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Some Norm Inequalities for Kronecker and Hadamard Products

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Dedication

To my mother , Mariem

To my father , Abdelfatah

To my wife , Duaa

To my children , Yazan, Mariem

To my brothers , Mustafa, Alian, Ahmed

To my sisters, Ansaf, Eman, Hanan

To my friend , Tariq

To my colleagues "teachers"

Declaration

I certify that the thesis, submitted for the degree of master, is the result of my own research except where otherwise acknowledged, and that the thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ………….

Alaa saleh

Date :

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Abstract

Many basic properties of the Kronecker products and Hadamard products are given , and many results for positive definite matrices are discussed. Moreover Holdert's inequality and the arthmetic, geometric mean inequalities are also applied for Kronecker and Hadamard products .

An analysis of inequalities concerning the spectral radius of Hadamard products of positive operators as l_p space have been done in all details, including some applications for the Kronecker products in matrix equations and differential matrix equations.

Furthermore we showed that these inequalities can be extended to infinite nonnegative matrices .

A development of inequalities for Kronecker products and Hadamard products of positive definite matrices involving Kronecker powers and Hadamard powers of linear combinat- ion of matrices are given in complete details.

"بعض أطوال المتباينات لضرب الكرونيكر والهادامارد"

إعداد: علاء عبد الفتاح مصطفى صالح

إشراف: د. جميل جمال إسماعيل

ملخص

تم استعراض خصائص كثيرة لضرب كرونيكر وهادامارد وكذلك بعض النتائج التي تتعلق بالمصفوفات الموجبة ومن ثم تطبيق متباينة هولدر والوسط الحسابي والهندسي لضرب كرونيكر وهادامارد.

وكذلك تم تحليل بعض المتباينات المتعلقة بنصف القطر الطبيعي للمؤثرات الموجبة على فضاءات l_{p} والتي تشمل بعض التطبيقات لضرب كرونيكر في المعادلات المصفوفية والتفاضلية وبيان ان هذه المتباينات يمكن توسيعها للمصفوفات اللانهائية غير السالبة وتطوير هذه المتباينات للمصفوفات الموجبة والتي تشتمل على قوى وتركيبات خطية من هذه المصفوفات بالتفصيل التام.

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Introduction

When most people multiply two matrices together, they generally use the conventional multiplication method.

We consider two types of matrix multiplication, that are very interesting, these multiplication are the Kronecker product and the Hadamard product.

In mathematics, the Kronecker product denoted by $\otimes$ is an operation on two matrices of arbitrary sizes resulting in a block matrix. The Kronecker product should not be confused with the usual matrix multiplication which is an entirely different operation.

The Hadamard product denoted by $\circ$ is a binary operation that takes two matrices of the same dimensions, and produces another matrix where each $ijth$ element is the product of the $ijth$ element of the original two matrices.

In chapter one, sections 1, 2 and 3, I give some basic concepts from matrix analysis.

In section 4, I give some of the basic properties of the Kronecker Product, and show The difference between matrix multiplication and Kronecker Products matrices, by comparing some basic properties, also, we present the Kronecker sum of matrices, the vec-vector.

At the end of this chapter in section 5, we present some properties of the Hadamard products of matrices.

In chapter two, we analyze some inequalities for Kronecker products and Hadamard products of positive definite matrices in all details.

In chapter three, we analyze the Hadamard product of matrices of operators on l_p, and inequalities for spectral radius of Hadamard products in all details.

Finally, in chapter four we put some applications of the Kronecker product, matrix equations, and matrix differential equations.

Index of Special Notation

$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
F	Usually field ($\mathbb{R}$ or $\mathbb{C}$)
M_n	Square matrix of size $n \times n$
$M_{m,n}$	Matrix of size $m \times n$
$\det(A)$	The determinant of the matrix $A = [a_{ij}] \in M_n$
A^T	The transpose matrix of a matrix A
$\bar{A}$	Conjugate of $A \in M_{m,n}$
A^*	Conjugate transpose of $A \in M_{m,n}$
A^{-1}	Inverse of a nonsingular $A \in M_n$
$A^{\frac{1}{2}}$	Square root of matrix such that $\left(A^{\frac{1}{2}}\right)^2 = A$
$\operatorname{tr} A$	Trace of $A \in M_n$
$\vert A \vert$	Absolute value $[\vert a_{ij} \vert]$ or $(AA^*)^{\frac{1}{2}}$
$\sigma(A)$	Spectrum of $A \in M_n$
$\rho(A)$	Spectral radius of $A \in M_n$
$A(\alpha, \beta)$	Submatrix of $A \in M_{m,n}$
$A(\alpha)$	Principal submatrix
$Vec(A)$	Vector of stacked columns of $A \in M_{m,n}$
$\otimes$	Kronecker product
$\circ$	Hadamard product
$\oplus$	Kronecker sum
$\Vert\cdot\Vert_1$	l_1 norm
$\Vert\cdot\Vert_2$	l_2 (Euclidean) norm, frobenius

$\|\cdot\|_\infty$	l_∞ (maximum absolute value) norm
$\|\cdot\|_p$	l_p norm
$\{\sigma_i(A)\}$	Singular value of $A \in M_{m,n}$
λ	eigenvalue of A
Cond A	Condition number
x	Column vector
U	Unitary matrix
$\widehat{\mathrm{A}}$	The Hadamard inverse
$[\mathrm{J}_{mn}]_{ij} = 1$	The Hadamard identity
$\sum$	Summation
$\prod$	Product
$A^{\otimes k}$	The k^{th} Kronecker power
$A^{(k)}$	The k^{th} Hadamard power
$\bullet$	The Hadamard sum
$\mathbb{P}_m$	The positive definite matrices

Chapter one

Preliminaries

1. 1 Introduction

The contents of sections 1.1, 1.2, and 1.3 can be found in ref. [11].

Definition 1.1.1 If $A = [a_{ij}] \in M_{m,n}$, then $A^T = [a_{ji}] \in M_{n,m}$ is Called the transporse of A and A^*=$[\bar{a}_{ji}] \in M_{n,m}$ is called the adjoint transpose of A, and the trace of A if $A \in M_n$ is defined by trace(A) = $\sum_{i=1}^{n} a_{ii}$.

Theorem 1.1.1 Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix then

trace(AB) = trace(BA).

Definition 1.1.2 If $A \in M_n$ then

(a) A is called Hermition If $A^* = A$.

(b) A is called normal If $A^*A = A\,A^*$.

(c) A is called unitary If $A^*A = A\,A^* = I_n$, where I_n is an identity matrix of order n.

(d) A is called orthogonal if $A^T = A^{-1}$. Therefore, $A^TA = A\,A^T = I_n$.

Remark 1.1.3 All unitany and Hermition matrices are normal.

Example 1.1.1 If $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in M_2$, then $A^*A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $AA^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, therefore $A^*A = AA^*$, thus A is normal.

Theorem 1.1.2 If A is a Hermition matrix, then its eigenvalues are real number.

Definition 1.1.3 A matrix $A \in M_n$ is called idempotent if $A^2 = A$, and is called nilpotent if $A^n = 0$ for positive integer n.

Definition 1.1.4 Let $A \in M_n$. A non-zero vector $x\ in\ \mathbb{C}^n$ is called an eigenvector corresponding to a scalar λ if $Ax = \lambda x$. The scalar λ is called an eigenvalue of A, the set of all eigenvalues of A is called the spectrum of A and is denoted by $\sigma(A)$.

Definition 1.1.5 The spectral radius of A is the non negative real number

$$\rho(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \}.$$

Example 1.1.2 Consider the matrix $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix} \in M_2$, then we have

$\begin{vmatrix} 7-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = 0$, thus (7- λ) $(1-\lambda) + 8 = 0$, which gives $\lambda = 3\,,5$, therefore

$\sigma(A) = \{ 3\ ,\ 5\}$, Hence $\rho(A) = 5$.

Theorem 1.1.3 Let $A \in M_n$, then trace (A) equals to the sum of the eigenvalues of A and det (A) equals to the product of the eigenvalues of A.

Theorerm 1.1.4 (Schurs unitary Triangularization theorem)

Given a matrix $A \in M_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ in any prescribed order, then there is a unitary matrix $U \in M_n$ such that $U^*AU = T$ where $T = [t_{i,j}] \in M_n$ is upper triangular matrix with diagonal entries $t_{ii} = \lambda_i\ ,\ i = 1\,,2\,,\ldots,$ n.

Definition 1.1.6 The matrix $P \in M_n$ is called a permutation matrix if each row and column has exactly one 1, and zeros elsewhere.

Example 1.1.3 Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, P and Q are permutation matrices**.**

Definition 1.1.8 (a) Let $A \in M_{m,n}$, for index sets $\alpha \subseteq \{1, \ \dots \ , m\}$ and $\beta \subseteq \{1, \ \dots \ , n\}$, we denote the submatrix that lies in the rows of A indexed by α and the columns indexed by β as $A(\alpha, \beta)$**.**

Example 1.1.4 $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}(\{1, 3\}, \{1, 2, 3\}) = \begin{bmatrix} a & b & c \\ g & h & i \end{bmatrix}$.

(b) If $m = n$ and $\alpha = \beta$, then the submatrix $A(\alpha)$ is called a principal submatrix of A.

1.2 Norms of vectors and matrices

Definition 1.2.1 Let V be a vector space over a field (ℝ or ℂ).

A function $\|.\|: V \longrightarrow \mathbb{R}$ is a vector norm if for all $x, y \in V$, we have:

(1) $\| x \| \geq 0$.

(2) $\| x \| = 0$ if and only if $x = 0$.

(3) $\| \alpha x \| = |\alpha| \ \| x \|$ for all scalars $\alpha \in$ □.

(4) $\| x + y \| \leq \| x \| + \| y \|$.

Definition 1.2.2 Let X be a complex (or real) linear space. Then the function

$$(.,.) : X \times X \longrightarrow \mathbb{C}\ (or\ \mathbb{R}) \text{ with the properties}$$

(1) $(x, x) \geq 0,$

(2) $(x, x) = 0$ if and only if $x = 0,$

(3) $(x, y) = \overline{(y, x)},$

(4) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z),$

for all $x, y, z \in X\ \ and\ \ \alpha, \beta \in \mathbb{C}\ (or\ \mathbb{R})$ is called an inner product space on X**.**

Example 1.2.1 (vector norms)

(a) The Euclidean norm (or l_2 norm) on $\mathbb{C}^n$ is

$$\| x \|_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$$

(b) The sum norm (or l_1 norm) on $\mathbb{C}^n$ is $\| x \|_1 = |x_1| + |x_2| + \cdots + |x_n| = \sum_{i=1}^{n} |x_i|$**.**

(c) The max norm (or l_∞ Norm) on $\mathbb{C}^n$ is $\| x \|_\infty = max\{ |x_1|, \ \ldots\ , |x_n| \}$**.**

(d) The l_p Norm on $\mathbb{C}^n$ is $\| x \|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ for $\infty > P \geq 1.$

Theorem 1.2.1 (Hölders Inequality)

If $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}, \text{ that is } \|x\, y\| \leq \|x\|_p \, \|y\|_q.$$

Theorem 1.2.2 (Cauchy – Schwarz Inequality)

If $< . \;, \; . >$ is an inner product on a vector space V over field □, then

$|< x, y >|^2 \leq \; < x, x >< y, y >$. For all $x \,, y \; \in V$, equality occurs if and only if

$x \; and \; y$ are linearly dependent.

Definition 1.2.3 A function $\|.\| : \; M_n \longrightarrow \mathbb{R}$ is said to be a matrix Norm if for all

$A, B \in M_n$ it satisfies the Following :

(a) $\| A \| \geq 0,$ $\qquad \| A \| = 0 \iff A = 0.$

(b) $\| \alpha A \| = |\alpha| \; \| A \|,$ for all scalars $\alpha \in$ □ .

(c) $\| A + B \| \leq \| A \| + \| B \|.$

(d) $\| A B \| \leq \| A \| \; \| B \|.$

Some important properties of matrix norm are **:**

(a) If $A \in M_n, \; then \; \| A^k \| \leq \| A \|^k \;, \quad k \geq 1.$

(b) $\| \mathrm{I}_n \| \geq 1.$

(c) If $A \in M_n$ is invertible matrix, then $\| A^{-1} \| \geq \| A \|^{-1}.$

(d) If $A \neq 0 \in M_n$ such that $A^2 = A \; then \; \| A \| \geq 1.$

Example 1.2.2 Let $A \in M_n$, the p-Norm is defined by $\| A \|_p = \left(\sum\limits_{i,j=1}^{n} |a_{ij}|^p \right)^{1/p}$

for $1 \leq P < \infty$, some special cases of the p-norm are **:**

(a) The l_1-Norm defined for $A \in M_n$ by $\| A \|_1 = \sum_{i,j=1}^{n} |a_{ij}|$. The maximum column sum matrix norm $\|\cdot\|_1$ is defined on M_n by $\| A \|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$.

(b) The l_∞-Norm defined for $A \in M_n$ by $\| A \|_\infty = \max_{1 \le i,j \le n} |a_{ij}|$. The maximum row sum matrix norm $\|\cdot\|_\infty$ is defined on M_n by $\| A \|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$.

(c) In particular , when p=2 then

$\| A \|_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{1/2} = (trace|A|^2)^{\frac{1}{2}} = \sqrt{trace\,(A^*A)}$, is called the Frobenius norm (Euclidean norm).

(d)The spectral Norm is defined by $\| A \|_{sp} = \max_{1 \le i \le n} \{ \sqrt{\lambda_i} : \lambda_i \in \sigma(A^*A) \}$.

Definition 1.2.4 Let X and Y be normed spaces and let $A : X \longrightarrow Y$ be a bounded linear operator with a bounded inverse $A^{-1} : Y \longrightarrow X$. Then Cond (A) = $\| A \| \, \| A^{-1} \|$, is called the condition number of A.

For example the $n \times n$ invertible matrix A we have

$$Cond_2(A) = \| A \|_2 \, \| A^{-1} \|_2 = \frac{\sqrt{|\lambda_{max}|}}{\sqrt{|\lambda_{min}|}}.$$

Definition 1.2.5 A matrix norm $\|\cdot\|$ is called unitarily invariant norm if $\| A \| = \| UAV \|$ For all $A \in M_n$ and all unitary matrices $U, V \in M_n$.

1.3 Positive definite matrices

Definition 1.3.1 A Hermition matrix $A \in M_n$, is said to be positive definite if $x^* A\, x > 0$ for all nonzero $x \in \mathbb{C}^n$, and it is called a positive semidefinite matrix if $x^* A\, x \geq 0$ for all $x \in \mathbb{C}^n$.

Properties of positive definite (semidefinite) matrices **:**

(a) Any principal submatrix of a positive definite matrix is positive definite**.**

(b) The sum of any two positive definite (semidefinite) matrices of the same size is positive definite (semidefinite)**.**

(c) Each eigenvalue of a positive definite (semidefinite) matrix is a positive (nonnegative) real number**.**

(d) For a Hermition matrices A, B we write $A > B$ if A – B is positive definite, similary we write $A \geq B$ if $A - B$ is positive semidefinite**.**

(e) A Hermation matrix with positive (nonnegative) eigenvalues is positive definite (semidefinite)**.**

Definition 1.3.2 Let $A, B \in M_n$, then B is a square root of A, if $B^2 = A$.

Example 1.3.1

Let $A = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \in M_2$, be a Hermition matrix**.**

Then $\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$, thus $(11-\lambda)(11-\lambda) - 1 = 0$, which gives

$\lambda = 10, 12$. The eigenvector for $\lambda = 12$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and for $\lambda = 10$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so the matrix of the eigenvectors is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Finally, we have to convert this matrix into an orthogonal matrix by applying the Gram-Schmidt orthonormalization process on the column vectors to give $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, which is a unitary matrix**.** Thus

$$A^{\frac{1}{2}} = UD^{\frac{1}{2}}U^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{12} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{10}+\sqrt{12}}{2} & \frac{\sqrt{10}-\sqrt{12}}{2} \\ \frac{\sqrt{10}-\sqrt{12}}{2} & \frac{\sqrt{10}+\sqrt{12}}{2} \end{bmatrix}.$$

Theorem 1.3.1 Let $A \in M_n$ be a positive semidefinite and let $r \geq 1$ be a given integer, then there exists a unique positive semidefinite Hermition matrix B such that $B^r = A$, written as $B = A^{\frac{1}{r}}$.

Example 1.3.2 (1) If $A \in P_n$ (positive definite matrix) with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $\mathrm{A} = \mathrm{U}\ \mathrm{dig}(\lambda_1, \lambda_2, \ldots, \lambda_n)\, U^*$, where U is a unitary matrix**.**

(2) If $k \geq 0$, $then\ A^k = U\ \mathrm{dig}\left(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k\right) \mathrm{U}^*$.

(3) The function calculus for A is defined as $f(A) = \mathrm{U}\ \mathrm{dig}(\mathrm{f}(\lambda_1), \mathrm{f}(\lambda_2), \ldots, \mathrm{f}(\lambda_n))\mathrm{U}^*$.

Definition 1.3.3 A map $\phi : M_n \longrightarrow M_m$ is unital if ϕ maps unit element to unit element, $i.e.$ $\phi(\mathrm{I}_n) = \mathrm{I}_m$. ϕ is positive if ϕ maps positive element to positive element, $i.e.$ $\mathrm{A} \geq 0 \Longrightarrow \phi(\mathrm{A}) \geq 0$.

Definition 1.3.4 A map $\psi : \mathbb{P}_n \times \mathbb{P}_n \longrightarrow \mathbb{P}_m$ is jointly concave if for any $\mathrm{A, B, C, D} \in \mathbb{P}_n$ and any $0 < \epsilon < 1$, $\psi(\epsilon A + (1-\epsilon)B,\ \epsilon C + (1-\epsilon)D)$

$$\geq \epsilon\, \psi(A, C) + (1-\epsilon)\, \psi(B, D).$$

Definition 1.3.5 Let $A \in M_{n,m}$, ($m \geq$ n). Let the eigenvalues of the $m \times m$ symmetric matrix A^*A be denoted by σ_i^2 , $i = 1, 2, \ldots, n$. Where $\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \geq \ldots \geq \sigma_n^2$, then σ_1 , σ_2 , ... , σ_n , are called the singular values of A.

Example 1.3.3 Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$, then $A^* = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$, thus $A^*A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 5 & 4 \\ 0 & 4 & 4 \end{bmatrix}$.

The eigenvalues of A^*A are 0 , 4 , 9. Thus the singular values are 0 , 2 , 3.

Theorem 1.3.2 (Singular value Decomposition)

Let $A \in M_{m,n}$ has rank r and let $\{\sigma_i\}_{i=1}^n$ be the nonzero singular value of A, then A can be represented in the form $\mathrm{A = U\, D\, V^*}$ where $\mathrm{U} \in \mathrm{M_m}$ and $\mathrm{V} \in \mathrm{M_n}$ are unitary and the matrix $\mathrm{D} = [\sigma_{\mathrm{i,j}}] \in \mathrm{M_{m,n}}$, $\sigma_{\mathrm{i,j}} = 0$ for all $i \neq j$, and $\sigma_{11} \geq \sigma_{22} \geq \ldots \geq \sigma_{\mathrm{rr}} \geq \sigma_{\mathrm{r+1,r+1}} = \cdots = \sigma_{qq} = 0$. where q = min{m,n}, the numbers $\{\sigma_{\mathrm{i,i}}\} = \{\sigma_{\mathrm{i}}\}$ are the singular values of $A \in M_{m,n}$.

