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Dynamic Set Theory and Extensions on Lagrange Squares: New Theorems, Philosophical Insights and Applications

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Dedication

I dedicate this work, with all my love, gratitude, and respect, to my dear parents who have been my greatest support. My heartfelt thanks go to the professors of the Mathematics Department at Al-Quds University, whose guidance and encouragement shaped my path. In memory of the spirit of Abu Hamid Al-Ghazali, whose wisdom and books remains a light for seekers of knowledge.

Ahmad Al-Awawdeh

Declaration

I certify that thesis submitted for the degree of master is the result of my own research except where otherwise acknowledge and this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Name: Ahmad Zeyad Mahmoud Al-Awawdeh

Signed: 

Date: 14 /5 / 2025

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Finally, I offer this work to everyone who encouraged me, supported me, and believed in me.

Ahmad Al-Awawdeh

Abstract

This thesis presents an innovative mathematical and philosophical development by establishing a connection between Lagrange Four Square Theorem and five well-known sequences: the sequence of natural numbers, square numbers, cubic numbers, triangular numbers, and the Fibonacci sequence. This integration resulted in the construction of five new sequences, each of which has been rigorously analyzed and accompanied by fourteen original theorems, all proven within a strict methodological framework..

This thesis reflects a multifaceted approach that bridges the rigorous structure of number theory with the philosophical perspective of mathematics. It introduces a novel principle in the philosophy of mathematics, termed "coexistence of opposites within a single entity", which opens new horizons for understanding contradiction and integration within the mathematical framework itself. Furthermore, the research establishes connections with fundamental issues in the philosophy of mathematics.

The developed sequences were also utilized in various applied domains, including a reimagining of the classical game of chess based on the framework of Lagrange's four-square theorem, as well as the resolution of real-world problems in fields such as physics, cryptography, economics, and engineering—demonstrating the practical value of the proposed theorems.

In the concluding part of the thesis, a new theoretical framework proposed under the title "Dynamic Set Theory," which reimagines mathematical sets as mutable entities rather than fixed ones—contrasting with the classical perspective established by Cantor. The logical, philosophical, and algebraic foundations of this approach presented, positioning it as a fertile ground for future research.

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Introduction:

1.1 Lagrange Four-Square Theorem

Lagrange Four-Square Theorem states that every positive integer expressed as the sum of four squared integers. This means that for any natural number n, there exist integers a,b,c and d, satisfying:

$n = a^2 + b^2 + c^2 + d^2$

For example, the number 30 represented as:

$30 = 4^2+3^2+2^2+1^2$

Note that, the case where (a=b=c=d) is possible, for example $4 =1^2+1^2+1^2+1^2$

Lagrange theorem is fundamental in number theory, as it establishes a universal property of integers that connects algebraic structures with geometric representations. [16]

The theorem belongs to a broader category of Warring problems, which study the minimum number of certain types of numbers (such as squares, cubes, or higher powers) needed to represent all integers. While some numbers require fewer than four squares, Lagrange theorem guarantees that no integer requires more than four. This result has significant implications in areas such as quadratic forms, lattice point enumeration, and modular arithmetic. [25]

Historical Context of Lagrange Theorem

Joseph-Louis Lagrange first formally proved the theorem in 1770, which trace back to earlier mathematicians. The Swiss mathematician Leonhard Euler had discovered a proof in 1748, but he never published his findings. Euler's contributions to quadratic forms laid much of the groundwork for Lagrange's proof, highlighting the deep connections between algebraic identities and number representations. [6]

Following Lagrange proof, Adrien-Marie Legendre refined the theorem in 1798 by introducing additional conditions on which numbers represented using fewer squares.

Later, Carl Friedrich Gauss extended these ideas in his 1801 work Disquisitions Arithmeticae, where he developed the law of quadratic reciprocity and introduced modular arithmetic methods to analyze sums of squares. These developments solidified the theorem's importance in classical number theory. [11]

Jacobi Sequence

Carl Gustav Jacobi, a 19th-century mathematician, explored the representation of numbers as sums of squares through his work on modular forms and theta functions. His Jacobi Four-Square Theorem provides an exact formula for counting the number of ways a given integer expressed as a sum of four squares. This result deeply connected to Lagrange's theorem, as it not only confirms the theorem's validity but also quantifies its solutions. [21]

The number of representations of an integer as a sum of four squares determined by a function involving divisor sums, demonstrating deep algebraic and combinatorial properties. This approach influenced later research in modular arithmetic and coding theory. [18]

Fermat Two-Square Theorem

Lagrange Four-Square Theorem deeply connected to an earlier result by Pierre de Fermat, who studied the representation of numbers as sums of squares. Fermat Two-Square Theorem states that a prime number expressed as the sum of two squares if and only if it takes the form for some integer as following:

If the prime number p gives one (mod 4), then $p= x^2+y^2$, where x and y are integers.

Euler, who explored sums involving more squares, laying the groundwork for Lagrange proof in 1770, later extended this theorem. Lagrange result generalized Fermat work by proving that every positive integer, not just certain primes, expressed as the sum of four squares. This generalization makes Lagrange theorem a direct extension of Fermat foundational work in number theory. [16]

The connection between Fermat and Lagrange theorems is part of a broader theme in quadratic forms and Waring's problem, which examines the minimum number of terms needed to represent integers using specific powers. Lagrange's theorem answers the question for squares, showing that four terms are always sufficient. Later, Gauss expanded on these ideas in Disquisitions Arithmeticae, where he developed modular arithmetic methods to analyze sums of squares more systematically. These contributions illustrate how Lagrange's result fits within a long-standing mathematical tradition initiated by Fermat, making it a cornerstone of classical number theory. [12]

Another critical theoretical link is with Murkowski theorem in geometry of numbers. Murkowski's theorem states that convex bodies in lattice spaces contain lattice points under certain conditions. This geometric viewpoint allows for a more profound understanding of why every integer decomposed into four squares, providing insights beyond pure algebraic reasoning.

(Lattice Space)

A lattice space is a discrete subgroup of the Euclidean space $\mathbb{R}^n$ which spans the real vector space. Formally,
a lattice Λ in $\mathbb{R}^n$ is the set of all linear combinations with integer coefficients of n linearly independent vectors v_1, ..., v_n in $\mathbb{R}^n$:

$\Lambda = \{ a_1v_1 + a_2v_2 + ... + a_nv_n \mid a_i \in \mathbb{Z} \text{ for } i = 1, ..., n \}$.

Definition (Lattice Points):
Lattice points are the individual elements (points) of a lattice Λ in $\mathbb{R}^n$. Each lattice point corresponds to a unique
integer linear combination of the basis vectors of the lattice.. [23]

Questions of the thesis

What new mathematical extensions can derived by integrating Lagrange Four Square Theorem with classical number sequences? In addition, how can these insights contribute to foundational and philosophical developments? Moreover, how we can link sets representation of number during Lagrange Four Squares with dynamic world?

1.2 Selected Sequences

Two famous sequences in number theory are used, the first is the sequence of triangular numbers, and the second is sequence of Fibonacci numbers.

1.2.1: Triangular Numbers

A triangular number is a number that represented as a triangle in a dot pattern, as in figure below.

It is known that the triangular numbers are written as $T_n = \frac{n(n+1)}{2}$

For n = 1, 2, 3, …

For example, the first five triangular numbers are:

1,3,6,10,15

The connection between triangular numbers and squares is significant in number theory, since every square number can be expressed as the sum of two consecutive triangular numbers, and this property has been explored in Diophantine equations and combinatorial number theory.[5]

Proof of this connection as following:

Taking a triangular number:

$T_n = \frac{n(n+1)}{2}$

Now adding the next one to this triangular number as following:

$T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+1+1)}{2}$

$= (n^2+n+n^2+2n+n+2)/2$

$= 2n^2+4n+2$

$= (n+1)^2$, which is a squared number.

So, x= n+1, then

$x^2 = \frac{n(n+1)}{2} + \frac{(n+1)(n+1+1)}{2}$

Triangular numbers appear frequently in problems related to arrangements, such as stacking spheres or organizing seats in rows. They also have connections to physics, where they describe energy levels in quantum mechanics. The relationship between triangular numbers and Lagrange theorem arises when considering how integers can be decomposed systematically. [3]

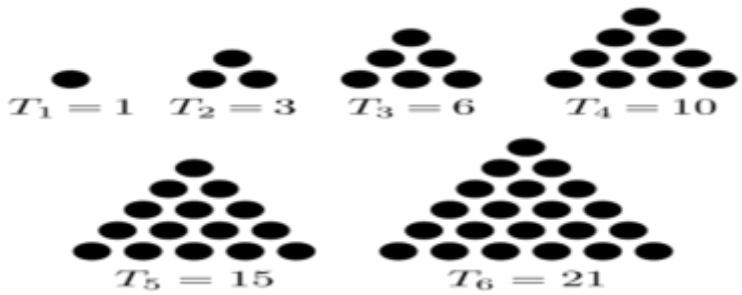

Figure 1: Triangular numbers dot representation

1.2.2: Fibonacci Sequence

Fibonacci sequence is one of the most famous numerical sequences, defined recursively as:

$F_n = F_{n-1} + F_{n-2}$

For clarification, the sequence begins as follows:

0,1,1,2,3,5,8,13,21,34,55,89,…

The Fibonacci sequence has deep connections to the golden ratio as following:

$F_n \approx (\varphi^n) / \sqrt{5}$, with $\varphi = (1 + \sqrt{5}) / 2 \approx 1.618$

Where φ is the golden ratio.

Each Fibonacci number is the summation of the previous two numbers, a property that has applications in art, architecture, and biological growth patterns. [20]

Fibonacci numbers also appear in solutions to Diophantine equations, including Pell's equation and sum-of-square problems. [8]

1.2.3 Methodology

This thesis adopts a theoretical and analytical approach within the domain of pure mathematics. The methodology involves several stages. First, a historical and philosophical framework presented to contextualize Lagrange's Four Square Theorem. Next, through deductive reasoning and structural analysis, five novel sequences constructed by integrating the theorem with classical sequences: natural numbers, square numbers, cubic numbers, triangular numbers, and the Fibonacci sequence.

Each resulting sequence rigorously examined, leading to the formulation and proof of fourteen theorems characterizing their properties. The methodology further includes the philosophical extrapolation of these results to explore a proposed mathematical-philosophical principle: the inclusion of contradiction within a consistent mathematical framework.

In addition, the study extends to practical applications, including the modification of the classical chess game using Lagrange-based logic, and solving real-world problems across physics, cryptography, engineering, and economics using the developed sequences. Finally, the methodology explores a new conceptual direction, termed Dynamic Set Theory, offering a philosophical and algebraic foundation for viewing mathematical sets as dynamic entities.

Chapter Two

New Sequences Based on Lagrange Four Squares

Throughout this chapter, five sequences given, which are new sequences that based on Lagrange Four Squares Theorem, present related theorems related to those sequences with examples and figures.

In the first section, three sequences, that building by Lagrange Four Squares pattern on selected natural numbers.

The second section concerns linking triangular numbers with Lagrange four squares theorem.

The third section concerns Fibonacci sequence with Lagrange four squares theorem.

2.1 Natural Sequences

In this section, new sequences are build; the first one is a sequence that its elements are the sum of four squares of consecutive natural numbers. We will show the properties of the elements of this sequence by four theorems. In addition, we will define a second sequence whose elements are sum of four squares of consecutive squared numbers and showing some properties by three theorems. Moreover, we will define a sequence whose elements are sum of four squares of consecutive cubic numbers and showing some properties by three theorems.

Definition 2.1.1: (Squared Sequence)

Let $\{a_n\}$ be a sequence of natural numbers defined by

$a_n = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$, where n is a natural number

By simplifying, we obtain that:

$a_n = 4n^2+12n+14$, and $n = 1, 2, 3, \ldots$

Squared Theorems

Theorem 2.1.1: (First Squared Theorem)

Let $a_n = 4n^2+12n+14$, where n is a natural number, the following properties are satisfies:

i) a_n is an even natural number.

ii) a_n ends with one element of the set $\{0, 4, 6\}$ as the number in units digit, and with the pattern: 0, 4,6,6,4,…

iii) The pattern of unit digit of sequence $\{a_n\}$ is a repeated pattern, according to the following periodicity: a_n and a_{n+5} both have the same number in units digits.

Proof

i) Since $a_n = 4n^2+12n+14$, where $n \in N$,

Then, $a_n = 2(2n^2+6n+7)$, which is an even number.

(ii) To prove this, there are the following cases:

Case 1: When n = 1+5k, where k=0, 1, 2, …

Then, $a_n = 4(1 + 5k)^2 + 12(1 + 5k) + 14 = 100k^2 + 100k + 30$

This number ends with zero, since it gives 0 (mod 10).

Case 2: When n= 2+5k, where k=0,1,2,3,…

Then, $a_n = 4(2 + 5k)^2 + 12(2 + 5k) + 14 = 100k^2 + 140k + 54$,

This number ends with 4, since it gives 4 (mod 10).

Case 3: When n= 3+5k, where k=0,1,2,3,…

Then, $a_n = 4(3 + 5k)^2 + 12(3 + 5k) + 14 = 100k^2 + 180k + 86$

This number ends with 6, since it gives 6 (mod 10) .

Case 4: When n= 4+5k, where k=0,1,2,3,…

Then, $a_n = 4(4 + 5k)^2 + 12(4 + 5k) + 14 = 100k^2 + 220k + 126$

This number ends with 6 , since it gives 6 (mod 10).

Case5: When n= 5k, where k=1,2,3,…

Then, $a_n = 4(5k)^2 + 12(5k) + 14 = 20k^2 + 60k + 14$

Which ends with 4, since this number gives 4 (mod 10)

(iii): Since $a_n = 4n^2 + 12n + 14$, then $a_{n+5} = 4(5 + n)^2 + 12(5 + n) + 14$

So, $a_{n+5} = 4n^2 + 112n + 174$

So, $a_{n+5} - a_n = 100n+160$, which gives 0 (mod 10)

Which means that both of them ends with the same number, so the pattern: 0, 4,6,6,4 is a repeated pattern.

