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Special Boundary Integral Equations For Potential Problems In Multi-Dimensional Regions

By:
Studant Name: Omar Abuloha
Registration Number: 20812243

Supervisor: Dr.Yousef Zahaykah

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The names and signatures of the examining committee members are as follows:

1-Dr. Yousef Zahaykah	Head of Committee, Signature	
2-Dr. Taha Abukaff	Internal Examiner, Signature	
3-Dr. Amjad Barham	External Examiner, Signature	

AL-Quds University
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Abstract

This thesis deals with the approximate solution of Laplace's equation in two-dimentional regions with circular holes and in three-dimentional regions with slender cavities with circular cross sections.

The propsed method, Special Boundary Integral Equations Method (SBIEM), introduces approximate solutions to the Laplace's equation by developing special boundary integral equations based on the representation formula of a $C^2(\overline{\Omega})$ function using its values on the boundary of the region Ω which derived from Green's identities.

The solution on the boundary of each circular hole is represented by a finite sum of circular harmonics with unknown coefficients.

The hole geometry is directly exploited in a new set of integral equations with special kernal functions which independently " pick out" these coefficients.

Each new integral equation contains only one coefficient relating to the particular hole and the resulting system of equations is solvable.

The accuracy of the obtainable approximate solution depends on the number of circular harmonics used in the representation series of the solution on the boundary of the hole.

We consider in this thesis two level of approximations. The first one is called the zeroth order level and the second is called the first order level.

Examples are given to demonstrate the proposed method (SBIEM).

Further, in this thesis, complete general numerical methods are propsed to solve Laplece's equation, namely the collocation and the Galerkin methods. These methods belong to the general frame work of projection methods. An important characterisation of these methods is that they are applicable regardless of the geometry of the domain of the Laplce's equation.

A comparison between the results obtained by these methods and these obtained by the special boundary inegral equations method are given in this thesis.

الملخص

قدمت هذه الأطروحة طريقة لتقريب حلول لمعادلة لابلاس في مناطق ثنائية الأبعاد تحوي فتحات دائرية أو مناطق ثلاثية الأبعاد تحوي تجويفات دقيقة مقاطعها عبارة عن دوائر.

الطريقة المعروضة تقدم حلا لمعادلة لابلاس من خلال تطوير معادلات تكاملية محيطية خاصة تعتمد أساسا على الصيغة الرياضية الخاصة بتمثيل الاقترانات المنتمية إلى الفضاء الاقتراني $C^2(\overline{\Omega})$ بدلالة القيم المحيطية لهذه الاقترانات والتي تنبثق عن ما يعرف بمتطابقات غرين .

إن الطبيعة الهندسية للمجالات (المناطق) موضع الدراسة مكنت من تمثيل حلول معادلة لابلاس بمتسلسلات دائرية هارمونية (Circular Harmonics) منتهية على المحيطات الداخلية للمنطقة معاملاتها غير معلومة والتي بتحديدها نحصل على تقريب للحل المطلوب .

إن هذا التمثيل أدى من خلال استخدام اقترانات (kernels) خاصة إلى الحصول على نظام من المعادلات التكاملية حلوله هي قيم المعاملات في المتسلسلات الدائرية الهارمونية الممثلة للحل التقريبي على المحيطات الداخلية وباستخدام هذه القيم المحيطية تم تقريب الحل لمعادلة لابلاس عند أي نقطة داخلية في مجال التعريف.

إن مستوى التقريب يرتبط ارتباطا وثيقا بعدد الحدود المأخوذة في المتسلسلات الهارمونية الممثلة للحلول على المحيطات الداخلية , ولقد تم التركيز على مستويين من التقريب:

الأول أطلق عليه مستوى ألرتبة الصفرية والثاني أطلق عليه مستوى ألرتبة الأولى.

ولقد عرضنا في هذه الأطروحة عدة أمثلة لتوضيح طريقة SBIEM.

محور أساسي آخر في هذه الأطروحة هو معالجة معادلة لابلاس بطرق عددية أساسية عامة وهي : طريقة أل Collocation وطريقة Galerkin وكلاهما تنتميان للإطار العام المعروف بالطرق الاسقاطية (Projection Methods) .

إن أهم ما يميز هذه الطرق هو التحرر من القيود الهندسية لمجالات معادلة لابلاس, ولقد تم عقد مقارنة ما بين نتائج هذه الطرق والطريقة الخاصة و المسماه طريقة المعادلات التكاملية المحيطية الخاصة والتي هي اللبنة الأساسية في هذه الاطروحة.

Contents

Introduction

One of the most important partial differential equations in applied mathematics is Laplace's equation.

It is the simplest example of elliptic partial differential equations of second order whose solutions are called harmonic functions and are important in many fields of sciences, notably the field of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational and fluid potential. The basic theory of elliptic equations presented in Gilbarg and Trudinger[11].

Many important problems require the solution of this equation, such as the problem of incomressible irrotational flow[1], flow of inviscid comressible fluid around circular cylinders[15], electrostatic field around gratings of charged wires[16].