Theorem 1.3.3 (Polar Decomposition)

Let $A \in M_{m,n}$, with $m \leq n$. Then A may be written in the form $A = PU$, where $P \in M_m$ is positive semidefinite, rank $p =$ rank A, and $U \in M_{m,n}$ has orthonormal rows (that is $UU^* = I$). The matrix P is always uniquely determined as $P = (AA^*)^{\frac{1}{2}}$, and U is uniquely determined when A has rank m . If A is real then P and U may be taken to be real.

1.4 The Kronecker product of matrices

Leopold Kronecker was a German mathematician was born in liegnitz, Prussia (December 7,1823-December 29,1891).

In mathematics, the Kronecker product denoted by ⊗ is an operation on two matrices of arbitrary size resulting in a block matrix. The Kronecker product should not be confused with the usual matrix multiplication which is an entirely different operation.

Definition 1.4.1 Let $A = [a_{ij}] \in M_{m,n}$, *and* $B = [b_{ij}] \in M_{p,q}$. Then the Kronecker product of A and B is defined as the matrix $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = [a_{ij}B] \in M_{mp,nq}$, and has mn blocks.

Example 1.4.1 Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, then

$$A\otimes B = \begin{bmatrix} B & 2B & 4B \\ 3B & 0 & B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 3 & 2 & 6 & 4 & 12 & 8 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 9 & 6 & 0 & 0 & 3 & 2 \end{bmatrix}.$$

And $B\otimes A = \begin{bmatrix} A & 0A \\ 3A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 6 & 12 & 2 & 4 & 8 \\ 9 & 0 & 3 & 6 & 0 & 2 \end{bmatrix}$, thus $A\otimes B \neq B\otimes A$, in general.

Also if $A = I_n$, then $A\otimes B = \begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & \cdots & 0 \\ 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & B \end{bmatrix}$, of size $n^2 \times n^2$ where $B \in M_n$.

And $B\otimes A = \begin{bmatrix} b_{11} & \ldots & 0 & & b_{1n} & \ldots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \ldots & b_{11} & & 0 & \ldots & b_{1n} \\ & \vdots & & \ddots & & \vdots & \\ b_{n1} & \ldots & 0 & & b_{nn} & \ldots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \ldots & b_{n1} & & 0 & \ldots & b_{nn} \end{bmatrix}$, of size $n^2 \times n^2$.

We note that if $A = I_n$, $B = I_m$, then $I_n\otimes I_m = I_{nm}$. For example

$$I_2\otimes I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

And if $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$, then $x\otimes y^T = [x_1 y\ ,\ x_2 y\ ,\ \ldots\ ,\ x_m y]^T$

$$= \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix} = xy^T = < x , y > \in M_{m,n}.$$

The following theorem states some basic properties of the Kronecker Product **:**

Theorem 1.4.1.[7] Let $A \in M_{m,n}$ then **:**

(a) $(\alpha A)\otimes B = \alpha\,(A\otimes B) = A\otimes(\alpha B)$, for all $\alpha \in F$ and $B \in M_{p,q}$.

(b) $(A\otimes B) \otimes C = A \otimes (B \otimes C)$, for $B \in M_{m,n}$ *and* $C \in M_{r,s}$.

(c) $(A+B)\otimes C = (A\otimes C) + (B\otimes C)$ for $B \in M_{m,n}$ and $C \in M_{r,s}$.

(d) $A\otimes(B + C) = (A\otimes B) + (A\otimes C)$ for $B , C \in M_{p,q}$.

(e) $(A\otimes B)^T = A^T\otimes B^T$ *for* $B \in M_{p,q}$.

(f) $(A\otimes B)^* = A^*\otimes B^*$ *for* $B \in M_{p,q}$.

(g) $O\otimes A = A\otimes O = O$.

Proof : a) $(\alpha A)\otimes B = \left[\alpha \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\right]\otimes B = \begin{bmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & \ddots & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{bmatrix}\otimes B$

$$= \begin{bmatrix} \alpha a_{11}B & \cdots & \alpha a_{1n}B \\ \vdots & \ddots & \vdots \\ \alpha a_{m1}B & \cdots & \alpha a_{mn}B \end{bmatrix} = \alpha \begin{bmatrix} a_{11}B & \cdots & a_{11}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \alpha\,(A\otimes B)$$

$$= \begin{bmatrix} \alpha a_{11}B & \cdots & \alpha a_{1n}B \\ \vdots & \ddots & \vdots \\ \alpha a_{m1}B & \cdots & \alpha a_{mn}B \end{bmatrix} = \begin{bmatrix} a_{11}\alpha B & \cdots & a_{1n}\alpha B \\ \vdots & \ddots & \vdots \\ a_{m1}\alpha B & \cdots & a_{mn}\alpha B \end{bmatrix} = A\otimes(\alpha B).$$

e) $(A\otimes B)^T = \left[a_{ij}B\right]^T = \left[a_{ji}B^T\right] = A^T\otimes B^T$.

f) $(A\otimes B)^* = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}^* = \begin{bmatrix} \overline{a_{11}}B^* & \cdots & \overline{a_{m1}}B^* \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}}B^* & \cdots & \overline{a_{mn}}B^* \end{bmatrix} = A^*\otimes B^*$.

g) $O \otimes A = \begin{bmatrix} 0A & \cdots & 0A \\ \vdots & \ddots & \vdots \\ 0A & \cdots & 0A \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = 0.$ ■

In the following, we will see the difference between AB and $A \otimes B$, it is known that if $A \in M_{m,k}$, $B \in M_{k,n}$ and $AB = 0$, it is not necessary that $A = 0$ or $B = 0$, but the following corollary shows that if $A \otimes B = 0$, then either $A = 0$ or $B = 0$.

Corollary 1.4.2 Let $A \in M_{m,n}$ and $B \in M_{p,q}$. Then $A \otimes B = 0$, if and only if either $A = 0$ or $B = 0$.

Proof : if $A \otimes B = 0$, then $[a_{ij} B] = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$.

So $B = 0$ or $a_{ij} = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, thus either $A = 0$ or $B = 0$.

Conversely, let either $A = 0$ or $B = 0$. Then by theorem (1.4.1 (g)) then $A \otimes B = 0$. ■

Theorem 1.4.3 (The mixed product rule)

Let $A \in M_{m,n}$, $B \in M_{p,q}$, $C \in M_{n,r}$ and $D \in M_{q,s}$ then $(A \otimes B)(C \otimes D) = (AC \otimes BD)$

Proof : (see ref [13]). ■

Theorem 1.4.4. [1] If $A \in M_n$ and $B \in M_n$ are normal matrices then,

$A \otimes B$ is normal.

Proof : $(A \otimes B)(A \otimes B)^* = (A \otimes B)(A^* \otimes B^*)$ (by theorem 1.4.1 (f))

$= AA^* \otimes BB^*$ (by theorem 1.4.3)

$= A^* A \otimes B^*B$ (since A and B are normal)

$= (A^*\otimes B^*)(A\otimes B)$ (by theorem 1.4.3)

$= (A\otimes B)^* (A\otimes B)$. ■

From the mixed rule product, we have the following corollaries **:**

Corollary 1.4.5. [7] If $A \in M_m$ and $B \in M_n$ are nonsingular, then $A\otimes B$ is also nonsingular, with $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

Proof : ($A\otimes B) (A^{-1}\otimes B^{-1}) = (AA^{-1})\otimes (BB^{-1})$ (by theorem1.4.3)

$= (I_m \otimes I_n) = I_{mn}$.

$(A^{-1}\otimes B^{-1}) (A\otimes B) = (A^{-1}A)\otimes (B^{-1}B) = (I_m \otimes I_n) = I_{mn}$.

Thus $A^{-1}\otimes B^{-1} = (A\otimes B)^{-1}$ under conventional matrix multiplication , so $A\otimes B$ is nonsingular.■

Corollary 1.4.6 If $A \in M_n$ is similar to $B \in M_n$ and $C \in M_m$ is similar to $D \in M_m$ then $A\otimes C$ is similar to $B\otimes D$.

Proof : Since A is similar to B and C is similar to D , there exist nonsingular matrices P, Q such that $A = PBP^{-1}$ and $C = QDQ^{-1}$, so

$A\otimes C = (PBP^{-1}) \otimes (QDQ^{-1})$

$= (P\otimes Q) (BP^{-1} \otimes DQ^{-1})$ (by mixed product rule)

$= (P\otimes Q)(B\otimes D)(P^{-1}\otimes Q^{-1})$ (by mixed product rule)

$= (P\otimes Q)(B\otimes D)\ (P\otimes Q)^{-1}$ (by corollary 1.4.5).■

The following corollaries present the orthogonal and unitary properties of Kronecker product in the usual sense **:**

Corollary 1.4.7 If $A \in M_n$ is orthogonal and $B \in M_m$ is orthogonal then A⊗B is orthogonal matrix**.**

Proof : *A* and *B* are orthogonal, so $AA^T = I_n$ and $BB^T = I_m$.

Using theorem (1.4.3), $(A\otimes B)(A\otimes B)^T = (A\otimes B)(A^T\otimes B^T) = AA^T\otimes BB^T$

$= I_n\otimes I_m = I_{nm}$.

Therefore A⊗B is orthogonal**.**■

Corollary 1.4.8 Let $U \in M_n$ and $V \in M_m$ be a unitary matrices , then U⊗V is a unitary matrix**.**

Proof : U and V are unitary implies $U^{-1} = U^*$ and $V^{-1} = V^*$. Using corollary (1.4.5)

$(U\otimes V)^{-1} = U^{-1} \otimes V^{-1} = U^* \otimes V^* = (U\otimes V)^*$. Therefore U⊗V is a unitary matrix**.**■

Theorem 1.4.9. [7] If $A \in M_n$ and $B \in M_n$, then $\mathrm{tr}(A\otimes B) = tr(A)\ \mathrm{tr}(B) = tr(B\otimes A)$.

Proof : $\mathrm{tr}\ (A\otimes B) = \mathrm{tr}\ (a_{11}B) + \mathrm{tr}\ (a_{22}B) + \ ... + \mathrm{tr}\ (a_{nn}B)$

$= a_{11}\mathrm{tr}\ B + a_{22}\mathrm{tr}\ B + \cdots + a_{nn}\mathrm{tr}\ B$

$= (\ a_{11} + a_{22} + \cdots + a_{nn}\ )\ \mathrm{tr}\ B$

$= \mathrm{tr}\, A\;\; \mathrm{tr}\, \mathrm{B}$.

Consequently, tr $(\mathrm{A}\otimes\mathrm{B}) = (\mathrm{tr}\, A)(\mathrm{tr}\ \mathrm{B}) = (\mathrm{tr}\ \mathrm{B})(\mathrm{tr}\ \mathrm{A}) = \mathrm{tr}\ (\mathrm{B}\otimes\mathrm{A})$. ■

Remark 1.4.1 By theorem (1.4.9) $\mathrm{tr}(\mathrm{A}\otimes\mathrm{B}) = tr(\mathrm{A})\ \mathrm{tr}(\mathrm{B})$, if A and B are square matrices, but if $\mathrm{A} \in \mathrm{M}_{\mathrm{nm}}, \mathrm{B} \in \mathrm{M}_{\mathrm{rs}}$, then $\mathrm{tr}(\mathrm{A}\otimes\mathrm{B}) \neq tr(\mathrm{B}\otimes\mathrm{A})$ in general as will see in the following example **:**

Example 1.4.2 Let $\mathrm{A} = [2 \quad -1]$, $\mathrm{B} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 5 \\ 1 & -1 \end{bmatrix}$, then $\mathrm{A}\otimes\mathrm{B} = \begin{bmatrix} 2 & 4 & -1 & -2 \\ 0 & 6 & 0 & -3 \\ 4 & 10 & -2 & -5 \\ 2 & -2 & -1 & 1 \end{bmatrix}$.

And $\mathrm{B}\otimes\mathrm{A} = \begin{bmatrix} 2 & -1 & 4 & -2 \\ 0 & 0 & 6 & -3 \\ 4 & -2 & 10 & -5 \\ 2 & -1 & -2 & 1 \end{bmatrix}$. Therefore tr $(\mathrm{A}\otimes\mathrm{B}) = 7$, and tr $(\mathrm{B}\otimes\mathrm{A}) = 13$. ■

The mixed product rule can be generalized in two ways as will see in the following theorem **:**

Theorem 1.4.10 If $\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_{\mathrm{P}} \in \mathrm{M}_{\mathrm{m}}$ and $\mathrm{B}_1, \mathrm{B}_2, \dots, \mathrm{B}_{\mathrm{P}} \in \mathrm{M}_{\mathrm{n}}$, then

(a) $(\mathrm{A}_1\otimes\mathrm{A}_2\otimes \dots \otimes\mathrm{A}_p)(\mathrm{B}_1\otimes\mathrm{B}_2\otimes \dots \otimes\mathrm{B}_p) = \mathrm{A}_1\mathrm{B}_1\otimes\mathrm{A}_2\mathrm{B}_2\otimes \dots \otimes\mathrm{A}_p\mathrm{B}_p$.

(b) $(\mathrm{A}_1\otimes\mathrm{B}_1)(\mathrm{A}_2\otimes\mathrm{B}_2) \dots (\mathrm{A}_p\otimes\mathrm{B}_{\mathrm{p}}) = (\mathrm{A}_1\mathrm{A}_2 \dots \mathrm{A}_p)\otimes(\mathrm{B}_1\mathrm{B}_2 \dots \mathrm{B}_p)$.

Proof : We use mathematical induction to prove (a) and (b)**.**

(a) Let $p = 2$, so by the mixed product property $(\mathrm{A}_1\otimes\mathrm{A}_2)(\mathrm{B}_1\otimes\mathrm{B}_2) = \mathrm{A}_1\mathrm{B}_1\otimes\mathrm{A}_2\mathrm{B}_2$.

Assume that $(\mathrm{A}_1\otimes\mathrm{A}_2\otimes \dots \otimes\mathrm{A}_n)(\mathrm{B}_1\otimes\mathrm{B}_2\otimes \dots \otimes\mathrm{B}_n) = \mathrm{A}_1\mathrm{B}_1\otimes\mathrm{A}_2\mathrm{B}_2\otimes \dots \otimes\mathrm{A}_n\mathrm{B}_n$.

Now , ($A_1 \otimes A_2 \otimes \ldots \otimes A_n \otimes A_{n+1}$) ($B_1 \otimes B_2 \otimes \ldots \otimes B_n \otimes B_{n+1}$)

$= [(A_1 \otimes A_2 \otimes \ldots \otimes A_n) \otimes A_{n+1}] [(B_1 \otimes B_2 \otimes \ldots \otimes B_n) \otimes B_{n+1}]$

$= [(A_1 \otimes A_2 \otimes \ldots \otimes A_n) (B_1 \otimes B_2 \otimes \ldots \otimes B_n)] \otimes [A_{n+1} B_{n+1}]$ (by theorem 1.4.3)

$= [A_1B_1 \otimes A_2B_2 \otimes \ldots \otimes A_nB_n] \otimes [A_{n+1} B_{n+1}] = A_1B_1 \otimes A_2B_2 \otimes \ldots \otimes A_nB_n \otimes A_{n+1}B_{n+1}$.

(b) Let $p = 2$, so by the mixed product property $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1A_2) \otimes (B_1B_2)$

Assume that $(A_1 \otimes B_1) (A_2 \otimes B_2) \ldots (A_n \otimes B_n) = (A_1A_2 \ldots A_n) \otimes (B_1B_2 \ldots B_n)$

Now $(A_1 \otimes B_1) (A_2 \otimes B_2) \ldots (A_n \otimes B_n) (A_{n+1} \otimes B_{n+1})$

$= [A_1 \otimes B_1) (A_2 \otimes B_2) \ldots (A_n \otimes B_n)] (A_{n+1} \otimes B_{n+1})$

$= [(A_1A_2 \ldots A_n) \otimes (B_1B_2 \ldots B_n)] (A_{n+1} \otimes B_{n+1})$

$= [(A_1A_2 \ldots A_n)A_{n+1}] \otimes [(B_1B_2 \ldots B_n)B_{n+1}]$ (by mixed product property)

$= (A_1A_2 \ldots A_nA_{n+1}) \otimes (B_1B_2 \ldots B_nB_{n+1})$. ■

Corollary 1.4.11 Let $A \in M_m$ and $B \in M_n$.

(a) if A and B are idempotent then $A \otimes B$ is an idempotent**.**

(b) If A and B are nilpotent then $A \otimes B$ is nilpotent**.**

Proof : (a) A and B are idempotent then $A^2 = A$, $B^2 = B$, so

$(A \otimes B)^2 = (A \otimes B) (A \otimes B) = (AA) \otimes (BB) = A^2 \otimes B^2 = A \otimes B$.

(b) A and B are nilpotent then $A^n = 0, B^n = 0$. So,

$(A \otimes B)^n = (A \otimes B)(A \otimes B) \ldots (A \otimes B) = (AA \ldots A) \otimes (BB \ldots B) = A^n \otimes B^n = 0 \otimes 0 = 0$.

Theorem 1.4.12. [13] Let $A \in M_m$ and $B \in M_n$, then

(a) $(A \otimes I)^k = A^k \otimes I$ and $(I \otimes B)^k = I \otimes B^k$, $k = 1, 2, \ldots$

(b) For any polynomial $p(t)$, $p(A \otimes I_m) = p(A) \otimes I_m$ and $p(I_n \otimes B) = I_n \otimes p(B)$.

Proof : (a) $(A \otimes I)^k = (A \otimes I)(A \otimes I) \ldots (A \otimes I)$

$$= (A\,A\ \ldots\ A) \otimes (I\,I\ \ldots\ I) \quad \text{(by theorem 1.4.10 (b))}$$

$$= A^k \otimes I.$$

And $(I \otimes B)^k = (I \otimes B)(I \otimes B) \ldots (I \otimes B) = (I\,I\ \ldots I) \otimes (B\,B\ \ldots B) = I \otimes B^k$.

(b) Let $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots$, so

$$p(A) = a_0 I_n + a_1 A + a_2 A^2 + \ldots = \sum_{k=0} a_k A^k \ , \quad A^0 = I_n.$$

Now, $p(A \otimes I_m) = \sum_{k=0} a_k (A \otimes I_m)^k = \sum_{k=0} a_k \left(A^k \otimes I_m\right)$ (by part a)

$= \sum_{k=0} ((a_k A^k) \otimes I_m)$ (by theorem 1.4.1 (a))

$= \left(\sum_{k=0} \left(a_k A^k\right) \right) \otimes I_m = p(A) \otimes I_m$.

Similarly, we can prove that $p(I_n \otimes B) = I_n \otimes p(B)$. ■

In the following lemma shows that the Kronecker product of two upper triangular matrices is also upper triangular.

