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Fractional Newton-Raphson Method

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Dedication

I present this as a way of gratitude to my mother whom I'm truly proud of and for them I am grateful as they stood by my side every day and moment.

Name: Mohammad Imad Hammo

Declaration

I certify that this thesis submitted for the degree of master is the result of my own research, except where otherwise acknowledged, and this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Abstract

Fractional Calculus has found numerous applications across various scientific fields.

In this thesis, we review key definitions of fractional derivatives and explore their applications. Specifically, we focus on three fractional derivative definitions and apply then to Newton-Raphson's method for solving equations. We implement these methods in python to compute polynomial roots and compare their performance in terms of iteration speed and accuracy. Our analysis highlights which definition yields faster convergence and better results.

In this thesis, we enhance the Newton-Raphson method by replacing the ordinary derivative with the proposed fractional derivatives and apply this modified approach to solve various equations numerically and compare the results with the original Newton-Raphson method.

Furthermore we conduct a comprehensive performance analysis to compare the numerical efficiency and accuracy of the modified methods.

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Chapter One

Introduction

Fractional calculus is a science with many applications in a wide variety of fields of engineering and sciences such as engineering conservation of mass and electrochemical analysis. In these areas, various analytical and numerical methods including their applications to new problems have been proposed in recent years.

Fractional derivatives are used Newton-Raphson's method to for solving equations. We will now study the definitions of derivatives and integrals and study their results.

The outline of the thesis is as follows:

In Chapter 2, a historical overview of the scientists who interpreted fractional derivatives in general is presented, a study of the basic functions used in fractional calculus, such as the gamma and beta functions, and an analysis of each of the three definitions of fractional derivatives in terms of properties, proofs, and examples.

In Chapter 3, we presented a number of the most important applications in which fractional calculus was used in various sciences such as engineering, biology and applied mathematics.

In Chapter 4 of this thesis,

First: General review of the Newton-Raphson method.

Second: A general study of the fractional Newton-Raphson method and the application of some examples of some definitions of fractional derivatives, including (Riemann-Liouvill's, deformable derivative, Classical derivatives), and we will apply these examples in chapter five.

In the chapter 5, we will apply these examples to a Python program to extract results for each of the three definitions of fractional derivatives using Fractional Newton-Raphson Method.

Finally, in chapter 6, the results extracted from the Python program will be discussed, through which the three definitions will be discussed, compared, and the fastest definition will be determined.

1.1 Problem of Studying

This study attempts to answer the following question:

1- Can we apply some definitions of fractional derivatives to Newton-Raphson's method?

2- Is it possible to compare them?

1.2 Objectives of Studying

Study objectives define learning outcomes and help clarify, organize, and prioritize study.

What are the objectives of this thesis?

First: To learn about fractional calculus and its different definitions.

Second: To identify the applications of fractional calculus in various types of sciences.

Third: Identify the Fractional Newton-Raphson method.

Fourth: A study of the use of some definitions of fractional derivatives in Fractional Newton-Raphson method.

Fifth: To extend some equations that have complex roots, finding the fractional derivatives for each of the three definitions of fractional calculus, applying the solutions to the Python program, to analyze the performance of a modified Fractional Newton method using existing numerical methods.

1.3 The importance of Studying

Fractional calculus becomes used in many applications and scientific and industrial fields. Therefore, it is necessary to study this science in order to be able to use its many applications in areas that serve human life and its development.

This study sheds light on the use of some of the definitions of fractional derivatives to calculate equations with complex roots using Fractional Newton-Raphson method (FNRM).

Chapter Two

Fractional Calculus

2.1 Introduction

The Fractional Calculus is a branch of mathematical analysis, which deals with the applications of derivatives and integrals. Its first appearance was in a letter written to Guillaume de L'Hospital by Gottfried Wilhelm Leibniz in 1695. Thanks to the notation $\frac{d^n}{dx^n} f(x), n \in N,$ L'Hospital could asked in a letter to Leibniz about the interpretation of taking $n = 1/2$ in a derivative, since at that moment Leibniz could not give a physical or geometrical interpretation to this question, he simply answered L'Hospital in a letter, "... this is an apparent paradox of which, one day, useful consequences will be drawn" , The name fractional calculus comes from a historical question, because in this branch of mathematical analysis, derivatives and integrals of a certain order $\alpha,$ with $\alpha \in \mathbb{R}$ or $\mathbb{C}$. [1]

There are many types of fractional integral and differential operators that are proposed by Riemann Liouville, Weyl, Riesz, Caputo and other scientists. Fractional Calculus has wide applications in applied mathematics and other sciences such as differential equations, physics, engineering, economics and many other sciences. [22]

2.2 Historical Notes of Fractional Calculus

The first mention of fractional calculus apparently by S. E Lacroix around 1819. He started by stating the common n^{th} derivative of the power function $y = x^m$ in term of the Gamma function. [7]

$$y^{(n)} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \qquad , \text{where } m \geq n .$$

He then let n be any real number to arrive at the first definition of the fractional derivative of a power function. After this period, a list of mathematicians who have provided important contributions up to the middle of the 20^{th} century included Fourier, N H Abel, J. Liouville, Riemann, Grunwald, Riesz, W. Feller and others. [7]

Among the most common definitions of fractional derivatives are the fractional derivative of Riemann-Liouville (R-L) and the fractional derivative of Caputo. [7]

2.3 Special Functions

In this section we present the special functions gamma and beta that we use to define an important kind of fractional derivatives and fractional integrals.

2.3.1 Gamma Function

Definition 2.3.1.1[7] The gamma function, denoted by $\Gamma(x)$, is an extension of the factorial function to real and complex numbers. Specifically, if $x \in \{1,2,3,\dots\}$, then

$$\Gamma(x) = \Gamma(n) = (n-1)!$$

More generally, for any positive real number n, $\Gamma(n)$ is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt\,, \quad x > 0$$

Example 2.3.1.1[8] Find $\Gamma(2)$ and $\Gamma\left(\frac{1}{2}\right)$?

Solution:

First we will find $\Gamma(2)$ by using the definition

$$\Gamma(2) = \lim_{t\to\infty} \int_0^t x^{2-1} e^{-x}\, dx$$

$$= \lim_{t\to\infty} \int_0^t x e^{-x}\, dx$$

$$= \lim_{t\to\infty} (-te^{-t} - e^{-t} + 0 + e^0) = 1$$

Therefore, $\Gamma(2) = 1$

Now we will find $\Gamma\left(\frac{1}{2}\right)$ by using the definition

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx$$

By using the substitution $x = y^2$, so when $x = 0$ then $y = 0$, and when

$x = \infty$ then $y = \infty$. Also, $dx = 2ydy$ we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty y^{-1} e^{-y^2} y dy$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy \quad \dots\dots\dots(1)$$

Also,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx \qquad \ldots\ldots\ldots(2)$$

Therefore,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^\infty e^{-x^2}dx \; 2\int_0^\infty e^{-y^2}dy$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}dxdy$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$$

Now, let $x = r\cos(\theta),\; y = r\sin(\theta)$, then

$$x^2 + y^2 = r^2(\cos^2(\theta) + \sin^2(\theta))$$

So, $x^2 + y^2 = r^2$ and

$$dxdy = rdrd\theta$$

Then θ changes from 0 to $\frac{\pi}{2}$ and r changes from 0 to ∞, so we have

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2}\, rdrd\theta$$

Using the substitution $r^2 = P$, then $2rdr = dP$ and so $rdr = \frac{dP}{2}$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2} \int_0^\infty e^{-P}\frac{dP}{2}d\theta$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^{\pi/2} \left[\frac{e^{-P}}{-1}\right]_0^\infty d\theta$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^{\pi/2} \left(\frac{e^{-\infty} - e^0}{-1}\right) d\theta$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^{\pi/2} d\theta \; = 2[\theta]_0^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{2} - 0\right) = \pi$$

Therefore, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Properties of Gamma Function: [7]

For any positive real number x

1. $\Gamma(x+1) = x\Gamma(x) \qquad \forall x \neq 0$
2. $\Gamma(x+1) = x! \qquad \text{for } x \in \mathbb{N}$

2.3.2 Beta Function

Definition 2.3.2.1[7] The Beta function is a unique function and is also called the first kind of Euler's integrals. The beta function is defined in the domains of real numbers. The notation to represent it is "β". The beta function is denoted by $\beta(p,q)$, where the parameters p and q should be real numbers.

More generally, for any positive real number p and q

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad , \quad p > 0\,, q > 0$$

Example 2.3.2.1: Find $\beta(2{,}3)$and $\beta\left(\frac{3}{2},\frac{1}{2}\right)$?

Solution:

$$\beta(2{,}3) = \int_0^1 x^{2-1}(1-x)^{3-1}\,dx$$

$$= \int_0^1 x(1-x)^2\,dx$$

$$= \int_0^1 x(1-2x+x^2)dx$$

$$= \int_0^1 (x-2x^2+x^3)\,dx$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

Using the identity

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Then,

$$\beta\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}+\frac{1}{2})}$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}$$

$$= \frac{1}{2}\sqrt{\pi}\sqrt{\pi}$$

$$= \frac{\pi}{2}$$

Example 2.3.2.2: Show that $\beta(p,q) = \beta(q,p)$.

Solution:

By definition of Beta Function

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

By using the substitution $u = 1 - x$, so when $u = 1$ then $x = 0$, and when $u = 0$ then $x = 1$. Also, $du = -dx$ we have

$$= \int_0^1 u^{q-1}(1-u)^{p-1}\,du$$

$$= \beta(q,p)$$

Properties of Beta Function: [7]

For any positive real numbers p and q, we have

1) $\beta(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt.$
2) $\beta(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin^{2p-1}\theta)(\cos^{2q-1}\theta)\,d\theta.$

2.4 Special Definitions in Fractional Calculus

There are several types of definitions of fractional derivatives. In this section, we will study special definitions that will be applied in the Newton-Raphson method.

2.4.1 Riemann-Liouville Definitions in Fractional Calculus

2.4.1 Introduction

The early attempt by Liouville was later improved by the scientist Holmgren, who in 1865 made important contributions to the study of fractional calculus. But it was Riemann who reconstructed it to fit Abel's integral equation, making it vastly more useful. Today there are many variations of types of infinite sums, but the Riemann-Liouville operator is still the most commonly used when performing fractional integration. [12]

2.4.1.1 Fractional Derivative for Riemann Liouville

There are many different definitions of fractional derivatives; the most popular ones are Riemann-Liouville and Caputo derivatives.

Definition 2.4.1.1[12] Riemann-Liouville derivative:

For any continuous function f defined on $\mathbb{R}^+$ and for any α, $\alpha \in \mathbb{R}^+$, we define the $\alpha -$ Riemann-Liouville Derivative by

$$D_{RL}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt$$

Definition 2.4.1.2[2] Caputo derivative:

For any continuous function f defined on $\mathbb{R}^+$ and for any $\alpha, n-1 < \alpha \leq n$, where n is a positive integer, the $\alpha -$Caputo Derivative is defined by

$$D_C^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt$$

Theorem 2.4.1.1 Let f continuous function f defined on $\mathbb{R}^+$ and for any $\alpha_i \in (0,1)$ $(i = 1,2)$, and $\alpha_1 + \alpha_2 \in (0,1]$, then the relation between $\alpha -$ Riemann-Liouville and Caputo Derivative is given by

$$D_C^{\alpha_1+\alpha_2} f(x) = D_{RL}^{\alpha_1+\alpha_2} f(x) - \frac{x^{-\alpha_1-\alpha_2}}{\Gamma(1-\alpha_1-\alpha_2)} f(0)$$

For the proof see reference [13].

Example 2.4.1.1[8]**:** Power Function

Suppose we want to find the α −Riemann-Liouville Derivative for the power function $f(x) = x^m$, $m \in \mathbb{N}$.

Solution:

$$D_{RL}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt \quad , \qquad \alpha \in \mathbb{R}^+$$

$$D_{RL}^{\alpha}(x^m) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} . t^m \, dt$$

Set $t = ux$ for $0 \leq u \leq 1$, then $dt = xdu$, we get

$$D_{RL}^{\alpha}(x^m) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^1 \left(x(1-u)\right)^{n-\alpha-1} . (xu)^m \, x \, du$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \int_0^1 u^m (1-u)^{n-\alpha-1} \, du$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \beta(m+1, n-\alpha)$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \frac{\Gamma(m+1)\Gamma(n-\alpha)}{\Gamma(m+n-\alpha+1)}$$

$$= \frac{\Gamma(m+1)}{\Gamma(m+n-\alpha+1)} \frac{d^n}{dx^n} x^{m+n-\alpha}$$

$$= \frac{\Gamma(m+1)}{\Gamma(m+n-\alpha+1)} \frac{\Gamma(m+n-\alpha+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Therefore,

$$D_{RL}^{\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Example 2.4.1.2[8]**:** Constant Function

Find the $\alpha -$ fractional derivative of any constant function K.