Theorem 2.1.2: (Second Squared Theorem)

Let $a_n = 4n^2+12n+14$, where n is a natural number, the following properties satisfies:

i) If the unit digit of a_n is zero, then the decimal digit is three and this is cyclic every five terms from $\{a_n\}$.

ii) If the unit digit of a_n is four, then the decimal digit is odd.

iii) If the unit digit of a_n is six in its unit digit, then the decimal digit is even.

Proof

(i) Since a_n ends with zero, then according the proof for part (ii) from theorem 2.1.1, can write n=1+5k, k=1,2,3,..

In addition, $a_n=100k^2 + 100k + 30$, which gives 30 (mod 100), that means it has 3 in it's decimal digits.

For the cyclic property, we proved it in part (iii) in theorem (2.1.1).

(ii) Given that a_n has 4 in its units digit, according to part (ii) from the proof of theorem (2.1.2) we have only two cases:

Case 1: When n= 2+5k, k=1, 2, 3…

Then $a_n = 100k^2 + 140k + 54$

Case 2: When n= 5k, k=1,2,3,…

Then $a_n = 20k^2 + 60k + 14$

In case 1: $(a_n-4)/10 = (100k^2 +140k+ 54 – 4) / 10 = 10k^2 + 14k + 5$

Which is odd, that means a_n has an odd number in its decimal digit.

In case 2: $(a_n-4) / 10 = (20k^2 + 60k + 14 – 4) / 10 = 10k^2 + 26k + 1$,

Which is an odd number.

(iii) Since a_n has six in its units digit, according part (2) from the proof, we have only two cases:

Case 1: when n = 3+5k, k=1, 2,3,…

Then $a_n = 100k^2 + 180k + 86, \text{and } n = 1,2,3, ...$

Case 2: when n = 4+5k, k=1,2,3,…

Then $a_n = 100k^2 + 220k + 126, \text{and } n = 1,2,3, ...$

In case 1: $(a_n-6) / 10 = (100k2 +180+86 – 6) / 10$

Then, $(a_n-6)/10 = 10k^2 + 18k + 8$

Which is even, that means a_n has even number in its decimal digit.

In case 2: $(a_n - 6) / 10 = (100k^2+220k+126-6)/ 10 = 10 K^2+22k+12$

Which is even, that means a_n has an even number in its decimal digit.

Theorem 2.1.3: (Third Squared Theorem)

Let $a_n = 4n^2+12n+14$, where n is a natural number, the following properties satisfies:

i) When examining the divisibility of a_n by three, the result is that every three consecutive terms from $\{a_n\}$; One of them leaves remainder 2 when divided by three , and the other two terms will be divisible by three without a remainder.

In other words, any number when divided by three has remainder one does not belong to this sequence.

ii) The numbers that are not divisible by three and that belong to sequence the $\{a_n\}$ can be determined as following:

If a_n comes as the sum of four consecutive natural numbers, and the largest of these numbers is a number divisible by three, considered d; then the term a_n will not be divisible by three. In the opposite case, if the number d is not divisible by three, then the number a_n will be divisible by three.

iii) Value of a_n divided by four, gives 2 (mod 4).

iv) Sequence $\{a_n\}$ is divisible by five once every five consecutive terms in the sequence $\{a_n\}$.

Proof

(i) Any element from a_n can be written as $a_n = a^2 + b^2+c^2 + d^2$, where a= n, b=n+1, c= n+2 and d = n+3

The number d in our choices satisfies: d > 3, so d written as:

Case 1: d=1+3m where m=0, 1, 2, 3, …

Then $a_n = (3m + 1 - 3)^2 + (1 + 3m - 2)^2 + (1 + 3m - 1)^2 + (1 + 3m)^2$

$= (3m - 2)^2 + (3m - 1)^2 + (3m)^2 + (1 + 3m)^2$

$= 9m^2+ 4 – 12m + 9m^2+1-6m+9m^2+9m^2+1+6m$

$= 36m^2+36m+15$

Which gives 0 (mod 3).

Case 2: d=2+3m where m=0,1,2,3,…

Then $a_n = (3m + 2 - 3)^2 + (2 + 3m - 2)^2 + (2 + 3m - 1)^2 + (2 + 3m)^2$

$= (3m - 1)^2 + (3m)^2 + (3m + 1)^2 + (2 + 3m)^2$

$= 9m^2+ 1 – 6m + 9m^2+9m^2 + 1 + 6m+9m^2+4+12m$

$= 36m^2+12m+6$

Which gives 0 (mod 3).

Case 3: d=3m, where m=1, 2, 3, …

Let $a_n = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$, let d = n, c =n-1,c= n-2,d =n-3

Then, $a_n = a^2 + b^2 + c^2 + d^2 = (d-3)^2 + (d-2)^2 + (d-1)^2 + d^2$

Now, since d is divisible by 3 and d > 3 then we can write it as: d=3+3m, m=1,2,3,…

Therefore, we get that:

$a_n = (3m-3)^2 + (3m-2)^2 + (3m-1)^2 + (3m)^2$

$= 36m^2 + 36m + 14$

Which gives 2 (mod 3).

(ii) Divisibility by 4

Since $a_n = 4n^2 + 12n + 14$, where n=1,2,3,…

Which gives 2 (mod 4).

(iii) Divisibility by five

Any number that is divisible by 5 end with 0 or 5, according to theorem (2.1.1), a_n will be even, so it doesn't end with five. Therefore, the only case in which the value a_n will be divisible by 5 is that when it ends with zero, and we proved in part (i) and part (ii) from theorem (2.1.1) that this case occurs once every five times and in periodically pattern .

Theorem 2.1.4: (Fourth Squared Theorem)

Let $a_n = 4n^2+12n+14$, where n is a natural number, then, the deference between two consecutive terms is linear.

Proof

Given that $a_n = 4n^2+10n+14$

So, $a_{n+1} = 4(n+1)^2 +10(n+1) + 14 = 4n^2+18n+28$

So, $a_{n+1} - a_n = (4n^2+ 18n+28) – (4n^2+10n+14) = 8n -14$,

Which is linear.

Example 2.1.1: calculate the first 24 terms in sequence $\{a_n\}$, *cheek properties of Squared Theorems, and find value of the difference between every two consecutive terms.*

Solution:

Consecutive Natural Numbers	a_n	$a_n - a_{n-1}$	Parity	Unit digit	Decimal digit	r (mod 3)	r (mod 4)	r (mod 5)
(1,2,3,4)	30	-	even	0	3	0 (mod 3)	2 (mod 4)	0 (mod 5)
(2,3,4,5)	54	24	even	4	5	0 (mod 3)	2 (mod 4)	4 (mod 5)
(3,4,5,6)	86	32	even	6	8	2 (mod 3)	2 (mod 4)	1 (mod 5)
(4,5,6,7)	126	40	even	6	2	0 (mod 3)	2 (mod 4)	1 (mod 5)
(5,6,7,8)	174	48	even	4	7	0 (mod 3)	2 (mod 4)	4 (mod 5)
(6,7,8,9)	230	56	even	0	3	2 (mod 3)	2(mod 4)	0 (mod 5)
(7,8,9,10)	294	64	even	4	9	0 (mod 3)	2 (mod 4)	4 (mod 5)
(8,9,10,11)	366	72	even	6	6	0 (mod 3)	2 (mod 4)	1 (mod 5)
(9,10,11,12)	446	80	even	6	4	2 (mod 3)	2 (mod 4)	1 (mod 5)
(10,11,12,13)	534	88	even	4	3	0 (mod 3)	2 (mod 4)	4 (mod 5)
(11,12,13,14)	630	96	even	0	3	0 (mod 3)	2 (mod 4)	0 (mod 5)
(12,13,14,15)	734	104	even	4	3	2 (mod 3)	2 (mod 4)	4 (mod 5)
(13,14,15,16)	846	112	even	6	4	0 (mod 3)	2 (mod 4)	1 (mod 5)
(14,15,16,17)	966	120	even	6	6	0 (mod 3)	2 (mod 4)	1 (mod 5)
(15,16,17,18)	1094	128	even	4	9	2 (mod 3)	2 (mod 4)	4 (mod 5)
(16,17,18,19)	1230	136	even	0	3	0 (mod 3)	2 (mod 4)	0 (mod 5)
(17,18,19,20)	1374	144	even	4	7	0 (mod 3)	2 (mod 4)	4 (mod 5)
(18,19,20,21)	1526	152	even	6	2	2 (mod 3)	2 (mod 4)	1 (mod 5)
(19,20,21,22)	1686	160	even	6	8	0 (mod 3)	2 (mod 4)	1 (mod 5)
(20,21,22,23)	1854	168	even	4	5	0 (mod 3)	2 (mod 4)	4 (mod 5)
(21,22,23,24)	2030	176	even	0	3	2 (mod 3)	2 (mod 4)	0 (mod 5)

Graph of Sequence $\{a_n\}$

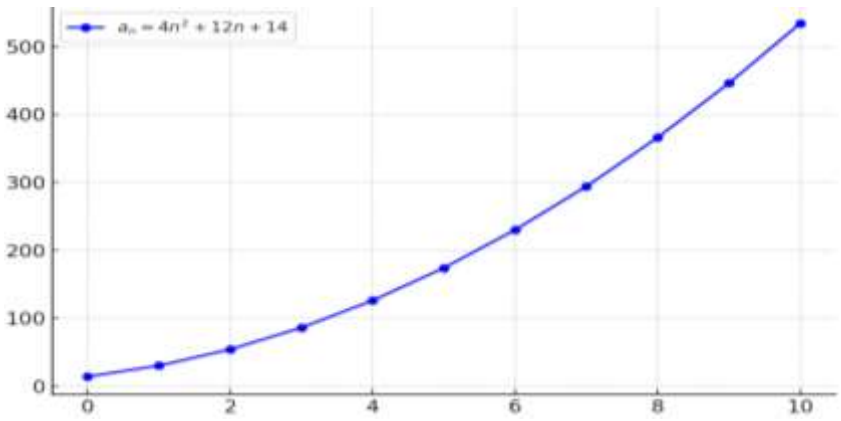

Figure 2: Graph of a_n vs

Definition 2.1.2: (Super Squared Sequence)

Let $\{s_n\}$ be a sequence of natural numbers defined by

$s_n = a^2 + b^2 + c^2 + d^2$

$a = n^2$, $b = (n+1)^2$, $c = (n+2)^2$, $d = (n+3)^2$

Where n=1, 2, 3, …

Equivalently,

$s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$, n=1, 2, 3, …

For Example:

$S_1 = 1^2+4^2+9^2+16^2 = 354$, $s_2 = 4^2+9^2+16^2+25^2 = 978$, and so on.

(Super Squared Theorems)

Theorem 2.1.5: (First Super Squared Theorem)

i) Each term in sequence $\{s_n\}$ is an even natural number.
ii) The unit digit of s_n is only 4 or 8
iii) Unit digit of terms taken from sequence $\{s_n\}$, appears in the following pattern: 4,8,8,8,8,4,…

Moreover, it is a repeated pattern, according to the following periodicity: s_n and s_{n+5} have the same number in units digits.

Proof:

(i) Given that $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$, *n=1, 2, 3,…*

Since all terms are even

Therefore, s_n is even.

(ii) Since s_n depends on n as $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$, by taking the cases for n as following:

Case 1: When n = 1+5k,k=1,2,3,…

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$= 4(1+5k)^4 + 24(1+5k)^3 + 84(1+5k)^2 + 144(1+5k) + 98$

$= 2500k^4 + 5000k^3 + 4500k^2 + 2000k + 354$

Which gives 4 (mod 10),

Which means that s_n in this case ends with 4.

Case 2: When n=2+5k, k=1, 2, 3,…

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$= 4(2+5k)^4 + 24(2+5k)^3 + 84(2+5k)^2 + 144(2+5k) + 98$

$$= 2500k^4 + 7000k^3 + 8100k^2 + 4480k + 978$$

Which gives 8 (mod 10), which means that s_n in this case ends with 8.

Case 3: When n=3+5k, k=1,2,3,…

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$$= 4(3 + 5k)^4 + 24(3 + 5k)^3 + 84(3 + 5k)^2 + 144(3 + 5k) + 98$$

$$= 2500k^4 + 9000k^3 + 12500^2 + 8640k + 2258$$

Which gives 8 (mod 10), which means that s_n in this case ends with 8.

Case 4: When n=4+5k, k=1,2,3,…

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$$= 4(4 + 5k)^4 + 24(4 + 5k)^3 + 84(4 + 5k)^2 + 144(4 + 5k) + 98$$
$$= 2500k^4 + 11000k^3 + 18900k^2 + 14960k + 4578$$

Which gives 8 (mod 10), which means that s_n in this case ends with 8.

Case 5: When n=5k, k=1,2,3,…

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$$= 4(5k)^4 + 24(5k)^3 + 84(5k)^2 + 144(5k) + 98$$

$$= 2500k^4 + 3000k^3 + 420k^2 + 720k + 98$$

Which gives 8 (mod 10), which means that s_n in this case ends with 8.

iii) Let $s_n = n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2$

$$= 4n^4 + 24n^3 + 84n^2 + 144n + 98$$

Then, $s_{n+5} = (n + 5)^2 + (n + 6)^2 + (n + 7)^2 + (n + 8)^2$

$$= 4n^4 + 104n^3 + 1044n^2 + 4784n + 8418$$

Now, $s_n - s_{n+5} = (4n^2+104n^3+1044n^2+4784n+8418) - (4n^2+24n^3+84n^2+144n+98)$

$$= 80n^3 + 960n^2 + 4640n + 8320$$

Which gives 0 (mod 10), so s_n and s_{n+5} have the same number in units digit.