Lemma 1.4.13 . [5] If $A \in M_n$ and $B \in M_m$ be upper triangular then $A \otimes B$ is upper triangular.

Proof : A and B are upper triangular, then $A = [a_{ij}]$ where $a_{ij} = 0$ for $i > j$ and

$B = [b_{pq}]$ where $b_{pq} = 0$ for $p > q$. By definition,

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn}B \end{bmatrix}. \text{ So, } [a_{ij}B] = 0 \quad \text{for } i > j$$

since $a_{ij} = 0$ for $i > j$. Now the block matrices $a_{ii}B$ are upper triangular since

B is upper triangular, hence $A \otimes B$ is upper triangular.■

The following theorem shows the relation between $\sigma(A)$, $\sigma(B)$ and $\sigma(A \otimes B)$:

Theorem 1.4.14.[13] Let $A \in M_n$ and $B \in M_m$, if λ is an eigenvalue of A with corresponding eigenvector $x \in F^n$ and if μ is an eigenvalue of B with corresponding eigenvector $y \in F^m$, then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y \in F^{nm}$. If $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(B) = \{\mu_1, \ldots, \mu_m\}$, then

$\sigma(A \otimes B) = \{\lambda_i \mu_j : i = 1, \ldots, n, j = 1, \ldots, m\}$ (including algebraic multiplicities).

In particular, $\sigma(A \otimes B) = \sigma(A)\,\sigma(B)$.

Proof : Suppose $Ax = \lambda x$ and $By = \mu y$, for $x, y \neq 0$. Now by the mixed product property

$(A \otimes B)(x \otimes y) = (Ax) \otimes (By) = \lambda x \otimes \mu y = \lambda\mu(x \otimes y)$.

By schurs triangularization theorem, there exist unitary matrices $U \in M_n$ and $V \in M_m$, such that $U^*AU = T_A$ and $V^*BV = T_B$ where T_A and T_B are upper triangular matrices

Then by theorems 1.4.1(f) and 1.4.10 (b) $(U \otimes V)^*(A \otimes B)(U \otimes V) = (U^*AV) \otimes (V^*AV)$

$= T_A \otimes T_B$, is

upper triangular and is similar to A⊗B. The eigenvalues of A, B and A⊗B are exactly the main diagonal entries of T_A , T_B and $T_A \otimes T_B$ respectively , and the main diagonal of $T_A \otimes T_B$ consists of pair wise products of the entries on the main diagonals of T_A and T_B.

Corollary 1.4.15 Let $A \in M_n$ and $B \in M_m$. Then $\rho(A \otimes B) = \rho(A)\, \rho(B)$.

Proof : Assume that $\{\lambda_1 , \ldots , \lambda_n\}$ and $\{\mu_1 , \ldots , \mu_m\}$ are the eigenvalues of $A \in M_n$ and $B \in M_m$, respectively**.** Then we have

$$\rho(A \otimes B) = \max_{i,\ j} \{ |\lambda_i \mu_j| \} = \left(\max_i |\lambda_i|\right) \left(\max_j |\mu_j|\right) = \rho(A)\, \rho(B). \blacksquare$$

Corollary 1.4.16.[7] If $A \in M_n$ and $B \in M_m$, then $\det(A \otimes B) = (\det A)^m (\det B)^n$.

Proof : $\det(A \otimes B) = \prod_{i=1}^{n} \prod_{j=1}^{m} (\lambda_i \mu_j) = \left(\lambda_1^m \prod_{j=1}^{m} \mu_j\right)\left(\lambda_2^m \prod_{j=1}^{m} \mu_j\right) \ . \ . \ . \ \left(\lambda_n^m \prod_{j=1}^{m} \mu_j\right)$

$$= (\prod_{i=1}^{n} \lambda_i)^m (\prod_{j=1}^{m} \mu_j)^n = (\lambda_1\lambda_2 \ldots \lambda_n)^m (\mu_1\mu_2 \ldots \mu_m)^n = (\det A)^m (\det B)^n . \blacksquare$$

Corollary 1.4.17 If $A \in M_n$ and $B \in M_m$ are positive (semi) definite Hermitian matrices Then A⊗B is also positive (semi) definite Hermitian **.**

Proof : (see ref [**13**]).■

In the following theorem prove the relation between (S.V.D) of A, B and $A\otimes B$ **:**

Theorem 1.4.18. [13] Let $A \in M_{m,n}$ and $B \in M_{p,q}$ have singular value decomposition $A = V_1 D_1 W_1^*$ and $B = V_2 D_2 W_2^*$, where $D_1 = [\sigma_{ij}(A)] \in M_{m,n}$, $D_2 = [\sigma_{ij}(B)] \in M_{p,q}$, and let rank $A = r_1$ and rank $B = r_2$. Then $A\otimes B = (V_1 \otimes V_2)(D_1 \otimes D_2)(W_1 \otimes W_2)^*$. The nonzero singular values of $A\otimes B$ are the $r_1 r_2$ positive numbers $\{ \sigma_i(A)\sigma_j(B) : 1 \le i \le r_1, 1 \le j \le r_2\}$ (including multiplicites). Zero is a singular value of $A\otimes B$ with multiplicity $\min\{mp , nq\} - r_1 r_2$. In particular, the singular values of $A\otimes B$ are the same as those of $B\otimes A$, and rank $(A\otimes B) =$ rank $(B\otimes A) = r_1 r_2$.

Theorem 1.4.19. [2] If $A \in M_{mn}$ and $B \in M_{mn}$ **.** Then for all p-norms $\| A\otimes B \| = \| A \| \| B \|$.

Proof : (Case 1) For Frobenius norm, $\| A\otimes B \|_F = \| A \|_F \| B \|_F$.

$\| A\otimes B \|_F^2 = tr[(A\otimes B)(A\otimes B)^*] = tr[(A\otimes B)(A^*\otimes B^*)]$ (by theorem 1.4.1 (f))

$= tr (AA^* \otimes BB^*)$ (by Theorem 1.4.3)

$= tr (AA^*) tr (BB^*) = tr (A^*A) tr (B^*B)$ (by theorem 1.4.9)

$= \| A \|_F^2 \| B \|_F^2 = (\| A \|_F \| B \|_F)^2$. Therefore $\| A\otimes B \|_F = \| A \|_F \| B \|_F$.

Now for the 2-norm ;

$\| A \|_2 \| B \|_2 = \sqrt{\lambda_{max}(A)\, \lambda_{max}(B)} = \sqrt{\lambda_{max}(A\otimes B)} = \| A\otimes B \|_2$.

(Case 2) The max-norm , $\| A\otimes B \|_{max} = \| A \|_{max} \| B \|_{max}$

$$\| A\otimes B \|_{\max} = \max_{1\le j_A \le n_A} \sum_{i_A=1}^{n_A} \left|a_{i_A j_A} B\right| = \max_{1\le j_A\le n_A,\ 1\le j_B\le m_B} \sum_{i_A=1}^{n_A} \sum_{i_B=1}^{m_B} \left|a_{i_A j_A} b_{i_B j_B}\right|.$$

$$= \max_{1\le j_A\le n_A} \sum_{i_A=1}^{n_A} \left|a_{i_A j_A}\right| \max_{1\le j_B\le n_B} \sum_{i_B=1}^{m_B} \left|b_{i_B j_B}\right| = \| A \|_{max} \| B \|_{\max}.$$

(Case 3) The ∞-norm is similar to the max-norm except the largest absolute row sum is used rather than the largest absolute column sum**,** by taking the transpose**.**

(Case 4) The spectral-Norm $\| A\otimes B \|_{sp} = \max_{i,\ j} \{ s_i(A)\, s_j(B) \}$

$$= \left(\max_{i} \{ s_i(A) \}\right) \left(\max_{j} \{ s_j(B) \}\right) = \| A \|_{sp} \| B \|_{sp}. \blacksquare$$

Corollary 1.4.20.[2] If $A \in M_n$ and $B \in M_m$ are nonsingular, then $\mathrm{cond}(A \otimes B) = \mathrm{cond}(A)\,\mathrm{cond}(B)$.

Proof : $\mathrm{cond}(A \otimes B) = \| A\otimes B \| \| (A\otimes B)^{-1} \|$

$$= \| A\otimes B \| \| A^{-1}\otimes B^{-1} \| \qquad \text{(by corollary 1.4.4)}$$

$$= \| A \| \| B \| \| A^{-1} \| \| B^{-1} \| = \mathrm{cond}(A)\,\mathrm{cond}(B). \blacksquare$$

The following will concern the Kronecker sum of matrices **:**

Definition 1.4.2 Let $A \in M_n$ and $B \in M_m$. Then the Kronecker sum of A and B is the mn-by-mn matrix denoted by $(A \oplus B)$ and defined as $A \oplus B = (I_m\otimes A) + (B\otimes I_n)$.

the following example shows that $(A \oplus B) \neq (B \oplus A)$ in general**.**

Example 1.4.3 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Then

$$A \oplus B = (I_2 \otimes A) + (B \otimes I_3) = \begin{bmatrix} 3 & 2 & 3 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 1 & 1 & 6 & 0 & 0 & 1 \\ 2 & 0 & 0 & 4 & 2 & 3 \\ 0 & 2 & 0 & 3 & 5 & 1 \\ 0 & 0 & 2 & 1 & 1 & 7 \end{bmatrix}$$

$$B \oplus A = (I_3 \otimes B) + (A \otimes I_2) = \begin{bmatrix} 3 & 1 & 2 & 0 & 3 & 0 \\ 2 & 4 & 0 & 2 & 0 & 3 \\ 3 & 0 & 4 & 1 & 1 & 0 \\ 0 & 3 & 2 & 5 & 0 & 1 \\ 1 & 0 & 1 & 0 & 6 & 1 \\ 0 & 1 & 0 & 1 & 2 & 7 \end{bmatrix}.$$

We saw the Kronecker product of two matrices A and B has as its eigenvalues all possible pairwise products of the eigenvalues of A and B**.** The following theorem shows that the Kronecker sum of A and B has as its eigenvalues all possible pairwise sums of the eigenvalues of A and B**.**

Theorem 1.4.21 Let $A \in M_n$ and $B \in M_m$. If $\lambda \in \sigma(A)$ and $x \in \mathbb{C}^n$ is a corresponding eigenvector of A, and if $\mu \in \sigma(B)$ and $y \in \mathbb{C}^m$ is a corresponding eigenvector of B, then $\lambda + \mu$ is an eigenvalue of the Kronecker sum $(I_m \otimes A) + (B \otimes I_n)$ and $y \otimes x \in \mathbb{C}^{nm}$ is a corresponding eigenvector of the Kronecker sum. In fact $\sigma(A \oplus B) = \sigma(A) + \sigma(B)$.

Proof : (see ref [13]). ■

Remark 1.4.2. [13] Let $A \in M_n$ and $B \in M_m$, then $I_m \otimes A$ commutes with $B \otimes I_n$.

Proof : $(I_m \otimes A)\,(B \otimes I_n) = (I_m B) \otimes (A\, I_n) = B \otimes A = (BI_m) \otimes (I_n A)$

$= (B \otimes I_n)\ (I_m \otimes A)$. ∎

Theorem 1.4.22 Let $A \in M_n$ and $B \in M_m$ be a matrices then $tr(A \oplus B) = m\, tr(A) + n\, tr(B)$.

Proof : $tr(A \oplus B) = tr\big((I_m \otimes A) + (B \otimes I_n)\big)$

$= tr(I_m \otimes A) + tr(B \otimes I_n)$

$= tr(I_m)\, tr(A) + tr(I_n)\, tr(B)$ (by theorem 1.4.9)

$= m\, tr(A) + n\, tr(B)$. ∎

Theorem 1.4.23 Let $A \in M_m$ and $B \in M_n$. Then for $1 < p < \infty$,

$$\| A \oplus B \|_p \leq \sqrt[p]{n}\ \| A \|_p + \sqrt[p]{m}\ \| B \|_p.$$

Proof : $\| A \oplus B \|_p = \| (I_n \otimes A) + (B \otimes I_m) \|_p \leq \| I_n \otimes A \|_p + \| B \otimes I_m \|_p$

$= \| I_n \|_p \| A \|_p + \| B \|_p \| I_m \|_p$ (by theorem 1.4.19)

$= \sqrt[p]{n}\ \| A \|_p + \sqrt[p]{m}\ \| B \|_p$. ∎

We consider members of M_{mn} as vectors by ordering their entries in a conventional way from left to right **,** which is given in the following definition **:**

Definition 1.4.3 Let $A = [a_{ij}] \in M_{mn}$, we associate the vector vec $A \in F^{mn}$ defined by

Vec $A = [a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn}]^T$.

Remark 1.4.3. [**6**] Let $A, B \in M_{mn}$ and $\alpha, \beta \in F$. Then $\mathrm{Vec}(\alpha A + \beta B) = \alpha \mathrm{Vec}(A) + \beta \mathrm{Vec}(B)$.

Proof : $$\mathrm{Vec}(\alpha A + \beta B) = \mathrm{Vec}\begin{bmatrix} \alpha a_{11} + \beta b_{11} & \dots & \alpha a_{1n} + \beta b_{1n} \\ \alpha a_{21} + \beta b_{21} & \dots & \alpha a_{2n} + \beta b_{2n} \\ \vdots & & \vdots \\ \alpha a_{m1} + \beta b_{m1} & \dots & \alpha a_{mn} + \beta b_{mn} \end{bmatrix}$$

$$= [\alpha a_{11} + \beta b_{11}, \dots, \alpha a_{m1} + \beta b_{m1}, \dots, \alpha a_{1n} + \beta b_{1n}, \dots, \alpha a_{mn} + \beta b_{mn}]^T$$

$$= [\alpha a_{11}, \dots, \alpha a_{m1}, \dots, \alpha a_{1n}, \dots, \alpha a_{mn}]^T + [\beta b_{11}, \dots, \beta b_{m1}, \dots, \beta b_{1n}, \dots, \beta b_{mn}]^T$$

$$= \alpha\, [a_{11}, \dots, a_{m1}, \dots, a_{1n}, \dots, a_{mn}]^T + \beta\, [b_{11}, \dots, b_{m1}, \dots, b_{1n}, \dots, b_{mn}]^T$$

$$= \alpha \mathrm{Vec}(A) + \beta \mathrm{Vec}(B). \blacksquare$$

The next theorem indicates to the close relationship between the Vec-vector and the Kronecker Product **:**

Theorem 1.4.24. [**13**] Let $A \in M_{mn}$, $B \in M_{pq}$ and $X \in M_{np}$, then $\mathrm{Vec}(AXB) = (B^T \otimes A)\,\mathrm{Vec}(X)$.

Proof : Denote the K-th column of AXB by $(AXB)_k$. Then $(AXB)_k = A(XB)_k = AXB_k$. This implies that $$(AXB)_k = A\begin{bmatrix} x_{11}b_{1k} + x_{12}b_{2k} + \dots + x_{1p}b_{pk} \\ \vdots \\ x_{n1}b_{1k} + x_{n2}b_{2k} + \dots + x_{np}b_{pk} \end{bmatrix}$$

$$= A\left[\begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} b_{1k} + \begin{bmatrix} x_{12} \\ \vdots \\ x_{n2} \end{bmatrix} b_{2k} + \dots + \begin{bmatrix} x_{1p} \\ \vdots \\ x_{np} \end{bmatrix} b_{pk}\right] = b_{1k}A\begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} + \dots + b_{pk}A\begin{bmatrix} x_{1p} \\ \vdots \\ x_{np} \end{bmatrix}$$

$= \left[b_{1k}A, b_{2k}A, \dots, b_{\boldsymbol{pk}}A\right] \mathrm{Vec}(X) = (B_k^T \otimes A)\, Vec(X)$ for $k = 1, 2, \dots, q$. So,

$$\mathrm{Vec}(AXB) = \begin{bmatrix} B_1^T \otimes A \\ B_2^T \otimes A \\ \vdots \\ B_q^T \otimes A \end{bmatrix} Vec(X) = (B^T \otimes A)\ \mathrm{Vec}(X). \blacksquare$$

Corollary 1.4.25.[7] Let $A \in M_n$, $B \in M_m$ and $X \in M_{nm}$. Then

(a) $\mathrm{Vec}(AX) = (I_m \otimes A)\ \mathrm{Vec}(X)$.

(b) $\mathrm{Vec}(XB) = (\ B^T \otimes I_n)\ \mathrm{Vec}(X)$.

(c) $\mathrm{Vec}(AX + XB) = (A \oplus B^T)\ \mathrm{Vec}(X)$.

Proof : (c) $\mathrm{Vec}(AX + XB) = \mathrm{Vec}(AX) + \mathrm{Vec}(XB)$ (by remark 1.4.3)

$= \mathrm{Vec}(AXI_m) + \mathrm{Vec}(I_n XB)$

$= (I_m \otimes A)\ \mathrm{Vec}(X) + (\ B^T \otimes I_n)\ \mathrm{Vec}(X)$ (by theorem 1.4.24)

$= \left((I_m \otimes A) + (B^T \otimes I_n)\right) \mathrm{Vec}(X)$

$= (A \oplus B^T)\ \mathrm{Vec}(X)$ (by definition 1.4.2).$\blacksquare$

Corollary 1.4.26.[7] Let $A \in M_{mn}$ and $B \in M_{np}$. Then $\mathrm{Vec}(AB) = \left(I_p \otimes A\right) \mathrm{Vec}(B) =$

$(B^T \otimes A)\ \mathrm{Vec}\ I_n = (B^T \otimes I_m)\ \mathrm{Vec}A$.

Proof : $\mathrm{Vec}(AB) = \mathrm{Vec}\left(ABI_p\right) = \left(I_P^T \otimes A\right) \mathrm{Vec}(B)$ (by theorem 1.4.24)

$= (\ I_p \otimes A)\ Vec(B)$.

Next, $AB = AI_n B$ is equivalent to $\mathrm{Vec}(AB) = (B^T \otimes A)\ \mathrm{Vec}\ I_n$. Finally $AB = I_m AB$

is equivalent to $\mathrm{Vec}(AB) = (B^T \otimes I_m)\ \mathrm{Vec}A$. $\blacksquare$

The following lemma describes the relation between $VecA$ and $Vec\mathrm{A}^{\mathrm{T}}$:

Lemma 1.4.27.[**13**] Let $\mathrm{A} \in \mathrm{M}_{mn}$. Then $Vec\mathrm{A}^{\mathrm{T}} = \mathrm{P}\,VecA$, where $P \in \mathrm{M}_{mn}$ is a permutation matrix this matrix P is given by P = $\sum_{i=1}^{m}\sum_{j=1}^{n}\left(E_{ij}\otimes E_{ij}^{T}\right)$ where each E_{ij} has entry 1 in position i,j and all other entries are zero.

The previous lemma leads us to the following theorem **:**

Theorem 1.4.28. [**13**] Let $\mathrm{A} \in \mathrm{M}_{mn}$ and $\mathrm{B} \in \mathrm{M}_{\mathrm{pq}}$. Then $A\otimes B = P_1(B\otimes A)P_2$ where P_1, P_2 are permutation matrices such that $P_1 \in \mathrm{M}_{mp}$, $P_2 \in \mathrm{M}_{\mathrm{nq}}$.