Solution:

a) For Riemann-Liouville derivative:

$$D_{RL}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt \quad , \qquad \alpha \in \mathbb{R}^+$$

$$D_{RL}^{\alpha} k = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} . k \, dt$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} k \left[\frac{(x-t)^{n-\alpha}}{n-\alpha} \right]_0^x$$

$$= \frac{1}{\Gamma(n-\alpha)} k \frac{d^n}{dx^n} \frac{x^{n-\alpha}}{n-\alpha}$$

$$= \frac{1.k}{(n-\alpha)\Gamma(n-\alpha)} \left[\frac{d^n}{dx^n} x^{n-\alpha} \right]$$

$$= \frac{1.k}{\Gamma(1+n-\alpha)} \frac{(n-\alpha)!}{(n-\alpha-n)!} x^{n-\alpha-n}$$

$$= \frac{1.k}{\Gamma(1+n-\alpha)} \frac{\Gamma(1+n-\alpha)}{\Gamma(1-\alpha)} x^{-\alpha}$$

$$D_{RL}^{\alpha} k = \frac{k x^{-\alpha}}{\Gamma(1-\alpha)}$$

b) For Caputo derivative:

$$D_C^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt \quad , n-1 < \alpha \le n$$

$$D_C^{\alpha} k = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} k \, dt$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} . 0 \, dt = 0$$

Properties Riemann-Liouville derivative: [8]

1) $D_{RL}^{\alpha} e^{\lambda x} = \lambda^n x^{n-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{\Gamma(k+n-\alpha+1)}$

2) $D_{RL}^{\alpha} \sin(\lambda x) = \frac{1}{2} (2\lambda)^n x^{n-\alpha} [E_1, (n-\alpha+1)(\lambda i k) + (-1)^n E_1, (n-\alpha-1)(-i\lambda x)]$

Where the Mittag-Leffler function is

$$E_{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \delta)}$$

2.4.1.3 Fractional Integral for Riemann-Liouville

In this section, we discuss the topic of fractional integration, focussing specifically on the Riemann-Liouville fractional integral, its definition, and its properties.

Definition 2.4.1.3[8] (Riemann-Liouville Operator). Let f be a continuous function on $\mathbb{R}^+$ and $\in \mathbb{R}$. The fractional integral of order α to the function f is defined by:

$$I_{RL}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

Example 2.4.1.3[8]**:** Power Function

Find the α −fractional Integral for the power function $f(x) = x^m,\ m \in \mathbb{N}$.

Solution:

By definition of Riemann-Liouville α −fractional integral we have

$$I_{RL}^{\alpha} x^m = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, t^m dt$$

Letting $t = ux$, for $0 \le u \le 1$, then $dt = xdu$, we get

$$I_{RL}^{\alpha} x^m = \frac{1}{\Gamma(\alpha)} \int_0^1 (x-ux)^{\alpha-1} (ux)^m x du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 x^{\alpha+m} (1-u)^{\alpha-1} u^m du$$

$$= \frac{x^{\alpha+m}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^m du$$

$$= \frac{x^{\alpha+m}}{\Gamma(\alpha)}\ \beta(\alpha, m+1)$$

$$= \frac{x^{\alpha+m}\Gamma(\alpha)\Gamma(m+1)}{\Gamma(\alpha)\Gamma(\alpha+m+1)}$$

$$I_{RL}^{\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} x^{\alpha+m}$$

Example 2.4.1.4[8]**:** Constant Function

Find the $\alpha -$ fractional Integral of any constant function k.

Solution:

By definition of Riemann-Liouville $\alpha -$fractional integral we have

$$I_{RL}^{\alpha} k = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, k\, dt$$

$$= \frac{k}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, dt$$

Using the substitution $t = xu$, Then $dt = xdu$, for $0 \leq u \leq 1$, we obtain

$$I_{RL}^{\alpha} k = \frac{k}{\Gamma(\alpha)} \int_0^1 (x-xu)^{\alpha-1} x\, du$$

$$= \frac{kx^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} du$$

$$= \frac{kx^{\alpha}}{\Gamma(\alpha)} \beta(1,\alpha)$$

$$= \frac{kx^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(\alpha+1)}$$

$$I_{RL}^{\alpha} k = \frac{k}{\Gamma(\alpha+1)} x^{\alpha}$$

Properties for Riemann-Liouville Integral: [8]

1) $I_{RL}^{\alpha} e^{ax} = \frac{e^{kx}}{k^{\alpha}}$
2) $I_{RL}^{\alpha} sinx = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1+\alpha}}{\Gamma(2k+2+\alpha)}$

2.4.2 Deformable Derivative Definitions in Fractional Calculus

A simple difference between some mathematical concepts may sometimes shed light on some hidden facts, for example continuity and differentiation are two specific concepts but are defined differently. Generalizing any mathematical concept is a great source of motivation, because it simplifies many complex facts and extends its application to a broader class of problems. There are many generalizations of the concept of the derivative to the fractional derivative since L'Hospital first asked this question to Leibniz in his treatise.

In 1695 about a meaningful interpretation of $\frac{d^{1/2}y}{dx^{1/2}}$. Different types of fractional derivatives have been studied so far, as most of the fractional derivatives have integral form, there are a few that attract the attention of mathematicians as they become famous in the world of fractional calculus such as Riemann-Liouville derivatives, Caputo, Hadamard and others to get a good idea For fractional calculus.

R. Khalil presented a term-based definition of the derivative of a fraction in 2014, and hereby he defined a new fractional derivative referred to as deformable derivative, and also extends it to broad functions. [9]

Definition 2.4.2.1[9] Let $f(t)$ be real valued function defined on interval (a, b). For a given number $\alpha,\ 0 \leq \alpha \leq 1$, we define deformable derivative by the following limit:

$$\lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta)f(t+\varepsilon\alpha) - f(t)}{\varepsilon}\ , \qquad \text{where } \alpha + \beta = 1$$

If this limit exists, we denote it by $D_D^{\alpha} f(t)$.

Remark 2.4.2.1

For $\alpha = 0$, $D_D^0 f(t) = f(t)$, which is the usual convention and if $\alpha = 1$, $D_D f(t) = f'(t)$. Therefore it can be deemed as a new fractional derivative with respect to parameter α. Throughout the thesis until unless specified we assume that $0 < \alpha < 1$.

Theorem 2.4.2.1[9] A differentiable function f at a point $t \in (a, b)$ is always $\alpha -$ Deformable differentiable at that point for any α. Moreover in this case we have

$$D_D^{\alpha} f(t) = \beta f(t) + \alpha D f(t) \quad \text{where } \alpha + \beta = 1$$

Proof:

By applying definition 2.4.2.1 of $\alpha -$ Deformable derivative to get

$$D_D^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta) f(t+\varepsilon\alpha) - f(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left(\frac{f(t+\varepsilon\alpha) - f(t)}{\varepsilon} + \beta f(t+\alpha\varepsilon) \right)$$

$$= \alpha D f(t) + \beta . \lim_{\varepsilon \to 0} f(t+\alpha\varepsilon)$$

$$D_D^{\alpha} f(t) = \beta f(t) + \alpha D f(t)$$

Both the terms exist as f, being differentiable at t is continuous as well. Hence the theorem follows.

Theorem 2.4.2.2[9] Let f be $\alpha -$Deformable differentiable at a point t for some α. Then it is continuous there.

Proof:

For continuity, it suffices to prove the following

$$\lim_{\varepsilon \to 0} \big(f(t+\varepsilon\alpha) - f(t) \big) = 0$$

The left hand side can also be written a

$$\lim_{\varepsilon \to 0} \big(f(t+\varepsilon\alpha) - f(t) \big) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta) f(t+\varepsilon\alpha) - f(t) - \varepsilon\beta f(t+\varepsilon\alpha)}{\varepsilon} \varepsilon$$

$$= \lim_{\varepsilon \to 0} \left(\frac{(1+\varepsilon\beta) f(t+\varepsilon\alpha) - f(t)}{\varepsilon} . \varepsilon - \varepsilon\beta f(t+\varepsilon\alpha) \right)$$

$$= D_D^{\alpha} f(t) . 0 - \beta \lim_{\varepsilon \to 0} \varepsilon f(t+\varepsilon\alpha) = 0$$

This completes theorem.

Corollary 2.4.2.1[9] An $\alpha -$Deformable differentiable function f defined in (a, b) is differentiable as well.

Proof:

For the existence of derivative by applying definition to get

$$D f(t) = \frac{1}{\alpha} \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon\alpha) - f(t)}{\varepsilon}$$

$$D f(t) = \frac{1}{\alpha} \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta) f(t+\varepsilon\alpha) - f(t) - \varepsilon\beta f(t+\alpha\varepsilon)}{\varepsilon}$$

$$D f(t) = \frac{1}{\alpha} \left(\lim_{\varepsilon \to 0} \frac{(1+\varepsilon\beta) f(t+\varepsilon\alpha) - f(t)}{\varepsilon} - \beta \lim_{\varepsilon \to 0} f(t+\alpha\varepsilon) \right)$$

$$Df(t) = \frac{1}{\alpha}\left(D_D^{\alpha} f(t) - \beta f(t)\right)$$

we summarise all these by saying that the two concepts $\alpha -$ Deformable differentiability and differentiability of a function defined in (a,b) are equivalent in the sense that one implies other.

2.4.2.3 Basic Properties of Deformable Derivative

Apart from discussing some fundamental properties of deformable derivative like linearity and commutativity the section deals with fundamental theorems: Rolle's, Mean-Value and Taylor's theorems. The geometric illustration of $D^{\alpha} f$ for some elementary functions f is also given.

Theorem 2.4.2.3[9] The operator D_D^{α} possesses the following properties for a differentiable function f and g:

a) Linearity : $D_D^{\alpha}(af + bg) = aD_D^{\alpha} f + bD_D^{\alpha} g$

b) Commutativity: $D_D^{\alpha_1}.D_D^{\alpha_2}(f) = D_D^{\alpha_2}.D_D^{\alpha_1}(f)$

c) For a constant function k, $D_D^{\alpha}(k) = \beta k$, where $\beta = 1 - \alpha$

d) $D_D^{\alpha}(f.g) = (D_D^{\alpha} f).g + \alpha f.Dg$

Proof:

a) By applying definition 2.4.2.1 of $\alpha -$Deformable derivative to get

$$D_D^{\alpha} f = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)f(t + \varepsilon\alpha) - f(t)}{\varepsilon}$$

$$D_D^{\alpha}(af + bg) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)(af + bg)(t + \varepsilon\alpha) - (af + bg)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)af(t + \varepsilon\alpha) + (1 + \varepsilon\beta)bg(t + \varepsilon\alpha) - af(t) - bg(t)}{\varepsilon}$$

$$= a\lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)f(t + \varepsilon\alpha)}{\varepsilon} + b\lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)g(t + \varepsilon\alpha)}{\varepsilon}$$

$$= aD_D^{\alpha} f + bD_D^{\alpha} g$$

b) By applying Theorem 2.4.2.1 to get

$$D_D^{\alpha_1} f = \beta_1 f + \alpha_1 Df$$

$$D_D^{\alpha_2} f = \beta_2 f + \alpha_2 Df$$

$$D_D^{\alpha_1}\left(D_D^{\alpha_2} f\right) = \beta_1(\beta_2 f + \alpha_2 Df) + \alpha_1 D(\beta_2 f + \alpha_2 Df)$$

$$= \beta_1\beta_2 f + \beta_1\alpha_2 Df + \alpha_1\beta_2 Df + \alpha_1\alpha_2 D^2 f$$

$$= \beta_1\beta_2 f + (\alpha_1\beta_2 + \alpha_2\beta_1)Df + \alpha_1\alpha_2 D^2 f$$

$$= \beta_2\beta_1 f + (\beta_2\alpha_1 + \alpha_2\beta_1)Df + \alpha_2\alpha_1 D^2 f$$

$$= \beta_2(\beta_1 f + \alpha_1 Df) + \alpha_2 D(\beta_1 f + \alpha_1 Df)$$

$$= D_D^{\alpha_2}(D_D^{\alpha_1} f)$$

c) By applying definition 2.4.2.1 of α −Deformable derivative to get

$$D_D^{\alpha} f = \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)f(t+\varepsilon\alpha) - f(t)}{\varepsilon}$$

$$D_D^{\alpha} k = \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)(k) - k}{\varepsilon}$$

$$D_D^{\alpha} k = \lim_{\varepsilon\to 0}\frac{k+\varepsilon\beta k - k}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\varepsilon\beta k}{\varepsilon}$$

$$\lim_{\varepsilon\to 0}\beta k = \beta k$$

d) By applying Theorem 2.4.2.1 to get

$$D_D^{\alpha}(f.g) = \beta(f.g)(t) + \alpha D(f.g)(t)$$

$$D_D^{\alpha}(f.g) = \beta f(t).\beta g(t) + \alpha\big(f(t).Dg(t) + g(t)Df(t)\big)$$

$$= \beta f(t).\beta g(t) + \alpha f(t).Dg(t) + \alpha g(t).Df(t)$$

$$= \beta f(t).\beta g(t) + \alpha g(t)Df(t) + \alpha f(t).Dg(t)$$

$$= \beta(f.g)(t) + \alpha g(t)Df(t) + \alpha f(t).Dg(t)$$

Hence,

$$D_D^{\alpha}(f.g) = (D_D^{\alpha} f).g + \alpha f.Dg$$

Example 2.4.2.1[9]**:** Power Function

Find the α − Deformable derivative for the power function $f(t) = t^r$, $r \in \mathbb{R}$.