So the pattern: (4,8,8,8,8,4,…) is a repeated pattern

Theorem 2.1.6: (Second Super Squared Theorem)

i) If unit digit of $\{s_n\}$ is four, then decimal digit is five, and this is repeated every five terms from $\{s_n\}$.

ii) The number s_n has an odd number in its decimal digit.

Proof

i) Since s_n ends with four, then according to the proof of case (1) in part (ii) in theorem (2.1.5), n=1+5k, k=0,1,2,3,..

Then, $s_n = 2500k^4 + 5000k^3 + 4500k^2 + 2000k + 354$

So, $(s_n - 4) / 10 = 250k^4 + 500k^3 + 450k^2 + 200k + 35$

Which means that s_n has 5 in decimal digit.

(ii) We have been proved that s_n ends with 4 or 8 in part (ii) from theorem (2.1.5), and in part (i) from theorem (2.1.6) we proved that s_n has 5 in its decimal digits whenever it has four in its units digits, and five is an odd number.

Now we will take the cases where s_n ends with 8, those cases are the same cases in part (ii) from the proof of theorem (2.1.5) except case (1)

Now:

Case 1:

When $s_n = 2500k^4 + 7000k^3 + 8100k^2 + 4480k + 978$

Then, $(S_n - 8) / 10 = 250k^4 + 700k^3 + 810k^2 + 448k + 97$

Which gives 7 (mod 10),

So, in this case s_n has an odd number in decimal digit.

Case 2:

When $s_n = 2500k^4 + 9000k^3 + 12500^2 + 8640k + 2258$

Then, $(s_n - 8) / 10 = 250k^4 + 900k^3 + 1250^2 + 864k + 225$

Which gives 5 (mod 10)

Therefore, s_n has an odd number in its decimal digits.

Case 3:

When $s_n = 2500k^4 + 11000k^3 + 18900k^2 + 14960k + 4578$

Then, $s_n - 8 / 10 = 250k^4 + 1100k^3 + 1890k^2 + 1496k + 457$

Which gives (7 mod 10).

Therefore, s_n has an odd number in decimal digit.

Case 4:

When $s_n = 2500k^4 + 13000k^3 + 26100k^2 + 23920k + 8418$

Then, $(s_n - 8) / 10 = 250k^4 + 1300k^3 + 2610k^2 + 2392k + 841$

Which gives 1 (mod 10)

Therefore, s_n has an odd number in its decimal digit.

Theorem 2.1.7: (Third Super Squared Theorem)
Let $s_n = 4n^4+24n^3 + 84n^2 + 144n + 98$, where $n \in \mathbb{N}$.

Then, the following properties are satisfy:

i) For every three consecutive terms in the sequence $\{s_n\}$, one term will be divisible by three with a remainder, which is two, and the other two terms will be divisible by three without any remainder.

In other words, any integer x such that $x \equiv 1 \pmod 3$ does not belong to the sequence $\{ s_n\}$.

A term s_n where it is not divisible by three can determined if it was the sum of four consecutives squares, and the biggest one is divisible by three.

ii) For every $s_n \in \{s_n\}$, when divided by 4, the remainder is always two. That is, $s_n \equiv 2 \pmod 4$, for all $n \in N$.

iii) No term in the sequence $\{s_n\}$ is divisible by five. That is, $s_n \not\equiv 0 \pmod 5$, for all $n \in N$.

Proof

(i) To prove this part, we have the following cases:

Case 1: When n=1+3k, k=1,2,3,…

Then $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$= 4(1 + 3k)^4 + 24(1 + 3k)^3 + 84(1 + 3k)^2 + 144(1 + 3k) + 98$

$= 324k^4 + 1080k^3 + 1620k^2 + 1200k + 354$

Which gives 0 (mod 3).

Case 2: When n=2+3k, k=1,2,3,… .

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$= 4(2+3k)^4+24(2+3k)^3+84(2+3k)^2+144(2+3k)+98$

$= 4(2 + 3k)^4 + 24(2 + 3k)^3 + 84(2 + 3k)^2 + 144(2 + 3k) + 98$

$= 324k^4 + 1512k^3 + 2916k^2 + 2688k + 978$

Which gives (0 mod 3).

Case 3: When n=3k, k=1, 2,3,… .

Then, $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$

$= 4(3 + 3k)^4 + 24(3 + 3k)^3 + 84(3 + 3k)^2 + 144(3 + 3k) + 98$

$= 324k^4 + 648k^3 + 756k^2 + 432k + 98$

Which gives 2 (mod 3).

(ii) Divisibility by four

Given that $s_n = 4n^4 + 24n^3 + 84n^2 + 144n + 98$, where n=1,2,3,…

Since all terms are divisible by four without reminder except the constant term which gives 2 (mod 4) ,Then, s_n gives 2 (mod 4).

(iii) Divisibility by five

Since we proved in part (ii) in theorem (2.1.5) that any terms in sequence s_n ends only with 4 or 8,Then, s_n isn't divisible by 5, since it hasn't 0 or 5 in its unit digit.

Example 2.1.2. Calculate the first 12 terms in $\{s_n\}$, and cheek the above properties in Super Squared Theorems,

Solution:

Four consecutive Square natural numbers	s_n	Parity	Unit digit	Decimal digit	r (mod 3)	r (mod 4)	r (mod 5)
(1,4,9,16)	354	even	4	5	0 (mod 3)	2 (mod 4)	4 (mod 5)
(4,9,16,25)	978	even	8	7	0 (mod 3)	2 (mod 4)	3 (mod 5)
(9,16,25,36)	2258	even	8	5	2 (mod 3)	2 (mod 4)	3 (mod 5)
(16,25,36,49)	4578	even	8	7	0 (mod 3)	2 (mod 4)	3 (mod 5)
(25,36,49,64)	8418	even	8	1	0 (mod 3)	2 (mod 4)	3 (mod 5)
(36,49,64,81)	14354	even	4	5	2(mod 3)	2 (mod 4)	4 (mod 5)
(49,64,81,100)	23058	even	8	5	0 (mod 3)	2 (mod 4)	3 (mod 5)
(64,81,100,121)	35298	even	8	9	0 (mod 3)	2 (mod 4)	3 (mod 5)
(81,100,121,144)	51938	even	8	3	2 (mod 3)	2 (mod 4)	3 (mod 5)
(100,121,144,169)	73938	even	8	3	0 (mod 3)	2 (mod 4)	3 (mod 5)
(121,144,169,196)	102354	even	4	5	0 (mod 3)	2 (mod 4)	4 (mod 5)
(144,169,196,225)	138338	even	8	3	2 (mod 3)	2 (mod 4)	3 (mod 5)

Graph of the Sequence $\{s_n\}$

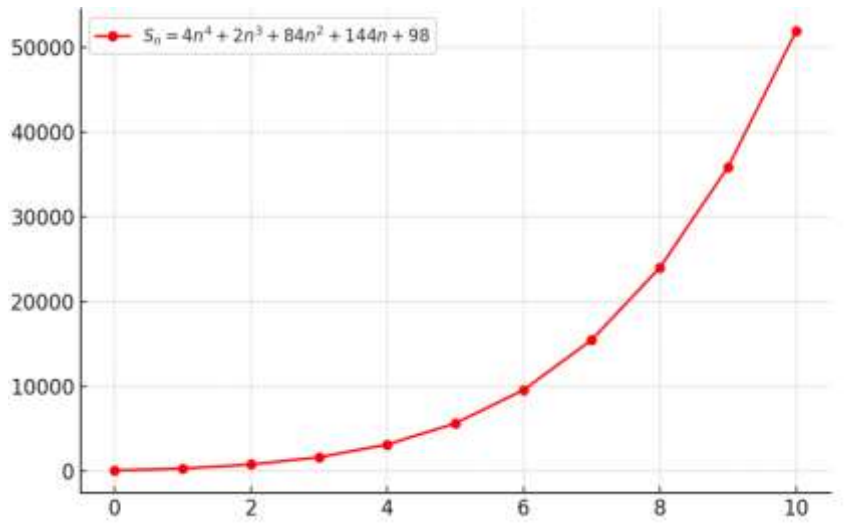

Figure 3: Graph of s_n vs. n

Definition 2.1.3: (Triple Squared Sequence)

Let $\{t_n\}$ be a sequence of natural numbers, such that

$t_n = a^2 + b^2 + c^2 + d^2$, *where* $a=n^3$, $b=(n+1)^3$, $c=(n+2)^3$, $d=(n+3)^3$, *and*

n=1,2,4,.....

So, $t_n = a^2 + b^2 + c^2 + d^2 = (n^3)^2 + ((n+1)^3)^2 + ((n+2)^3)^2 + (n+3)^3)^2$

Then, $t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$

(Triple Squared Theorems)

Theorem 2.1.8: (First Triple Squared Theorem)

For each t_n that written as:

$t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$, where n=1,2,3,…

The following properties hold:

i) All terms in sequence $\{t_n\}$ are even natural numbers.

ii) The term t_n ends with one element from the set (0, 4, 6) as the number on unit digit.

iii) Units digit of numbers which taken from the sequence$\{t_n\}$ have the following pattern: 0, 4,6,6,4,…,and it is a repeated pattern, according to the following periodicity: t_n and t_{n+5} have the same number in unit digit.

Proof

(i) Since $t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$

Then, t_n gives 0 (mod 2)

Which means that t_n is even.

(ii) Given that $t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$, n=1,2,3,….

Now we will take these cases:

Case 1: When n = 1+5k, k=0,1,2,…

$t_n = 4(1+5k)^6 + 36(1+5k)^5 + 210(1+5k)^4 + 720(1+5k)^3$

$+1470(1+5k)^2 + 1656(1+5k) + 794$

Then, $t_n = 62500k^6 + 187500k^5 + 281250k^4 + 250000k^3 + 132750k^2 + 39000k + 4890$

This number ends with zero, since it gives 0 (mod 10)

Case 2: When n= 2+5k, k=1,2,3,…

Then, $t_n = 4(2+5k)^6 + 36(2+5k)^5 + 210(2+5k)^4 + 720(2+5k)^3$

$+1470(2+5k)^2 + 1656(2+5k) + 794$

So, t_n= $62500k^6$+$262500k^5$ +506250 k^4+ 560000 k^3+366750 k^2+ 132720k +20514

This number ends with 6 , since it gives 6 (mod 10).

Case 3: When n= 3+5k, k=1,2,3,…

Then, $t_n = 4(3+5k)^6 + 36(3+5k)^5 + 210(3+5k)^4 + 720(3+5k)^3$

$+1470(3+5k)^2 + 1656(3+5k) + 794$

So, t_n= $62500k^6$+$337500k^5$ + $806250k^4$ +1080000 k^3+ 846750 k^2+365040k +67106

This number ends with 6 , since it gives 6 (mod 10).

Case 4: When n= 4+5k k=1,2,3,…

Then, $t_n = 4(4+5k)^6 + 36(4+5k)^5 + 210\,(4+5k)^4 + 720(4+5k)^3$

$+1470(4+5k)^2 + 1656(4+5k) + 794$

So t_n = $62500k^6$+$412500k^5$ + $1181250k^4$ +1870000k^3+ 1716750 k^2

+861960k+184026

This number ends with 6 , since it gives 6 (mod 10).

Case 5: When n = 5+5k, k=1,2,3,…

Then, $t_n = 4(5+5k)^6 + 36(5+5k)^5 + 210(5+5k)^4 + 720(5+5k)^3$

$+1470(5+5k)^2 + 1656(5+5k) + 794$

So, t_n = $62500k^6$+$487500k^5$ + $1631250k^4$ +2990000 k^3+ 3156750k^2

+1814280k +442074

Which ends with four, since this number gives 4 (mod 10).

(iii) Showing that t_n ends with the same number that t_{n+5} being as following:

Since $t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$

Then, $t_{n+5} = 4(n+5)^6 + 36(n+5)^5 + 210(n+5)^4 + 720(n+5)^3 + 1470(n+5)^2$

$+1656(n+5) + 794$

So, $t_{n+5} = 4n^6 + 156n^5 + 2610n^4 + 23920n^3 + 126270n^2 + 362856n + 442074$

So, $t_{n+5} - t_n = 120n^5 + 2400n^4 + 23200n^3 + 115800n^2 + 361200n + 441280$

Which gives 0 (mod 10), which means that both of them ends with the same number, so the pattern: 0, 4,6,6,4,…, is a repeated pattern.

Theorem 2.1.9: (Second Triple Squared Theorem)

Let, $t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$, where n=1,2,3,…

The following properties hold:

i) If the unit digit of $\{t_n\}$ is zero, then the decimal digit is nine.
ii) If the number t_n has four in its units digits, then the number in decimal digit is odd.

iii) If the number t_n has six in its units digit, then the number in the decimal digit is even.

Proof

(i) From part (ii) in proof of theorem (2.1.8), the only case for t_n to be ended with zero is when n = 1+5k, k=0,1,2,3,…

So, $t_n = 62500k^6+187500k^5 +281250\ k^4+ 250000\ k^3+132750\ k^2+ 39000k +4890$

$t_n /10 = 6250k^6+18750k^5 +28125\ k^4+ 25000\ k^3+13275\ k^2+ 3900k +489$

Which gives 9 (mod 10), which means that t_n has nine in its decimal digits.