Proof : Let $\mathrm{Y} = \mathrm{AXB}^{\mathrm{T}}$, where $X \in \mathrm{M}_{\mathrm{nq}}$. Then $Y^T = \mathrm{BX}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$. So $\mathrm{VecY} = (B\otimes A)\,Vec\mathrm{X}$

And $\mathrm{Ve}cY^T = (A\otimes B)Vec\mathrm{X}^{\mathrm{T}}$ (by theorem 1.4.24)**.**

But $\mathrm{Ve}cY^T = P_1\mathrm{VecY}$, where $P_1 \in \mathrm{M}_{mp}$ is a permutation matrix, and $\mathrm{VecX} = P_2\,Vec\mathrm{X}^{\mathrm{T}}$

Where $P_2 \in \mathrm{M}_{\mathrm{nq}}$, is a permutation matrix**.** So,

$(A\otimes B)Vec\mathrm{X}^{\mathrm{T}} = \mathrm{Ve}cY^T = P_1\mathrm{VecY} = P_1(B\otimes A)\,Vec\mathrm{X}$, i.e

$(A\otimes B)Vec\mathrm{X}^{\mathrm{T}} = P_1(B\otimes A)\,Vec\mathrm{X}$. But $\mathrm{VecX} = P_2\,Vec\mathrm{X}^{\mathrm{T}}$, so

$(A\otimes B)Vec\mathrm{X}^{\mathrm{T}} = P_1(B\otimes A)P_2\,Vec\mathrm{X}^{\mathrm{T}}$, for all $X^T \in \mathrm{M}_{\mathrm{qn}}$ and this implies

$A\otimes B = P_1(B\otimes A)P_2$. ■

Corollary 1.4.29 Let $\mathrm{A} \in \mathrm{M}_{mn}$ and $\mathrm{B} \in \mathrm{M}_{\mathrm{pq}}$. Then $\| A\otimes B \| = \| B\otimes A \|$ for any unitarily invariant norm $\|\cdot\|$ on $\mathrm{M}_{mp,nq}$.

Proof : $\| A\otimes B \| = \| P_1(B\otimes A)P_2 \|$

$$= \| B\otimes A \|$$

Since P_1 , P_2 are unitary matrices. ■

1.5 The Hadamard product of matrices

The Hadamard product is a binary operation that takes two matrices of the same size, and produces another matrix where each element ij is the product of element ij of the original two matrices.

Definition 1.5.1 The Hadamard product of $\mathrm{A} = [\mathrm{a_{ij}}] \in \mathrm{M_{mn}}$ and $\mathrm{B} = [\mathrm{b_{ij}}] \in \mathrm{M_{mn}}$ is defined by $\mathrm{A} \circ \mathrm{B} = [\mathrm{a_{ij}b_{ij}}] \in \mathrm{M_{mn}}$.

Example 1.5.1 If $\mathrm{A} = \begin{bmatrix} 2 & 3 & \mathrm{i} \\ -1 & 7 & 9 \\ 3\mathrm{i} & 0 & -5 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} -1 & 9 & 6 \\ 2 & -5 & 0 \\ -\mathrm{i} & 1 & -2 \end{bmatrix}$. Then

$$\mathrm{A} \circ \mathrm{B} = \begin{bmatrix} -2 & 27 & 6\mathrm{i} \\ -2 & -35 & 0 \\ 3 & 0 & 10 \end{bmatrix}.$$

The following theorem Shows the set of $m \times n$ matrices with nonzero entries form an abelian group under the Hadamard product :

Theorem 1.5.1 $[\mathbf{14}]$ Let $A, B \in M_{mn}$. Then $\mathrm{A} \circ \mathrm{B} = \mathrm{B} \circ \mathrm{A}$.

Proof : Let A and B be $m \times n$ matrices with entries in $\mathbb{C}$. Then $[\mathrm{A} \circ \mathrm{B}]_{ij} = [a_{ij}b_{ij}]$

$=[b_{ij}a_{ij}] = [\mathrm{B} \circ \mathrm{A}]_{\mathrm{ij}}$ and therefore $\mathrm{A} \circ \mathrm{B} = \mathrm{B} \circ \mathrm{A}$. ■

Definition 1.5.2 The Hadamard identity is the $m \times n$ matrix J_{mn} defined by $[\mathrm{J}_{mn}]_{ij} = 1$ for all $1 \le i \le m$, $1 \le j \le n$.

Theorem 1.5.2 [**14**] Let $A \in M_{mn}$. Then $\mathrm{A} \circ \mathrm{J}_{mn} = \mathrm{J}_{mn} \circ \mathrm{A} = A$.

Proof : $[\mathrm{A} \circ \mathrm{J}_{mn}]_{ij} = [\mathrm{J}_{mn} \circ \mathrm{A}]_{ij}$ (by theorem 1.5.1)

$= [\mathrm{J}_{mn}]_{ij}\,[A]_{ij}$ (by definition H.P)

$= (1)\,[A]_{ij}$ (by definition HID)

$= [A]_{ij}$. Therefore $\mathrm{A} \circ \mathrm{J}_{mn} = A$. ■

Definition 1.5.3 Let $A \in M_{mn}$ and suppose $[A]_{\mathrm{ij}} \neq 0$ for all $1 \le i \le m$, $1 \le j \le n$.

Then the Hadamard inverse denoted by $\widehat{\mathrm{A}}$ is $[\widehat{\mathrm{A}}]_{ij} = ([A]_{ij})^{-1} = \frac{1}{a_{ij}}, a_{ij} \neq 0$ for $1 \le i \le m$, $1 \le j \le n$.

Theorem 1.5.3 [**14**] Let $A \in M_{mn}$ such that $[A]_{\mathrm{ij}} \neq 0$ for all $1 \le i \le m,\ 1 \le j \le n$. Then $\mathrm{A} \circ \widehat{\mathrm{A}} = \widehat{\mathrm{A}} \circ \mathrm{A} = \mathrm{J}_{mn}$.

Proof : $[\mathrm{A} \circ \widehat{\mathrm{A}}]_{ij} = [\widehat{\mathrm{A}} \circ \mathrm{A}]_{ij}$ (by theorem 1.5.1)

$= [\widehat{\mathrm{A}}]_{ij}\,[A]_{ij}$ (by definition H.P)

$= ([A]_{ij})^{-1}\,[A]_{ij} = 1 = [\mathrm{J}_{mn}]_{ij}$.

Therefore $\mathrm{A} \circ \widehat{\mathrm{A}} = \widehat{\mathrm{A}} \circ \mathrm{A} = \mathrm{J}_{mn}$. ■

The following theorem states some basic properties of the Hadamard Product **:**

Theorem 1.5.4 [14] Suppose $A, B, C \in M_{mn}$, then

(a) $\alpha\,(A \circ B) = (\alpha A) \circ B = A \circ (\alpha B)$, for all $\alpha \in F$.

(b) $C \circ (A + B) = C \circ A + C \circ B$.

(c) $(A \circ B)^T = A^T \circ B^T$.

Proof : (a) $[\alpha\,(A \circ B)]_{ij} = \alpha\,[A \circ B]_{ij} = \alpha\,[A]_{ij}\,[B]_{ij} = [\alpha A]_{ij}\,[B]_{ij} = [(\alpha A) \circ B]_{ij}$

So, $\alpha\,(A \circ B) = (\alpha A) \circ B$. And

$[\alpha\,(A \circ B)]_{ij} = \alpha\,[A \circ B]_{ij} = \alpha\,[A]_{ij}\,[B]_{ij} = [A]_{ij}\,\alpha\,[B]_{ij} = [A]_{ij}[\alpha B]_{ij}$

$= [A \circ (\alpha B)]_{ij}$. Therefore $\alpha\,(A \circ B) = A \circ (\alpha B)$.

(b) $[C \circ (A + B)]_{ij} = [C]_{ij}\,[A + B]_{ij} = [C]_{ij}\left([A]_{ij} + [B]_{ij} \right)$

$$= [C]_{ij}[A]_{ij} + [C]_{ij}\,[B]_{ij}$$

$$= [C \circ A]_{ij} + [C \circ B]_{ij}$$

$$= [C \circ A + C \circ B]_{ij}.$$

Therefore $C \circ (A + B) = C \circ A + C \circ B$.

(c) $(A \circ B)^T = [A \circ B]_{ij}^T = [A \circ B]_{ji} = [A]_{ji}\,[B]_{ji} = [A]_{ij}^T\,[B]_{ij}^T = A^T \circ B^T$. ■

From the previous results, we conclude the following corollary **:**

Corollary 1.5.5 If $A, B \in M_{mn}$, then $(\widehat{A \circ B}) = \widehat{A} \circ \widehat{B}$ such that $[A]_{ij} \neq 0$ and

$[B]_{ij} \neq 0$.

Proof : $[A \circ B]_{ij} \left[\widehat{A} \circ \widehat{B}\right]_{ij} = \left([A]_{ij}\, [B]_{ij}\right) \left(\left[\widehat{A}\right]_{ij} \left[\widehat{B}\right]_{ij}\right)$

$$= \left([A]_{ij} \left[\widehat{A}\right]_{ij} \right) \left([B]_{ij} \left[\widehat{B}\right]_{ij} \right)$$

$$= [J_{mn}]_{ij}\, [J_{mn}]_{ij} \ \text{(by theorem 1.5.3)}$$

$$= 1 \,.\, 1 = 1 = J_{mn}.$$

Therefore, $\left(\widehat{A} \circ \widehat{B}\right) = ([A \circ B]_{ij})^{-1} = (\widehat{A \circ B})$. ■

Remark 1.5.1 Let $A, B \in M_n$, if A and B are diagonal matrices then $A \circ B = AB$.

Proof : $A \circ B = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} \circ \begin{bmatrix} b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_{nn} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn}b_{nn} \end{bmatrix} = A\,B. \blacksquare$$

The following theorem gives the relation between diagonal matrices and the matrix products on the Hadamard multiplication **:**

Theorem 1.5.6 [14] If $A, B \in M_{mn}$ and if $D \in M_m$ and $E \in M_n$ are diagonal then

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) .$$

Proof : $[D(A \circ B)E]_{ij} = \sum_{k=1}^{m} [D]_{ik}\, [(A \circ B)E]_{kj}$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} [D]_{ik}\, [A \circ B]_{kl}\, [E]_{lj}$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} [\mathrm{D}]_{\mathrm{ik}} [\mathrm{A}]_{\mathrm{kl}} [\mathrm{B}]_{\mathrm{kl}} \ [\mathrm{E}]_{\mathrm{lj}} \quad (\text{ by definition HP })$$

$$= \sum_{k=1}^{m} [\mathrm{D}]_{\mathrm{ik}} \ [\mathrm{A}]_{\mathrm{kj}} [\mathrm{B}]_{\mathrm{kj}} \ [\mathrm{E}]_{\mathrm{jj}} \quad (\ [\mathrm{E}]_{\mathrm{lj}} = 0 \ \text{ for all } l \neq j)$$

$$= [\mathrm{D}]_{\mathrm{ii}} \ [\mathrm{A}]_{\mathrm{ij}} [\mathrm{B}]_{\mathrm{ij}} \ [\mathrm{E}]_{\mathrm{jj}} \qquad (\ [\mathrm{D}]_{\mathrm{ik}} = 0 \ \text{ for all } i \neq k \ )$$

$$= [\mathrm{D}]_{\mathrm{ii}} \ [\mathrm{A}]_{\mathrm{ij}} [\mathrm{E}]_{\mathrm{jj}} [\mathrm{B}]_{\mathrm{ij}}$$

$$= [\mathrm{D}]_{\mathrm{ii}} \left(\sum_{l=1}^{n} [\mathrm{A}]_{\mathrm{il}} [\mathrm{E}]_{\mathrm{lj}} \right) [\mathrm{B}]_{\mathrm{ij}} \quad (\ [\mathrm{E}]_{\mathrm{lj}} = 0 \ \text{ for all } l \neq j)$$

$$= [\mathrm{D}]_{\mathrm{ii}} \ [\mathrm{AE}]_{\mathrm{ij}} [\mathrm{B}]_{\mathrm{ij}} \quad (\text{ by theorem entries matrix products })$$

$$= \left(\sum_{k=1}^{m} [\mathrm{D}]_{\mathrm{ik}} \ [\mathrm{AE}]_{\mathrm{kj}} \ \right) [\mathrm{B}]_{\mathrm{ij}} \qquad (\ [\mathrm{D}]_{\mathrm{ik}} = 0 \ \text{ for all } i \neq k \ )$$

$= [DAE]_{ij} \ [\mathrm{B}]_{\mathrm{ij}} \ = \ [(\mathrm{DAE}) \circ \mathrm{B})]_{\mathrm{ij}}$. Therefore $\mathrm{D}(A \circ B)\mathrm{E} = (DAE) \circ \mathrm{B}$.

Also,

$$[(\mathrm{D}AE) \circ \mathrm{B}]_{ij} = [DAE]_{ij} [\mathrm{B}]_{\mathrm{ij}} = \left(\sum_{k=1}^{n} [DA]_{ik} \, [E]_{kj} \right) [\mathrm{B}]_{\mathrm{ij}}$$

$$= [DA]_{ij} \, [E]_{jj} \ [\mathrm{B}]_{\mathrm{ij}} \quad (\ [\mathrm{E}]_{\mathrm{kj}} = 0 \ \text{ for all } k \neq j)$$

$$= [DA]_{ij} \, [\mathrm{B}]_{\mathrm{ij}} [E]_{jj}$$

$$= [DA]_{ij} \left(\sum_{k=1}^{n} [\mathrm{B}]_{\mathrm{ik}} [E]_{kj} \right), \quad [\mathrm{E}]_{\mathrm{kj}} = 0 \ \text{ for all } k \neq j .$$

$$= [DA]_{ij} \, [BE]_{ij} = [(DA) \circ (BE)]_{ij}.$$

Therefore, $\mathrm{D}(A \circ B)\mathrm{E} = (\mathrm{DA}) \circ (\mathrm{BE})$. ■

Definition 1.5.4 Define the diagonal matrix $D_x \in \mathrm{M}_{\boldsymbol{n}}$ with entries from a vector $x \in \mathbb{C}^n$

by $[D_x]_{ij} = .\begin{cases} [\mathrm{x}]_\mathrm{i} & \text{if } \mathrm{i} = \mathrm{j} \\ 0 & \text{if } \mathrm{i} \neq \mathrm{j} \end{cases}$

Theorem 1.5.7 [13] Let $A, B \in M_{mn}$ and let $x \in \mathbb{C}^n$. Then the i th diagonal entry of the matrix AD_xB^T coincides with the i th entry of the vector $(A \circ B)x$, $i = 1$, ... , m.

Proof : If $A = [a_{ij}]$, $B = [b_{ij}]$ and $x = [x_i]$, then

$$(AD_xB^T)_{ii} = \sum_{j=1}^{n} a_{ij}x_jb_{ij} = \sum_{j=1}^{n} a_{ij}b_{ij}x_j = [(A \circ B)x]_i \,, \text{ for } i = 1 \,, \ldots \,, m. \blacksquare$$

The following lemma relate the Hadamard product to the Kronecker product by identifying $A \circ B$ as a submatrix of $A \otimes B$.

Lemma 1.5.8 [13] If $A, B \in M_{mn}$ then $A \circ B = (A \otimes B)\,(\alpha\,,\beta)$ in which $\alpha = \{\,1\,,\ \mathrm{m} + 2\,, 2\mathrm{m} + 3\,, \ \ldots \ , m^2\,\}$ and $\beta = \{\,1\,, \mathrm{n} + 2\,, 2\mathrm{n} + 3\,, \ \ldots \ , n^2\,\}$. In particular if $m = n$, $A \circ B$ is a principal submatrix of $A \otimes B$. $\blacksquare$

Theorem 1.5.9 If $A, B \in M_{mn}$ then $rank(A \circ B) \leq (rank\ A\,)(rank\ B)$.

Proof : By lemma 1.5.8 the Hadamard product is a submatrix of the Kronecker product, but the rank of the submatrix is not greater than the rank of the matrix, thus

$rank(A \circ B) \leq rank(A \otimes B) = (rank\ A\,)(rank\ B)$. (by theorem 1.4.18)

Therefore $rank(A \circ B) \leq (rank\ A\,)(rank\ B)$. $\blacksquare$

Theorem 1.5.10 [13] Let $A, B \in M_n$, $A \geq 0$, and $B \geq 0$, then $\rho(A \circ B) \leq \rho(A)\,\rho(B)$.

Proof : We have $\rho(A \otimes B) = \rho(A)\, \rho(B)$, by corollary (1.4.16). But $A \otimes B \geq 0$ and

$A \circ B$ is a principal submatrix of $A \otimes B$ by Lemma (1.5.8),

$\rho(A \circ B) \leq \rho(A \otimes B) = \rho(A)\, \rho(B)$. Therefore,

$\rho(A \circ B) \leq \rho(A)\, \rho(B)$. ■

Based on lemma (1.5.8) we will give the proof of the schur's product theorem in a new style as follows **:**

Theorem 1.5.11 (schur's product theorem)

If $A, B \in M_n$ are positive semidefinite, then $A \circ B$ is also positive semidefinite.

Proof : $A\,, B \geq 0$ given, it follows that $A \otimes B \geq 0$ (by corollary 1.4.17) , but

$A \circ B$ is a principal submatrix of $A \otimes B$ (by lemma 1.5.8). So, $A \circ B \geq 0$. ■

The following theorem compares the determinant of the matrices A, B and $A \circ B$ **:**

Theorem 1.5.12 [12] (Oppenheim's inequality)

If $A, B \in M_n$ are positive semidefinite, then

1) $\det(A) \prod_{i=1}^{n} b_{ii} \leq \det(A \circ B)$.

2) $\det(B) \prod_{i=1}^{n} a_{ii} \leq \det(A \circ B)$.

Theorem (1.5.12) implies the Hadamar$d$'s inequality in the usual way as follows **:**

Theorem 1.5.13 $[\mathbf{12}]$ (Hadamar$d$'s inequality)

If $A \in M_n$ is positive semidefinite , then $\det(A) \leq \prod_{i=1}^{n} a_{ii}$.

Proof : Let A be any positive semidefinite matrix of size n**.** Note that I_n is positive semidefinite matrix of size n**.** Now we have the following

$$\det(A) = [\mathrm{I_n}]_{11} \ \dots \ [\mathrm{I}_n]_{nn} \det(A) \ \leq \ \det(\mathrm{I}_n \circ A) \quad \text{(by theorem 1.5.12)}$$

$$= [A]_{11} \ \dots \ [A]_{nn} = \prod_{i=1}^{n} a_{ii}. \ ■$$

Corollary 1.5.15 $[\mathbf{12}]$ Let $A, B \in M_n$ are positive semidefinite . Then

$$\det(A) \det(\mathrm{B}) \leq \det(A \circ B).$$

Proof : $\det(A \circ B) \geq [A]_{11} \ \dots \ [A]_{nn} \det(\mathrm{B})$ (by theorem 1.5.12)

$\geq \det(A) \det(\mathrm{B})$ (by theorem 1.5.14)**.** ■

Chapter two

Inequalities for Kronecker products and Hadamard products of positive definite matrices

In this chapter, we will see some inequalities for Kronecker products and Hadamard products of positive definite matrices**.** The contents of this chapter can be found in [10]**.**

2.1 Introduction

The following property involving Kronecker products of matrices can be derived from The mixed-product property (1.4.3)**.**

Theorem 2.1.1 Let $A \in M_n$ and $B \in M_m$, then $(A \otimes B)^k = A^k \otimes B^k$ for any natural number k**.**

Proof : $(A \otimes B)^k = (A \otimes B)\,(A \otimes B)\ \dots\ (A \otimes B)$ (k- times)

$= (A\,A\ \dots\ A) \otimes (B\,B\ \dots\ B)$ (by theorem 1.4.3)

$= A^k \otimes B^k$. ■

Corollary 2.1.2 For any $A, B \in \mathbb{P}_n$ and $q \in \mathbb{Q}$, we have $(A \otimes B)^q = A^q \otimes B^q$.