Solution:

By applying definition 2.4.2.1 of α −Deformable derivative to get

$$D_D^{\alpha} f = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta) f(t + \varepsilon\alpha) - f(t)}{\varepsilon}$$

$$D_D^{\alpha}(t^r) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)(t + \varepsilon\alpha)^r - t^r}{\varepsilon}$$

$$D_D^{\alpha}(t^r) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta) \sum_{k=0}^{r} \binom{r}{k} t^{r-k} (\varepsilon\alpha)^k - t^r}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)\left(t^r + \varepsilon\alpha r t^{r-1} + \binom{r}{2} t^{r-2} (\varepsilon\alpha)^2 + \cdots + \binom{r}{r} \varepsilon^{r-1} \alpha^r\right) - t^r}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left((1 + \varepsilon\beta)(\alpha r t^{r-1}) + \beta t^r\right)$$

$$D_D^{\alpha} f = \beta t^r + r\alpha t^{r-1}$$

2.4.2.4 Some Useful Theorems on Deformable Derivative

In this section we extend Rolle′s and Taylor’s theorems to deformable derivative with respect to α.

Theorem 2.4.2.4[9] (Rolle’s theorem on deformable derivative):

Let $f : [a, b] \to R$ be a function satisfying:

(i) f is continuous on $[a, b]$

(ii) f is α-differentiable in (a, b)

(iii) $f(a) = f(b)$.

Then, there exists a point $c \in (a, b)$ such that $D^{\alpha} f(c) = \beta f(c)$

Proof:

f is α −differentiable on (a, b), then f is differentiable on (a, b), by Rolle’s theorem $\exists\, c \in (a, b)$, such that $Df(c) = 0$. Hence $D_D^{\alpha} f(c) = \beta f(c) + \alpha Df(c) = \beta f(c)$.

Theorem 2.4.2.5[9] Taylor's theorem

Suppose f is n-times α −differentiable such that all α −derivatives are continuous on $[a, a + h]$. Then

$$f(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k}\left(D_k^\alpha f(a) - \beta \frac{h(1-t)^{k-n+1}}{\alpha n} D_k^\alpha f(a+th)\right) + \frac{h^n}{n!\,\alpha^n} D_n^\alpha f(a+th)$$

where $D_k^\alpha = D^\alpha D^\alpha \dots \dots D^\alpha$, (k-times), $0 < t < 1$.

Proof:

Consider a function φ defined by:

$$\varphi(t) = \sum_{k=0}^{n-1} \frac{(a+h-t)^k}{k!\,\alpha^k} D_k^\alpha f(t) + \frac{A}{n!\,\alpha^n}(a+h-t)^n \quad \rightarrow \quad (1)$$

Where A is a constant to be chosen A such that $\varphi(a + h) = \varphi(a)$. This yields

$$\varphi(a) = \sum_{k=0}^{n-1} \frac{(a+h-a)^k}{k!\,\alpha^k} D_k^\alpha f(a) + \frac{A}{n!\,\alpha^n}(a+h-a)^n$$

$$\varphi(a) = \sum_{k=0}^{n-1} \frac{(h)^k}{k!\,\alpha^k} D_k^\alpha f(a) + \frac{A}{n!\,\alpha^n}(h)^n$$

$$\frac{A}{n!\,\alpha^n} h^n = f(a+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a) \quad \rightarrow \quad (2)$$

Now by hypothesis, φ is α −differentiable in$(a, a + h)$.

By equation (1),

$$\varphi(t) = \sum_{k=0}^{n-1} \frac{(a+h-t)^k}{k!\,\alpha^k} D_k^\alpha f(t) + \frac{A}{n!\,\alpha^n}(a+h-t)^n$$

$$\varphi(t) = \frac{(a+h-t)^0}{0!\,\alpha^0} D_0^\alpha f(t) + \frac{(a+h-t)^1}{1!\,\alpha^1} D_1^\alpha f(t) + \dots + \frac{(a+h-t)^{n-2}}{(n-2)!\,\alpha^{n-2}} D_{n-2}^\alpha f(t) + \frac{(a+h-t)^{n-1}}{(n-1)!\,\alpha^{n-1}} D_{n-1}^\alpha f(t) + \frac{A}{n!\,\alpha^n}(a+h-t)^n$$

Using part (d) of theorem 2.4.2.3 the $\alpha -$ deformable derivative $D^\alpha \varphi$ is given by

$$D^\alpha \varphi(t) = D^\alpha\left(\frac{(a+h-t)^0}{0!\,\alpha^0} D_0^\alpha f(t)\right) + D^\alpha\left(\frac{(a+h-t)^1}{1!\,\alpha^1} D_1^\alpha f(t)\right) + \cdots + D^\alpha\left(\frac{(a+h-t)^{n-1}}{(n-1)!\,\alpha^{n-1}}\right) + D^\alpha\left(\frac{A}{n!\,\alpha^n}(a+h-t)^n\right)$$

$$= D_1^\alpha f(t) + \frac{(a+h-t)}{1!\alpha^1} D_2^\alpha f(t) + \frac{-1}{1!\alpha^1}\alpha D_1^\alpha f(t) + \frac{(a+h-t)^2}{2!\alpha^2} D_3^\alpha f(t) + \frac{-2(a+h-t)}{2!\alpha^2}\alpha D_2^\alpha f(t)$$
$$+ \cdots + \frac{(a+h-t)^{n-2}}{(n-2)!\alpha^{n-2}} D_{n-1}^\alpha f(t) + \frac{-(n-2)(a+h-t)^{n-3}}{(n-2)!\,\alpha^{n-2}} \alpha D_{n-2}^\alpha + \frac{(a+h-t)^{n-1}}{(n-1)!\alpha^{n-1}} D_n^\alpha f(t)$$
$$+ \frac{-(n-1)(a+h-t)^{n-2}}{(n-1)!\,\alpha^{n-1}} \alpha D_{n-1}^\alpha f(t) + \frac{A}{n!\,\alpha^n}[\beta(a+h-t)^n - n\alpha(a+h-t)^{n-1}]$$

Therefore,

$$D^\alpha \varphi(t) = \frac{(a+h-t)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(t) + \frac{A}{\alpha^n n!}(\beta(a+h-t)^n - \alpha n(a+h-t)^{n-1}) \ldots.(3)$$

Hence φ satisfies all the conditions of Rolle's Theorem. So there is some $\theta \in (0, 1)$ such that

$$D^\alpha \varphi(a+\theta h) = \beta\varphi(a+\theta h)$$

Using equations (1), (2) and (3), we have

$$\frac{A}{n!\,\alpha^n} h^n = f(a+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a)$$

$$f(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a) + \frac{A}{n!\,\alpha^n} h^n$$

$$\varphi(a+\theta h) = \sum_{k=0}^{n-1} \frac{(h-\theta h)^k}{k!\,\alpha^k} D_k^\alpha f(a+\theta h) + \frac{A}{n!\,\alpha^n}(h-\theta h)^n$$

$$\beta\varphi(a+\theta h) = \sum_{k=0}^{n-1} \beta\frac{(h-\theta h)^k}{k!\,\alpha^k} D_k^\alpha f(a+\theta h) + \frac{A}{n!\,\alpha^n}\beta(h-\theta h)^n$$

$$\beta\varphi(a+\theta h) = \sum_{k=0}^{n-1} \frac{(h)^k}{k!\,\alpha^k}\beta(1-\theta)^k D_k^\alpha f(a+\theta h) + \frac{A}{n!\,\alpha^n}\beta h^n(1-\theta)^n$$

$$D^\alpha \varphi(a+\theta h) = \frac{(h-h\theta)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(a+\theta h) + \frac{A}{\alpha^n n!}(\beta(h-h\theta)^n - \alpha n(h-h\theta)^{n-1}))$$

Then,

$$\frac{(h-h\theta)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(a+\theta h) + \frac{A}{\alpha^n n!}(\beta(h-h\theta)^n - \alpha n(h-h\theta)^{n-1})) =$$

$$\sum_{k=0}^{n-1} \frac{(h)^k}{k!\,\alpha^k}\beta(1-\theta)^k D_k^\alpha f(a+\theta h) + \frac{A}{n!\,\alpha^n}\beta h^n(1-\theta)^n$$

$$\frac{(h-h\theta)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(a+\theta h) + \frac{A}{\alpha^n n!}\ \beta h^n (1-\theta)^n - \ \frac{A}{\alpha^n n!}\ \alpha n h^{n-1}(1-\theta)^{n-1}$$

$$= \ \sum_{k=0}^{n-1} \frac{h^k}{k!\alpha^k} \beta (1-\theta)^k D_k^\alpha f(a+\theta h) + \frac{A}{n!\alpha^n} \beta h^n (1-\theta)^n$$

If $h^{n-1} = \frac{h^n}{h}$, and is substituted $\frac{A}{n!\alpha^n} h^n$ into the equation (2)

$$\frac{(h-h\theta)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(a+\theta h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} \beta (1-\theta)^k D_k^\alpha f(a+\theta h)$$
$$= \frac{\alpha n}{h}(1-\theta)^{n-1} \left[f(a+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a) \right]$$

$$\frac{(h-h\theta)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(a+\theta h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} \beta (1-\theta)^k D_k^\alpha f(a+\theta h)$$
$$= \ \frac{\alpha n}{h}(1-\theta)^{n-1} f(a+h) - \frac{\alpha n}{h}(1-\theta)^{n-1} \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a)$$

$$\frac{\alpha n}{h}(1-\theta)^{n-1} f(a+h) = \ \frac{(h-h\theta)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha f(a+\theta h)$$
$$-\sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} \beta (1-\theta)^k D_k^\alpha f(a+\theta h) + \ \frac{\alpha n}{h}(1-\theta)^{n-1} \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a)$$

Multiply by $\frac{h}{n\alpha}(1-\theta)^{1-n}$, we get

$$f(a+h) = \frac{h^n}{\alpha^n n!} D_n^\alpha f(a+\theta h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} \frac{\beta h}{n\alpha} (1-\theta)^{k-n+1} D_k^\alpha f(a+\theta h) + \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} D_k^\alpha f(a)$$

Therefore,

$$f(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\,\alpha^k} \left(D_k^\alpha f(a) - \beta \frac{h(1-\theta)^{k-n+1}}{\alpha n} D_k^\alpha f(a+\theta h) \right) + \frac{h^n}{n!\,\alpha} D_n^\alpha f(a+\theta h)$$

This completes proof of the theorem.

Proposition for Deformable Derivatives: [9]

By using Theorem 1.4.2.1[a], we can derive the following properties

1) $D_D^\alpha(e^t) = e^t$
2) $D_D^\alpha(\log(t)) = \beta\log(t) + \frac{\alpha}{t}\ , t > 0$
3) $D_D^\alpha(\sin(t)) = \beta\sin(t) + \alpha\cos(t)$

2.4.2.2 α −Fractional Integral

The fractional integral, as an inverse operator of the fractional derivative, plays an equally important role as the fractional derivative in the field of fractional calculus. The section defines the fractional integral as an inverse operator of the deformable derivative. We'll also learn some basic properties of this type of fractional integration. All functions covered in this section are assumed to be continuous.