(ii) The term t_n ends with four in two cases as we have been shown in part (ii) from proof of theorem (2.1.8), which are:

Case 1: when n= 2+5k,k=1,2,3

Then, $t_n=62500k^6+262500k^5 +506250\ k^4+ 560000\ k^3+366750\ k^2+ 132720k +20514$

Now, $(t_n-4) / 10 = 62500k^6+26250k^5 +50625\ k^4+ 56000\ \boldsymbol{k^3}+36675\ \boldsymbol{k^2}$

$+ 13272k +2051$

Which is an odd number

Then, t_n has an odd number in its decimal digit

Case 2: When n= 5k, k=1,2,3

Then, $t_n = 62500\boldsymbol{k^6}+112500\boldsymbol{k^5} + 131250\boldsymbol{k^4} +90000\ \boldsymbol{k^3}+ 36750\boldsymbol{k^2}$

$+8280k+794$

Now, $(t_n-4)/10 = 6250\boldsymbol{k^6}+11250\boldsymbol{k^5} + 131250\boldsymbol{k^4} +9000\ \boldsymbol{k^3}+ 3675\boldsymbol{k^2}$

$+828k+79$

Which is an odd number

Then, t_n has an odd number in its decimal digits

(iii) The general term t_n ends with 6 in two cases, as we have been shown in the second part in this proof, the cases are:

Case 1: when

n= 3+5k, k=1,2,3,…

Then, $t_n = 4(3+5k)^6 + 36(3+5k)^5 + 210(3+5k)^4 + 720(3+5k)^3$

$+1470(3+5k)^2 + 1656(3+5k) + 794$

So, $t_n = 62500k^6+337500k^5 + 806250k^4 +1080000\ k^3+ 846750\ k^2+365040k +67106$

Now, $(t_n – 6) / 10 = 6250k^6+33750k^5 + 80625k^4 +108000\ k^3+ 84675\ k^2$

$+36504k +6710$

Which is an even number.

Then, t_n has an even number in decimal digits.

Case 2: When n= 4+5k, k=1,2,3,…

Then, $t_n= 4(4 + 5k)^6 + 36(4 + 5k)^5 + 210(4 + 5k)^4 + 720(4 + 5k)^3$

$+1470(4 + 5k)^2 + 1656(4 + 5k) + 794$

So, $t_n = 62500k^6+412500k^5 + 1181250k^4 +1870000k^3+ 1716750\ k^2$

$+861960k +184026$

So, $(t_n – 6) / 10 = 6250k^6+41250k^5 + 118125k^4 +187000k^3$

$+ 171675\ k^2+86196k +18402$

Which is an even number, which means that t_n has an even number in its decimal digits

Theorem 2.1.10: (Third Triple Squared Theorem)

Let $t_n= 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$, where n=1,2,3,…

The following properties hold:

i) When examining the divisibility of t_n by three; for every three consecutive terms from $\{t_n\}$; One of them is not divisible by three. Moreover, the remainder is one when n=3,9,…, and two when n=6,12,…

ii) When term t_n is divided by four, the remainder is two.

iii) Sequence $\{t_n\}$ is divisible by five once every five consecutive terms in t_n.

(i) To prove this, we have the following cases:

Case 1: when n=1+3m, m=0,1,2,3,…

$t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$

So, $t_n = (1 + 3m)^6 + +36(1 + 3m)^5 + 210(1 + 3m)^4 + 720(1 + 3m)^3$

$+1470(1 + 3m)^2 + 1656(1 + 3m) + 794$

Then, $t_n =2916m^6 + 14580m^5 + 3645m^4 + 54000m^3 + 47790m^2 + 23400m + 4890$

Which gives 0 (mod 3), since each term from t_n is divisible by 3

Case 2: When n=2+3m, m=0,1,2,3,… .

Then, $t_n= 4(2 + 3m)^6 + 36(2 + 3m)^5 + 210(2 + 3m)^4 + 720(2 + 3m)^3 + 1470(2 + 3m)^2$

$+1656(2 + 3m) + 794$

So, $t_n = 2916m^6 + 20412m^5 + 65610m^4 + 120960m^3 + 132030m^2 + 7963m + 20514$

Which gives 0 (mod 3), *since each term is divisible by 3.*

Case 3: When n=3m, m=0,1,2,3,… .

Then, $t_n = 4(3m)^6 + 36(3m)^5 + 210(3m)^4 + 720(3m)^3$

$+1470(3m)^2 + 1656(3m) + 794$

So, $t_n = 2916m^6 + 8748m^5 + 17010m^4 + 19440m^3$

$+13230m^2 + 4968m + 794$

Which gives 1 (mod 3) when m=0,2,4,…, and gives 2 (mod 3) when m=1,3,5,…

So, t_n is not divisible by three every three terms from $\{t_n\}$.

(ii) Divisibility by four:

Since $t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 794$, n=1,2,3,…

There are two cases, which are:

Case 1: When n is an even natural number:

All terms are divisible by four except last term, which gives 2 (mod 4), so, the sum of them in this case gives 2 (mod 4).

Case 2: When n is an odd natural number:

The first, the second, the fourth terms are all divisible by three, and for the other terms, they are not divisible by three and each of them gives 2 (mod 4), and they are three terms, So the sum of them will give 2 (mod 4), So the result that the sum of all terms when n is an odd natural number gives 2 (mod 4).

Therefore, the result from the two cases is that t_n gives 2 (mod 4).

(iii) Divisibility by five:

Any number is divisible by five should ends with zero or five in its units digit. According to proof of part (i) from theorem (2.1.8), t_n is even, and part (ii) from the same proof show that t_n ends with 0 only every five terms, So t_n is divisible by five every five terms from the sequence$\{t_n\}$.

Example 2.1.3 Calculate the first 12 terms in sequence $\{t_n\}$, cheek Triple Squared Theorems and find values of $(t_n - t_{n-1})$.

Solution

Consecutives Cubic Natural Numbers	t_n	Parity	Unit digit	Decimal digit	r (mod 3)	r (mod 4)	r (mod 5)
(1,8,27,64)	4890	even	0	9	0 (mod 3)	2 (mod 4)	0 (mod 5)
(8,27,64,125)	20514	even	4	1	0 (mod 3)	2 (mod 4)	4 (mod 5)
(27,64,125,216)	67106	even	6	0	1 (mod 3)	2 (mod 4)	1 (mod 5)
(64,125,216,343)	184026	even	6	2	0 (mod 3)	2 (mod 4)	1 (mod 5)
(125,216,343,512)	442074	even	4	7	0 (mod 3)	2 (mod 4)	4 (mod 5)
(216,343,512,729)	957890	even	0	9	2 (mod 3)	2 (mod 4)	0 (mod 5)

2.2 Four Lagrange Squares with Triangular Numbers

This section, concerns linking four Lagrange squares theorem with the sequence of triangular numbers. In addition, analyzing this sequence through specific theorems related to it, providing the proofs.

Definition 2.2.1: (Tetra-Triangular Sequence)

We define $\{v_n\}$ to be a sequence of natural numbers, such that each element is the sum of four squares of consecutive triangular numbers.

So, $v_n = (\frac{n^2+n}{2})^2+(\frac{(n+1)^2+n+1}{2})^2 + (\frac{(n+2^2)+n+2}{2})^2 + (\frac{(n+3^2)+n+3}{2})^2$

So, $v_n = n^4 + 8n^3 + 31n^2 + 60n + 46$

Tetra-Triangular Theorems

Theorem 2.2.11: (First Tetra Triangular Theorem)

The sequence $\{v_n\}$ satisfies the following properties:

i) The general term v_n is an even natural number.

ii) The general term v_n ends with one element of the set (0,2,6) as number in unit digit.

iii) Units digit for numbers taken from the sequence $\{v_n\}$, have the following pattern: (6,0,2,0,6, …), and it is a repeated pattern, according to the following periodicity:

v_k and v_{k+5} both have the same number in unit digit.

Proof

(i) Given that $v_n = n^4 + 8n^3 + 31n^2 + 60n + 46$, we have two cases:

Case 1: When n is an odd natural number:

In this case, first and third terms are odd, so sum of them is even, while the other terms are even, so sum of all terms is even.

Case 2: When n is an even natural number:

In this case, first and third terms are even, so sum of them is even, while the other terms are even, so sum of all terms is even.

(ii) We have the following five cases:

Case 1: When n = 1+5k, k=0,1,2,…

Then, $v_n = (1+5k)^4 + 8(1+5k)^3 + 31\,(1+5k)^2 + 60(1+5k) + 46$

So, $v_n = 625\,k^4 + 1500k^3 + 1525k^2 + 750k + 146$

This number ends with 6, since it gives 6 (mod 10).

Case 2: When n= 2+5k, k=0,1,2,3,…

Then, $v_n = (2+5k)^4 + 8(2+5k)^3 + 31\,(2+5k)^2 + 60(2+5k) + 46$

So, $v_n = 625\,k^4 + 2000k^3 + 2575k^2 + 1560k + 370$

This number ends with 0, since it gives 0 (mod 10).

Case 3: When n= 3+5k, k=0,1,2,3,…

Then, $v_n = (3+5k)^4 + 8(3+5k)^3 + 31\,(3+5k)^2 + 60(3+5k) + 46$

So, $v_n = 625\,k^4 + 2500k^3 + 3925k^2 + 2850k + 802$

This number ends with2 , since it gives 2 (mod 10)

Case 4: When n= 4+5k k=0,1,2,3,…

Then, $v_n = (4+5k)^4 + 8(4+5k)^3 + 31\,(4+5k)^2 + 60(4+5k) + 46$

So, $v_n = 625\,k^4 + 3000k^3 + 5575k^2 + 4740k + 1550$

This number ends with 0 , since it gives 0 (mod 10).

Case5: When n= 5k, k=,1,2,3,…

Then, $v_n = (5k)^4 + 8(5k)^3 + 31\,(5k)^2 + 60(5k) + 46$

So, $v_n = 625\,k^4 + 1000k^3 + 775k^2 + 300k + 46$

Which ends with 6, since this number gives 6 (mod 10).

So in all cases: v_n ends only with 0 or 2 or 6,with the pattern: 6,0,2,0,6,…,

(iii) The number of elements of this pattern is five, so to prove it is a repeated pattern; we need to prove that v_n and v_{n+5} have that same number in unit digit, as following:

$v_{n+5} = (n+5)^4 + 8(n+5)^3 + 31(5+n)^2 + 60(5+n) + 46$

$v_{n+5} = n^4 + 28n^3 + 301n^2 + 1470n + 2746$

$v_n = n^4 + 8n^3 + 31n^2 + 60n + 46$

So,

$v_{n+5} - v_n = (n^4 + 28n^3 + 301n^2 + 1470n + 2746) - (n^4 + 8n^3 + 31n^2 + 60n + 46)$

So, $v_{n+5} - v_n = 20n^3 + 270n^2 + 1410n + 2700$

Which gives $0 \pmod{10}$, which means that the pattern is repeated every 5 terms in $\{v_n\}$

Theorem 2.2.12: (Second Tetra Triangular Theorem)

i) If the unit digit of v_n is two, then the decimal digit is zero and this is cyclic every five numbers from $\{v_n\}$.
ii) If the unit digit of v_n is zero, then the number in the decimal digit is an odd number.

iii) If the unit digit of v_n is zero, then the number in the decimal digit is an even number.

Proof

(i) According to part (i) in first Tetra Triangular Theorem, v_n ends with two when

$v_n = (3 + 5k)^4 + 8(3 + 5k)^3 + 31\ (3 + 5k)^2 + 60(3 + 5k) + 46$

And it is the only case that make v_n ends with 2, since other cases make v_n ends with 0 or 6 as shown in part (1) from the proof of First Tetra Triangular Theorem, now:

$v_n = (3 + 5k)^4 + 8(3 + 5k)^3 + 31\ (3 + 5k)^2 + 60(3 + 5k) + 46$

$v_n = 625\ k^4 + 2500k^3 + 3925k^2 + 2850k + 802$

$(v_n - 2) / 10 = 62{,}5\ k^4 + 250k^3 + 392{,}5k^2 + 285k + 80$

Which gives 0 (mod 10), so when v_n ends with 2 then it has 0 in decimal digit.

(ii) In part (i) in the proof of First Tetra Triangular Theorem, v_n ends with zero in two cases as following:

Case 1: $v_n = (2 + 5k)^4 + 8(2 + 5k)^3 + 31\ (2 + 5k)^2 + 60(2 + 5k) + 46$

So $v_n = 625\ k^4 + 2000k^3 + 2575k^2 + 1560k + 370$

So $(v_n - 0) /10 = 62{,}5\ k^4 + 200k^3 + 257{,}5k^2 + 156k + 37$

Which is odd, so it ends with odd number, so v_n has odd number in its decimal digit.

Case 2:

$v_n = (4 + 5k)^4 + 8(4 + 5k)^3 + 31\ (4 + 5k)^2 + 60(4 + 5k) + 46$

Then, $v_n = 625\ k^4 + 3000k^3 + 5575k^2 + 4740k + 1550$

So, $v_n /10 = 62{,}5\ k^4 + 300k^3 + 557{,}5k^2 + 474k + 155$

Which is odd, so it ends with odd number, so v_n has an odd number in its decimal digit.

Therefore, if v_n has six in its unit digit, then it will have an even number in its decimal digit.

(iii) According to part (ii) in proof of First Tetra Triangular Theorem, v_n ends with zero in two cases, now:

Case 1:

$v_n = (1+5k)^4 + 8(1+5k)^3 + 31(1+5k)^2 + 60(1+5k) + 46$

So, $v_n = 625\,k^4 + 1500n^3 + 1525k^2 + 750k + 146$

Now $(v_n-6)/10 = 62.5\,k^4 + 150n^3 + 152.5k^2 + 75k + 140$

Which is even, then v_n has even in its decimal digits.

Case 2:

$v_n = (5k)^4 + 8(5k)^3 + 31(5k)^2 + 60(5k) + 46$, k =1,2,3,…

$= 625\,k^4 + 1000k^3 + 775k^2 + 300k + 46$

Now, $(v_n-6)/10 = 62.5\,k^4 + 100k^3 + 77.5k^2 + 30k + 4$, which is even

Then, v_n has an even number in its decimal digit.

Therefore, if v_n has six in its unit digit, then it have an even number in its decimal digit.

Theorem 2.2.13: (Third Tetra Triangular Theorem)

i) The general term v_n is not divisible by three.

ii) The general term v_n is divisible by five once every five consecutive terms.