Proof : $A, B \in \mathbb{P}_n$, so

$(A \otimes B) = (\mathrm{A}^{1/\mathrm{n}} \otimes B^{1/n})^n$, for any positive integer n, so it follows that

$(A \otimes B)^{1/n} = \mathrm{A}^{1/\mathrm{n}} \otimes B^{1/n}$. Now $(A \otimes B)^{m/n} = \mathrm{A}^{\mathrm{m}/\mathrm{n}} \otimes B^{m/n}$ for any positive integer m, n**.** Therefore $(A \otimes B)^q = A^q \otimes B^q$ for any $q = \frac{m}{n} \in \mathbb{Q}$. ■

The following lemma generalizing theorem (2.1.1) **:**

Lemma 2.1.3 Let $A \in M_n$ and $B \in M_m$, are positive definite matrices. Then for any non-zero real number r

$$(A \otimes B)^r = A^r \otimes B^r.$$

Proof : A, B are positive definite matrices, assures that there exists unitary matrix U and V, such that

$A = UD_AU^*$, where U is a unitary matrix and $D_A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

$B = VD_BV^*$, where V is a unitary matrix and $D_B = \text{diag}(\mu_1, \mu_2, \ldots, \mu_m)$.

Thus, $(A \otimes B)^r = [(UD_AU^*) \otimes (VD_BV^*)]^r$

$$= [(U \otimes V)(D_A \otimes D_B)(U^* \otimes V^*)]^r \quad \text{(by theorem 1.4.3)}$$

$$= (U \otimes V)(D_A \otimes D_B)^r (U^* \otimes V^*) \quad \text{(by (2) in example 1.3.2)}$$

$$= (U \otimes V)({D_A}^r \otimes {D_B}^r)(U^* \otimes V^*)$$

$$= (U\, {D_A}^r\, U^*) \otimes (V\, {D_B}^r\, V^*) \quad \text{(by theorem 1.4.3)}$$

$$= A^r \otimes B^r. \blacksquare$$

Remark 2.1.4 Let $A \in M_n$ and $B \in M_m$ are matrices with polar decomposition (i.e) $A = U_A|A|$ and $B = U_B|B|$. Then $A \otimes B = U_A|A| \otimes U_B|B|$

$$= (U_A \otimes U_B)(\,|A| \otimes |B|) \quad \text{(by theorem 1.4.3)}$$

$$= (U_A \otimes U_B)\left[(A^*A)^{\frac{1}{2}} \otimes (B^*B)^{\frac{1}{2}}\right] \quad \text{(where } |A| = (A^*A)^{\frac{1}{2}},\ |B| = (B^*B)^{\frac{1}{2}} \text{)}$$

$= (U_A \otimes U_B)\,[(A^*A)\otimes(B^*B)]^{\frac{1}{2}}$, $[(A\otimes B)^r = A^r \otimes B^r$ for any positive real number $r]$

$= (U_A \otimes U_B)\,[(A^* \otimes B^*)(A\otimes B)]^{\frac{1}{2}} = (U_A \otimes U_B)\,[(A\otimes B)^*(A\otimes B)]^{\frac{1}{2}} = (U_A \otimes U_B)\,|A\otimes B|.$

Lemma 2.1.5.[3] A map ϕ defined by $\phi(A, B) = (A^{-1} + B^{-1})^{-1}$ for $A, B \in \mathbb{P}_n$ is jointly concave**.**

Theorem 2.1.6. [3] The following identity holds for any $A, B \in \mathbb{P}_n$ and $s > 0$:

$((s^{-1}A\otimes I)^{-1} + (I\otimes B)^{-1})^{-1} = (A\otimes B^{-1})\left((A\otimes B^{-1}) + (sI\otimes I)\right)^{-1}(I\otimes B).$

Proof : $A, B \in \mathbb{P}_n$ and s is positive, take $X = s^{-1}A\otimes I$, $Y = I\otimes B$, $Z = A\otimes B^{-1}$ and $P = X + Y$. It follows from the mixed-product property of the Kronecker product that

$\left(Z + (sI\otimes I)\right)\left((s^{-1}I\otimes I) - (s^{-1}Y)P^{-1}(s^{-1}Z)\right)$

$= Z(s^{-1}I\otimes I) + (sI\otimes I)\,(s^{-1}I\otimes I) - Z(s^{-1}Y)(X+Y)^{-1}(s^{-1}Z)$

$-(sI\otimes I)(s^{-1}Y)(X+Y)^{-1}(s^{-1}Z)$

$= (s^{-1}Z) + (I\otimes I) - X(X+Y)^{-1}(s^{-1}Z) - Y(X+Y)^{-1}(s^{-1}Z)$

$= (s^{-1}Z) + I_{n^2} - (X+Y)(X+Y)^{-1}(s^{-1}Z) = I_{n^2}.$

That is

$(s^{-1}I\otimes I) - (s^{-1}Y)P^{-1}(s^{-1}Z) = \left(Z + (sI\otimes I)\right)^{-1}.$

Again, the mixed-product property yields

$Z^{-1}(X^{-1} + Y^{-1})^{-1}Y^{-1} = Z^{-1}(X^{-1}(X+Y)Y^{-1})^{-1}\,Y^{-1}$

$= Z^{-1}((Y(X+Y)^{-1}X)^{-1})^{-1}Y^{-1}$

$= Z^{-1}(Y(X+Y)^{-1}X)Y^{-1}$

$= Z^{-1}[(X+Y)(X+Y)^{-1}X - X(X+Y)^{-1}X]\,Y^{-1}$

$= Z^{-1}[X - X(X+Y)^{-1}X]\,Y^{-1}$

$= (A^{-1}\otimes B)X(Y^{-1}) - (A^{-1}\otimes B)X(X+Y)^{-1}X\,Y^{-1}$

$= (s^{-1}\mathrm{I}\otimes \mathrm{I}) - (s^{-1}Y)(\mathrm{X}+\mathrm{Y})^{-1}(\mathrm{s}^{-1}\mathrm{Z})$

$= \left(Z + (s\mathrm{I}\otimes\mathrm{I})\right)^{-1}.$

Thus, $(X^{-1}+Y^{-1})^{-1} = Z\left(Z+(s\mathrm{I}\otimes\mathrm{I})\right)^{-1}Y$. Which is

$((s^{-1}\mathrm{A}\otimes\mathrm{I})^{-1} + (\mathrm{I}\otimes\mathrm{B})^{-1})^{-1} = (\mathrm{A}\otimes\mathrm{B}^{-1})\left((\mathrm{A}\otimes\mathrm{B}^{-1}) + (\mathrm{sI}\otimes\mathrm{I})\right)^{-1}(\mathrm{I}\otimes\mathrm{B})$. ■

2.2 Inequalities for Kronecker products

In this section we drive inequalities for the Kronecker product of positive definite matrices in the form $(\alpha A+\beta B)^r\otimes(\alpha C+\beta D)^s$ and $\alpha(A^r\otimes \mathrm{C}^{\mathrm{s}}) + \beta(\mathrm{B}^{\mathrm{r}}\otimes\mathrm{D}^{\mathrm{s}})$ where A, B, C, D are positive definite matrices and α, β, r, s are positive real numbers such that $r+s = 1$.

Theorem 2.2.1.[10] For $A, B, C, D \in \mathbb{P}_n$ and $\alpha, \beta, r, s > 0$ such that $r+s=1$,

$(\alpha A+\beta B)^r\otimes(\alpha C+\beta D)^s \geq \alpha(A^r\otimes \mathrm{C}^{\mathrm{s}}) + \beta(\mathrm{B}^{\mathrm{r}}\otimes\mathrm{D}^{\mathrm{s}}).$

Proof : Let f be a real-valued function defined by $f(t) = t^r$ for $t > 0$ and $0 < r < 1$.

Clearly, f is continuous, and f is representation for

$t^r = \frac{\sin r\pi}{\pi} \int_0^\infty \frac{s^{r-1}t}{s+t}\, ds$. write $Y = \mathrm{I} \otimes \mathrm{B}$ and $Z = \mathrm{A} \otimes \mathrm{B}^{-1}$. Hence, the functional calculus

for $\mathrm{A} \otimes \mathrm{B}^{-1}$ is $f(\mathrm{A} \otimes \mathrm{B}^{-1}) = (\mathrm{A} \otimes \mathrm{B}^{-1})^{\mathrm{r}}$ can be written as

$\frac{\sin r\pi}{\pi} \int_0^\infty (s\mathrm{I} \otimes \mathrm{I})^{r-1} Z\big(Z + (s\mathrm{I} \otimes \mathrm{I})\big)^{-1} ds.$ It follows from lemma 2.1.3 that

$A^r \otimes B^{1-r} = (A^r \mathrm{I}) \otimes (B^{-r} B) = (A^r \otimes B^{-r})(\mathrm{I} \otimes B) = (A \otimes B^{-1})^r (\mathrm{I} \otimes B).$

Hence, by lemma 2.1.6 we obtain

$$A^r \otimes B^{1-r} = \frac{\sin r\pi}{\pi} \int_0^\infty (s\mathrm{I} \otimes \mathrm{I})^{r-1} Z\big(Z + (s\mathrm{I} \otimes \mathrm{I})\big)^{-1} ds \ \mathrm{Y}$$

$$= \frac{\sin r\pi}{\pi} \int_0^\infty \mathrm{s}^{\mathrm{r}-1} \mathrm{Z}\big(Z + (s\mathrm{I} \otimes \mathrm{I})\big)^{-1} ds\, Y$$

$$= \frac{\sin r\pi}{\pi} \int_0^\infty \mathrm{s}^{\mathrm{r}-1} \mathrm{Z}\big(Z + (s\mathrm{I} \otimes \mathrm{I})\big)^{-1} Y\, ds$$

$$= \frac{\sin r\pi}{\pi} \int_0^\infty \mathrm{s}^{\mathrm{r}-1} ((s^{-1} A \otimes \mathrm{I})^{-1} + Y^{-1})^{-1}\, ds. \quad \text{(by lemma 2.1.6)}$$

Since $s^{-1} A \otimes \mathrm{I}$ and $\mathrm{I} \otimes \mathrm{B}$ are positive definite, by lemma 2.1.5 we have that the map

$\phi : \mathbb{P}_{n^2} \times \mathbb{P}_{n^2} \longrightarrow \mathbb{P}_{n^2}$ defined by

$\phi(s^{-1} A \otimes \mathrm{I}, \mathrm{I} \otimes \mathrm{B}) = ((s^{-1} A \otimes \mathrm{I})^{-1} + (\mathrm{I} \otimes \mathrm{B})^{-1})^{-1}$ is jointly concave. It is well-known that the positive linear combination of the jointly concave maps is jointly concave. Hence, from the viewpoint of the Riemann integral, the integrand is also jointly concave and so is $A^r \otimes B^{1-r}$. This means that for any $A, B, C, D \in \mathbb{P}_n$ and scalar $0 < \epsilon < 1$,

$(\epsilon A + (1-\epsilon) B)^r \otimes (\epsilon C + (1-\epsilon) D)^s \geq \epsilon (A^r \otimes C^s) + (1-\epsilon)(B^r \otimes D^s)$. For $s > 0$ and $r + s = 1$. Let $\epsilon = \alpha/(\alpha + \beta)$, thus $0 < \epsilon < 1$.

So, $\left(\frac{\alpha}{\alpha+\beta}A+\left(1-\frac{\alpha}{\alpha+\beta}\right)B\right)^{r}\otimes\left(\frac{\alpha}{\alpha+\beta}C+\left(1-\frac{\alpha}{\alpha+\beta}\right)D\right)^{s}$

$$=\left(\frac{\alpha\, A}{\alpha+\beta}+B-\frac{\alpha\, B}{\alpha+\beta}\right)^{r}\otimes\left(\frac{\alpha\, c}{\alpha+\beta}+D-\frac{\alpha\, D}{\alpha+\beta}\right)^{s}$$

$$=\left(\frac{\alpha\, A+\alpha\, B+\beta\, B-\alpha\, B}{\alpha+\beta}\right)^{r}\otimes\left(\frac{\alpha\, C+\alpha\, D+\beta\, D-\alpha\, D}{\alpha+\beta}\right)^{s}$$

$$=\left(\frac{1}{\alpha+\beta}\right)^{r+s}\ [(\alpha\, A+\beta\, B)^{r}\otimes(\alpha\, C+\beta D)^{s}]$$

$$=\left(\frac{1}{\alpha+\beta}\right)[(\alpha\, A+\beta\, B)^{r}\otimes(\alpha\, C+\beta D)^{s}]\quad (\text{ since } r+s=1)$$

$$\geq\frac{\alpha}{\alpha+\beta}\,(A^{r}\otimes\mathrm{C}^{\mathrm{s}})+\frac{\beta}{\alpha+\beta}\,(\mathrm{B}^{\mathrm{r}}\otimes\mathrm{D}^{\mathrm{s}})$$

$$=\left(\frac{1}{\alpha+\beta}\right)[\alpha\,(A^{r}\otimes\mathrm{C}^{\mathrm{s}})+\beta\,(\mathrm{B}^{\mathrm{r}}\otimes\mathrm{D}^{\mathrm{s}})\,]$$

Therefore $(\alpha A+\beta B)^{r}\otimes(\alpha C+\beta D)^{s}\geq\ \alpha(A^{r}\otimes\mathrm{C}^{\mathrm{s}})+\beta(\mathrm{B}^{\mathrm{r}}\otimes\mathrm{D}^{\mathrm{s}})$. ■

From theorem (2.2.1), we obtain the Hölder inequality for positive definite matrices as a special case.

Recall that the real numbers p,q are conjugate exponents if p,q are positive and $\frac{1}{p}+\frac{1}{q}=1.$

Corollary 2.2.2 For $A,B,C,D\in\mathbb{P}_{n}$ and conjugate exponents p,q, we have

$$(A\otimes B)+(C\otimes D)\leq(A^{p}+C^{p})^{\frac{1}{p}}\otimes(B^{q}+D^{q})^{\frac{1}{q}}.$$

Proof : take $\alpha=\beta=1,\quad r=\frac{1}{p}$ and $s=\frac{1}{q}$ in theorem 2.2.1. Then

$(A+B)^{\frac{1}{p}} \otimes (C+D)^{\frac{1}{q}} \geq (\mathrm{A}^{\frac{1}{\mathrm{p}}} \otimes \mathrm{C}^{\frac{1}{\mathrm{q}}}) + (B^{\frac{1}{p}} \otimes D^{\frac{1}{q}})$

Replacing B with C, Hence

$(A+C)^{\frac{1}{p}} \otimes (B+D)^{\frac{1}{q}} \geq (\mathrm{A}^{\frac{1}{\mathrm{p}}} \otimes \mathrm{B}^{\frac{1}{\mathrm{q}}}) + (C^{\frac{1}{p}} \otimes D^{\frac{1}{q}})$

Finally we replace A, B, C, D with A^p, B^q, C^p, D^q respectively we have

$(A^p + C^p)^{\frac{1}{p}} \otimes (B^q + D^q)^{\frac{1}{q}} \geq ((A^p)^{\frac{1}{p}} \otimes (B^q)^{\frac{1}{q}}) + ((C^p)^{\frac{1}{p}} \otimes (D^q)^{\frac{1}{q}})$

Therefore $(A \otimes B) + (C \otimes D) \leq (A^p + C^p)^{\frac{1}{p}} \otimes (B^q + D^q)^{\frac{1}{q}}$. ■

Remarks 2.2.1 The Cauchy-Schwarz inequality is obtained from corollary (2.2.2) by taking $p = 2$, since $(A \otimes B) + (C \otimes D) \leq (A^2 + C^2)^{\frac{1}{2}} \otimes (B^2 + D^2)^{\frac{1}{2}}$.

Corollary 2.2.3 For $A, B \in \mathbb{P}_n$ and conjugate exponents p, q, we have

$A \oplus B \leq (A^p + \mathrm{I})^{\frac{1}{p}} \otimes (B^q + \mathrm{I})^{\frac{1}{q}}$.

Proof : Let $B = C = \mathrm{I}$. By corollary (2.2.2) we get

$(A \otimes \mathrm{I}) + (\mathrm{I} \otimes D) \leq (A^p + \mathrm{I}^p)^{\frac{1}{p}} \otimes (\mathrm{I}^q + D^q)^{\frac{1}{q}}$. Now let $D = B$ then

$(A \otimes \mathrm{I}) + (\mathrm{I} \otimes B) \leq (A^p + \mathrm{I}^p)^{\frac{1}{p}} \otimes (\mathrm{I}^q + B^q)^{\frac{1}{q}}$

Hence $A \oplus B \leq (A^p + \mathrm{I})^{\frac{1}{p}} \otimes (B^q + \mathrm{I})^{\frac{1}{q}}$ (since $\mathrm{I}^p = \mathrm{I}^q = \mathrm{I}$).■

For $A, B, C, D \in \mathbb{P}_n$ and $\alpha, \beta, r, s > 0$ such that $r + s = 1$. Pattrawut Chansangiam, Patcharin Hemchote, Praiboon Pantaragphong in[10], developed the following results **:**

(1) $(\alpha A+\beta B)^r \otimes (\alpha A+\beta B)^s \ \geq \ \alpha(A^r \otimes \mathrm{A}^s)+\beta(\mathrm{B}^r \otimes \mathrm{B}^s)$.

Proof : Take $C=A,\ D=B$ in theorem 2.2.1, we get the inequality

$$(\alpha A+\beta B)^r \otimes (\alpha A+\beta B)^s \geq \ \alpha(A^r \otimes \mathrm{A}^s)+\beta(\mathrm{B}^r \otimes \mathrm{B}^s).$$

(2) $(\alpha A+\beta B)^r \otimes (\beta A+\alpha B)^s \ \geq \ \alpha(A^r \otimes \mathrm{B}^s)+\beta(\mathrm{B}^r \otimes \mathrm{A}^s)$.

Proof : Let $\alpha C=\beta A,\ \ \beta D=\alpha B$ in theorem 2.2.1 then we get the inequality

$$(\alpha A+\beta B)^r \otimes (\beta A+\alpha B)^s \ \geq \ \alpha(A^r \otimes \mathrm{B}^s)+\beta(\mathrm{B}^r \otimes \mathrm{A}^s).$$

(3) $\left((\alpha A+\beta B) \otimes (\alpha C+\beta D)\right)^{\frac{1}{2}} \geq \alpha(\mathrm{A} \otimes \mathrm{C})^{\frac{1}{2}}+\beta(\mathrm{B} \otimes \mathrm{D})^{\frac{1}{2}}$.