Definition 2.4.2.2[9] Let f be a continuous function defined on $[a,b]$. We define $\alpha -$ fractional Integral of f, denoted by $I_a^\alpha f,$ by the integral

$$I_D^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_a^t e^{\frac{\beta}{\alpha}x} f(x)dx \quad ,where\ \alpha + \beta = 1$$

$\alpha \in (0,1]$.

Theorem 2.4.2.6[9] The operator I_a^α a possesses the following properties:

(a) Linearity: $I_D^\alpha(bf + cg) = bI_D^\alpha f + cI_D^\alpha g$

(b) Commutativity: $I_D^{\alpha_1} I_D^{\alpha_2} = I_D^{\alpha_2} I_D^{\alpha_1}\ ,where\ \alpha_i + \beta_i = 1 \qquad ,i = 1,2.$

Proof:

(a) By applying definition 2.4.2.2 of α −Deformable Integral to get

$$I_D^\alpha(bf + cg) = \frac{1}{\alpha} e^{-\frac{\beta}{\alpha}t} \int_a^t e^{\frac{\beta}{\alpha}x} (bf + cg)dx$$

$$= \frac{1}{\alpha} e^{-\frac{\beta}{\alpha}t} \left(\int_a^t e^{\frac{\beta}{\alpha}x}\, b\, f dx + \int_a^t e^{\frac{\beta}{\alpha}x}\, c\, g dx \right)$$

$$= b.\frac{1}{\alpha} e^{-\frac{\beta}{\alpha}t} \int_a^t e^{\frac{\beta}{\alpha}x} f\, dx + c.\frac{1}{\alpha} e^{-\frac{\beta}{\alpha}t} \int_a^t e^{\frac{\beta}{\alpha}x}\, g\, dx$$

$$= b\, I_D^\alpha f + c\ \ I_D^\alpha g$$

(b) By applying definition 2.4.2.2 of α −Deformable Integral to get

$$I_D^{\alpha_1} I_D^{\alpha_2} f(t) = I_D^{\alpha_1} \left(\frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}t} \int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta)\ d\theta \right)$$

$$I_D^{\alpha_1} I_D^{\alpha_2} f(t) = \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha}t} \int_a^t e^{\frac{\beta_1}{\alpha_1}x} \left(\frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}x} \int_a^x e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta)d\theta \right) dx$$

$$= \frac{1}{\alpha_1\alpha_2} e^{-\frac{\beta_1}{\alpha_1}t} \int_a^t \int_a^x e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)x} e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) d\theta dx \;\; \rightarrow \;\; 1$$

$$= \frac{1}{\alpha_1\alpha_2} e^{-\frac{\beta_1}{\alpha_1}t} \int_a^t \int_\theta^t e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)x} e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) dx d\theta \;\; \rightarrow \;\; 2$$

$$= \frac{1}{\alpha_1\alpha_2} e^{-\frac{\beta_1}{\alpha_1}t} \int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) \left(\int_\theta^t e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)x} dx \right) d\theta$$

$$= \frac{1}{\alpha_1\alpha_2} e^{-\frac{\beta_1}{\alpha_1}t} \int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) \left[\frac{1}{\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}} e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)x} \right]_\theta^t d\theta$$

$$= \frac{1}{\alpha_1\alpha_2} e^{-\frac{\beta_1}{\alpha_1}t} \int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) \left[\frac{\alpha_1\alpha_2}{\beta_1\alpha_2 - \beta_2\alpha_1} \left(e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)t} - e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)\theta} \right) \right]$$

$$= \frac{1}{\beta_1\alpha_2 - \beta_2\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \left[\int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) \left(e^{\frac{\beta_1}{\alpha_1}t} . e^{-\frac{\beta_2}{\alpha_2}t} \right) d\theta - \int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) \left(e^{\frac{\beta_1}{\alpha_1}\theta} . e^{-\frac{\beta_2}{\alpha_2}\theta} \right) d\theta \right]$$

$$= \frac{1}{\beta_1\alpha_2 - \beta_2\alpha_1} \left(e^{-\frac{\beta_2}{\alpha_2}t} \int_a^t e^{\frac{\beta_2}{\alpha_2}\theta} f(\theta) d\theta - e^{-\frac{\beta_1}{\alpha_1}t} \int_a^t e^{\frac{\beta_1}{\alpha_1}\theta} f(\theta)\, d\theta \right)$$

$$= \frac{1}{\beta_1\alpha_2 - \beta_2\alpha_1} \left(\alpha_2 I_D^{\alpha_2} - \alpha_1 I_D^{\alpha_1} \right) f(t)$$

Interchanging the role of α_1 and α_2, we have

$$I_D^{\alpha_2} I_D^{\alpha_1} f(t) = \frac{1}{\beta_1\alpha_2 - \beta_2\alpha_1} (\alpha_1 I^{\alpha_1} - \alpha_2 I^{\alpha_2}) f(t) =$$

$$= \frac{1}{\beta_1\alpha_1 - \beta_2\alpha_2} (\alpha_2 I^{\alpha_2} - \alpha_1 I^{\alpha_1}) f(t) =$$

$$= I_D^{\alpha_1} I_D^{\alpha_2} f(t)$$

This completes the proof.

Example 2.4.2.2[9]**:** Constant Function

Find the $\alpha -$ fractional Integral of any constant function $f(t) = \lambda$.

Solution:

By applying definition 2.4.2.2 of α −Deformable Integral to get

$$I_D^{\alpha}\lambda = \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_a^t e^{\frac{\beta}{\alpha}x}\lambda dx$$

$$= \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t} .\ \lambda\int_a^t e^{\frac{\beta}{\alpha}x}dx$$

$$= \frac{\lambda}{\alpha}e^{\frac{-\beta}{\alpha}t}\left[\frac{e^{\frac{\beta}{\alpha}t} - e^{\frac{\beta}{\alpha}a}}{\frac{\beta}{\alpha}}\right]$$

$$= \frac{\lambda}{\beta}\left[e^0 - e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}a}\right]$$

$$I_D^{\alpha}\lambda = \frac{\lambda}{\beta}\left(1 - e^{\frac{\beta}{\alpha}(a-t)}\right)$$

Proposition Deformable Integration: [9]

1) $I_D^{\alpha}e^t = \left(e^t - e^{\frac{(a-\beta t)}{\alpha}}\right)$

2) $I_D^{\alpha}\sin(t) = \frac{1}{\alpha^2+\beta^2}\left(\beta\sin(t) - \alpha\cos(t) + e^{\frac{\beta}{\alpha}(a-t)}(\alpha\cos(a) - \beta\sin(a))\right)$

2.4.2.3 Applications to Fractional Differential Equations

We solve some simple linear fractional differential equations using deformable derivative as D^{α} operator. In first example we discuss method of solving homogeneous linear, while in second non-homogeneous linear fractional differential equations.

Example 2.4.2.3[9]**:** Solve the fractional differential equation:

$$D_D^{\alpha} y(t) + P(t)y(t) = 0$$

where $P(t)$ is continuous.

Solution:

Using Theorem 1.4.2.1[a], then the equation gets transformed to

$$D_D^{\alpha} y(t) = \beta y + \alpha D y$$

$$\alpha D y + \beta y + P(t)y = 0$$

$$Dy + \frac{\big(\beta + P(t)\big)}{\alpha} y = 0$$

The first order homogeneous liner differential equation is one of the forms $Dy + S(t)y = 0$, $Q(t) = \int S(t)dt$, and then the solution is $= Ce^{-Q(t)}$, so

$$Q(t) = \int \frac{\big(\beta + P(t)\big)}{\alpha} dt$$

$$= \frac{1}{\alpha}\left(\int \beta dt \; + \; \int P(t)dt\right)$$

$$= \frac{\beta t + \int P(t)dt}{\alpha}$$

So, $y = Ce^{-Q(t)}$, then

$$y = Ce^{\frac{-(\beta t + \int P(t)dt)}{\alpha}}$$

Example 2.4.2.4[9]**:** Solve the fractional differential equation

$$D_D^{0.5}[D_D^{0.5}y(t)] = 0$$

Solution:

Using Theorem 2.4.2.1

$$D_D^{0.5}y(t) = 0.5y(t) + 0.5Dy(t)$$

$$D_D^{0.5}[D_D^{0.5}y(t)] = D_D^{0.5}\big(0.5y(t) + 0.5Dy(t)\big)$$

$$= \quad 0.5\big(0.5y(t) + 0.5Dy(t)\big) + 0.5D\big(0.5y(t) + 0.5Dy(t)\big)$$

$$= \tfrac{1}{4}D^2y + \tfrac{1}{2}Dy + \tfrac{1}{4}y = 0$$

$$D^2y + \frac{(\alpha_1\beta_2 + \alpha_2\beta_1)}{\alpha_1\alpha_2}Dy + \frac{\beta_1\beta_2}{\alpha_1\alpha_2}y = 0$$

$$D^2y + 2Dy + y = 0$$

The characteristic equation is

$$q^2 + \frac{(\alpha_1\beta_2 + \alpha_2\beta_1)}{\alpha_1\alpha_2}q + \frac{\beta_1\beta_2}{\alpha_1\alpha_2} = 0$$

$$q^2 + 2q + 1 = 0$$

$$(q + 1)^2 = 0$$

$$q = -1$$

The general solution is

$$y = C_1e^{-t} + C_2te^{-t}$$

2.4.3 Classical Derivative Definitions in Fractional Calculus

The derivative of order has been an interesting research topic for centuries. The idea was driven by a question, "What does it mean by $\frac{d^{\frac{1}{2}}f}{dx^{1/2}}$, asked by L'Hospital in 1695 in his letters to Leibniz. Mathematicians have tried to answer this question for centuries from many points of view. Different types of fractional derivatives were introduced: Riemann-Liouville, Caputo, and many others. Most fractional derivatives are defined via fractional integrals.

Some of the inconsistencies of the existing fractional derivatives For example the constant fractional differentiation of Riemann is not zero.

To overcome most of these and other difficulties, Khalil et al, they came up with an interesting idea that extends the definition of the familiar limit, and called it classical derivative. [20]

Definition 2.4.3.1[20] For a function $f:[0,\infty) \longrightarrow R,$ the classical $\alpha-$fractional derivative of $f(t)$ at $t > 0$, where $\alpha \in (0,1)$ is defined by

$$D_{CL}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}$$

If f is classical $\alpha -$ differentiable for some t in the interval $(0, a),\ a > 0$ then at $t = 0$ the fractional derivative is defined as follows $D^{\alpha} f(0) = \lim_{\varepsilon \to 0^+} D^{\alpha} f(t)$.

Theorem 2.4.3.1[20] If a function $f:[0,\infty) \longrightarrow \mathbb{R}$, is classical $\alpha-$differentiable function at $a > 0$, where $\alpha \in (0,1]$, then f is continuous at a.

Proof:

As the function f is $\alpha-$differentiable function at $t = a$, the definition states

$$D_{CL}^{\alpha} f(a) = \lim_{\varepsilon \to 0} \frac{f\left(ae^{\varepsilon a^{-\alpha}}\right) - f(a)}{\varepsilon}$$

We can look into

$$f\left(ae^{\varepsilon a^{-\alpha}}\right) - f(a) = \frac{f\left(ae^{\varepsilon a^{-\alpha}}\right) - f(a)}{\varepsilon}\varepsilon$$

Taking the limit as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0}\left(f\left(ae^{\varepsilon a^{-\alpha}}\right) - f(a)\right) = D_{CL}^{\alpha} f(a) \, .(0)$$

It is known that $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$

Hence

$$ae^{\varepsilon a^{-\alpha}} = a(1 + \varepsilon a^{-\alpha} + \frac{(\varepsilon a^{-\alpha})^2}{2!} + \cdots)$$

We assume $h = \varepsilon a^{1-\alpha} + \frac{\varepsilon^2}{2!} a^{1-2\alpha} + \cdots$, it's clear that $h \to 0 \ as \ \varepsilon \to 0$, then

$$\lim_{h \to 0} f(a+h) - f(a) = 0$$

Or

$$lim_{h \to 0} f(a+h) = f(a)$$

Hence f is continuous at a.