Further the solution of boundary values problems of partial differential equations is an important field of application of integral equations, see Atkinson[6], Kress[4], Kanwal[17].

Depending on the nonlinerarity of the differential equation and the geometry of the domain of the problem analytical solutions are not always available the best alternative is to find approximate solution. Many numerical methods were developed based on finite differences, finite elements and boundary integrals methods, see Kress[5], and LeVeque[8],[6],[10].

The technique ,,special boundary integral equations method" (SBIEM) was proposed by Barone and Caulk for potential problems in regions with circular holes and regions with slender cavities, see [2,3] .

The same technique was applied to three dimenional regions with spherical cavities [9]. In [9] Rania derived SBIEM where the potential or its outward normal derivative is not constant.

In this thesis we reanalyze the work of Barone and Caulk [2,3]. We approach the problem by formulating special boundary integral equations which take explicit a count of the hole geometry and the corresponding characteristic of the solution.

First we approximate the field on the boundary of each hole by a trigonometric polynomial of finite degree (finite series of circular harmonics).

The unknown coefficients are calculated as follows:

corresponding to each hole we choose a special kernal function which independently pick out respective coefficient at that hole. Taken together, the equations at any one hole express the coefficients on the boundary of the hole in terms of integrals over the outer boundary and the other holes in the domain. Since each equation corresponds to one coefficient at its associated hole the system is well-conditioned and hence it is solvable.

Further we consider Laplace's equation in the frame of projection methods, collocation and Galerkin , see [4,5,6] and [10]. These methods are comletely general numerical methods that used with general domains. We apply these methods to the representation formula of the solution of the Laplace's equation. In both cases we used $\{1, \sin\theta, \cos\theta, ..., \sin n\theta, \cos n\theta\}$ as a basis.

The thesis is organized as follows:

In Chapter One we present the scheme of special boundary integral equations method (SBIEM). We introduce the domain of the Laplec's equation, namely two-dimensional regions with circular holes. We construct the basic boundary integral equations on the boundary of each circular hole. We assume that the field represented by a finite series of circular harmonics. We introduce special kernal functions and use the geometry of the holes to determine the unknown coefficients. We treat Dirichlet's problem when the potential is constant on the boundary of the circular holes. Here we take care about two cases, when the normal field is considered to be constant then we obtained the zeroth order solution and when it is represented by a trigonometric polynomial of order one then we get the first order solution.

In Chapter Two we present and analyze the SBIEM scheme if the domain is three-dimentional region with slender cavities. Here we suppose that the cross-sections of the slender cavities are circular and the fields are axis of symmetry, hence we reduce the surface integrals (appear in the formulated integral equations) to line integrals along the space curves passing through

:he centre of each slender cavity. As an application we present the coaxial :ori case.

:n Chapter Three we present the projection methods collocation and Galerkin. We apply these methods on the integral representation formula of the Laplac's equation.

Further we apply these methods as well as SBIEM to a problem whose solu- :ion is know and carried out a comparison between the results of SBIEM and collocation and Galerkin methods. We conclude the thesis by an outlook.

Chapter 1

Special Boundary Integral Equations for Approximate Solution of Laplace's Equation in Two-Dimensional Regions with Circular Holes

1.1 Formulation

Suppose that $\mathcal{R}$ is the interior of a two-dimentional connected bounded region (with external boundary denoted by $\partial\mathcal{R}$) containing N circular holes centred at ξ^{α}, $\alpha = 1, 2..., N$ points. Let a_{α} denotes the radius of hole α. On each circular boundary ∂C^{α} either a potential function(harmonic in $\mathcal{R}$ and satisfies irregular condion) or its normal derivative is prescribed. Want to determine the potential at any point in $\mathcal{R}$.

This means want ϕ that satisfies $\Delta\phi = 0$ in $\mathcal{R}$ and on ∂C_{α} either ϕ or $\frac{\partial\phi}{\partial \mathbf{n}}$ is prescribed.

Define the function

$$g(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}| \tag{1.1.1}$$

and let ϕ be a harmonic function in $\mathcal{R}$ which satisfies regular boundary conditions on $\partial\mathcal{R}$ and ∂C^{α}. Then by using Green's second identity in the usual way, we can write

$$\lambda\phi(\mathbf{y}) + \int_{\partial R}\left(\phi\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial\phi}{\partial \mathbf{n}}\right)d\,s + \sum_{\alpha=1}^{\mathrm{N}}\int_{\partial C^{\alpha}}\left(\phi^{\alpha}\frac{\partial g}{\partial \mathbf{n}} - gq^{\alpha}\right)d\,s = 0 \tag{1.1.2}$$

where ϕ^{α} and q^{α} are the values of ϕ and its outward normal derivative on the hole α. The constant λ is given by

$$\lambda = \begin{cases} 1 & , \ \mathbf{y} \in \mathcal{R} \\ \frac{1}{2} & , \ \mathbf{y} \in \{\partial\mathcal{R}, \partial C^\alpha\} \\ 0 & , \ \mathbf{y} \notin \{\mathcal{R}, \partial\mathcal{R}, \partial C^\alpha\} \end{cases} \tag{1.1.3}$$