Proof : Let $r=s$ in theorem (2.2.1) we get

$$(\alpha A+\beta B)^{\frac{1}{2}} \otimes (\alpha C+\beta D)^{\frac{1}{2}} \geq \ \alpha\left(A^{\frac{1}{2}} \otimes \mathrm{C}^{\frac{1}{2}}\right)+\beta\left(\mathrm{B}^{\frac{1}{2}} \otimes \mathrm{D}^{\frac{1}{2}}\right).$$

Then by corollary (2.1.2), we get the inequality

$$\left((\alpha A+\beta B) \otimes (\alpha C+\beta D)\right)^{\frac{1}{2}} \geq \alpha(\mathrm{A} \otimes \mathrm{C})^{\frac{1}{2}}+\beta(\mathrm{B} \otimes \mathrm{D})^{\frac{1}{2}}.$$

(4) $(\mathrm{A}+\mathrm{B})^{r} \ \otimes \ (\mathrm{C}+\mathrm{D})^{s} \ \geq (A^r \otimes C^s)+(B^r \otimes D^s)$.

Proof : Take $\beta=\alpha$ in theorem 2.2.1 we get to

$(\alpha A+\alpha B)^r \otimes (\alpha C+\alpha D)^s \geq \ \alpha(A^r \otimes \mathrm{C}^s)+\alpha(\mathrm{B}^r \otimes \mathrm{D}^s)$, then

$\alpha^r\ (\mathrm{A}+\mathrm{B})^{r} \otimes \alpha^{\mathrm{s}}\ (\mathrm{C}+\mathrm{D})^{s} \ \geq \ \alpha\ [(A^r \otimes \mathrm{C}^s)+(\mathrm{B}^r \otimes \mathrm{D}^s)]$, then

$\alpha^{r+s}\ [(\mathrm{A}+\mathrm{B})^{r} \otimes \ (\mathrm{C}+\mathrm{D})^{s}] \ \geq \ \alpha\ [(A^r \otimes \mathrm{C}^s)+(\mathrm{B}^r \otimes \mathrm{D}^s)]$, (by theorem 1.4.1 (a))

Then $\alpha\ [(\mathrm{A}+\mathrm{B})^{r} \otimes \ (\mathrm{C}+\mathrm{D})^{s}] \ \geq \ \alpha\ [(A^r \otimes \mathrm{C}^s)+(\mathrm{B}^r \otimes \mathrm{D}^s)]$, (since $r+s=1$)

Hence $(\mathrm{A}+\mathrm{B})^{r} \ \otimes \ (\mathrm{C}+\mathrm{D})^{s} \ \geq (A^r \otimes C^s)+(B^r \otimes D^s)$.

(5) $\left((A+B)\otimes(C+D)\right)^{\frac{1}{2}} \geq (\mathrm{A}\otimes\mathrm{C})^{\frac{1}{2}} + (\mathrm{B}\otimes\mathrm{D})^{\frac{1}{2}}$.

Proof : Let $r = s$ in result (4) then

$(\mathrm{A}+\mathrm{B})^{\frac{1}{2}} \otimes (\mathrm{C}+\mathrm{D})^{\frac{1}{2}} \geq \left(A^{\frac{1}{2}}\otimes C^{\frac{1}{2}}\right) + \left(B^{\frac{1}{2}}\otimes D^{\frac{1}{2}}\right)$, but $(A\otimes B)^r = A^r\otimes B^r$, hence

$\left((A+B)\otimes(C+D)\right)^{\frac{1}{2}} \geq (\mathrm{A}\otimes\mathrm{C})^{\frac{1}{2}} + (\mathrm{B}\otimes\mathrm{D})^{\frac{1}{2}}$.

(6) $(\mathrm{A}+\mathrm{B})^{\boldsymbol{r}} \otimes (\mathrm{A}+\mathrm{B})^{\boldsymbol{s}} \geq (A^r\otimes A^s) + (B^r\otimes B^s)$.

Proof : Let $\beta = \alpha$ in result (1) we get to

$(\alpha A + \alpha B)^r \otimes (\alpha A + \alpha B)^s \geq \alpha(A^r\otimes \mathrm{A}^{\mathrm{s}}) + \alpha(\mathrm{B}^{\mathrm{r}}\otimes\mathrm{B}^{\mathrm{s}})$, then

$\alpha^{r+s}\,[(A+B)^r\otimes(A+B)^s] \geq \alpha\,[(A^r\otimes \mathrm{A}^{\mathrm{s}}) + (\mathrm{B}^{\mathrm{r}}\otimes\mathrm{B}^{\mathrm{s}})]$, then

$\alpha\,[(A+B)^r\otimes(A+B)^s] \geq \alpha\,[(A^r\otimes \mathrm{A}^{\mathrm{s}}) + (\mathrm{B}^{\mathrm{r}}\otimes\mathrm{B}^{\mathrm{s}})]$, (since $r+s=1$)

Hence $(\mathrm{A}+\mathrm{B})^{\boldsymbol{r}} \otimes (\mathrm{A}+\mathrm{B})^{\boldsymbol{s}} \geq (A^r\otimes A^s) + (B^r\otimes B^s)$.

(7) $(\mathrm{A}+\mathrm{B})^{\boldsymbol{r}} \otimes (\mathrm{A}+\mathrm{B})^{\boldsymbol{s}} \geq (A^r\otimes B^s) + (B^r\otimes A^s)$.

Proof : Let $\beta = \alpha$ in result (2), then we get the inequality

$(\mathrm{A}+\mathrm{B})^{\boldsymbol{r}} \otimes (\mathrm{A}+\mathrm{B})^{\boldsymbol{s}} \geq (A^r\otimes B^s) + (B^r\otimes A^s)$.

(8) $\left((\alpha A+\beta B)\otimes(\beta A+\alpha B)\right)^{\frac{1}{2}} \geq \alpha(\mathrm{A}\otimes\mathrm{B})^{\frac{1}{2}} + \beta(\mathrm{B}\otimes\mathrm{A})^{\frac{1}{2}}$

Proof : Let $r = s$ in result (2) and by $(A\otimes B)^r = A^r\otimes B^r$ then we get the inequality

$\left((\alpha A+\beta B)\otimes(\beta A+\alpha B)\right)^{\frac{1}{2}} \geq \alpha(\mathrm{A}\otimes\mathrm{B})^{\frac{1}{2}} + \beta(\mathrm{B}\otimes\mathrm{A})^{\frac{1}{2}}$.

Definition 2.2.1 Let $A \in M_{m,n}$. The k^{th} Kronecker power $A^{\otimes k}$ is defined inductively for all positive integer k by $A^{\otimes 1} = A$ and $A^{\otimes k} = A \otimes A^{\otimes (k-1)}$ for $k = 2, 3, \dots$ (i. e) $A^{\otimes k} = A \otimes A \otimes \dots \otimes A$ (k-times). This definition implies that $A \in M_{m,n}$, the matrix $A^{\otimes k} \in M_{m^k, n^k}$.

Theorem 2.2.4 For any $A \in \mathbb{P}_n$, positive integer k, and real number r, then $(A^{\otimes k})^r = (A^r)^{\otimes k}$.

Proof : Let $p(k)$ be the statement $(A^{\otimes k})^r = (A^r)^{\otimes k}$. If $k = 2$, then $(A^{\otimes 2})^r = (A \otimes A)^r = A^r \otimes A^r$, which is true. Therefore $p(2)$ is satisfies.

Assume that $p(t)$ is satisfies, $(A^{\otimes t})^r = (A^r)^{\otimes t}$. Now

$$(A^{\otimes (t+1)})^r = (A^{\otimes t} \otimes A)^r = (A^{\otimes t})^r \otimes A^r = (A^r)^{\otimes t} \otimes A^r = (A^r)^{\otimes (t+1)}.$$

Thus, $p(t+1)$ is true, thus $p(k)$ is true for all k. ■

Corollary 2.2.5.[9] Let $\{A_i\}_{i=1}^m$ be a set of arbitrary square matrices with the same size. Then the Kronecker product has the following

$$tr\,(A_{i_1} A_{i_2} \dots A_{i_l})^{\otimes k} = tr^k\left(A_{i_1} A_{i_2} \dots A_{i_l}\right),$$ For any positive integer k.

Proof : $tr\,(A_{i_1} A_{i_2} \dots A_{i_l})^{\otimes k} = tr\left[(A_{i_1} A_{i_2} \dots A_{i_l}) \otimes \dots \otimes (A_{i_1} A_{i_2} \dots A_{i_l})\right]$ (k-times)

$= tr\left(A_{i_1} A_{i_2} \dots A_{i_l}\right) tr(A_{i_1} A_{i_2} \dots A_{i_l}) \dots tr(A_{i_1} A_{i_2} \dots A_{i_l})$ (by corollary 1.4.9)

$= \left[tr\left(A_{i_1} A_{i_2} \dots A_{i_l}\right)\right]^k = tr^k\left(A_{i_1} A_{i_2} \dots A_{i_l}\right)$. ■

Corollaries 2.2.6 If $A, B \in \mathbb{P}_n$, and $\alpha, \beta > 0$, then

(1) $((\alpha A + \beta B)^{\frac{1}{2}})^{\otimes 2} \geq \alpha(A^{\frac{1}{2}})^{\otimes 2} + \beta(B^{\frac{1}{2}})^{\otimes 2}$.

(2) $((A + B)^{\frac{1}{2}})^{\otimes 2} \geq (A^{\frac{1}{2}})^{\otimes 2} + (B^{\frac{1}{2}})^{\otimes 2}$.

(3) $((A + B)^{\frac{1}{2}})^{\otimes 2} \geq (A \otimes B)^{\frac{1}{2}} + (B \otimes A)^{\frac{1}{2}}$.

Proof : (1) Take $r = s$ in result (1) we get to

$(\alpha A + \beta B)^{\frac{1}{2}} \otimes (\alpha A + \beta B)^{\frac{1}{2}} \geq \alpha\left(A^{\frac{1}{2}} \otimes \mathrm{A}^{\frac{1}{2}}\right) + \beta\left(\mathrm{B}^{\frac{1}{2}} \otimes \mathrm{B}^{\frac{1}{2}}\right)$, then

$((\alpha A + \beta B)^{\frac{1}{2}})^{\otimes 2} \geq \alpha(A^{\frac{1}{2}})^{\otimes 2} + \beta(B^{\frac{1}{2}})^{\otimes 2}$ (by definition 2.2.1).

(2) From 1 in corollary 2.2.6 with $\alpha = \beta = 1$.

(3) Take $r = s$ in result (7) we get to

$(A + B)^{\frac{1}{2}} \otimes (A + B)^{\frac{1}{2}} \geq \left(A^{\frac{1}{2}} \otimes \mathrm{B}^{\frac{1}{2}}\right) + \left(\mathrm{B}^{\frac{1}{2}} \otimes \mathrm{A}^{\frac{1}{2}}\right)$, then

$((A + B)^{\frac{1}{2}})^{\otimes 2} \geq (A \otimes B)^{\frac{1}{2}} + (B \otimes A)^{\frac{1}{2}}$ (by definition 2.2.1 and lemma 2.1.3).■

The next result is the AM-GM inequality for the Kronecker product of matrices **:**

Corollary 2.2.7 If $A, B \in \mathbb{P}_n$ commute under the Kronecker product**,** then

$(A \otimes B)^{\frac{1}{2}} \leq \frac{1}{2}((A + B)^{\frac{1}{2}})^{\otimes 2}$, with equality iff $A = B$.

Proof : $((A + B)^{\frac{1}{2}})^{\otimes 2} \geq (A \otimes B)^{\frac{1}{2}} + (B \otimes A)^{\frac{1}{2}}$ (by corollary 2.2.6 (3))

$= (A \otimes B)^{\frac{1}{2}} + (A \otimes B)^{\frac{1}{2}}$ ($A \otimes B = B \otimes A$, given

$= 2\ (A \otimes B)^{\frac{1}{2}}$

Hence, $\frac{1}{2}\ ((A+B)^{\frac{1}{2}})^{\otimes 2}\ \geq\ (A \otimes B)^{\frac{1}{2}}$.

For the equality,

$\Longleftarrow$) When $A = B$, we have $\frac{1}{2}\ ((A+B)^{\frac{1}{2}})^{\otimes 2} = \frac{1}{2}\ ((A+A)^{\frac{1}{2}})^{\otimes 2} = \frac{1}{2}\ ((2A)^{\frac{1}{2}})^{\otimes 2}$

$= \frac{1}{2}\left((2A)^{\frac{1}{2}} \otimes (2A)^{\frac{1}{2}}\right) = \frac{\sqrt{2}\times\sqrt{2}}{2}\ \left(A^{\frac{1}{2}} \otimes \mathrm{A}^{\frac{1}{2}}\right) = ((A)^{\frac{1}{2}})^{\otimes 2}$ ... $(*)$

$(A \otimes B)^{\frac{1}{2}} = (A \otimes A)^{\frac{1}{2}} = (\mathrm{A}^{\otimes 2})^{\frac{1}{2}} = (A^{\frac{1}{2}})^{\otimes 2}$ ... $(**)$.

From $(*)$ and $(**)$, we have $\frac{1}{2}\ ((A+B)^{\frac{1}{2}})^{\otimes 2} = (A \otimes B)^{\frac{1}{2}}$.

$\Rightarrow$) Assume that $\frac{1}{2}\ ((A+B)^{\frac{1}{2}})^{\otimes 2} = (A \otimes B)^{\frac{1}{2}}$, then

$2(A \otimes B)^{\frac{1}{2}} = ((A+B)^{\frac{1}{2}})^{\otimes 2} = ((A+B)^{\otimes 2})^{\frac{1}{2}}$ (by theorem 2.2.4)

$= [(A+B) \otimes (A+B)]^{\frac{1}{2}}$, then

$4\ (A \otimes B) = (A+B) \otimes (A+B)$, then

$(A \otimes B) + (A \otimes B) + (A \otimes B) + (A \otimes B) = (A \otimes A) + (A \otimes B) + (A \otimes B) + (B \otimes B)$, then

$(A \otimes B) - (A \otimes A) = (B \otimes B) - (A \otimes B)$, then

$A \otimes (B - A) = B \otimes (B - A)$, then

$A \otimes (B - A) - B \otimes (B - A) = 0$, then

$(A - B) \otimes (B - A) = 0$, then

$(A - B) = 0$ or $(B - A) = 0$ (by corollary 1.4.2), so $A = B$. ■

2.3 Inequalities for Hadamard products

In this section we drive inequalities for Hadamard products of positive definite matrices

Theorem 2.3.1.[10] For $A, B, C, D \in \mathbb{P}_n$ and $\alpha, \beta, r, s > 0$ such that $r + s = 1$,

$(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \geq \alpha(A^r \circ \mathrm{C^s}) + \beta(\mathrm{B^r} \circ \mathrm{D^s})$.

Proof : Define $\phi : \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_{n^2}$ by $\phi(\mathrm{A}, \mathrm{B}) = \mathrm{A^r} \otimes \mathrm{B^s}$. The Hadamard product of matrices is a principal submatrix of the Kronecker product of matrices**.** Consequently, there exists a unital positive linear map $\varphi : \mathbb{P}_{n^2} \to \mathbb{P}_n$ such that $\varphi(A \otimes B) = A \circ B$. Hence, $(\varphi \circ \phi)(\mathrm{A}, \mathrm{B}) = \varphi\big(\phi(\mathrm{A}, \mathrm{B})\big) = \varphi(\mathrm{A^r} \otimes \mathrm{B^s}) = \mathrm{A^r} \circ \mathrm{B^s}$.Since ϕ is jointly concave by theorem 2.2.1 and φ is positive and linear, the composition $\varphi \circ \phi$ is also jointly concave. This means that for any $A, B, C, D \in \mathbb{P}_n$ and any scalar $0 < \epsilon < 1$,

$(\epsilon A + (1 - \epsilon)B)^r \circ (\epsilon C + (1 - \epsilon)D)^s \geq \epsilon(A^r \circ C^s) + (1 - \epsilon)(B^r \circ D^s)$. Since

$0 < \alpha/(\alpha + \beta) < 1$, by replacing $\epsilon \; with \; \alpha/(\alpha + \beta)$, we get

$$\left(\frac{\alpha}{\alpha+\beta} A + \left(1 - \frac{\alpha}{\alpha+\beta}\right) B\right)^r \circ \left(\frac{\alpha}{\alpha+\beta} C + \left(1 - \frac{\alpha}{\alpha+\beta}\right) D\right)^s$$

$$= \left(\frac{\alpha\, A}{\alpha+\beta} + B - \frac{\alpha\, B}{\alpha+\beta}\right)^r \circ \left(\frac{\alpha\, c}{\alpha+\beta} + D - \frac{\alpha\, D}{\alpha+\beta}\right)^s$$

$$= \left(\frac{\alpha\, A + \alpha\, B + \beta\, B - \alpha\, B}{\alpha+\beta}\right)^r \circ \left(\frac{\alpha\, C + \alpha\, D + \beta\, D - \alpha\, D}{\alpha+\beta}\right)^s$$

$$= \left(\frac{1}{\alpha+\beta}\right)^{r+s} \left[(\alpha\, A + \beta\, B)^r \circ (\alpha\, C + \beta D)^s\right]$$

$$= \left(\frac{1}{\alpha+\beta}\right) \left[(\alpha\, A + \beta\, B)^r \circ (\alpha\, C + \beta D)^s\right] \quad (\text{ since } r + s = 1)$$

$\geq \frac{\alpha}{\alpha+\beta}\,(A^r \circ \mathrm{C^s}) + \frac{\beta}{\alpha+\beta}\,(\mathrm{B^r} \circ \mathrm{D^s}) \;\; = \left(\frac{1}{\alpha+\beta}\right)\,[\alpha\,(A^r \circ \mathrm{C^s}) + \beta\,(\mathrm{B^r} \circ \mathrm{D^s})\,]$

Therefore $(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \geq \alpha(A^r \circ \mathrm{C^s}) + \beta(\mathrm{B^r} \circ \mathrm{D^s})$. ■

From this theorem(2.3.1), we obtain the Hölder inequality for positive definite matrices as a special case.