2.4.3.2 Basic Properties of Classical Derivative

Theorem 2.4.3.2[20] Let $\alpha \in (0,1]$, $t > 0$ and f, g be classical $\alpha -$ differentiable function then,

1) $D_{CL}^{\alpha}(af + bg)(t) = aD_{CL}^{\alpha}\big(f(t)\big) + bD_{CL}^{\alpha}\big(g(t)\big); a, b$ constant
2) $D_{CL}^{\alpha}(fg)(t) = f(t)D_{CL}^{\alpha}\big(g(t)\big) + g(t)D_{CL}^{\alpha}\big(f(t)\big)$
3) $D_{CL}^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{g(t)D_{CL}^{\alpha}(f)(t) - f(t)D_{CL}^{\alpha}(g)(t)}{(g(t))^2}$
4) $D_{CL}^{\alpha}(f \circ g)(t) = \left(\frac{d\big(f\big(g(t)\big)}{dt} D_{CL}^{\alpha}\big(g(t)\big)\right)$, for f differentiable at $g(t)$.
5) if f is differentiable then $D_{CL}^{\alpha} f(t) = t^{1-\alpha}\frac{df}{dt}$

Proof:

1) By applying definition 2.4.3.1 of $\alpha -$Classical derivative to get

$$D_{CL}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\big(te^{\varepsilon t^{-\alpha}}\big) - f(t)}{\varepsilon}$$

$$D_{CL}^{\alpha}(af + bg)(t) = \lim_{\varepsilon \to 0} \frac{(af + bg)\big(te^{\varepsilon t^{-\alpha}}\big) - (af + bg)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(af)\big(te^{\varepsilon t^{-\alpha}}\big) + (bg)\big(te^{\varepsilon t^{-\alpha}}\big) - (af)(t) - (bg)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(af)\big(te^{\varepsilon t^{-\alpha}}\big) - (af)(t)}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{(bg)\big(te^{\varepsilon t^{-\alpha}}\big) - (bg)(t)}{\varepsilon}$$

$$= aD_{CL}^{\alpha}(f(t)) + bD_{CL}^{\alpha}(g(t))$$

2) By applying definition of $\alpha -$Classical derivative to get

$$D_{CL}^{\alpha}(fg)(t) = \lim_{\varepsilon \to 0} \frac{(fg)\big(te^{\varepsilon t^{-\alpha}}\big) - (fg)(t)}{\varepsilon}$$

we need to add and subtract the value $f(t)(g)(te^{\varepsilon t^{-\alpha}})$

$$= \lim_{\varepsilon \to 0} \frac{(f)\left(te^{\varepsilon t^{-\alpha}}\right)(g)\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)(g)\left(te^{\varepsilon t^{-\alpha}}\right) + f(t)(g)\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)g(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(f)\left(te^{\varepsilon t^{-\alpha}}\right)(g)\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)(g)(te^{\varepsilon t^{-\alpha}})}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{f(t)(g)\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)g(t)}{\varepsilon}$$

$$= \left(D_{CL}^{\alpha} f(t)\right) \lim_{\varepsilon \to 0} g\left(te^{\varepsilon t^{-\alpha}}\right) + f(t)\left(D_{CL}^{\alpha} g(t)\right)$$

But $\lim_{\varepsilon \to 0} g\left(te^{\varepsilon t^{-\alpha}}\right) = g(\lim_{\varepsilon \to 0} te^{\varepsilon t^{-\alpha}}) = g(t)$

$$= \left(D_{CL}^{\alpha} f(t)\right) g(t) + f(t)\left(D_{CL}^{\alpha} g(t)\right)$$

3) By applying definition of α −Classical derivative to get

$$D_{CL}^{\alpha}\left(\frac{f}{g}\right)(t) = \lim_{\varepsilon \to 0} \frac{\left(\frac{f}{g}\right)\left(te^{\varepsilon t^{-\alpha}}\right) - (\frac{f}{g})(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\dfrac{f\left(te^{\varepsilon t^{-\alpha}}\right)g(t) - g\left(te^{\varepsilon t^{-\alpha}}\right)f(t)}{g(te^{\varepsilon t^{-\alpha}})g(t)}}{\varepsilon}$$

Then it's treated similarly the product case but using $f(t)g(t)$

$$= \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right)g(t) - f(t)g(t) + f(t)g(t) - g\left(te^{\varepsilon t^{-\alpha}}\right)f(t)}{\varepsilon} \lim_{\varepsilon \to 0} \frac{1}{g(te^{\varepsilon t^{-\alpha}})g(t)}$$

$$= \frac{g(t)D_{CL}^{\alpha} f(t) - f(t)D_{CL}^{\alpha} g(t)}{(g(t))^2}$$

4) By applying definition of α −Classical derivative to get

$$D_{CL}^{\alpha}(f \circ g)(t) = \lim_{\varepsilon \to 0} \frac{(f \circ g)\left(te^{\varepsilon t^{-\alpha}}\right) - (f \circ g)(t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f\left(g\left(te^{\varepsilon t^{-\alpha}}\right)\right) - f\left(g(t)\right)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f\left(g\left(te^{\varepsilon t^{-\alpha}}\right)\right) - f\left(g(t)\right)}{g(te^{\varepsilon t^{-\alpha}}) - g(t)} \lim_{\varepsilon \to 0} \frac{g(te^{\varepsilon t^{-\alpha}}) - g(t)}{\varepsilon}$$

But $\lim\limits_{\varepsilon \to 0} g\left(te^{\varepsilon t^{-\alpha}}\right) = g(t)$ *or* $\lim_{\varepsilon \to 0} g\left(te^{\varepsilon t^{-\alpha}}\right) - g(t) = 0$

Hence, we could say $g\left(te^{\varepsilon t^{-\alpha}}\right) - g(t) = h$ with $h \to 0$ *as* $\varepsilon \to 0$

$$= \lim_{h \to 0} \frac{f(g(t)+h) - f\big(g(t)\big)}{h} \; D_{CL}^{\alpha}\big(g(t)\big)$$

$$= f'\big(g(t)\big) D_{CL}^{\alpha}\big(g(t)\big)$$

5) By applying definition 2.4.3.1 of α −Classical derivative to get

$$D_{CL}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}$$

For exponential Taylor expansion

$$te^{\varepsilon t^{-\alpha}} = t\left(1 + \varepsilon t^{-\alpha} + \frac{(\varepsilon t^{-\alpha})}{2!} + \cdots\right)$$

Substituting in the definition we get

$$D_{CL}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha} + \frac{(\varepsilon)^2}{2!} t^{1-2\alpha} + \cdots\right) - f(t)}{\varepsilon}$$

We assume $h = \varepsilon t^{1-\alpha}\left(1 + \frac{\varepsilon}{2!} t^{-\alpha} + \cdots\right)$, it's clear that $h \to 0$ as $\varepsilon \to 0$, and $\varepsilon = \frac{h}{t^{1-\alpha}(1+\frac{\varepsilon}{2!}t^{-\alpha}+\cdots)}$, to get

$$D_{CL}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \left(\frac{\frac{f(t+h) - f(t)}{h}}{t^{1-\alpha}\left(1 + \frac{\varepsilon}{2!} t^{-\alpha} + \cdots\right)} \right)$$

$$= \lim_{\varepsilon \to 0} \frac{f(t+h) - f(t)}{h} \; \lim_{\varepsilon \to 0} t^{1-\alpha}\left(1 + \frac{\varepsilon}{2!} t^{-\alpha} + \cdots\right)$$

Hence

$$D_{CL}^{\alpha} f(t) = t^{1-\alpha} \frac{df(t)}{dt}$$

Example 2.4.3.1[20]**:** Constant Function

Find the $\alpha -$ Classical derivative for the constant function C.

Solution:

Using Theorem 2.4.3.2 (5), then

$$D_{CL}^{\alpha} f(t) = t^{1-\alpha} \frac{df(t)}{dt}$$

$$D_{CL}^{\alpha} C = t^{1-\alpha} \frac{d}{dt} C$$

$$D_{CL}^{\alpha} C = t^{1-\alpha}.(0)$$

$$D_{CL}^{\alpha} C = 0$$

Example 2.4.3.2[20]**:** Power Function

Find the $\alpha -$ Classical derivative for the Power function $f(x) = t^r$, $r \in \mathbb{R}$.

Solution:

Using Theorem 2.4.3.2 (5), then

$$D_{CL}^{\alpha} f(t) = t^{1-\alpha} \frac{df(t)}{dt}$$

$$D_{CL}^{\alpha} t^r = t^{1-\alpha} \frac{d}{dt} t^r$$

$$D_{CL}^{\alpha} t^r = t^{1-\alpha}.rt^{r-1}$$

$$D_{CL}^{\alpha} t^r = rt^{r-\alpha}$$

2.4.3.3 Some Useful Theorems on Classical Derivative

In this section we extend Rolle′s and Taylor's theorems to Classical derivative with respect to α.

Theorem 2.4.3.3[20] (Rolle's Theorem on Classical derivative):

Let $a > 0$ and $f: [a, b] \longrightarrow \mathbb{R}$ be a given function which satisfies the following:

(i) f is continuous on $[a, b]$
(ii) f is classical α $-$differentiable on (a, b) for some $\alpha \in (0, 1)$
(iii) $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $D_{CL}^{\alpha} f(c) = 0$.

Proof:

Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there exists $c \in (a, b)$, that is a point of local extrema. So, we have

$$D_{CL}^{\alpha} f(c^{+}) = \lim_{\varepsilon \to 0^{+}} \frac{f\left(ce^{\varepsilon c^{-\alpha}}\right) - f(c)}{\varepsilon}$$

=

$$D_{CL}^{\alpha} f(c^{-}) = \lim_{\varepsilon \to 0^{-}} \frac{f\left(ce^{\varepsilon c^{-\alpha}}\right) - f(c)}{\varepsilon}$$

However, $D_{CL}^{\alpha} f(c^{+})$ and $D_{CL}^{\alpha} f(c^{-})$ have opposite signs. Hence by contradiction $D_{CL}^{\alpha} f(c) = 0$.

Theorem 2.4.3.4 [20] Taylor Theorem

For a differentiable function $f(t)$ which expands about zero point such as $D_{CL}^{\alpha} f(t) = \sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} t^{k}$, the index rule does not apply $D^{\alpha} D^{\beta}\left(f(t)\right) \neq D^{\alpha+\beta}\left(f(t)\right)$.

Proof:

For R.H.S

$$D^{\alpha+\beta}\left(f(t)\right) = \sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} D^{\alpha+\beta} t^{k}$$

$$= \sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} k t^{(k-(\alpha+\beta))}$$

For L.H.S

$$D^{\beta}\left(f(t)\right) = \sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} \left[D^{\beta} t^{k}\right]$$

$$= \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} kt^{(k-(\beta))}$$

$$D^{\alpha}\left(D^{\beta}(t^k)\right) = D^{\alpha}\left(\sum_{k=0}^{\infty} \frac{f^k(0)}{k!} kt^{(k-(\beta))}\right)$$

$$= \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} kD^{\alpha}(t^{(k-(\beta))})$$

$$= \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} k\left[(k-\beta)t^{(k-\beta)-(\alpha)}\right]$$

$\neq$R.H.S

Proposition for Classical Derivatives: [20]

By using Theorem 2.4.3.2 (5), we can derive the following properties

1) $D_{CL}^{\alpha} e^{ax} = ax^{1-\alpha} e^{ax}$
2) $D_{CL}^{\alpha} \sin(ax) = ax^{1-\alpha} \cos(ax)$
3) $D_{CL}^{\alpha} \ln(x) = x^{1-\alpha} (\frac{1}{x})$

2.4.3.2 α −Fractional Integral

Definition 2.4.3.2[20] For a function $: [0, t) \to \mathbb{R}$, $t > 0$ and $\alpha \in \mathbb{R}$ then the $\alpha -$ Classical integral $I_{CL}^{\alpha} f(t)$ is defined by

$$I_{CL}^{\alpha} f(t) = \int_0^t \frac{f(x)}{x^{1-\alpha}} dx$$

provided the Riemann improper integral exists.

Theorem 2.4.3.5[20] Let $f: [0, \infty) \to \mathbb{R}$ be continuous function such that $I_{CL}^{\alpha} f(t)$ exists, and $\alpha \in (0, 1)$, then

$$D_{CL}^{\alpha} I_{CL}^{\alpha} f(t) = f(t)$$

Proof:

As f is continuous, $I_{CL}^{\alpha} f(t)$ is differentiable.