Proof

(i) Since: $v_n = n^4 + 8n^3 + 31n^2 + 60n + 4$

Then, we have the following cases:

Case 1: when n=1+3m, where m=1,2,3,…

Then, $v_n = (1+3m)^4 + 8(1+3m)^3 + 31(1+3m)^2 + 60(1+3m) + 46$

$= 81m^4 + 324m^3 + 549m^2 + 450m + 146$

Which gives 2 (mod 3)

Case 2: when n=2+3m, where m=1,2,3,…

Then, $v_n = (2+3m)^4 + 8(2+3m)^3 + 31(2+3m)^2 + 60(2+3m) + 46$

$= 81m^4 + 432m^3 + 927m^2 + 936m + 37$

Which gives 1 (mod 3)

Since 81 gives 0 (mod 3) and 432, 927, 936 also give 0 (mod 3), but 370 gives 1 (mod 3).

Case 3: when n=3m, m=1,2,3,…

Then, $v_n = (3m)^4 + 8(3m)^3 + 31(3m)^2 + 60(3m) + 46$

$$= 81m^4 + 202m^3 + 279m^2 + 180m + 46$$

Which gives 1 (mod 3)

ii) Divisibility by five

Any number to be divisible by five should ends with zero or five, and in this sequence, v_n ends with 0 two times every five terms, and no term ends with five ,so the conclusion for divisibility by five is that, the sequence divisible by five two times every five terms, which are the cases where v_n ends with zero.

Example 2.2.1 Calculate the first 12 terms in $\{v_n\}$ and cheek the properties of Tetra Triangular Theorems, and find values of $(v_n - v_{n)}$

Solution

Consecutive Triangular Numbers	v_n	$v_n - v_{n-1}$	Parity	Unit digit	Decimal digit	v (mod 3)	v (mod 4)	v (mod 5)
(1,3,6,10)	146	-	even	6	4	2 (mod 3)	2 (mod 4)	1 (mod 5)
(3,6,10,15)	370	224	even	0	7	1 (mod 3)	2 (mod 4)	0 (mod 5)
6,10,15,21)	802	432	even	2	0	1 (mod 3)	2 (mod 4)	2 (mod 5)
(10,15,21,28)	1550	432	even	0	5	2 (mod 3)	2 (mod 4)	0 (mod 5)
(15,21,28,36)	2746	1196	even	6	4	1 (mod 3)	2 (mod 4)	1 (mod 5)
(21,28,36,45	4546	1800	even	6	4	1 (mod 3)	2 (mod 4)	1 (mod 5)
(28,36,45,55)	7130	2584	even	0	3	2 (mod 3)	2 (mod 4)	0 (mod 5)
(36,45,55,66)	10702	3572	even	2	0	1 (mod 3)	2 (mod 4)	2 (mod 5)
(45,55,66,78)	15490	4788	even	0	9	1 (mod 3)	2 (mod 4)	0 (mod 5)
(55,66,78,91)	21746	6256	even	6	4	2 (mod 3)	2 (mod 4)	1 (mod 5)

Graph of Tetra Triangular Sequence

The given sequence defined as:

$v_n = n^4 + 8n^3 + 31n^2 + 60n + 46$, n=1,2,3,…

The graph is:

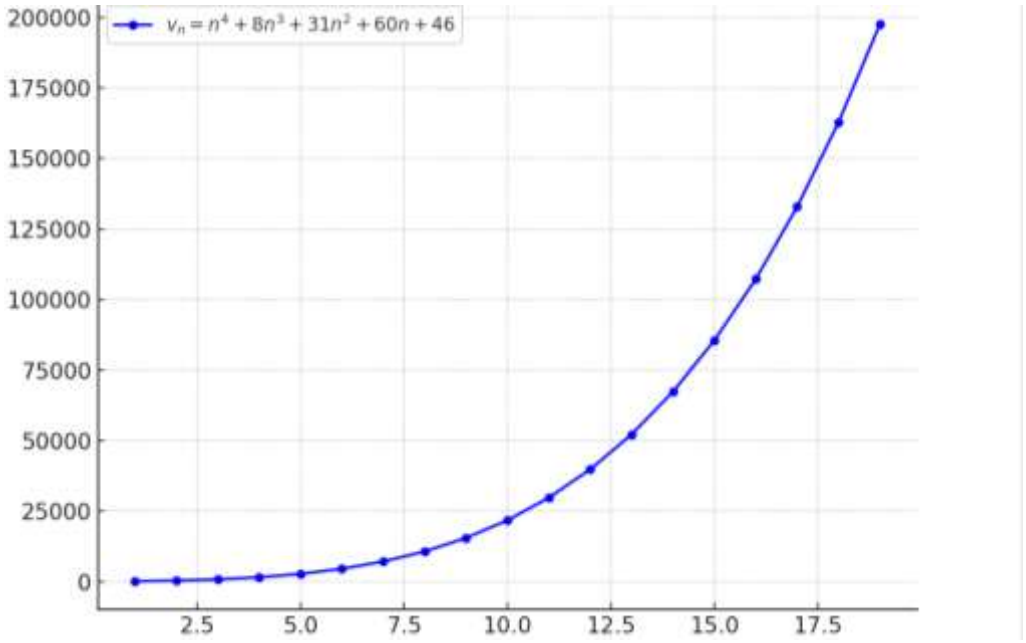

Figure 2.5: Graph of v_n vs. n

From the graph, we observe the following characteristics:

The sequence exhibits a rapid growth rate due to the dominance of the quartic term (n^4). The values of r_n increase steeply as n grows, confirming that the sequence follows an exponential-like trend for large n, and the points on the graph represent discrete values, as the sequence defined for natural numbers.

The function is strictly increasing, meaning each successive term is greater than the previous one. This behavior aligns with the expectation for a polynomial of degree 4, where the leading term dictates the overall growth pattern.

2.3 Linking Fibonacci Sequence with Lagrange Squares

This section concerns linking Fibonacci sequence With Lagrange squares, by taking a new sequence, such that every term in this new sequence is the sum of squares of four consultative Fibonacci numbers.

Definition 2.3.1: (Lagrange – Fibonacci Sequence)

We define a Lagrange- Fibonacci Sequence $\{m_n\}$ as following:

Let $m_n = a^2+b^2+c^2+d^2$, where $a = (f_n)^2$, $b = (f_{n+1})^2$, $c = (f_{n+2})^2$, $d = (\mathbf{f_n}_{+3})^2$

According to Fibonacci sequence , $f_{n+2} = f_n + f_{n+1}$, and $f_{n+3} = f_{n+1} + f_{n+2}$

So, $m_n = (f_n)^2+(f_{n+1})^2+(f_{n+2})^2+(f_{n+3})^2$

So, $m_n = (f_n)^2+(f_{n+1})^2+(f_n + f_{n+1})^2+(f_{n+1} +f_{n+2})^2$

$m_n = (f_n)^2 + (f_{n+1})^2 + (f_n)^2 + (f_{n+1})^2 +(2f_nf_{n+1})+ (f_{n+1})^2+ (f_{n+2})^2 + (2f_{n+1} f_{n+2})$

So, $m_n = 2(f_n)^2 + 3(f_{n+1})^2 + (f_{n+2})^2 + (2f_nf_{n+1})+ (2f_{n+1}f_{n+2})$

Theorem 2.5.14: (Lagrange and Fibonacci Theorem)

Let $m_n = 2(f_n)^2 + 3(f_{n+1})^2 + (f_{n+2})^2 + (2f_nf_{n+1})+ (2f_{n+1}f_{n+2})$, where f_{n+1},f_{n+2} and f_{n+3} taken from Fibonacci Sequence, then the following properties hold

(1) Sequence $\{m_n\}$ follows an exponential growth pattern with leading order $\varphi(2n)$.

(2) Sequence $\{m_n\}$ follows a predictable even-odd pattern based on n.

Proof

(i) $f_n \approx (\varphi^n)/\sqrt{5}$, where φ is the golden ratio

We approximate m_n as:

$m_n \approx (\varphi^{(2n)} + \varphi^{(2(n+1)} + \varphi^{2(n+2)} + \varphi^{(2(n+3)}) / 5$

Thus, the sequence follows an exponential growth pattern with leading order $\varphi^{(2n).}$

(ii) By observing the parity of Fibonacci numbers, we note that:

Odd and even Fibonacci numbers alternate periodically. Squaring them maintains the parity structure. Thus, m_n follows a predictable even-odd pattern based on n.

This section introduced a sequence derived from Fibonacci numbers and Lagrange's theorem. We proved its exponential growth, and its parity. Future research may involve deeper combinatorial analysis and algebraic expressions for specific cases.

Example 2.5 Calculate the first 13 terms in Lagrange- Fibonacci sequence, determine the parity and the difference between two consecutive terms and find the divisibility on 3,4 and 5.

Solution:

Numbers to be Squared	m_n	Parity	$m_n - m_{n-1}$	r (mod 3)	r (mod 4)	r (mod 5)
(0, 1, 1, 2)	6	Even	-	0 (mod 3)	0 (mod 4)	0 (mod 5)
(1, 1, 2, 3)	15	Odd	9	0 (mod 3)	0 (mod 4)	0 (mod 5)
(1, 2, 3, 5)	39	Odd	24	0 (mod 3)	0 (mod 4)	0 (mod 5)
(2, 3, 5, 8)	102	Even	63	0 (mod 3)	0 (mod 4)	0 (mod 5)
(3, 5, 8, 13)	267	Odd	165	0 (mod 3)	0 (mod 4)	0 (mod 5)
(5, 8, 13, 21)	699	Odd	432	0 (mod 3)	0 (mod 4)	0 (mod 5)
(8, 13, 21, 34)	1830	Even	1131	0 (mod 3)	0 (mod 4)	0 (mod 5)
(13, 21, 34, 55)	4791	Odd	1961	0 (mod 3)	0 (mod 4)	0 (mod 5)
(21, 34, 55, 89)	12543	Odd	7752	0 (mod 3)	0 (mod 4)	0 (mod 5)
(34, 55, 89, 144)	32838	Even	12295	0 (mod 3)	0 (mod 4)	0 (mod 5)
(55, 89, 144, 233)	85971	Odd	53133	0 (mod 3)	0 (mod 4)	0 (mod 5)
(89, 144, 233, 377)	225075	Odd	139104	0 (mod 3)	0 (mod 4)	0 (mod 5)
(144, 233, 377, 610)	589254	Even	364179	0 (mod 3)	0 (mod 4)	0 (mod 5)

Graph of Lagrange – Fibonacci Sequence

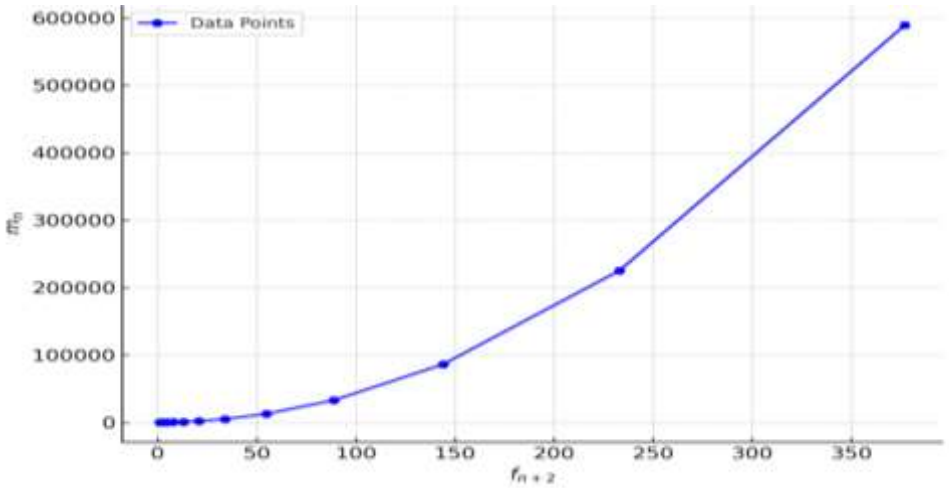

Figure 2.4: m_n vs. f_{n+2}

Geometric Analysis

The plotted curve represents the relationship between the Fibonacci sequence index f_{n+2} and the corresponding values m_n. Observing the curve, we note the following characteristics: The function exhibits an exponential-like growth pattern, indicating a rapid increase in values.

The curve is strictly increasing, showing that as f_{n+2} grows, m_n increases significantly. The concavity of the curve suggests an accelerating growth rate, meaning the second derivative is positive. The function appears to follow a polynomial or exponential trend, requiring further analysis for precise modeling.

In conclusion, the relationship between m_n and $f_{\{n+2\}}$suggests an underlying mathematical structure that could be analyzed further using logarithmic transformations or polynomial fitting.

Chapter Three

Philosophical Insights in Mathematics

This chapter deal with ideas in mathematical philosophy, based on what discovered in the Second Chapter from this thesis.

3.1 Coexistence of Opposites within a Single Entity

Throughout the ages, philosophers and scientists have pointed to a fundamental concept in the universe and life: the coexistence of opposites within a single entity.

The green trees, mostly composed of water, from which fire can be extracted; the darkness at the end of the night, from which the light of dawn emerges; the sick individual who begins to recover gradually, carrying the seeds of healing even while still ill. These, and many other examples, illustrate the profound principle that opposites coexist and complement one another within the same entity.

Heraclitus Said:

"The road up and the road down are the same."[19]

AL Hazen Said:

"The contradiction inherent in the natural world, when properly understood, reveals the deeper truths of existence and the fundamental laws governing the cosmos."[17]

AL-Biruni Said:

"The nature of things is revealed through their opposites; the greater the contradiction, the closer we come to understanding the true essence."[1]

One of the central ideas in the philosophy of mathematics is its correspondence with the universe. What we aim to demonstrate here is that the principle of containing the opposite exists in mathematics just as it does in the real universe, using Fibonacci-Lagrange Squares for this purpose.