Corollary 2.3.2 For $A, B, C, D \in \mathbb{P}_n$ and conjugate exponents p, q, we have

$$(A \circ B) + (C \circ D) \leq (A^p + C^p)^{\frac{1}{p}} \circ (B^q + D^q)^{\frac{1}{q}}.$$

Proof : Take $\alpha = \beta = 1$, $r = \frac{1}{p}$ and $s = \frac{1}{q}$ in theorem 2.3.1. Then

$$(A + B)^{\frac{1}{p}} \circ (C + D)^{\frac{1}{q}} \geq \left(\mathrm{A}^{\frac{1}{\mathrm{p}}} \circ \mathrm{C}^{\frac{1}{\mathrm{q}}}\right) + \left(B^{\frac{1}{p}} \circ D^{\frac{1}{q}}\right).$$

Replacing B with C, hence

$$(A + C)^{\frac{1}{p}} \circ (B + D)^{\frac{1}{q}} \geq \left(\mathrm{A}^{\frac{1}{\mathrm{p}}} \circ \mathrm{B}^{\frac{1}{\mathrm{q}}}\right) + \left(C^{\frac{1}{p}} \circ D^{\frac{1}{q}}\right).$$

Finally we replace A, B, C, D with A^p, B^q, C^p, D^q respectively we have

$$(A^p + C^p)^{\frac{1}{p}} \circ (B^q + D^q)^{\frac{1}{q}} \geq ((A^p)^{\frac{1}{p}} \circ (B^q)^{\frac{1}{q}}) + ((C^p)^{\frac{1}{p}} \circ (D^q)^{\frac{1}{q}}).$$

Therefore, $(A \circ B) + (C \circ D) \leq (A^p + C^p)^{\frac{1}{p}} \circ (B^q + D^q)^{\frac{1}{q}}$. ■

Remarks 2.3.1 The Cauchy-Schwarz inequality is obtained from corollary (2.3.2) by taking $p = 2$, since $(A \circ B) + (C \circ D) \leq (A^2 + C^2)^{\frac{1}{2}} \circ (B^2 + D^2)^{\frac{1}{2}}$.

Definition 2.3.1 The Hadamard sum of $A, B \in M_n$ is denoted by $A \bullet B$ where

$A \bullet B = A \circ \mathrm{I} + \mathrm{I} \circ \mathrm{B}.$

By corollary (2.3.2), and taking $B = C = I$ we obtain the following :

Corollary 2.3.3 For $A, B, C, D \in \mathbb{P}_n$ and conjugate exponents p, q, we have

$$A \bullet B \leq (A^p + I)^{\frac{1}{p}} \circ (B^q + I)^{\frac{1}{q}}.$$

Proof : $A \bullet B = A \circ \mathrm{I} + \mathrm{I} \circ \mathrm{B} \leq (A^p + \mathrm{I}^p)^{\frac{1}{p}} \circ (B^q + \mathrm{I}^q)^{\frac{1}{q}}$ (by corollary 2.3.2)

$$= (A^p + I)^{\frac{1}{p}} \circ (B^q + I)^{\frac{1}{q}}.$$

For $A, B, C, D \in \mathbb{P}_n$ and $\alpha, \beta, r, s > 0$ such that $r + s = 1$. Pattrawut Chansangiam, Patcharin Hemchote, Praiboon Pantaragphong in[10], developed the following results :

(1) $(\alpha A + \beta B)^r \circ (\alpha A + \beta B)^s \geq \alpha(A^r \circ \mathrm{A^s}) + \beta(\mathrm{B^r} \circ \mathrm{B^s})$.

Proof : Take $A = C,\ B = D$ in theorem 2.3.1, we get

$$(\alpha A + \beta B)^r \circ (\alpha A + \beta B)^s \geq \alpha(A^r \circ \mathrm{A^s}) + \beta(\mathrm{B^r} \circ \mathrm{B^s}).$$

(2) $(\alpha A + \beta B)^r \circ (\beta A + \alpha B)^s \geq \alpha(A^r \circ \mathrm{B^s}) + \beta(\mathrm{A^s} \circ \mathrm{B^r})$.

Proof : Let $\alpha C = \beta A,\ \beta D = \alpha B$ in theorem (2.3.1) then we get the inequality

$$(\alpha A + \beta B)^r \circ (\beta A + \alpha B)^s \geq \alpha(A^r \circ \mathrm{B^s}) + \beta(\mathrm{A^s} \circ \mathrm{B^r}).$$

(3) $(\alpha A + \beta B)^{\frac{1}{2}} \circ (\alpha C + \beta D)^{\frac{1}{2}} \geq \alpha\left(A^{\frac{1}{2}} \circ \mathrm{C}^{\frac{1}{2}}\right) + \beta\left(\mathrm{B}^{\frac{1}{2}} \circ \mathrm{D}^{\frac{1}{2}}\right)$.

Proof : Let $r = s$ in theorem (2.3.1) then we get the inequality

$$(\alpha A + \beta B)^{\frac{1}{2}} \circ (\alpha C + \beta D)^{\frac{1}{2}} \geq \alpha\left(A^{\frac{1}{2}} \circ \mathrm{C}^{\frac{1}{2}}\right) + \beta\left(\mathrm{B}^{\frac{1}{2}} \circ \mathrm{D}^{\frac{1}{2}}\right).$$

(4) $(A+B)^r \circ (C+D)^s \geq (A^r \circ C^s) + (B^r \circ D^s)$.

Proof : Take $\beta = \alpha$ in theorem 2.3.1 we get to

$(\alpha A + \alpha B)^r \circ (\alpha C + \alpha D)^s \geq \alpha(A^r \circ C^s) + \alpha(B^r \circ D^s)$, then

$\alpha^r (A+B)^r \circ \alpha^s (C+D)^s \geq \alpha\,[(A^r \circ C^s) + (B^r \circ D^s)]$, then

$\alpha^{r+s}\,[(A+B)^r \circ (C+D)^s] \geq \alpha\,[(A^r \circ C^s) + (B^r \circ D^s)]$, (by theorem 1.5.3 (a))

Then $\alpha\,[(A+B)^r \circ (C+D)^s] \geq \alpha\,[(A^r \circ C^s) + (B^r \circ D^s)]$, (since $r + s = 1$)

Hence $(A+B)^r \circ (C+D)^s \geq (A^r \circ C^s) + (B^r \circ D^s)$.

(5) $(A+B)^r \circ (A+B)^s \geq (A^r \circ B^s) + (A^s \circ B^r)$.

Proof : Let $\beta = \alpha = 1$ in result (2), we have

$(A+B)^r \circ (A+B)^s \geq (A^r \circ B^s) + (A^s \circ B^r)$.

(6) $(\alpha A + \beta B)^{\frac{1}{2}} \circ (\beta A + \alpha B)^{\frac{1}{2}} \geq (\alpha + \beta)\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)$.

Proof : Let $r = s$ in result (2) then we get the inequality

$(\alpha A + \beta B)^{\frac{1}{2}} \circ (\beta A + \alpha B)^{\frac{1}{2}} \geq \alpha\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right) + \beta\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)$.

Thus, $(\alpha A + \beta B)^{\frac{1}{2}} \circ (\beta A + \alpha B)^{\frac{1}{2}} \geq (\alpha + \beta)\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)$.

Definition 2.3.2 Let $A \in M_n$ then the kth Hadamard power of A is $A^{(k)} = \left[a_{ij}^k\right] =$

$A \circ A^{(k-1)}, k = 2, 3, \dots$.

Hence, if $A \in M_n$, then $(\alpha A)^{(k)} = \left[(\alpha a_{ij})^k\right] = \left[\alpha^k a_{ij}^k\right] = \alpha^k\left[a_{ij}^k\right] = \alpha^k A^{(k)}$.

Corollaries 2.3.4 If $A, B \in \mathbb{P}_n$, and $\alpha, \beta > 0$, then

(1) $\left((\alpha A + \beta B)^{\frac{1}{2}}\right)^{(2)} \geq \alpha(A^{\frac{1}{2}})^{(2)} + \beta(B^{\frac{1}{2}})^{(2)}$.

(2) $\left((A + B)^{\frac{1}{2}}\right)^{(2)} \geq (A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)}$.

(3) $\left((A + B)^{\frac{1}{2}}\right)^{(2)} \geq 2\,(A^{\frac{1}{2}} \circ B^{\frac{1}{2}})$.

Proof : (1) Take $r = s$ in result (1) then we get to

$(\alpha A + \beta B)^{\frac{1}{2}} \circ (\alpha A + \beta B)^{\frac{1}{2}} \geq \alpha\left(A^{\frac{1}{2}} \circ \mathrm{A}^{\frac{1}{2}}\right) + \beta\left(\mathrm{B}^{\frac{1}{2}} \circ \mathrm{B}^{\frac{1}{2}}\right)$. Then we have

$\left((\alpha A + \beta B)^{\frac{1}{2}}\right)^{(2)} \geq \alpha\left(A^{\frac{1}{2}}\right)^{(2)} + \beta\left(B^{\frac{1}{2}}\right)^{(2)}$ (by definition 2.3.2).

(2) Let $\beta = \alpha = 1$ in (1), we get

$$\left((A + B)^{\frac{1}{2}}\right)^{(2)} \geq \left(A^{\frac{1}{2}}\right)^{(2)} + \left(B^{\frac{1}{2}}\right)^{(2)}.$$

(3) Let $\beta = \alpha = 1$ in result (6), we get

$$\left((A + B)^{\frac{1}{2}}\right)^{(2)} \geq 2\,(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}). \blacksquare$$

The next result is the AM-GM inequality for matrices involving the Hadamard product :

Corollary 2.2.5 For $A, B \in \mathbb{P}_n$, we have the following inequality

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \leq \frac{1}{2}\left((A + B)^{\frac{1}{2}}\right)^{(2)}.$$

Proof : From (3) in Corollary (2.3.4), dividing both sides on 2, we get the inequality

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \leq \frac{1}{2}\left((A + B)^{\frac{1}{2}}\right)^{(2)}. \blacksquare$$

Chapter three

Bounds on the Spectral Radius of Hadamard Products of Positive Operators on l_p- Spaces

3.1 Hadamard product of matrices of operators on l_p

Definition 3.1.1 The space l_p is the space of all sequences $x = (\xi_i) = (\xi_1, \xi_2, \dots)$ of numbers such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges, thus

$$\sum_{i=1}^{\infty} |\xi_i|^p < \infty.$$

Definition 3.1.2 A linear operator $A : X \longrightarrow Y$ from a normed space X into a normed space Y is called bounded if there exists a positive numbers C such that

$\|Ax\| \leq C\|x\|$, for all $x \in X$.

We write $x \geq 0$ for $x = (\xi_n) \in l_p$, whenever $\xi_n \geq 0$ for all $n \geq 1$, and we denoted by l_p^+ the set of all $x \geq 0$ in l_p. Abounded linear operator $A : l_p \longrightarrow l_p$ is called positive (denoted by $A \geq 0$) if $Ax \geq 0$ for all $x \in l_p^+$. As we assume $p < \infty$, every bounded operator on l_p has a matrix representation with respect to the standard basis, and we will identify the operator with its matrix.

In case $A \geq 0$, we have $A = [a_{ij}]$, where each $a_{ij} \geq 0$. We will use frequently that if $0 \leq A \leq B$ on l_p^+ ($i.e.,\ B - A \geq 0$), then $\| A \| \leq \| B \|$.

Theorem 3.1.1.[11] Let $\|\cdot\|$ be a matrix norm on M_n. Then

$\rho(A) = \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}}$, for all $A \in M_n$.

In the following theorems we will see upper bounds for some Hadamards products of positive operator on l_p **:**

Theorem 3.1.2 [4] Let A, B, C and D be positive operators on l_p. Then we have

$(A \circ B)(C \circ D) \leq \big((A \circ A)(C \circ C)\big)^{\frac{1}{2}} \circ \big((B \circ B)(D \circ D)\big)^{\frac{1}{2}}.$

Proof : Let $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$, and $[d_{ij}]$ denote the matrices of the operators $A, B, C,$ and D respectively**.** Then the matrix of the operator product $(A \circ B)(C \circ D)$ is given by $\sum_{l=1}^{\infty} a_{il} b_{il} c_{lj} d_{lj}$**.** From Cauchy-Schwarz inequality we get

$$\sum_{l=1}^{\infty} a_{il} b_{il} c_{lj} d_{lj} \leq \left(\sum_{l=1}^{\infty} a_{il}^2 c_{lj}^2\right)^{\frac{1}{2}} \left(\sum_{l=1}^{\infty} b_{il}^2 d_{lj}^2\right)^{\frac{1}{2}},$$

$$= \big((A \circ A)(C \circ C)\big)^{\frac{1}{2}} \circ \big((B \circ B)(D \circ D)\big)^{\frac{1}{2}}. \blacksquare$$

Corollary 3.1.3 Let A and B be positive linear operator on l_p. Then we have

$$(A \circ B)^2 \leq \big((A \circ A)(B \circ B)\big)^{\frac{1}{2}} \circ \big((B \circ B)(A \circ A)\big)^{\frac{1}{2}}.$$

Proof : Take $D = A$ and $C = B$ in theorem (3.1.2) so,

$(A \circ B)(B \circ A) \leq \big((A \circ A)(B \circ B)\big)^{\frac{1}{2}} \circ \big((B \circ B)(A \circ A)\big)^{\frac{1}{2}}.$

Thus, $(A \circ B)^2 \leq \big((A \circ A)(B \circ B)\big)^{\frac{1}{2}} \circ \big((B \circ B)(A \circ A)\big)^{\frac{1}{2}}$ (since $A \circ B = B \circ A$)**.**$\blacksquare$

Corollary 3.1.4 Let A and B be positive linear operators on l_p. Then we have

$$\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^2 \leq (A^2)^{\frac{1}{2}} \circ (B^2)^{\frac{1}{2}}.$$

Proof : We substitute $A^{\frac{1}{2}}$ for both A and C, and $B^{\frac{1}{2}}$ for B and D in theorem (3.1.2) then,

$$\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right) \leq \left(\left(A^{\frac{1}{2}} \circ A^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} \circ A^{\frac{1}{2}}\right)\right)^{\frac{1}{2}} \circ \left(\left(B^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)\left(B^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}.$$

So, $\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^2 \leq \left(\left(A^{\frac{1}{2}} \circ A^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \circ \left(\left(B^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} = \left(A^{\frac{1}{2}} \circ A^{\frac{1}{2}}\right) \circ \left(B^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right).$

Thus, $\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^2 \leq \left(A^{\frac{1}{2}}\right)^{(2)} \circ \left(B^{\frac{1}{2}}\right)^{(2)}$ (by definition (2.3.2)).

Therefore, $\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^2 \leq (A^2)^{\frac{1}{2}} \circ (B^2)^{\frac{1}{2}}$. ■

Theorem 3.1.5 [4] Let A and B be positive linear operators on l_p. Then we have

$$(A \circ A)(B \circ B) \leq AB \circ AB.$$

Proof : $X_i \geq 0$, $\forall i = 1, \ldots$, n.

Let $p(n)$ be the statement, $\sum_{i=1}^{n} X_i^2 \leq \left(\sum_{i=1}^{n} X_i\right)^2$, $n = 1, 2, \ldots$.

For $n = 2$, $\sum_{i=1}^{2} X_i^2 = X_1^2 + X_2^2$.

$$\left(\sum_{i=1}^{2} X_i\right)^2 = (X_1 + X_2)^2 = X_1^2 + X_2^2 + X_1X_2 + X_2X_1.$$

But, $X_1X_2 + X_2X_1 \geq 0$. Therefore, $\sum_{i=1}^{2} X_i^2 \leq \left(\sum_{i=1}^{2} X_i\right)^2$.

Assume that $p(k)$ is true, so $\sum_{i=1}^{k} X_i^2 \le \left(\sum_{i=1}^{k} X_i\right)^2$.

$$\sum_{i=1}^{k+1} X_i^2 = (X_1^2 + X_2^2 + \cdots + X_k^2) + X_{k+1}^2 .$$

$$\left(\sum_{i=1}^{k+1} X_i\right)^2 = \left((X_1 + X_2 + \cdots + X_k) + X_{k+1}\right)^2$$

$$= (X_1 + X_2 + \cdots + X_k)^2 + (X_{k+1})^2 + (X_1 + X_2 + \cdots + X_k)\,(X_{k+1})$$

$$+ (X_{k+1})\,(X_1 + X_2 + \cdots + X_k).$$

But , $(X_1 + X_2 + \cdots + X_k)\,(X_{k+1}) + (X_{k+1})\,(X_1 + X_2 + \cdots + X_k) \ge 0$. Therefore,

$p(k+1)$ is true**.**

Hence, $p(n)$ is true $\forall n$.

Take $X_i = a_{ik}\, b_{kj}$.

Let $[a_{ij}]$ and $[b_{ij}]$ denote the matrices of A and B, respectively.

Then the (i,j) entry of $(A \circ A)\,(B \circ B)$ is $\sum_{k=1}^{\infty} a_{ik}^2\, b_{kj}^2$, and

$$(A \circ A)\,(B \circ B) = \sum_{k=1}^{\infty} a_{ik}^2\, b_{kj}^2 \le \left(\sum_{k=1}^{\infty} a_{ik} b_{kj}\right)^2$$

$$= \left(\sum_{k=1}^{\infty} a_{ik} b_{kj}\right)\left(\sum_{k=1}^{\infty} a_{ik} b_{kj}\right)$$

$$= AB \circ AB. \blacksquare$$

The following lemma shows that the Hadamard product of two positive linear operators on l_p is bounded :

Lemma 3.1.6 [4] Let A and B be a positive linear operators on l_p. Then $A \circ B$ is a positive linear operator on l_p and $\| A \circ B \| \leq \| A \| \| B \|$.

Proof : It is sufficient to prove that $\| X \circ Y \| \leq 1$, whenever $\| X \| = \| Y \| = 1$.

Assume $Y = [b_{ij}]$. From $\| Y \| = 1$ it follows that $b_{ij} \leq 1$ for all $i\, , j,$ so that $0 \leq X \circ Y \leq X$. This implies immediately that $X \circ Y$ is a positive operator from l_p to l_p and $\| X \circ Y \| \leq 1$. Thus we take

$X = \frac{A}{\|A\|}, \qquad Y = \frac{B}{\|B\|}$. Then $\| X \| = 1$ and $\| Y \| = 1$,

thus $\| X \circ Y \| \leq 1$. Therefore

$\| \frac{A}{\|A\|} \circ \frac{B}{\|B\|} \| \leq 1,$ then $\frac{1}{\|A\| \, \|B\|} \| A \circ B \| \leq 1.$

Therefore, $\| A \circ B \| \leq \| A \| \| B \|$. ■

Lemma 3.1.7 [4] Let A and B be positive linear operators on l_p. Then $A^{\frac{1}{2}} \circ B^{\frac{1}{2}}$ is a positive operator on l_p and $\left\| A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \right\| \leq \| A \|^{\frac{1}{2}} \| B \|^{\frac{1}{2}}$.

Proof : By the identity $(ab)^{\frac{1}{2}} = \min\left\{\frac{t^2}{2} a + \frac{1}{2t^2} b;\ t > 0\right\}$, where a, b are positive numbers, which refers to Krivine calculus in Banach lattices, we get

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \leq \frac{t^2}{2} A + \frac{1}{2t^2} B, \text{ for all } t > 0.$$

This implies that $A^{\frac{1}{2}} \circ B^{\frac{1}{2}}$ is a positive operator on l_p, and

$$\left\|A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right\| \leq \frac{t^2}{2}\|A\| + \frac{1}{2t^2}\|B\|, \text{ for all } t > 0.$$

By taking the minimum over t, we get

$$min\left\|A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right\| \leq min\left\{\frac{t^2}{2}\|A\| + \frac{1}{2t^2}\|B\|, \text{ for all } t > 0\right\}. \text{ Then,}$$

$$\left\|A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right\| \leq (\|A\|\|B\|)^{\frac{1}{2}}$$

$$= \|A\|^{\frac{1}{2}}\|B\|^{\frac{1}{2}}. \blacksquare$$

3.2 Inequalities for spectral radius of Hadamard products

In this section, we will see some inequalities for spectral radius of Hadamard products of positive operators on l_p.