Using Theorem 2.4.3.2 (5), then

$$D_{CL}^{\alpha}\left(I_{CL}^{\alpha}f(t)\right) = t^{1-\alpha}\frac{d\left(I_{CL}^{\alpha}f(t)\right)}{dt}$$

$$= t^{1-\alpha}\frac{d}{dt}\left(\int_0^t \frac{f(x)}{x^{1-\alpha}}\,dx\right)$$

$$= t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}} = f(t)$$

Example 2.4.3.3[20]**:** Let's consider the $\alpha -$ Classical integral of $f(x) = 3$.

Solution:

Use the Definition 2.4.3.2, then

$$I_{CL}^{\alpha}(3) = \int_0^t \frac{3}{x^{1-\alpha}}dx$$

$$= 3\int_0^t x^{-1+\alpha}dx$$

$$= 3\frac{t^{\alpha}}{\alpha}$$

We can see that the half integral of $f(x) = 3$ is $I_0^{0.5}(3) = 6x^{0.5}$

Example 2.4.3.3[20]**:** Let's consider the $\alpha -$ Classical integral of $f(t) = t^n$.

Solution:

Use the Definition 2.4.3.2, then

$$I_{CL}^{\alpha}(t^n) = \int_0^t \left(\frac{x^n}{x^{1-\alpha}}\right)dx$$

$$= \int_0^t x^{n+\alpha-1}\,dx$$

$$= \frac{t^{n+\alpha}}{n+\alpha}$$

2.4.3.4 Applications to Fractional Differential Equations

The definition simplifies the fractional differential equations by reducing them into ordinary differential equations using the fact that,

$$D_{CL}^{\alpha} f(t) = t^{1-\alpha} \frac{df}{dt}$$

Example 2.4.3.4[21]**:** Let's consider the fractional differential equations

$$D^{1/3} y = -y + x$$

Solution:

The fractional differential equation above converts by Theorem 2.4.3.2 (5) to

$$x^{2/3} y' = -y + x$$

Rearranging the equation to get the general first order linear differential equation form

$$y' + x^{-2/3} y = x^{1/3}$$

The integrating factor $= e^{\int x^{-2/3} dx} = e^{3x^{1/3}}$

The Integral

$$\int x^{1/3} e^{3x^{1/3}} dx$$

Can be solved using the substitution $x^{\frac{1}{3}} = u$, then $u^3 = x$, also $\frac{1}{3} x^{-\frac{2}{3}} dx = du$,

$dx = 3u^2 du$, we have

$$\int u e^{3u} (3u^2) du = \int 3u^3 e^{3u} du$$

Which can be easily solved using tabular integration to get

$$\int 3u^3 e^{3u} du = u^3 e^{3u} - u^2 e^{3u} + \frac{2}{3} u e^{3u} - \frac{2}{9} e^{3u} + c$$

Hence the solution to

$$D^{\frac{1}{3}} y = -y + x$$

is

$$y = \left(x - x^{\frac{2}{3}} + \frac{2}{3} x^{\frac{1}{3}} - \frac{2}{9} \right) + c e^{-3x^{\frac{1}{3}}}$$

Chapter Three

Applications of Fractional Calculus

Fractional Derivative models are used for accurate modeling of those systems that require accurate modeling of damping. In these areas, various analytical and numerical methods have been proposed. In this section their applications to new problems that have emerged in recent years will be studied.

3.1 Fractional conservation of mass

Scientists Whitcraft and Meerschaert (2008) described that partial conservation of the mass equation is needed in order to model fluid flow when the control volume is not large enough compared to the heterogeneity scale. However, if the flow within the control volume is nonlinear, we use partial conservation of the differential, integral, and fractional equations. [22]

3.2 Electrochemical analysis

When studying the redox behavior of a substrate in solution, a voltage is applied to the electrode surface to force electron transfer between the electrode and the substrate. The resulting electron transfer is measured as a current. The current depends on the concentration of the substrate on the electrode surface. When the substrate is consumed, the fresh substrate diffuses to the electrode as described by Fick's laws of diffusion. Taking the Laplace transform of Fick's second law yields a second-order ordinary differential equation

$$\frac{d^2}{dx^2}C(x,s) = sC(x,s)$$

Whose solution $C(x,s)$ contains a dependence of half the power on s. taking the derivative of $C(x,s)$ and then the inverse Laplace transform produces the following relationship:

$$\frac{d}{dx}C(x,t) = \frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}C(x,t)$$

This relates the substrate concentration on the electrode surface to the current. This relationship is applied in electrochemical kinetics to explain mechanical behavior. [17]

3.3 Time-space Fractional diffusion equation models

Anomalous diffusion processes in complex media are well described using partial order diffusion equation models. Fractional Calculus is applied in modeling anomalous diffusion.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -K(-\Delta)^\beta u$$

K: Mechanical Stiffness, Δ: vector of discrete fractional derivatives

Whereas the simple extension of the fractional derivative is the variable order fractional derivative, where α and β are changed to $\alpha(x,t)$ and $\beta(x,t)$. Its applications are implemented in modeling anomalous propagation. [14]

3.4 Proportional Integral Derivative (PID) controllers

Generalizing proportional integral derivative (PID) controllers to use fractional orders can increase their degree of freedom. The new equation relating the control variable $u(t)$ in terms of a measured error value $e(t)$ can be written as

$$u(t) = K_P e(t) + K_i D_t^{-\alpha} e(t) + K_d D_t^\beta e^t$$

$e(t)$: Error value, K_p, K_i, K_d: fractional order stiffness

Where $\alpha \; and \; \beta$ are positive fractional orders and $K_P,\; K_i,$ and $K_d,$ all non-negative denote the coefficients for the proportional, integral, and derivative terms, respectively (sometimes denoted $P,\; i, and\; d$). [18]

3.5 Acoustic wave equations for complex media

The propagation of sound waves in complex media, such as those found in biological tissues, usually obeys the power-frequency law. This type of phenomenon can be described using a causal wave equation involving fractional time derivatives:

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \tau_\sigma^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 u - \frac{\tau_\varepsilon^\beta}{c_0^2}\frac{\partial^{\beta+2} u}{\partial t^{\beta+2}} = 0$$

Fractional differential equations were given a physical meaning by extracting them from physical principles and interpreting the fractional arrangement in terms of acoustic media by the two scientists Bandy and Holm, as they deduced Lomnitz's law in the world of earthquakes and Nutting's law in rheology using fractional calculus. [11]

3.6 Variable-order fractional Schrödinger equation

The variable-order fractional Schrödinger equation has been exploited so that we can study fractional quantum phenomena:

$$ih\frac{\partial\psi^{\alpha(r)}(r,t)}{\partial t^{\alpha(r)}} = (-h^2\Delta)^{\frac{\beta(t)}{2}}\psi(r,t) + V(r,t)\psi(r,t)$$

Where $\Delta = \frac{\partial^2}{\partial r^2}$ are the Laplace operator and the operator $(-h^2\Delta)^{\frac{\beta(t)}{2}}$ is the variable-order fractional quantum Riesz derivative. [5]

3.7 Groundwater flow problem

In 2013-2014 Atangana et al. Some groundwater flow problems were analysed using fractional calculus, where Darcy's law was generalized to look at water flow as a function of an incorrect derivative.

The concept of the partial order derivative and the law of conservation of mass are then used to derive a new equation for groundwater flow.

This generalized law and the law of conservation of mass are then used to derive a new equation for groundwater flow. [4]

Chapter Four

Fractional Newton-Raphson Method

We know that the Newton-Raphson's method (N-R) is useful for finding the roots of a $n-$degree polynomials, with $n \in \mathbb{N}$. In this chapter this method will be used with some definitions of fractional derivatives, namely Riemann-Liouville, Deformable derivative and Classical derivative. First we will consider the usual Newton-Raphson method.

4.1 Newton-Raphson Method

4.1.1 Introduction

The Newton-Raphson method, or Newton's method, is a method for solving equations numerically. It is based on the simple idea of linear approximation. For the one-dimensional case, the N-R method is one of the most widely used methods for finding the roots x_*.

The N-R method is expressed in terms of an iteration function $\phi : R \to R$, as follows

$$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} , \quad n = 0,1, \ldots$$

The derivative of f at x_n cannot be equal to zero since division by zero will not be possible. [6]

4.1.2 Historical

The Newton-Raphson method is named after Isaac Newton; the man who discovered the method in 1736, and Joseph Raphson, the man who described the method back in 1690. Both mathematicians utilized calculus in this method in order to find the roots of an equation. [6]

4.1.3 The Newton-Raphson Iteration

Let $f \in C^2[a,b]$, $\{x_n\}$ be a sequence of approximate roots for f, such that:

$f'(x_n) \neq 0$, and $|x_n - \alpha| < \epsilon, \forall n$, where α is the exact root for $f(x) = 0$

Using Taylor expansion for f, around x_n, we get

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{(x-x_n)^2}{2!}f''(x_n) + \cdots$$

Substitute $x = \alpha$

$$0 = f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + (small\ terms)$$

So,

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n)$$

Solving for α gives

$$\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$$

Hence starting from an initial approximation x_0, Newton Raphson method generates the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2, \ldots$$

4.1.4 A Geometric Interpretation of the Newton-Raphson Iteration

In Figure 4.1.4 below, the curve $y = f(x)$ meets the x −axis at x_*. Let x_1 be the current estimate of r. The tangent line to $y = f(x)$ at the point $(x_1, f(x_1))$ has equation

$$y = f(x_1) + (x - x_1)f'(x_1)$$

Let x_2 be the x −intercept of the tangent line. Then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

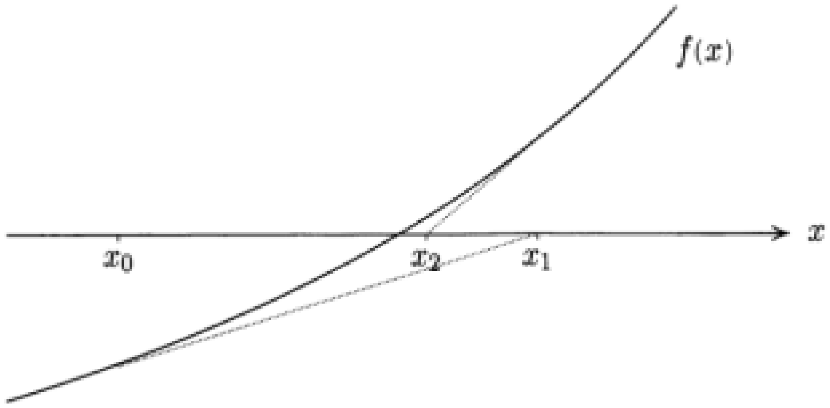

Figure 4.1.4: Illustration of the Newton-Raphson Method

The N-R method is based on creating a sequence $\{x_n\}$ by means of the intersection of the tangent line of the function $f\ (x)$ at the x_n point with the x axis; if the initial condition x_0 is close enough to the root x_* then the sequence $\{x_n\}$ should be convergent to the root x_*.

In the case of polynomials if we have a root x_* with a certain multiplicity m, with $m \in N$, i.e

$$f(x) = (x - x_*)^m g(x)\ ,\quad g(x_*) \neq 0,$$

N-R method converges linearly. [6]

4.1.5 The Convergence theorem of Newton Method

Newton's method can be written as a functional iteration that converges to a fixed point.

Theorem 4.1.5.1[10] Let f be a function that is twice continuously differentiable and suppose there exists a α such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then there exists a δ such that for any $x_0 \in (\alpha - \delta\ , \alpha + \delta)$, the sequence

$$x_n = g(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Converges to α.

Proof:

Note that

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

Therefore, $g'(\alpha) = 0$ because we assume $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Further we know g' is continuous because we assumed f was twice continuously differentiable.

Therefore, given $K < 1$, there exists $\delta > 0$ such that for all $x \in (\alpha - \delta, \alpha + \delta)$, we have $|g'(x)| < K$. For any $a, b \in (\alpha - \delta, \alpha + \delta)$ we can also write

$$|g(a) - g(b)| \leq |g'(c)||a - b| \leq K|a - b|$$

In the interval of $\alpha \pm \delta$ we have that g is a shrinking map. Therefore, there exist a unique fixed point α such that $g(\alpha) = \alpha$. This value α is a root of f.

Steps of Newton-Raphson algorithm:

Step 1: Choose appropriate initial guess $x_0 \in [a, b]$

Step 2: Set $n = 0$

Step 3: Calculate, $x_1 = x_0 - \frac{f(x_1)}{f'(x_1)}$

Step 4: Set $n = n + 1$ and continue in the iterative processes,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 4.1.5.1: Use Newton-Raphson algorithm to find the approximate root of the following equation $f(x) = x^3 - x$, with $x_0 = 1.5$.