The Fibonacci sequence represents chaos and dynamic changes, while Lagrange squares represent strict and static order. Since the sequence connected to Lagrange Four Squares theory in the manner detailed in Chapter 2, we were able to demonstrate the containment of change and dynamism within strict and static order. Moreover, we expressed the strict order through the fact that its four components are dynamic and variable. Here, a mathematical system has been contained within another mathematical system of an opposing nature, namely, the containment of change within stability.

The Fibonacci sequence grows linearly, while Lagrange Four Squares theory presents a quadratic growth, which appears to be a seeming contradiction. However, through the Fibonacci-Lagrange theory, linear growth is contained within quadratic growth, representing one aspect of the containment of opposites.

The Fibonacci sequence based on composition, where each new term is the sum of the previous two terms, whereas Lagrange's Four Squares represent analysis, where a number is expressed as the sum of squares of integers. Here, composition is contained within analysis, two seemingly contradictory properties.

The Fibonacci sequence represents a simple progression, while Lagrange Four Squares theory represents a more complicated, accelerated system.

The Fibonacci sequence does not provide a probabilistic aspect, where each number has a unique representation expressed as the sum of the previous two terms. In contrast, Lagrange's Four Squares theory provides a high probabilistic aspect, as numbers not belonging to the Jacobi sequence expressed in multiple ways. Thus, through the presented theory, determinism is contained within probability.

Summary: The Fibonacci-Lagrange theory provides evidence for the containment of opposites in mathematics, demonstrating that a system with certain properties transformed into another system with different properties.

By linking the Fibonacci sequence and Lagrange Four Squares, as demonstrated in Chapter 2, we show that the same number can appear in a geometric form and a sequential growth pattern. This represents the overlap of numbers in a complex mathematical structure that reflects both natural sequence and geometric symmetry.

While the geometric symmetry in Lagrange’s theory might seem contradictory to the growing natural sequence in Fibonacci, the connection between them proves that what appears as a mathematical contradiction in numerical structures can be contained by transitioning from one property to its opposite.

This containment of opposites also reflected in how a number may appear static in algebra and geometry, yet grows sequentially in the Fibonacci sequence, combining stability and

change simultaneously. Lagrange squares show the fixed structure of numbers, while the Fibonacci sequence indicates continuous evolution, suggesting that even systems with seemingly opposing properties can transform into one another, integrating both rigidity and dynamism in the mathematical universe.

Question: Can a number, which considered a static mathematical entity, and the dynamic process represented by natural growth, both express a part of the greater truth?

Answer: We discover that the universe does not move in a single direction, but rather contains within itself an overlap of opposites. Mathematics has great power to represent existence in different states: both static and dynamic.

Thus, we conclude that mathematics is not just a set of theoretical calculations or something slightly broader than that. It can also provide reinforcement for deep philosophical ideas that explain the universe, such as the concept of containing opposites. Mathematics directed to become more humanistic and enhance our understanding of existential questions and cosmic concepts.

Gödel spoke about the cognitive and cosmic limits of mathematics. Even in the most complete mathematical systems, every external system requires its opposite. Thus, all of mathematics interconnected, as every mathematical theory or system is incomplete. Therefore, the idea of linking mathematical stability with mathematical change, as represented in the Fibonacci sequence, based on a fundamental principle: Gödel's incompleteness theorem.

Gödel said:

"The more I studied, the more I became aware of the fact that mathematics, far from being a static system of knowledge, is a growing, living organism."[14]

Based on this discussion, it affirmed that the connection between Lagrange Four Squares and the Fibonacci sequence goes beyond the boundaries of mathematics as an abstract tool. It expresses a cosmic pattern based on the balance between stability and change. This suggests that the mathematical existence, in essence, be a containment of opposites: between stability and movement, between perfection and change, between order and chaos.

Closing statement: "The connection between the Fibonacci sequence and Lagrange Four Squares theory demonstrates the ability of mathematics to contain the opposites of stability and growth, geometric symmetry and natural expansion, reflecting the philosophical truth that the universe and mathematics are not mere contradictory representations, but rather integrated entities in an ongoing process.

3.2 Creative Adjacency

The connection between triangular numbers and Lagrange Four Squares Theorem reflects a profound philosophical idea: "creative adjacency". This concept suggests that mathematical innovation arises when seemingly distinct systems interact and give rise to entirely new structures. By linking triangular numbers, which represent a geometric

progression of accumulation, with Lagrange's theorem, which is rooted in numerical decomposition, a bridge formed between geometric order and numerical universality.

This interaction illustrates how mathematical systems, like the universe, thrive on the interplay of distinct entities, producing harmony and insight where isolation once prevailed.

On Creativity in Mathematic

Jacques Hadamard, in The Psychology of Invention in the Mathematical Field, emphasizes the creative essence of mathematics:

"Great ideas are born when separate worlds unexpectedly meet in a moment of shared understanding."[15]

This underscores how linking triangular numbers and Lagrange's theorem epitomizes the unification of distinct mathematical domains.

On the Unity of Mathematical Structures

Philip J. Davis and Reuben Harsh, in The Mathematical Experience, argue:

""Mathematics is not merely a collection of isolated theorems but a network of interwoven ideas، creating a new architecture of thought."[4]
The work on this connection between Lagrange Four Squares and Triangular Numbers aligns with this view, presenting a harmonious interplay between geometry and number theory.

On the Philosophical Role of Numbers in Structure

Stephan Körner, in The Philosophy of Mathematics, discusses how numbers form the essence of structural and philosophical thought:

"Numbers are not just tools but symbols of order and growth, bridging abstract logic with tangible reality."[22]

This view supports the idea that triangular numbers embody accumulation and order, while Lagrange theorem reflects a universal decomposition.

On Mathematical Innovation as a Product of Interplay

John Nash, in his works on systems theory reflects:

"Mathematical creativity lies in transforming complexity into unity allowing disparate systems to converse in a shared language."[24]

The work mirrors this idea by enabling a "conversation" between triangular numbers and Lagrange theorem.

3.3 A Journey from Simplicity to Complexity in Numerical Structures

Galilei said:

"Mathematics is the language in which God has written the universe." [9]

Mathematics, much like philosophy, is not merely a tool for computation but a profound language that reveals the inherent order of the cosmos. In this section, we explore three distinct sequences that not only demonstrate numerical relationships but also symbolize an evolutionary journey—from the basic elements of nature to more refine and abstract forms.

These sequences defined as follows:

Sequence $\{a_n\}$: Each term defined as the sum of the squares of four consecutive natural numbers.

Sequence $\{s_n\}$: Each term defined as the sum of the squares of four consecutive squares.

Sequence $\{t_n\}$: Each term defined as the sum of the squares of four consecutive cubes.

These sequences serve as metaphors for the transformation from elemental simplicity to complex perfection, mirroring the very evolution of cosmic order.

Sequence $\{a_n\}$: Embodiment of Simplicity and Natural Order

Sequence $\{a_n\}$ constructed by summing the squares of four consecutive natural numbers. Natural numbers represent the primary building blocks in any mathematical system; they are the fundamental elements from which complexity arises. Here, squaring each natural number and summing them reflects an inherent harmony—a basic yet essential interplay that mirrors the ordered structure of nature.

The sequential progression of these numbers emphasizes regularity and balance. Each term in $\{a_n\}$ can be interpreted as a stepping-stone in the transformative process of numerical abstraction. In this light, $\{a_n\}$ stands as a symbol of pure, unadulterated simplicity an initial state from which more sophisticated forms emerge.

Sequence $\{s_n\}$: Transition from Natural Squares to Ideal Perfection.

In Sequence $\{s_n\}$, the building blocks are no longer mere natural numbers but perfect squares. Here, each term is the sum of the squares of four consecutive perfect squares. Perfect squares regarded as icons of geometric and numerical perfection. Their inherent symmetry and balance elevate the discussion from basic arithmetic to a realm of ideal forms.

Einstein said:

"Pure mathematics is, in its way, the poetry of logical ideas."[7]

"By squaring perfect squares‹ Sequence $\{s_n\}$ amplifies the notion of refinement‹ transitioning from tangible numbers to their idealized counterparts. This process symbolizes the evolution of ordinary elements into forms that exude an almost transcendent beauty—a journey from the concrete to the abstract, from the mundane to the sublime.

Sequence $\{t_n\}$: The Pinnacle of Complexity and Mathematical Creativity

Sequence $\{t_n\}$ reaches the zenith of this transformative journey by utilizing cubes—entities that introduce a third dimension into our numerical understanding. Here, each term is the sum of the squares of four consecutive cubes. Unlike squares, cubes suggest a volumetric and spatial complexity, providing a richer and more intricate mathematical tapestry.

""Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."[27]

In sequence $\{t_n\}$, squaring cubes fuses spatial intuition with rigorous algebraic operations. This synthesis highlights the convergence of creativity and systematic thought. It is an exemplar of mathematical ingenuity, where layers of abstraction coalesce to provide insights into the structure of reality itself.

Integration of the Sequences: An Evolutionary Journey through Levels of Mathematical Existence

When viewed collectively, Sequences $\{a_n\}$, $\{s_n\}$, and $\{t_n\}$ illustrate an evolutionary narrative in numerical form. The journey begins with the elementary simplicity of $\{a_n\}$, ascends through the ideal perfection of $\{s_n\}$, and culminates in the multidimensional complexity of $\{t_n\}$. This progression reflects the manner in which the universe itself develops—from simple, natural elements to intricate, interwoven structures.

Poincare said:

"Mathematics is the art of giving the same name to different things". [26]

This philosophical synthesis suggests that the apparent diversity in numerical forms is underpinned by a fundamental unity. Each sequence, though distinct in its construction, forms an integral part of a broader, harmonious system illustrating the interplay between simplicity and complexity, matter and ideality.

3.4. Mathematics as a Mirror of Philosophy and the Cosmos

In conclusion, the study of Sequences $\{a_n\}$, $\{s_n\}$, and $\{t_n\}$ demonstrates that mathematics transcends mere computation; it is a medium for exploring the profound philosophical questions of existence. Each sequence represents a phase in the grand narrative of transformation from the elemental simplicity of natural numbers to the sublime intricacies of abstract forms.

Gauss Said:

"Mathematics is the queen of the sciences"[13]

This final reflection encourages us to view mathematics not just as a collection of formulas and computations but as a universal language, that articulates the deepest truths of the cosmos. It is through these sequences that we glimpse the interplay between the tangible and the abstract a mirror reflecting both the logical and the poetic aspects of reality

Chapter Four

Applications

This chapter, present applications on Lagrange squares, especially on the sequences that discovered in Chapter two from this thesis.

These applications come from various fields. Some applications supported with specific explanations about the causality that links the application to the chosen sequence.

4.1. Lagrange Chees

Lagrange Chess is an innovative game concept that fuses the strategic depth of classical chess with the numerical precision of Lagrange Four Square Theorem. According to this theorem, every natural number expressed as:

$N= a^2+b^2+c^2+d^2$, (a,b,c,d ∈ natural numbers U {0})

In addition, the steps as following:

a represent number of steps to the north, b represents number of steps to the south, **c** represents number of steps to the east, and d represents number of steps to the west.

The sum or specific combinations of these values determine the magnitude of a piece's move, thereby introducing a dual layer of tactical planning that intertwines mathematical reasoning with traditional chess strategy.

Theoretical Foundations and Game Mechanism

In Lagrange Chess, each turn is not solely about moving pieces based on fixed rules; rather, it involves a mini mathematical puzzle. The drawn number is analyzed and decomposed into its four-square representation, and the resulting square roots are used as "movement units" in the corresponding cardinal directions. The player must decide how to best allocate these values according to the piece they intend to move, balancing both numerical optimization and board strategy.

For example, a card drawn with the number 29 might be decomposed

$29= 25+4+0+0=5^2+2^2+0^2+0^2$

Or

$29= 16+9+4+0 = 4^2+3^2+2^2+0^2$

Detailed Piece Movements in Lagrange Chess

1) King

The King remains a highly sensitive piece. Its movement is determined by the smallest non-zero square root among the four derived values—i.e. The King moves exactly that many squares in a chosen direction, ensuring cautious advancement and emphasizing its vulnerability on the board.

2) Queen

As the most versatile piece, the Queen utilizes the sum of all non-zero square roots. With a total movement capacity of, the player can distribute this total between horizontal, vertical, or even diagonal directions based on open paths on the board. This grants the Queen extensive control and flexibility, reflective of its classical dominance in chess.

3) Rook (Castle)

The Rook's movements remain confined to horizontal and vertical paths. In this variant, the vertical movement is determined by the values assigned to the north and south (for example, and), while the horizontal movement is guided by the east and west values (and). The player chooses one of the two axes per turn and moves the Rook by summing the appropriate pair of square roots, ensuring that the chosen path is unobstructed.

4) Bishop

The Bishop continues to move diagonally. However, its range now depends on a combination of two directional values. For instance, moving northeast might use the sum, while a southwest move might rely on. This dual-component move requires the player to strategically balance the two components to maximize the Bishop's diagonal reach.

5) Knight

The Knight redefines its classic "L-shaped" move by utilizing two non-adjacent square roots. Typically, one square root—preferably the larger value—determines the length of the primary move in one direction (either vertical or horizontal), while the smaller value, drawn from the remaining non-zero options, dictates the perpendicular offset. This method preserves the Knight's unique ability to jump over pieces while introducing a calculated numerical twist.

6) Pawn

The Pawn's forward movement determined by the smallest available square root, commonly, moving it forward by that number of squares. In situations where the Pawn may capture an opposing piece diagonally, a combination of two square roots used to

facilitate this diagonal strike, thus allowing the Pawn to engage dynamically in tactical maneuvers.

Strategic and Tactical Considerations:

Chess Lagrange introduces a dual-layer challenge that compels players to engage in both numerical and strategic analysis. Key aspects include:

Tactical Flexibility: Players must continuously adjust their strategies based on the numerical representation drawn each turn, transforming what was once a static move set into a dynamic puzzle.

Rapid Numerical Analysis: The game incentivizes quick mental calculations, forcing players to determine optimal representations for each drawn number in real time.