Lemma 3.2.1 [4] Let A and B be positive linear operator on l_p. Then we have

$$\rho\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right) \leq \rho(A)^{\frac{1}{2}}\,\rho(B)^{\frac{1}{2}}.$$

Proof : From corollary (3.1.4), it follows that

$$\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^{2n} \leq (A^{2n})^{\frac{1}{2}} \circ (B^{2n})^{\frac{1}{2}}.$$

Taking norms on both sides we get,

$$\left\|\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^{2n}\right\| \leq \left\|(A^{2n})^{\frac{1}{2}} \circ (B^{2n})^{\frac{1}{2}}\right\|, \text{ then}$$

$$\left\|\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^{2n}\right\| \leq \|(A^{2n})\|^{\frac{1}{2}}\,\|(B^{2n})\|^{\frac{1}{2}} \quad \text{(by lemma (3.1.7)).}$$

Taking $(2n)th$ roots on both sides we get,

$$\left\|\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^{2n}\right\|^{\frac{1}{2n}} \leq \|(A^{2n})\|^{\left(\frac{1}{2}\right)\left(\frac{1}{2n}\right)}\,\|(B^{2n})\|^{\left(\frac{1}{2}\right)\left(\frac{1}{2n}\right)}.$$

And taking limit for $(n \longrightarrow \infty)$ on both sides we have,

$$\lim_{n\to\infty}\left\|\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right)^{2n}\right\|^{\frac{1}{2n}} \leq \lim_{n\to\infty}\|(A^{2n})\|^{\left(\frac{1}{2}\right)\left(\frac{1}{2n}\right)}\ \lim_{n\to\infty}\|(B^{2n})\|^{\left(\frac{1}{2}\right)\left(\frac{1}{2n}\right)}.$$

So, $\rho\left(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\right) \leq \rho(A)^{\frac{1}{2}}\,\rho(B)^{\frac{1}{2}}$ (by theorem (3.1.1)).■

Lemma 3.2.2 [4] Let A and B be positive linear operators on l_p. Then we have

$$\rho(A \circ B) \leq \rho(A)\, \rho(B).$$

Proof : Take $B = A$ in theorem (3.1.5) we get,

$$(A \circ A)^2 \leq A^2 \circ A^2.$$

Then $(A \circ A)^{2n} \leq A^{2n} \circ A^{2n}$, taking norms in both sides we get,

$$\|(A \circ A)^{2n}\| \leq \|A^{2n} \circ A^{2n}\| \leq \|A^{2n}\|\, \|A^{2n}\| \quad \text{(by lemma(3.1.6)).}$$

Taking $(2n)th$ root and limit as $n \longrightarrow \infty$, on both sides we have

$$\rho(A \circ A) \leq \rho(A)^2 \quad \ldots (a)$$

Similarly, $\rho(B \circ B) \leq \rho(B)^2 \quad \ldots\ (b)$.

In theorem (3.1.2), take $C = A,\ D = B$ we get

$$(A \circ B)^2 \leq \left((A \circ A)(A \circ A)\right)^{\frac{1}{2}} \circ \left((B \circ B)(B \circ B)\right)^{\frac{1}{2}}.$$

Thus, $(A \circ B)^2 \leq ((A \circ A)^2)^{\frac{1}{2}} \circ ((B \circ B)^2)^{\frac{1}{2}}$.

So, $(A \circ B)^2 \leq (A \circ A) \circ (B \circ B)$.

Then, $(A \circ B)^{2n} \leq (A \circ A)^n \circ (B \circ B)^n$.

Taking norms in both sides we get,

$$\|(A \circ B)^{2n}\| \leq \|(A \circ A)^n \circ (B \circ B)^n\|$$

$$\leq \|(A \circ A)^n\|\, \|(B \circ B)^n\|.$$

Taking $(2n)th$ root and limit as $n \longrightarrow \infty$ on both sides we have

$\rho(A \circ B) \leq \rho(A \circ A)^{\frac{1}{2}} \rho(B \circ B)^{\frac{1}{2}}.$

From (a) and (b) we get, $\rho(A \circ B) \leq \rho(A \circ A)^{\frac{1}{2}} \rho(B \circ B)^{\frac{1}{2}} \leq \rho(A)\,\rho(B).$

Therefore, $\rho(A \circ B) \leq \rho(A)\,\rho(B)$. ■

Theorem 3.2.3 [4] Let A and B be positive linear operator on l_p. Then,

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}\big((A \circ A)\,(B \circ B)\big) \leq \rho^{\frac{1}{2}}(AB \circ AB) \leq \rho(AB).$$

Proof : From corollary (3.1.3), it follows that

$$(A \circ B)^{2n} \leq \left(\big((A \circ A)(B \circ B)\big)^n\right)^{\frac{1}{2}} \circ \left(\big((B \circ B)(A \circ A)\big)^n\right)^{\frac{1}{2}}.$$

Taking norms in both sides we get,

$$\|(A \circ B)^{2n}\| \leq \left\| \left(\big((A \circ A)(B \circ B)\big)^n\right)^{\frac{1}{2}} \circ \left(\big((B \circ B)(A \circ A)\big)^n\right)^{\frac{1}{2}} \right\|$$

$$\leq \left\|\big((A \circ A)(B \circ B)\big)^n\right\|^{\frac{1}{2}} \left\|\big((B \circ B)(A \circ A)\big)^n\right\|^{\frac{1}{2}} \quad \text{(by lemma (3.1.7)).}$$

Taking $(2n)th$ root and limit as $n \longrightarrow \infty$ on both sides we have

$$\rho(A \circ B) \leq \rho^{\frac{1}{4}}\big((A \circ A)\,(B \circ B)\big)\,\rho^{\frac{1}{4}}\big((B \circ B)\,(A \circ A)\big)$$

$$= \rho^{\frac{1}{2}}\big((A \circ A)\,(B \circ B)\big) \qquad \text{(since } \rho(AB) = \rho(BA) \text{).}$$

From $(A \circ A)\,(B \circ B) \leq AB \circ AB$, we get

$$\rho\big((A \circ A)\,(B \circ B)\big) \leq \rho(AB \circ AB) \leq \rho(AB)\,\rho(AB) \qquad \text{(by lemma (3.2.2)).}$$

$$= \rho^2(AB).$$

Therefore, $\rho^{\frac{1}{2}}\big((A \circ A)\,(B \circ B)\big) \leq \rho^{\frac{1}{2}}(AB \circ AB) \leq \rho(AB)$. ■

Chapter four

Applications on Kronecker product.

In this section we present application of the Kronecker product to matrix equations, matrix differential equations :

4.1 Matrix equations

Knowledge of the Kronecker product and its application facilitates our analysis of matrix equations, since the Kronecker product can be used to give a convenient represntation for linear matrix equations.

We start by studying the simplest matrix equation as the following theorem **:**

Theorem 4.1.1 [7] Let $A \in M_n$, $B \in M_m$, $C \in M_{n,m}$ and $X \in M_{n,m}$, such that $AXB = C$, then the system $(B^T \otimes A)\, Vec(X) = Vec(C)$. Has a unique solution if and only if $B^T \otimes A$ is invertible if and only if B and A both are invertible.

If either A or B are not invertible, then there exist a solution X if and only if rank $(B^T \otimes A) =$ rank $([B^T \otimes A : \ Vec(C)])$. Where $[B^T \otimes A : \ Vec(C)]$ is the augmented matrix of $B^T \otimes A$ and $Vec(C)$; otherwise the system has no solution.■

This equation $AXB = C$ can be generalized as follows **:**

$A_1XB_1 + A_2XB_2 + \cdots + A_pXB_p = C$, where $A_j \in M_n$, $B_j \in M_m$ $(j = 1, \dots ,p)$, and $X, C \in M_{n,m}$.

With the same technique we can rewrite this equation as **:**

$Vec(A_1XB_1) + Vec(A_2XB_2) + \cdots + Vec\big(A_pXB_p\big) = Vec(C).$

So, $(B_1^T \otimes A_1)\, Vec(X) + \cdots + \big(B_p^T \otimes A_p\big)\, Vec(X) = Vec(C).$ (by theorem 1.4.26)**.**

$i.e,\ \sum_{j=1}^{p} \big(B_j^T \otimes A_j\big)\, Vec(X) = Vec(C).$

The unique solution is obtained if and only if $\sum_{j=1}^{p} \big(B_j^T \otimes A_j\big)$ is invertible.■

The following theorem examine if the $AXB = C$ has a unique X. By using eigenvalue of the Kronecker sum**.**

Theorem 4.1.2 [13] Let $A \in M_n$ and $B \in M_m$. The equation $AX + XB = C$ has a unique solution $X \in M_{n,m}$ for each $C \in M_{n,m}$ if and only if $\sigma(A) \cap \sigma(-B) = \phi$.

Proof : The eigenvalue of B^T are the same as those of B. Now, if we take the Vec(.) of both sides in equation $AX + XB = C$ we get $(A \oplus B^T)\, Vec(X) = Vec(C)$ (by coroll-ary (1.4.27))**.** And this system of equations has a unique solution if and only if $A \oplus B^T$ is invertible, that is if and only if non of the eigenvalues of $A \oplus B^T$ is zero**.** But $\sigma(A \oplus B^T) = \{\lambda_i + \mu_j : i = 1, \dots, n,\ \ j = 1, \dots, m\}$, where $\sigma(A) = \{\lambda_i : i = 1, \dots, n\}$ and $\sigma(B) = \{\ \mu_j :\ \ j = 1, \dots, m\ \}$. So The equation $AX + XB = C$ has a unique solution if and only if $\lambda_i + \mu_j \neq 0$ for all i, j, $i.e.$, if and only if $\lambda_i \neq -\mu_j$ if and only if (A) and $(-B)$ have no common eigenvalue if and only if $\sigma(A) \cap \sigma(-B) = \phi$. ■

If on other hand A and $-B$ have an eigenvalue in common, the existence of the solution depends on the rank of the augmented matrix $[A \oplus B^T : Vec(C)]$. If the rank of this matrix is equal to the rank of $A \oplus B^T$, then the solution exist otherwise they do not.

Theorem 4.1.3 [7] If $A \in M_n$ and $B \in M_m$. The equation $AX - XA = \mu X$, which has a nontrivial solution if and only if μ is an eigenvalue of $-A^T \oplus A$. But the eigenvalues of $-A^T \oplus A$ are $\{\lambda_i - \lambda_j : \lambda_i \in \sigma(A)\}$. Hence $AX - XA = \mu X$ has a nontrivial solution if and only if $\mu = \lambda_i - \lambda_j$ for some i, j. ∎

Lemma 4.1.1.4 [8] Let $\mathrm{A, B, C}$ and $\mathrm{D} \in \mathrm{M_n}$ such that $CD = DC$. Then $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is Invertible if and only if $AD - BC$ is invertible and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC)$. ∎

Theorem 4.1.1.5 [8] Let $A_1,\ B_1,\ C_1,\ D_1,\ A_2,\ B_2,\ C_2,\ D_2,\ E$, and $F \in M_n$ be given matrices such that $C_1D_1 = D_1C_1$ and $C_2D_2 = D_2C_2$. Then the system

$$A_1XA_2 + B_1YB_2 = E$$

$$C_1XC_2 + D_1YD_2 = F$$

has a unique solution if and only if $A_2^T D_2^T \otimes A_1 D_1 - B_2^T C_2^T \otimes B_1 C_1$ is invertible. ∎

Corollary 4.1.6 Let $\mathrm{A, B, C, D, E}$, and $\mathrm{F} \in \mathrm{M_n}$ be given matrices. Then the system

$$AX + YB = E$$

$$CX + YD = F$$

has a unique solution if and only if $D^T \otimes A - B^T \otimes C$ is invertible. ∎

Corollary 4.1.7 Let A, B, C, D, E, and F $\in \mathrm{M_n}$ be given matrices**.** Then the system

$$XA + BY = E$$

$$XC + DY = F$$

has a unique solution if and only if $A^T \otimes D - C^T \otimes B$ is invertible.■

If we assume that $CD = DC$, then the system

Corollary 4.1.8 Let A, B, C, D, E, and F $\in \mathrm{M_n}$ be given matrices**.** Then the system

$$AX + BY = E$$

$$CX + DY = F$$

has a unique solution if and only if $AD - BC$ is invertible.■

Corollary 4.1.9 Let A, B, C, D, E, and F $\in \mathrm{M_n}$ be given matrices**.** Then the system

$$XA + YB = E$$

$$XC + YD = F$$

has a unique solution if and only if $DA - CB$ is invertible.■

The important application of the theorem (1.4.12.(b)) are for $p(t) = e^t, g(t) = \sin t,$

$h(t) = \cos t,$ lead to the following result **:**

Corollary 4.1.10 [7] Let $A \in M_n$, be a scalar matrix**.** Then

(1) $e^{(A \otimes \mathrm{I}_m)} = e^A \otimes \mathrm{I_m}$.

(2) $\sin(\mathrm{A} \otimes \mathrm{I_m}) = \sin(\mathrm{A}) \otimes \mathrm{I_m}$.

Proof (1) : We can write e^A as a power series such as **:**

$$e^A = \mathrm{I_n} + A + \frac{1}{2!}A^2 + \cdots$$

so, $$\mathrm{e}^{(A\otimes \mathrm{I}_m)} = (\mathrm{I_n}\otimes\mathrm{I_m}) + (A\otimes\mathrm{I_m}) + \frac{1}{2!}(A\otimes\mathrm{I_m})^2 + \cdots$$

$$= (\mathrm{I_n}\otimes\mathrm{I_m}) + (A\otimes\mathrm{I_m}) + \frac{1}{2!}(A^2\otimes\mathrm{I_m}) + \cdots$$

$$= (\mathrm{I_n} + A + \frac{1}{2!}A^2 + \cdots) \otimes \mathrm{I_m}$$

$$= e^A\otimes\mathrm{I_m}.$$

Proof (2) : We can write sin A as a power series such as **:**

$$\sin \mathrm{A} = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$$

so, $$\sin(\mathrm{A}\otimes \mathrm{I_m}) = (\mathrm{A}\otimes\mathrm{I_m}) - \frac{(A\otimes\mathrm{I_m})^3}{3!} + \frac{(A\otimes\mathrm{I_m})^5}{5!} - \frac{(A\otimes\mathrm{I_m})^7}{7!} + \cdots$$

$$= (A\otimes\mathrm{I_m}) - \frac{(A^3\otimes\mathrm{I_m})}{3!} + \frac{(A^5\otimes\mathrm{I_m})}{5!} - \frac{(A^7\otimes\mathrm{I_m})}{7!} + \cdots$$

$$= (A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots)\otimes\mathrm{I_m}$$

$$= \sin \mathrm{A} \otimes\mathrm{I_m}.$$

4.2 Matrix differential equations

In this section we present another application of the Kronecker product that deals with matrix differential equations of the form $\dot{X} = AX + XB$.

Definition 4.2.1 Given the matrix $A(t) = \left[a_{ij}(t)\right] \in M_{m,n}$, where each $a_{ij}(t)$ is a differentiable function, then the derivative of the matrix A with respect to the scalar t is defined as : $\frac{d}{dt}\, A(t) = \left[\frac{d}{dt}\, a_{ij}(t)\right] = \dot{A}$.

Similarly, the integral of the matrix is defined as : $\int A(t)dt = \left[\int a_{ij}(t)dt\right]$.

Theorem 4.2.1 [7] Let $A(t) \in M_{m,n}$ and $B(t) \in M_{p,q}$, be differentiable matrices (each matrix is assumed to be a function of t). Then

$$\frac{d}{dt}[A(t) \otimes B(t)] = \left[\frac{d}{dt}A(t)\right] \otimes B + A \otimes \left[\frac{d}{dt}B(t)\right].$$

Proof : On differentiating the (i,j)th block of $A(t) \otimes B(t)$, we obtain

$$\frac{d}{dt}[A(t) \otimes B(t)] = \frac{d}{dt}\left[a_{ij}B(t)\right] = \frac{da_{ij}}{dt}B(t) + a_{ij}\frac{d}{dt}B(t).$$

$$= \left[\frac{d}{dt}A(t)\right] \otimes B + A \otimes \left[\frac{d}{dt}B(t)\right]. \blacksquare$$

Corollary 4.2.2 Let $A(t) \in M_m$ and $B(t) \in M_n$ be differentiable matrices (each matrix is assumed to be a function of t). Then

$$\frac{d}{dt}[A(t) \oplus B(t)] = \frac{d}{dt}A(t) \oplus \frac{d}{dt}B(t).$$

Proof : $\frac{d}{dt}\mathrm{I}_n = 0$, and using definition 1.4.2, then we have

$$\frac{d}{dt}[A(t) \oplus B(t)] = \frac{d}{dt}[\mathrm{I}_n \otimes A(t)] + \frac{d}{dt}[B(t) \otimes \mathrm{I}_m]$$

$$= \mathrm{I}_\mathrm{n} \otimes \left[\frac{d}{dt}A(t)\right] + \left[\frac{d}{dt}B(t)\right] \otimes \mathrm{I}_m$$

$$= \frac{d}{dt}A(t) \oplus \frac{d}{dt}B(t). \blacksquare$$

The simplest form of matrix differential equations as the following theotem :

Theorem 4.2.3 [7] $\dot{x} = Ax;\ x(0) = c$, where $A \in M_n$ ... (1)

This equation has the following solution **:** $x = e^{At}c$.

Using this fact we can solve the matrix differential equation **:**

$\dot{X} = AX + XB;\ X(0) = C$ ... (2), where $A \in M_n,\ B \in M_m, X \in M_{n,m}$, and

$C \in M_{m,n}$.

Proof: use the $Vec(.)$-notation, then we get $Vec\dot{X} = (\mathrm{I_m} \otimes \mathrm{A} + \mathrm{B^T} \otimes \mathrm{I_n})\ VecX$, and

$Vec\,X(0) = Vec\,C$. Let $Vec\,X = x$, and $Vec\,C = c$.

Then (1) becomes $\dot{x} = (\mathrm{I_m} \otimes \mathrm{A} + \mathrm{B^T} \otimes \mathrm{I_n})\ x;\ \ x(0) = c$. By the solution (2) we have

$x = (\exp(\mathrm{I_m} \otimes \mathrm{A} + \mathrm{B^T} \otimes \mathrm{I_n})\mathrm{t})c$. But $\exp(\mathrm{I_m} \otimes \mathrm{A} + \mathrm{B^T} \otimes \mathrm{I_n}) = \exp(I_m \otimes A)\ \exp(\mathrm{B^T} \otimes \mathrm{I_n})$

$= (I_m \otimes expA)\ (exp\mathrm{B^T} \otimes \mathrm{I_n})$

$= \exp\ (B^T) \otimes\ expA$

so, $x = (expB^Tt \otimes expAT)\ c;\ i.e. Vec\,X = (expB^Tt \otimes expAT)\ Vec\,C$

$= Vec\ (expAt\,.\,C.\,expBt)$.

Thus, $X = expAt\,.\,C.\,expBt. \blacksquare$

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