Solution:

Since,

$$f'(x) = 3x^2 - 1$$

The Newton-Raphson method formula is given by

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}$$

$$= \frac{3x_n^3 - x_n - x_n^3 + x_n}{3x_n^2 - 1}$$

$$= \frac{2x_n^3}{3x_n^2 - 1}$$

Iterating this formula from initial guess $x_0 = 1.5$ yields

$$x_1 = \frac{2x_0^3}{3x_0^2 - 1} = \frac{2(1.5)^3}{3(1.5)^3 - 1}$$

$$x_1 = 1.5 - \frac{1.875}{5.75} = 1.1739130434782608$$

We continue, iteratively is satisfied:

Table 4.1.5.2: Iteration with Newton-Raphson Method

n	x_n	x_{n+1}	e_n	$\frac{e_n}{e_{n-1}^2}$
1	1.173913043	1.032307127498	0.173913043	0.695652172
2	1.0323071275	1.00145595378	0,173913043	1,068154411
3	1.001455954	1.00000316894	$1.455954e^{-3}$	1,39492538
4	1.000003169	1.000000000015	$3.169e^{-6}$	1.494950908
5	1.000000000015	1.0	$1.5e^{-11}$	1.493642906

The convergence of Newton's method to $x_* = 1$, is very fast and this is clear from the second column. The last column indicates that the convergence is of second order in agreement with the theoretical result, since $x_* = 1$ is a simple zero.

4.1.6 Application of the Newton-Raphson Method

4.1.6.1 Solving transcendental equations

Many transcendental equations can be solved using Newton's method. So, if Newton's method is applied to a transcendental equation, and it is observed that it converges to a solution to the equation, this means that the solution is a computable number as it is exactly represented by the pair consisting of an initial approximation and an algorithm number to increase the accuracy of any approximate number. [23]

4.1.6.2 Energy flow

The Newton-Raphson method is a vital tool for analyzing energy flow, but it is sometimes inefficient for this use due to constantly evolving energy generation strategies and errors in the model on which it is used. But if it is inefficient, there are other tools that can be used, such as the Gauss-Seidel method, but in general the Newton-Raphson method is the standard due to its relative lack of complexity compared to the latter. [23]

4.1.6.3 Newton-Raphson Method in practice

The application of the NR method in practice is diverse. Applications are practical Makes it easier for students to visualize various applications. Wang et al. In 2002 it was used The NR method was used to estimate logistic model parameters with a real bladder cancer dataset, set up at the Fred Hutchinson Cancer Research Center in 1990. In 2012 the NR method was used to estimate logistic model parameters using data from the 2004 cable television survey in Taiwan. [23]

4.2 Fractional Newton-Raphson Method

4.2.1 Introduction

The N-R method is useful for finding the roots of a polynomial of degree n, with $n \in \mathbb{N}$, however it is limited to only being able to find real roots of the polynomial if a real initial condition is taken, to solve this problem and to develop a method that is able to find both the real and complex roots of a polynomial is made use of the method of N-R with the implementation of the fractional derivative.

The Fractional Newton-Raphson (F N-R) method can be defined for $f \in P_n(x)$, where $P_n(x)$ is the set of polynomials of degree $\leq n$, as follows

$$x_{n+1} = \Phi(\alpha, x_n) = x_n - \frac{f(x_n)}{D^\alpha f(x_n)} \quad , \qquad n = 0,1, \dots \dots$$

where $0 < \alpha < 2$. [6]

4.2.2 The Convergence theorem of the Fractional Newton-Raphson Method

Proposition 4.2.2.1[19] Let $f: \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function with a zero $\xi \in \Omega$. Then, any sequence $\{x_i\}_{i=0}^{\infty}$ generated by the iteration function of the F N–R method, such that $x_i \to \xi$, fulfills the following condition:

$$|x_{i+1} - \xi| \leq \frac{\left|\Phi^{(p)}(\alpha, \xi)\right|}{p!} |x_i - \xi|^p$$

Where

$$p = \begin{cases} 1, & if\ \alpha \neq 1\ and\ f^\alpha(\xi) \neq 0 \\ 2, & if\ \alpha = 1\ and\ f^1(\xi) \neq 0 \end{cases}$$

Proof:

Considering the iteration function of the fractional Newton Raphson method

$$\Phi(\alpha, x) = x - \left(f^\alpha(x)\right)^{-1} f(x),$$

And calculating its first and second derivative

$$\Phi^{(1)}(\alpha, x) = 1 - \left(f^\alpha(x)\right)^{-1} f^{(1)}(x) + f(x)\left[\left(f^\alpha(x)\right)^{-2} D_x f^\alpha(x)\right]$$

$$\begin{aligned}\Phi^{(2)}(\alpha, x) = f(x)\left[\left(f^\alpha(x)\right)^{-2} D_x^2 f^\alpha(x) - 2\left(f^\alpha(x)\right)^{-3}\left(D_x f^\alpha(x)\right)^2\right] \\ + 2\left(f^\alpha(x)\right)^{-2} f^{(1)}(x)\, D_x f^\alpha(x) - \left(f^\alpha(x)\right)^{-1} f^{(2)}(x)\end{aligned}$$

Then, assuming that $f^\alpha(\xi) \neq 0$, $\forall \alpha \in (\mathbb{R} \backslash \mathbb{Z}) \cup \{1\}$, the fact that ξ is a zero of f, we obtain that

$$\lim_{x\to\xi} \Phi(\alpha, x) = \xi$$

$$\lim_{x\to\xi}\left|\Phi^{(1)}(\alpha, x)\right| = \begin{cases} \left|1 - \left(f^{\alpha}(\xi)\right)^{-1} f^{(1)}(\xi)\right| & , if\ \alpha \neq 1 \\ 0 & , if\ \alpha = 1 \end{cases}$$

$$\lim_{x\to\xi} |\Phi^{(2)}(\alpha, x)| = \begin{cases} \left|2\left(f^{\alpha}(\xi)\right)^{-2} f^{1}(\xi)\, D_x f^{\alpha}(\xi) - \left(f^{\alpha}(\xi)\right)^{-1} f^{(2)}(\xi)\right| & , if\ \alpha \neq 1 \\ \left|\left(f^{\alpha}(\xi)\right)^{-1} f^{(2)}(\xi)\right| & , if\ \alpha = 1 \end{cases}$$

So the F N–R method has an order of convergence at least linear, with $p \geq 1$.

4.2.3 Fractional Newton Raphson method with respect to some Fractional Derivative Definitions

In this section, you will learn how to find the α −fractional derivative equations $(D^{\alpha} f)$ of polynomials for each of the three definitions. Note that these equations will be applied to the Python program for the three laws in order to compare them and find the best law.

Example 4.2.3.1: Consider the function

$f(x) = x^3 + x$, $x_0 = 0.5$ and $\alpha = 0.9$.

Find:

$a)$ $D^{\alpha}_{RL} f(x)$

$b)$ $D^{\alpha}_{D} f(x)$

$c)$ $D^{\alpha}_{CL} f(x)$

Solution:

a)

We know that the α −fractional derivative with Riemann-Liouville is

$$D^{\alpha}_{RL} x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

So,

$$D^{0.9}_{RL} x^3 = \frac{\Gamma(4)}{\Gamma(3-0.9+1)}\ x^{3-0.9}$$

$$= \frac{3!}{\Gamma(3.1)}\ x^{2.1}$$

$$= \frac{6}{2.19762}\ x^{2.1} = 2.73023\ x^{2.1}$$

$$D_{RL}^{0.9}\, x = \frac{\Gamma(1)}{\Gamma(1-0.9+1)}\ x^{1-0.9}$$

$$= \frac{1!}{\Gamma(1.1)}\ x^{0.1}$$

$$= \frac{1}{0.951351}\ x^{0.1} = 1.05114\, x^{0.1}$$

So,

$$D_{RL}^{\alpha} f = 2.73023\, x^{2.1} + 1.05114\, x^{0.1}$$

b)

We know that the $\alpha -$ fractional derivative with Deformable derivative is

$$D_{D}^{\alpha} f(x) = \beta f(x) + \alpha f'(x) \quad , where\ \alpha + \beta = 1$$

So,

$$D_{D}^{0.9}(x^3 + x) = 0.1(x^3 + x) + 0.9(3x^2 + 1)$$

$$= 0.1x^3 + 0.1x + 2.7x^2 + 0.9$$

So,

$$D_{D}^{\alpha} f = 0.1x^3 + 2.7x^2 + 0.1x + 0.9$$

c)

We know that the $\alpha -$ fractional derivative with Classical derivative is

$$D_{CL}^{\alpha} f(x) = x^{1-\alpha} f'(x)$$

So,

$$D_{CL}^{0.9}(x^3 + x) = x^{1-0.9}(3x^2 + 1)$$

$$= x^{0.1}(3x^2) + x^{0.1}$$

So,

$$D_{CL}^{\alpha} f = 3x^{2.1} + x^{0.1}$$

Example 4.2.3.2: Consider the function

$$f(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.12$$

$x_0 = 1\ ,\ \ and\ \ \ \alpha = 0.8\ .$

Find:

$a)\ D^{\alpha}_{RL}f(x)$

$b)\ D^{\alpha}_{D}f(x)$

$c)\ D^{\alpha}_{CL}f(x)$

Solution:

a)

We know that the α −fractional derivative with Riemann-Liouville is

$$D^{0.8}_{RL}x^6 = \frac{\Gamma(7)}{\Gamma(6-0.8+1)}x^{5.2} = \frac{720}{\Gamma(6.2)}x^{5.2} = \frac{720}{169.406}x^{5.2}$$

$$D^{0.8}_{RL}x^6 = 4.250145x^{5.2}$$

$$D^{0.8}_{RL}x^5 = \frac{\Gamma(6)}{\Gamma(5-0.8+1)}x^{4.2} = \frac{120}{\Gamma(5.2)}x^{4.2} = \frac{120}{32.5781}x^{4.2}$$

$$D^{0.8}_{RL}x^5 = 3.6835x^{4.2}$$

$$D^{0.8}_{RL}x^4 = \frac{\Gamma(5)}{\Gamma(4-0.8+1)}x^{3.2} = \frac{24}{\Gamma(4.2)}x^{3.2} = \frac{24}{7.75669}x^{3.2}$$

$$D^{0.8}_{RL}x^4 = 3.0941033x^{3.2}$$

$$D^{0.8}_{RL}x^3 = \frac{\Gamma(4)}{\Gamma(3-0.8+1)}x^{2.2} = \frac{6}{\Gamma(3.2)}x^{2.2} = \frac{6}{2.42397}x^{2.2}$$

$$D^{0.8}_{RL}x^3 = 2{,}47528x^{2.2}$$

$$D^{0.8}_{RL}x^2 = \frac{\Gamma(3)}{\Gamma(2-0.8+1)}x^{1.2} = \frac{2}{\Gamma(2.2)}x^{1.2} = \frac{2}{1.1018}x^{1.2}$$

$$D^{0.8}x^2 = 1.8152115x^{1.2}$$

$$D^{0.8}_{RL}x = \frac{\Gamma(2)}{\Gamma(1-0.8+1)}x^{0.2} = \frac{1}{\Gamma(1.2)}x^{0.2} = \frac{1}{0.918169}x^{0.2}$$

$$D^{0.8}x = 1.089124x^{0.2}$$

$$D^{0.8}_{RL}1 = \frac{\Gamma(1)}{\Gamma(0-0.8+1)}x^{-0.8} = \frac{1}{\Gamma(0.2)}x^{-0.8} = \frac{1}{4.59084}x^{-0.8}$$

$$D^{0.8}1 = 0.21783x^{-0.8}$$

So,

$$D_{RL}^{\alpha}f(x) = -12.84(4.250145)x^{5.2} - 25.6(3.6835)x^{4.2} + 16.55(3.0941033)x^{3.2} - 2.21(2.47528)x^{2.2} + 26.71(1.8152115)x^{1.2} - 4.29(1.089124)x^{0.2} - 15.12\ (0.21783)x^{-0.8}$$

So,

$$D_{RL}^{\alpha}f = -54.5719x^{5.2} - 94.2965x^{4.2} + 51.20741x^{3.2} - 5.470365x^{2.2} + 48.4843x^{1.2} - 4.67234x^{0.2} - 3.31312x^{-0.8}$$

b)