Creative Planning: The interplay between mathematical decomposition and piece movement opens opportunities for unconventional tactics and innovative maneuvers that differ substantially from standard chess play.

Conclusion

The proposed Lagrange chess system represents a pioneering fusion of classical chess strategy and advanced mathematical theory. By integrating the principles of Lagrange's Four Square Theorem into the movement mechanics, the game challenges players to think both logically and creatively. This approach not only deepens the strategic complexity inherent in traditional chess but also offers a stimulating playground for mathematical ingenuity, making it a compelling subject of study and an engaging pursuit for intellectual gamers.

4.2 Applications: Solved Problem

This section uses the equations of the sequences that in chapter 2 to solve many problems from different fields.

Example 4.2.1: (Kinetic Energy)

A ball fell towards the ground and collided with it four consecutive times. Its velocity during the last impact was 4 m/s, while its velocities during the previous impacts were gradually greater as follows: 5, 6, and 7 m/s. calculate the total kinetic energy for all the impacts, given that the mass of the ball was 2 kg.

Solution:

$K_E = 0.5\ mv^2$

Total K.E = $K.E_1 + K.E_2 + K.E_3 + K.E_4$

$= 0{,}5m\ (v_1^2 + v_2^2 + v_3^2 + v_4^2)$

But $v_3 = v_4 + 1$ and $v_2 = v_3 + 1$ and $v_1 = v_2 + 1$

Using the forth term in Squared Sequence as following:

Total K.E = $0.5\ m\ (4n^2 + 12n + 14)$, where $n = v_1 = 4$

$= (0.5)\ (2)\ (64+48+14) = 63$ joule

Example 4.2.2: (Encryption)

A new encryption method proposed using a sequence whose general term is the sum of the squares of four consecutive square numbers.

The sequence defined as:

$s_n = n^4 + 24\,n^3 + 84\,n^2 + 110n + 98$, n = 1,2,3,… (which is Super Squared Sequence)

Encrypt the word "SAFE" using this sequence. The encryption process involves the following steps:

1) Convert each letter to its corresponding alphabetical position (A=1, B=2,... ,Z=26)

2) Apply the sequence formula to each number obtained

3) Send the encrypted numbers as the cipher text.

Solution

Step 1: Convert Letters to Numbers

Using the standard alphabetical encoding:

S = 19

A = 1

F = 6

E = 5

Thus, the numerical representation of the word is [5, 6, 1, 19]

Step 2: Encrypt Using the Sequence

We will use the sequence formula for each number

For the first number

$s_n = n^4 + 24\,n^3 + 84\,n^2 + 110n + 98$

$s_{19}=19^4+(24)(19^3)+(84)(19^2)+(110)19+98=327641$

For the second term

$s_1=1^4+(24)(1^3)+(84)(1^2)+(110)(1)+98=317$

For the third term

$s_6=6^4+(24)(6^3)+(84)(6^2)+(110)6+98=10262$

For the fourth term

$s_5=5^4+(24)(5^3)+(84)(5^2)+(110)(0.5)+98=6373$

Step 3: Transmit the Encrypted Message

The cipher text to be sent is:

6373, 10262, 317, 327641

Step 4: Decryption Process

To decrypt the message, the receiver needs to know the sequence formula.

They would then solve from the equation:

$s_n = n^4 + 24\,n^3 + 84\,n^2 + 110n + 98$

Example 4.2.3: (Economic)

A tech startup specializes in developing advanced artificial intelligence (AI) systems. Due to rapid technological advancements and growing market demand, the company experiences exponential growth in its revenue. Their revenue model follows the sequence:

$t_n = 4n^6 + 36n^5 + 210n^4 + 720n^3 + 1470n^2 + 1656n + 7744$, n = 1,2,3,…

The company's revenue grows rapidly because:

They continuously upgrade their AI products, enhancing performance and attracting more clients, they expand into new markets with high demand for AI solutions, including healthcare, finance, and autonomous vehicles, and they secure significant contracts with leading tech firms and government agencies.

Questions:

1) Calculate the total revenue for the first 4 quarters

2) Determine the average quarterly revenue over this period.

3) Predict the revenue for the 5th quarter and analyze the trend in terms of market expansion and product evolution.

Solution:

Part 1: Total Revenue Calculation:

First quarter:

$t_1 = 4(1)^6 + 36(1)^5 + 210(1)^4 + 720(1)^3 + 1470(1)^2 + 1656(1) + 774 = 4870$ thousand dollars

Second quarter:

$t_2 = 4(2)^6 + 36(2)^5 + 210(2)^4 + 720(2)^3 + 1470(2)^2 + 1656(2) + 774 = 22782$ thousand dollars

Third quarter:

$t_3 = 4(3)^6 + 36(3)^5 + 210(3)^4 + 720(3)^3 + 1470(3)^2 + 1656(3) + 774 = 62838$ thousand dollars

Fourth quarter:

$t_4 = 4(4)^6 + 36(4)^5 + 210(4)^4 + 720(4)^3 + 1470(4)^2 + 1656(4) + 774 = 139046$ thousand dollars

Total Revenue for 4 quarters:

t_{total} = 4870+22782+62832+139046 = 229536 thousand dollars

Part 2: Average quarterly Revenue

Average revenue = 229536/4= 57384 thousand dollars

Part 3: Prediction for 5th quarter

5^{th} quarter prediction revenue = $4.5^6+36.5^5+210.5^4+720.5^3+1470.5^2+1656.5+774$

= 264310 thousand dollars

Example 4.2.4: (Architectural)

Consider a hallway in an architectural design with a total length of 123 meters. divide this hallway into four squares sections based on the Fibonacci sequence, determine the length of each section.

Solution:

The Fibonacci sequence is generated by the relation $F_n= F_{n-1} + F_{n-2}$

Starting with F_0=0, F_1=1, F_2=1, F_3=2, F_4=3, F_5=5, F_6=8, F_7=13, F_8=21, F_9= 34

To build number 123 from summation we write 123=13+21+34+55

Section 1: 13 meters, Section 2: 21 meters, Section 3: 34 meters, Section 4: 55 meters

Example 4.2.5: (Visual Pattern)

Imagine you are creating a visual pattern using four concentric circles. The areas of the circles are equal to the squares of four consecutive triangular numbers. If the first triangular number in the sequence is T_4 (the fourth triangular number), calculate the total area of all four circles combined.

Solution:

Triangular numbers formula: $T_n= n (n+1) / 2$, So $T_4= 4(4+1) / 2 = 10$, n = 1,2,3,…

Now the four consecutive triangular numbers are 10, 15, 21, and 28

Calculating their squares

10^2 = 100

15^2 = 225

21^2 = 441

28^2 = 784

Then, the result that is: 100+225+441+784= 1550

Solution by Tetra Triangular Sequence.

$r_n=n^4+8n^3+31n^2+60n+46$, where n= 4

So $r_n= 4^4+8(4)^3+31(4)^2+60(4)+46$ = 1550

Example 4.2.6: (Strength for a Beam)

In architectural design, the reinforcement strength for a beam is determined by the sum of the squares of four consecutive natural numbers, calculating the total reinforcement strength when n = 5.

Solution

Using Squared Sequence:

$a_n = 4n^2+12n+14$, n = 1,2,3,…

So, $a_n = 4(5)^2 +12 (5)+14 = 174$

Thus the total reinforcement strength for the beam at n= 5 is 174.

The choice to model strength as the sum of four consecutive squares reflects the way physical systems, such as thermal expansion or structural loads, behave under non-linear scaling. In thermal expansion, when materials are heated, the dimensional change scales quadratically with the size of the object. Similarly, in structural engineering, when forces are applied to multi-layered systems, the resulting strength often increases non-linearly with the addition of each layer. This is why the sum of squares is a natural mathematical model for these types of systems, representing the cumulative, non-linear effects of force or expansion across successive layers.

Example 4.2.7: (Planetary Orbital)

When studying planetary motions, it observed that their trajectories follow complex mathematical patterns due to the gravitational influences of surrounding celestial bodies.

An approximation sequence is:

$s_n = n^4 + 24 n^3 + 84 n^2 + 11n + 98$, n = 1,2,3,… (Which is Super Squared Sequence).

Calculate the distance traveled over 6 month.

Solution:

Using Super Squared Sequence:

$s_n = n^4 + 24 n^3 + 84 n^2 + 11n + 98$, n = 1,2,3,…

Then, $S_6 = 6^4 + 24.6^3 + 84.6^2 + 11n + 98 = 19426$ astro unit

Which is the distance traveled over 6 months

Realism of the Application: In astronomy, planetary orbits known to follow non-linear patterns due to varying gravitational influences. Therefore, using sequences with multiple terms of different orders, like this one, is reasonable.

Relation to the Sequence: This sequence includes terms of different degrees (from first to fourth), reflecting the complexity of celestial motion equations governed by Kepler's and Newton's laws.

Example 4.2.8: (Neural Network Training Improvement)

Deep neural networks exhibit complex learning patterns influenced by multiple layers of processing. To model the improvement in accuracy after each training epoch, the sequence:

$t_n = 4\,n^6 + 36\,n^5 + 210\,n^4 + 720\,n^3 + 1470\,n^2 + 1656\,n + 774$, where n= 1, 2, 3,…

Calculate the improvement over 3 epochs.

Solution

$a_n = 4\,n^6 + 36\,n^5 + 210\,n^4 + 720\,n^3 + 1470\,n^2 + 1656\,n + 774$, n = 1,2,3,…

$a_3 = 4\,.3^6 + 36\,.3^5 + 210\,.3^4 + 720\,.3^3 + 1470\,.3^2 + 1656\,.3 + 774$

a_3 = 179290 Accuracy Points.

Realism of the Application: Neural networks improve non-linearly due to cumulative learning effects from different layers, necessitating non-linear mathematical models.

Relation to the Sequence: The presence of multiple-degree terms reflects the complex hierarchical interactions between layers in neural networks.

Example 4.2.9: (Cancerous Tumors)

Cancerous tumors grow in a complex manner that influenced by factors such as nutrient availability, genetic mutations, and interactions with surrounding tissues. Instead of using a simple exponential model, we propose a polynomial-based growth model where the tumor's size (in cubic millimeters) at time (days or weeks) follows the pattern.

Where represents the estimated tumor volume at time.

1) Calculate the tumor volume at tent

2) Predict the tumor volume at day 15.

3) Explain why this model might be more appropriate than traditional exponential models

Solution:

The tumor volume at any given time given by:

$t_n = (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2$, n = 1,2,3,…

<u>For 1:</u>

$t_{10} = (9)^2 + (10)^2 + (11)^2 + (12)^2 = 446\ mm^3$

<u>For 2:</u>

$t_{15} = (14)^2 + (15)^2 + (16)^2 + (17)^2 = 966\ mm^3$

Unlike an exponential model, which assumes unrestricted growth, tumors face biological constraints such as oxygen supply (angiogenesis), immune response, and spatial limitations.

The polynomial structure reflects an initial acceleration of tumor growth, followed by a slowdown due to micro environmental resistance and treatment interventions. Helps in

predicting tumor progression without overestimating size at later stages, unlike purely exponential models.

Can be useful in designing chemotherapy or radiation schedules by estimating the tumor's response over time. Offers a simplified mathematical approach that oncologists can use to approximate tumor behavior without requiring complex computational models

Chapter Five

Dynamic Set Theory

5.1 Introduction

Classical set theory assumes that sets are static collections of elements, unchanged unless explicitly modified by set operations.

However, many natural and mathematical systems exhibit dynamic behaviors, where elements appear, disappear, or transform under specific rules.

Definition 5.1.1: (Dynamic Set)

A Dynamic Set is a set where element membership changes over time:

$D(t) = \{x_i (t) \mid i \in I\}$, where $x_i (t)$ represents the i-th element at time t, and I is an index set

Note:

Each element $x_i (t)$ may: Join the set under certain conditions, leave the set due to internal or external rules and transform into another element within or outside the set.

Dynamic Set Theory (DST), which studies dynamic sets whose elements change due to internal or external influences. (DST) Theory introduces deal with: Leader Elements, which influence how sets evolve, Dynamic Subsets, where membership is conditional, Dynamic Union and Intersection, where element membership is not fixed, And Existence and Non-Existence of an element in the set at time elemental Opposition, where sets contain dynamically defined opposites.

Dynamic Set Theory can deal with Lagrange Squares Theorem, and there as an example about this linking, will be shown in this chapter later.

A Leader Element is a special element that dictates how changes occur. Formally:

$L(D) \in D(t)$, where $F_L (D)$ governs element transitions, noting that the leader may change based on size, order, or external influence.

Definition 5.1.2: (The Empty Dynamic Set)

The empty dynamic set Ø (t) is a dynamic set with no elements at time t. formally,

$Ø(t) = \{\ \}$.

Definition 5.1.3: (Dynamic Subset)

Let D(t) be a dynamic set, $S(t) \subseteq D(t)$, then S(t) called a dynamic subset of D(t).

5.2 Dynamic Set Operations

This section introduce some operations on dynamic sets.

Definition 5.2.1: (Union of Two Dynamic Sets)

The union of two dynamic sets D_1 and D_2 at a given state defined as:
$D_1 \cup D_2 = \{x \mid x \in D_1(t) \text{ or } x \in D_2(t)\}$.

However, since elements in D_1 and D_2 are not necessarily constant, an element x may: Appear in the union at some times but not others, leave the union if removed from both D_1 and D_2, or be influenced by a leader element from either set.

Definition 5.2.2: (Intersection of Two Dynamic Sets)

Intersection of two dynamic sets D_1 and D_2 defined as
$D_1 \cap D_2 = \{x \mid x \in D_1 \text{ at t and } x \in D_2 \text{ at t}\}$.
However, due to elemental evolution, an element x may: appear in the intersection only at specific times, exit the intersection if it transitions in either set, or be influenced by leader-based modifications.

Definition 5.2.3: (The Negation of a Dynamic Set)

The negation of a dynamic set D(t), denoted by D*(t), is the set of elements not in D(t) at time t.

