$$\frac{O(O^{\lambda} f(z))}{g(z)} = p(z), \quad p(z) \in P$$

Thus we get $O(O^{\lambda} f(z)) = p(z) \cdot g(z)$. But

$$O(O^{\lambda} f(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} \cdot n \cdot a_n \cdot z^n. \qquad (1)$$

Also, since $p(z) \in P$ and $g(z) \in S^*$

$$p(z)g(z) = (z + b_2 z^2 + \ldots + b_n z^n \ldots)(1 + p_1 z + p_2 z^2 + \ldots + p_n z^n + \ldots). \quad (2)$$

By equating Equations (1) and (2) we get

$$\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} n a_n = (b_n + b_{n-1} p_1 + \ldots. + b_2 p_{n-2} + b_1 p_{n-1}).$$

Hence

$$\left| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} n a_n \right| = \left| b_n + b_{n-1} p_1 + \ldots. + b_2 p_{n-2} + b_1 p_{n-1} \right|.$$

By property

$$\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} n|a_n| \le |b_n| + |b_{n-1}||p_1| + \ldots. + |b_2||p_{n-2}| + |b_1||p_{n-1}|$$

$$\le n + (n-1)2 + (n-2)2 + \ldots. + 2.2 + 1.2$$

$$= n + 2(1 + 2 + \ldots. + n - 1) = n^2.$$

Therefore

$$\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} n|a_n| \le n^2.$$

This implies to

$$|a_n| \le \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} n.$$

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## تعميم الاقترانات شبه المحدبة

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ملخص:

في هذه الرسالة قمنا بدراسة تعميم مجموعة الاقترانات العقدية شبه المحدبة ودراسة خصائصها. وتعتبر مجموعة الاقترانات العقدية شبه المحدبة مجموعة جزئية من عائلة (س) التي تضم الاقترانات المعقدة التي توصف بانها تحليلية، واوحد لواحد وصورة الصفر هي صفر والمشتقة الأولى عند الصفر تساوي واحد. وتسمى هذه الاقترانات بشبه المحدبة اذا كان الجزء الحقيقي لحاصل قسمة مشتقة الاقتران على مشتقة اقتران محدب موجبة.

ففي الفصل الاول قمنا بدراسة الاقترانات المعقدة التي توصف بانها واحد لواحد حيث تم عرض تعريف الاقترانات المركبة الواحد لواحد وبعض الامثلة والنظريات ذات الصلة، ومن ثم دراسة عائلة (س) وبعض خصائصها وتم تقديم عرض مختصر حول تقريب المعاملات للاقترانات في هذه العائلة. اضافة الى ذلك تم دراسة نظريتا النمو ونظرية الانحراف لتقديم حدود عليا ودنيا للاقترانات الموجودة في عائلة (س) ولمشتقاتها، كما تم دراسة مجموعتين جزئيتين من عائلة (س) هما الاقترانات المحدبة والاقترانات مثيلة النجمة وتم تقديم امثله عليها وبعض النظريات ودراسة بعض الخصائص.

أما في الفصل الثاني فقد تم عرض فكرة التفاضل والتكامل الكسري، حيث قدمنا ملخصا حول تاريخ التفاضل الكسري العقدي وتم مناقشة اقترانين هامين هما اقتران جاما واقتران بيتا وعدد من العلاقات الخاصة بهما. وبعد ذلك تم تعريف التكاملات الكسرية ودراسة خصائصها. ومن ثم تم استخدام التكامل الكسري لتعريف المشتقات الكسرية وتم دراسة خصائصها. وفي نهاية الفصل الثاني تم تعريف المعاملات الكسرية من خلال المشتقات الكسرية ودراسة وخصائصها.

واخيرا في الفصل الثالث، قمنا بدراسة معمقة لمجموعة الاقترانات المركبة شبه المحدبة. ومن ثم تم تعميم هذه المجموعة باستخدام المعاملات الكسرية وتم ودراسة خصائصها وتقديم نظرية جديدة لتعريف تعميم الاقترانات المركبة القريبة من المحدبة، ومن ثم تم وضع حدود لهذه الاقترانات وتم ايجاد تقدير لمعاملاتها.