We know that the α −fractional derivative with Deformable derivative is

$$D_{D}^{\alpha}f(x) = \beta f(x) + \alpha f'(x) \quad , where\ \alpha + \beta = 1$$

So,

$$D_{D}^{0.8}f(x) = 0.2(-12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.12) + 0.8(\ -77.04x^5 - 128x^4 + 66.2x^3 - 6.63x^2 + 53.42x - 4.29)$$

$$= -2.568x^6 - 5.12x^5 + 3.31x^4 - 0.442x^3 + 5.342x^2 - 0.858x - 3.04 + 61.632x^5 - 102.4x^4 + 52.96x^3 - 5.304x^2 + 42.736x - 3.432$$

So,

$$D_{D}^{\alpha}f = -2.568x^6 + 56.512x^5 - 99.09x^4 + 52.518x^3 + 0.038x^2 + 41.878x - 6.472$$

c)

We know that the fractional derivative with Classical derivative is

$$D_{CL}^{\alpha}f(x) = x^{1-\alpha}f'(x)$$

So,

$$D_{CL}^{0.8}f(x) = x^{0.2}(-77.04x^5 - 128x^4 + 66.2x^3 - 6.63x^2 + 53.42x - 4.29)$$

So,

$$D_{CL}^{0.8}f = -77.04x^{5.2} - 128x^{4.2} + 66.2x^{3.2} - 6.63x^{2.2} + 53.42x^{1.2} - 4.29x^{0.2}$$

4.2.4 Advantages of the Fractional Newton Raphson method

One of the main advantages of the fractional Newton Raphson method is that the initial condition is constant, changing the alpha order to obtain the complex root (real part and Imaginary part) of a polynomial. [6]

Ince the order α of the derivative is varied, different values of α can throw the same root, but with a different number of iteration, then to optimize the method, it is possible to implement a filter in which once we have obtained the roots, only those whose orders of the derivatives have generated a smaller number of iterations are extracted. [6]

The method does not guarantee that all roots of the polynomial are found by leaving an initial condition fixed and by varying the orders alpha of the derivative, as in the classical N-R method, finding the roots will depend on giving an appropriate initial condition, as a consequence, one more advantage that the F N-R method possesses because it has the advantage of working in the complex space, is that the initial condition has the freedom to be both real and complex, as in the classical N-R method. [6]

Chapter Five

Implementing the Fractional Newton-Raphson Method in Python

5.1 Introduction

Python is a programming language widely used in networking applications, software development, data literacy, and machine learning. It is effective and easy to learn. It was created by Guido van Rossum, and first released on February 20, 1991. While you may know the python as a large snake, the name of the Python programming language comes from an old BBC television comedy sketch series called Monty Python's Flying Circus. [1]

In this chapter, we replace the usual derivative in Newton-Raphson method with the fractional derivatives Riemann Liouville, Deformable and Classical fractional derivatives.

We implement the modified version using python.

5.2 Applying difference Fractional derivatives in modified version

Example 5.2: Consider the function

$$f(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.21$$

$x_0 = 1.$

a) For Newton-Raphson Method

For $\alpha = 1$, the results are stated in the following table

Table 5.2[a]: Results of the equation for Newton-Raphson Method

n	$Re(x)$	$Im(x)$	e_n	e_n/e_{n-1}^2
1	0.82468	0.0	1.40869	0.561438
2	0.33285	0.0	0.91685	0.462031
3	1.36046	0.0	1.94446	2.313129
4	1.14957	0.0	1.73358	0.458506
5	0.97369	0.0	1.55769	0.518316
6	0.79133	0.0	1.37533	0.566821
7	−0.22447	0.0	0.35953	0.190073
8	−0.95108	0.0	0.36708	2.839823
9	−0.71223	0.0	0.12822	0.951573
10	−0.60517	0.0	0.02117	1.287640
11	−0.58467	0.0	0.00067	1.490412
12	−0.58400	0.0	$6.8451e^{-7}$	1.534044
13	−0.58400	0.0	$7.1942e^{-13}$	1.534044
14	−0.58400	0.0	$1.1102e^{-16}$	1.535654
15	−0.58400	0.0	−	−
16	−0.58400	0.0	−	−

b) Riemann-Liouville

For $\alpha = 0.9$, the results are stated in the following table

Table 5.2[b]: Results of the equation for Riemann-Liouville

n	$Re(x)$	$Im(x)$	e_n	e_n/e_{n-1}
25	0.88235	−0.2476915	$0.1633273e^{-11}$	0.2354403
26	0.82366	−0.2476915	$0.38581193e^{-12}$	0.2354403
27	0.82366	−0.2476915	$0.90024751e^{-13}$	0.23334
28	0.82366	−0.2476915	$0.21670760e^{-13}$	0.2407
29	0.82366	−0.2476915	$0.52080732e^{-14}$	0.24033
30	0.82366	−0.2476915	$0.10299525e^{-14}$	0.1978
31	0.82366	−0.2476915	$0.10299525e^{-14}$	1.0833124
32	0.82366	−0.2476915	$0.81253232e^{-15}$	0.728232
33	0.82366	−0.2476915	$0.86042284e^{-15}$	1.05894
34	0.82366	−0.2476915	$0.91593399e^{-15}$	1.06452
35	0.82366	−0.2476915	$0.88817842e^{-15}$	0.9697
36	0.82366	−0.2476915	$0.94368957e^{-15}$	1.0625
37	0.82366	−0.2476915	$0.83266727e^{-15}$	0.8824
38	0.82366	−0.2476915	$0.94368957e^{-15}$	1.13333
39	0.82366	−0.2476915	$0.83266727e^{-15}$	0.8824

For $\alpha = 0.95$, the results are stated in the following table

Table 5.2[b]: Results of the equation for Riemann-Liouville

n	$Re(x)$	$Im(x)$	e_n	e_n/e_{n-1}
25	-2.62297	0.0	$0.181201e^{-07}$	0.186229
26	-2.62297	0.0	$0.337448e^{-08}$	0.186229
27	-2.62297	0.0	$0.628425e^{-9}$	0.186229
28	-2.62297	0.0	$0.117031e^{-9}$	0.186229
29	-2.62297	0.0	$0.217948e^{-10}$	0.186231
30	-2.62297	0.0	$0.405882e^{-11}$	0.186228
31	-2.62297	0.0	$0.756325e^{-12}$	0.1863412
32	-2.62297	0.0	$0.140838e^{-12}$	0.1862135
33	-2.62297	0.0	$0.263531e^{-13}$	0.1871165
34	-2.62297	0.0	$0.492978e^{-14}$	0.1870662
35	-2.62297	0.0	$0.144069e^{-14}$	0.18706624
36	-2.62297	0.0	$0.492833e^{-15}$	0.99999993
37	-2.62297	0.0	$0.446619e^{-15}$	0.99999993
38	-2.62297	0.0	$0.445462e^{-15}$	0.999999999
39	-2.62297	0.0	$0.445389e^{-15}$	0.99999999
40	-2.62297	0.0	$0.445389e^{-15}$	0.99999999
41	-2.62297	0.0	$0.445389e^{-15}$	1.0

For $\alpha = 1.45$, the results are stated in the following table

Table 5.2[b]: Results of the equation for Riemann-Liouville

$\boldsymbol{n}$	$\boldsymbol{Re}(\boldsymbol{x})$	$\boldsymbol{Im}(\boldsymbol{x})$	$\boldsymbol{e_n}$	$\boldsymbol{e_n/e_{n-1}}$
130	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
131	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
132	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
133	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
134	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
135	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
136	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
137	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
138	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
139	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
140	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
141	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
142	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
143	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
144	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
145	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
146	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0
147	-0.2170569	-0.999111	$0.2775558e^{-16}$	1.0

c) For Deformable Derivative

For $\alpha = 0.95$, the results are stated in the following table

Table 5.2[c]: Results of the equation for Deformable Derivative

n	$Re(x)$	$Im(x)$	e_n	e_n/e_{n-1}
28	-2.622979	0.0	$0.119921e^{-05}$	0.0617892
29	-2.622979	0.0	$0.734002e^{-07}$	0.0612071
30	-2.622979	0.0	$0.449245e^{-08}$	0.0612048
31	-2.622979	0.0	$0.274961e^{-09}$	0.0612051
32	-2.622979	0.0	$0.168292e^{-10}$	0.0612059
33	-2.622979	0.0	$0.103028e^{-11}$	0.0612202
34	-2.622979	0.0	$0.626165e^{-13}$	0.0612202
35	-2.622979	0.0	$0.444089e^{-14}$	0.0709219
36	-2.622979	0.0	$0.444089e^{-14}$	0.1
37	-2.622979	0.0	$0.444089e^{-14}$	0.1

d) For Classical Derivative

For $\alpha = 1.45$, the results are stated in the following table

Table 5.2[d]: Results of the equation for Classical Derivative

n	$Re(x)$	$Im(x)$	e_n	e_n/e_{n-1}
159	-0.21705	-0.99911	$0.56732e^{-14}$	0.78807
160	-0.21705	-0.99911	$0.44367e^{-14}$	0.78529
161	-0.21705	-0.99911	$0.35135e^{-14}$	0.78612
162	-0.21705	-0.99911	$0.27336e^{-14}$	0.78597
163	-0.21705	-0.99911	$0.21762e^{-14}$	0.78381
164	-0.21705	-0.99911	$0.17422e^{-14}$	0.78588
165	-0.21705	-0.99911	$0.13463e^{-14}$	0.78341
166	-0.21705	-0.99911	$0.10235e^{-14}$	0.77746
167	-0.21705	-0.99911	$0.81631e^{-15}$	0.78204
168	-0.21705	-0.99911	$0.64736e^{-15}$	0.5
169	-0.21705	-0.99911	$0.51178e^{-15}$	1.0

Chapter Six

Results and Recommendations

Through our study of fractional calculus, we see has many important applications in various types of science.

We also noted that The Newton-Raphson (N-R) method is useful to find the roots of a polynomial of degree n However, this method is limited since it diverges for the case in which polynomials only have complex roots if a real initial condition is taken, For this reason, some definitions of fractional derivatives were used to express equations that have complex roots, we used three definitions of fractional derivatives:

(Riemann-Liouvill's, Deformable derivative, Classical derivatives)

The idea in this message is which of the three definitions gives us better results and faster Iteration than applying Newton-Raphson method.

After we analyzed the fractional derivatives and studied their results, as in chapter five, what did we notice?

Example page 56 when we chose a specific alpha value, and applied it Riemann-Liouville Fractional derivative, so that the result we obtained the complex roots the last column indicates that the modified version convergence linearly, and Iteration was faster compared to the other two definitions.

7. Conclusions

The fractional Newton-Raphson method is very effective at finding roots of polynomials since it does not present the problems of divergence as the classical N-R method for a polynomial with only complex roots, however the really interesting thing is that this method opens up the possibility of creating new fractional iterative methods by combining the F N-R method with the existing iterative methods.

In this work another application of fractional calculus is presented and opens the possibility of extending iterative methods that allow us to find roots of more general functions of polynomials.

We end the paper with some important questions that can be implemented in future research

1) Can we get a definition of fractional derivatives that is better than the Riemann definition?
2) If we use this definition($D^{\alpha}x^{p} = px^{p-\alpha}$), will it be better than Riemann?
3) Is there any similarity between the classical fractional derivative and deformable derivative?

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طريقة نيوتن رافسون الكسرية

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الملخص

لقد وجد حساب التفاضل والتكامل الكسري تطبيقات عديدة في مختلف المجالات العلمية.

في هذه الرسالة، نستعرض التعريفات الرئيسية للمشتقات الكسرية ونستكشف تطبيقاتها. نركز تحديدًا على ثلاثة تعريفات للمشتقات الكسرية، ونطبقها على طريقة نيوتن- رافسون لحل المعادلات. نُطبّق هذه الطرق في بايثون لحساب جذور كثيرات الحدود، ونقارن أدائها من حيث سرعة التكرار ودقته. يُسلّط تحليلنا الضوء على التعريف الذي يُعطي تقاربًا أسرع ونتائج أفضل.

في هذه الأطروحة، قمنا بتعزيز طريقة نيوتن-رافسون عن طريق استبدال المشتق العادي بالمشتقات الكسرية المقترحة وتطبيق هذا النهج المعدل لحل المعادلات المختلفة عدديًا ومقارنة النتائج مع طريقة نيوتن-رافسون الأصلية.

علاوة على ذلك، نقوم بإجراء تحليل شامل للأداء لمقارنة الكفاءة العددية ودقة الطرق المعدل.