Definition 5.2.4: (The Subtraction of Dynamic Sets)

The subtraction of dynamic sets $D_1(t)$ and $D_2(t)$, denoted $D_1(t) - D_2(t)$, is the set of elements in set $D_1(t)$ but not in set $D_2(t)$ at time t.

5.3 Advanced Functions in Dynamic Set Theory

In this section, thesis show many advanced functions that needed to make dynamic set theory realistic.

Definition 5.3.1: (Moon Function)

Let D(t) be a dynamic set, we define the Moon function as:

$M(x, t) = \{1, \text{if } x \in D \text{ at time t; } 0, \text{otherwise}\}$

This function determines whether a given element exists in a set at a specific time.

The name is token from that moon appears in different presences during month

Example 5.3.1:

Given the set D = {3, 5, 7} at t = 0 and D = {2, 3, 7} at t = 1.

Is 5 in the set D at t = 1?

Answer: No, since M(5,1) = 0, since 5 is not in set D at t = 1.

One of the basics assumptions of dynamic set theory, is the leader element existence, this assumptions make this theory very useful in politics and economics and social sciences.

The leader has many ways to be chosen, the two choices that built are in the following:

Definition 5.3.2: (Leader Selection by Maximum Value)

$L_{max}(D, t) = \{ \max |x| , x \in S(t)\}$

Definition 5.3.3: (Leader Selection by Proximity to the Mean)

$L_{mean}(D, t) = \min \{ |x - \mu(D, t)|: x \in D(t) \}$

Where $\mu(D, t)$ is the mean of elements in D at time t.

Definition 5.3.4: (Future Leader Selection by maximum)

Let $L_{f.max}(D, t) = \{x \in D \mid \text{based on } L(D,t)_{max}\}$

Definition 5.3.5: (Future Leader Selection by maximum)

Let $L_{f\,mean}(D, t) = \{x \in D \mid \text{based on } L(D,t)_{mean}\}$

Example 5.3.2:

Given the set D = {8 , 5 , 4 , 3 } at time t=0 and D = {7,6,5,2} at time t= 1,

Find the maximum leader at t=0 and find the maximum leader at t =1

Solution:

$L_{max}(D, t) = \{\max |x|, x \in D(t)\}$

Now,

1) $L_{max}(D, 0) = \{\max |x|, x \in D(0)\}$

$L_{max}(D, 0) = \max \{|8|,|5|,|4|,|3|\}$

$L_{max}(D, 0) = 8$

2) $L_{max}(D, 1) = \{\max x, x \in D(1)\}$

$L_{max}(D, 1) = \max \{|7|, |6|, |5|, |2|\}$

$L_{max}(D, 1) = 7$

This example done on Lagrange squares and realistically by:

There is a house with a fixed area of 114 square meters, consisting of a living room and three other rooms, all rooms are square, with areas: { 8^2,5^2,4^2,3^2} and the living room is the largest room, what is the length of the side of the living room when it was built?

The answer is 8 according to L_{max} (D, 0) in the above example

The man wanted to modify the internal division of the rooms after a period of one year, as making the area in another suitable way, what is the length of the living room after one year?

The answer is 7 according to L_{max} (D, 1) in the above example

Example 5.3.3:

After 2 year of construction, the municipality ordered the man to remove the living room because a street would pass through it. What is area of the new living room is he nominated based on his primary consideration in choosing a living room as the biggest room?

Solution:

D(1)={7,6,5,2}

At t=2, D(2)={6,5,2}, since the man demolish the living room

So, the new living room is 6 and it represent by:

$L_{future\ max}(D, t) = \{ x \in D \mid x = max (D,t)\}$

So, $L_{future\ max}(D, 2) = \{ x \in D \mid x = max (D,2)\}$

$= max \{6,5,4\} = 6$

So, the new living room is the room that its area is $6^2 = 36$ meters.

Definition 5.3.6: (Group Cohesion Function)

Group Cohesion function measures the cohesion of elements within a set by evaluating their relative proximity.

$C (D, t) = 1 - (1/n) \Sigma |x_i - x_j|$, for all $x_i, x_j \in D$

For Example:

D_1 = {10, 11, 12} → High cohesion.

D_2 = {5, 50, 100} → Low cohesion.

Definition 5.3.7: (Set Compatibility Function)

Set Compatibility function evaluates the compatibility between two sets, particularly when selecting an alternative set.

This function finds the best replacement set when the preferred set is unavailable.

Comp (D_1, D_2) = C (D_1, D_2) ,where C is a similarity metric such as Jacquard Index or Euclidean Distance.

Example 5.3.4:

Given target set $D_1 = \{1, 2, 3, 4\}$ and candidate sets:

$D_2 = \{1, 2, 3, 5\}$, $D_3 = \{7, 8, 9, 10\}$, if we cannot obtain the original set, which set is the appropriate one to choose if the decision based on the highest similarity?

Solution:

$C(D_2, D_1)$ = (75% similarity)

$C(D_3, D_1)$ = (0% similarity)

Best choice is D_2, as it has the highest similarity to set D_1.

Definition 5.3.8 (Set Stability Function)

Let $S(D, t) = 1 - ([Comp((D(t), D(t-1))] \cdot [(LD(t) - LD(t-1)) / LD(t-1))])$,

S(D,T) called set stability function.

Determine stability as following:

IF S (D,t) more than 60% , then set D called stable set at time t.

IF S (D,t) = 60% ,then the set D called half stable set at time t.

IF S (D,t) less than 60% , then the set D called unstable set at time t.

Example 5.3.6:

If D changes from D (0) = {1,2,3,4} to D (1) = {1,2,3,5} after 1 unit of time, find the stability, according maximum leader.

Solution

$S_{Lmax}(D, t) = 1 - ([Comp[(D(t), D(t-1)]] \cdot [(L_{max} D(t) - L_{max} D(t-1)) / L_{max} D(t-1))])$

$S_{Lmax}(D,1) = 1 - ([Comp[(D(1), D(1-1)]] \cdot [(L_{max} D(1) - L_{max} D(1-1)) / L_{max} D(1-1))])$

$S(D,1) = 1 - ([Comp[(D(1), D(0)]] \cdot [(L_{max} D(1) - L_{max} D(0)) / L_{max} D(0))])$

$= 1 - ([0,75] \cdot [(5-4)/4])$

$= 1 - (0,75 \cdot 0,25)$

$= 1 - (0,1875) = (0,8125) = 81.25\ \%$

So the set D is stable at time t = 1

5.4. The Philosophical Imperative for Dynamic Sets

This section taking about the philosophical imperative for dynamic sets theory, and given logical ground for it, the idea of this section is about embracing change as an essential structure.

5.4.1: The Illusion of Static Certainty in the World of Change

Throughout history, mathematics has sought to provide a sense of permanence in an otherwise transient world. Cantor's set theory, rooted in the notion of static and well-defined collections, echoed a longing for certainty in the face of chaos. Yet, to perceive sets as rigid, unchanging entities is to deny the very nature of existence, which is characterized by transformation, flux, and evolution.

Just as ancient philosophers, from Heraclitus to Bergson, have asserted, "change is the only constant," our mathematical frameworks must also evolve to reflect this undeniable truth. A static conception of sets fails to capture the dynamic essence of nature, a nature that is always in a state of becoming, and not just being. This is where dynamic set theory emerges—not merely as a mathematical extension, but as a philosophical necessity to bridge the chasm between static abstraction and living reality.[2]

5.4.2: The Imperative for Internal Dynamics

To view sets as simple collections of elements without accounting for their internal evolution is to reduce their essence to a mere shadow of reality. Life itself, in all its forms, from the movement of particles to the development of human consciousness, is a series of continuous transformations. Why should mathematics, the language of nature, remain imprisoned in a world of stillness, while everything around us moves?

Philosophically, the notion of stasis in set theory is not just a limitation; it is an abstraction that obscures the deeper, living principles that govern dynamic systems. Just as Aristotle viewed potentiality as an inherent aspect of being, the dynamic nature of sets calls for a new conception: one where internal change, interrelations, and the passage of time are integrated into the very fabric of their structure.

5.4.3: The Quest for Stability in a World of Constant Change

In the quest for truth, stability has often been equated with permanence, but the true nature of stability is found not in unchanging uniformity but in the ability to adapt, evolve, and return to equilibrium amidst disturbance. This is the essence of dynamic stability—an active form of balance that reflects the nature of complex systems in motion.

In dynamic set theory, stability is not defined merely by the unchanging presence of elements, but by their relative coherence, their resilience in the face of internal and external perturbations. This type of stability can be philosophically understood as a form of harmony within change, akin to the chi in Eastern philosophy, which maintains balance without resisting the flow of time.

Mathematical Argument: Stability in dynamic sets requires measuring not only the difference between elements but also the change in each element's value over time. It is a deeper, more organic kind of stability, rooted in the ongoing dance of internal dynamics.

Application Example: Much like a living organism, a set might exhibit stability not by remaining unchanged but by adjusting in response to both internal relationships and external forces. Consider a biological system whose components—cells, organs, molecules—shift, adapt, and re-align over time. True stability, in this case, is not the absence of change but the ability to maintain function amidst it.

5.4.4: Leadership in Dynamic Systems: The Philosophical Implications

In every dynamic system, certain elements rise to leadership, guiding the system through changes. However, leadership in a dynamic set is not an absolute concept based solely on an element's value. Rather, leadership emerges through interactions, dependencies, and adaptability to the system is evolving needs.

This philosophical inquiry into leadership parallels the concept of virtue ethics, where virtue defined not by static moral actions but by the ability to adapt one's actions to the context of evolving circumstances. Leadership in dynamic sets, similarly, emerges not from fixed dominance but through an element's responsiveness to the shifting conditions of the system.

Mathematical Argument: Leadership modeled as a dynamic process, reflecting an element's capacity to influence the set's state based on its position in the overall structure and its relational dynamics.

Application Example: In a social network, leadership may not always belong to the highest-status member but to those who can adapt, respond, and guide others through changing circumstances. In dynamic set theory, this idea captured mathematically by considering not only the highest value but also the influence an element exerts over time.

5.4.5: Bridging Mathematical Abstraction with the Fluidity of Existence

Dynamic set theory is more than an extension of mathematical thought; it is an attempt to align the abstract language of mathematics with the fluidity and temporality of existence. From the philosopher's standpoint, this transition is not just mathematical but metaphysical.

In essence, dynamic set theory aims to build a framework that can model not just the content of reality, but the very processes that shape it. It aspires to describe systems that are always becoming rather than simply being systems whose structure emerges from interactions, time, and change. This theory does not reject the static, but rather embraces it as part of a larger dynamic whole.

5.5 Conclusion

The birth of dynamic set theory marks the beginning of a new philosophical chapter in mathematics—one that acknowledges the fluidity of existence and the inevitability of change. It is a theory not about what is static, but about what is becoming.

Recommendations and Future Research Prospects

After investigating several existing sequences and applying Lagrange Four-Square Theorem to consecutive numbers within these sequences, the same approach can be extended to other sequences such as: amicable numbers, perfect numbers, pentagonal numbers, and even political numbers.

Furthermore, an interesting research direction would be to explore the possibility of reformulating number theory so that its primary foundation relies on Lagrange Four-Square Theorem rather than prime numbers.

Additionally, more development on Dynamic Sets Theory project, expanding its scope significantly.

Another proposition is a research focus on constructing new vector spaces and advancing the field of mathematical logic—particularly temporal logic and causal logic—while integrating these developments with Lagrange Four-Square Theorem.

Finally, recommending guiding the engineering community toward leveraging number theory, and this thesis specifically, in areas related to optimizing space utilization, cost efficiency, and aesthetic control.

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نظرية المجموعات الديناميكية وتوسيعات على نظرية مربعات لاغرانج: نظريات جديدة، إضاءات فلسفية، وتطبيقات

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ملخص الرسالة

تتناول هذه الرسالة تطويرًا رياضيًا وفلسفيًا مبتكرًا من خلال ربط نظرية مربعات لاغرانج بخمس متتاليات عددية شهيرة، وهي: متتالية الأعداد الطبيعية، الأعداد المربعة، الأعداد المكعبة، الأعداد المثلثية، ومتتالية فيبوناتشي. وقد نتج عن هذا الربط إنتاج خمس متتاليات جديدة، تم تحليلها بدقة وقدمت حولها أربع عشرة مبرهنة تم برهنتها بالكامل ضمن سياق منهجي صارم.

تعكس هذا الرسالة مقاربة متعددة الجوانب تربط ما بين البنية الصارمة لنظرية الأعداد من جهة، والرؤية الفلسفية للرياضيات من جهة أخرى، من خلال طرح مبدأ جديد في فلسفة الرياضيات يُدعى: "احتواء النقيض في النظام الرياضي"، بما يفتح آفاقًا جديدة لفهم التناقض والتكامل داخل البناء الرياضي ذاته، وكذلك تم الربط مع قضايا جوهرية في فلسفة الرياضيات.

كما استُخدمت هذه المتتاليات المطوّرة في مجالات تطبيقية متنوعة، من ضمنها: إعادة تصميم لعبة الشطرنج التقليدية انطلاقًا من منظور قائم على مربعات لاغرانج، وكذلك حل مسائل واقعية من ميادين الفيزياء، التشفير، الاقتصاد، الهندسة، وغير ذلك، بما يبرهن على القيمة التطبيقية للنظريات المطروحة.

وفي ختام الرسالة، يُقترح إطار نظري جديد يُسمى "نظرية المجموعات الديناميكية"، وهو توجّه يُعيد تصور المجموعات الرياضية بوصفها كيانات متغيرة لا ثابتة، خلافًا للطرح التقليدي الذي طرحه العالم كانتور في نظرية المجموعات الكلاسيكية، مع تقديم الأسس المنطقية والفلسفية والجبريّة التي تؤهله ليكون ميدانًا خصبًا للبحث المستقبلي.