In applying boundary integral equations method, see[18] to regions with holes, the boundary of each hole must divided into a fairly large number of finite segments to gain an adequate representation of the local field variation. We avoid this situation by developing special boundary integral equations which take advantage of circular geometry of the hole. We first represent ϕ^α and q^α by a finite sum of circular harmonics on ∂C^α as follows

$$\left.\begin{aligned} \phi^\alpha &= \phi_0^\alpha + \textstyle\sum_{n=1}^K (\phi_{1n}^\alpha \sin n\theta^\alpha + \phi_{2n}^\alpha \cos n\theta^\alpha) \\ q^\alpha &= q_0^\alpha + \textstyle\sum_{n=1}^K (q_{1n}^\alpha \sin n\theta^\alpha + q_{2n}^\alpha \cos n\theta^\alpha) \end{aligned}\right\} \tag{1.1.4}$$

where $\phi_0^\alpha, \phi_{\lambda n}^\alpha, q_0^\alpha, q_{\lambda n}^\alpha$, α=1,2...,N, λ=1,2 , are constants and θ^α is the polar angle centred at ξ^α measured relative to the positive x-axis.
We evaluate the second integral in (1.1.2).
First we take the case when $\mathbf{y} \notin \partial C^\alpha$. Let $\mathbf{x} \in \partial C^\alpha$ then

$$\mathbf{x} - \mathbf{y} = (a_\alpha \cos\theta^\alpha - r_\alpha \cos\psi^\alpha, a_\alpha \sin\theta^\alpha - r_\alpha \sin\psi^\alpha)$$

and

$$|\mathbf{x} - \mathbf{y}|^2 = (a_\alpha^2 + r_\alpha^2 - 2a_\alpha r_\alpha \cos(\psi^\alpha - \theta^\alpha))$$

Thus

$$g(\mathbf{x}, \mathbf{y}) = \frac{-1}{4\pi} \log\left(a_\alpha^2 + r_\alpha^2 - 2a_\alpha r_\alpha \cos(\psi^\alpha - \theta^\alpha)\right)$$

where $\psi^\alpha = \theta^\alpha(\mathbf{y})$, $r_\alpha = |\mathbf{y} - \xi^\alpha|$
Now

$$\frac{\partial g}{\partial \mathbf{n}} = \nabla g.\mathbf{n} = \frac{-1}{2\pi\,|\mathbf{x} - \mathbf{y}|^2}(\mathbf{x} - \mathbf{y}).\mathbf{n}$$

Since the vector $(a_\alpha \cos\theta^\alpha, a_\alpha \sin\theta^\alpha)$ is normal vector to the ∂C^α with norm equal a_α. Then a unit normal vector to ∂C^α is $\mathbf{n} = (\cos\theta^\alpha, \sin\theta^\alpha)$.
Hence

$$\begin{aligned} \frac{\partial g}{\partial \mathbf{n}} = \nabla g.\mathbf{n} &= \frac{-1}{2\pi\,|\mathbf{x} - \mathbf{y}|^2} \times (a_\alpha \cos\theta^\alpha - r_\alpha \cos\psi^\alpha, a_\alpha \sin\theta^\alpha - r_\alpha \sin\psi^\alpha).(\cos\theta^\alpha, \sin\theta^\alpha) \\ &= \frac{1}{2\pi} \frac{(a_\alpha - r_\alpha \cos(\psi^\alpha - \theta^\alpha))}{(a_\alpha^2 + r_\alpha^2 - 2a_\alpha r_\alpha \cos(\psi^\alpha - \theta^\alpha))} \\ &= \frac{1}{2\pi r_\alpha} \frac{((b_\alpha - \cos(\psi^\alpha - \theta^\alpha))}{(1 + b_\alpha^2 - 2b_\alpha \cos(\psi^\alpha - \theta^\alpha))} \end{aligned}$$

where $b_\alpha = \frac{a_\alpha}{r_\alpha}$.
Therefore

$$\begin{aligned}
\int_{\partial c^\alpha} \phi^\alpha \frac{\partial g}{\partial \mathbf{n}} d\,s &= \int_{\partial c^\alpha} \left(\phi_0^\alpha + \sum_{n=1}^{K} (\phi_{1n}^\alpha \sin n\theta^\alpha + \phi_{2n}^\alpha \cos n\theta^\alpha) \right) \frac{\partial g}{\partial \mathbf{n}} d\,s = \phi_0^\alpha \int_{\partial C^\alpha} \frac{\partial g}{\partial \mathbf{n}} d\,s \\
&+ \sum_{n=1}^{K} \frac{\phi_{1n}^\alpha}{2\pi} \frac{a_\alpha}{r_\alpha} \int_0^{2\pi} \sin n\theta^\alpha \frac{(b_\alpha - \cos(\psi^\alpha - \theta^\alpha))}{(1 + b_\alpha^2 - 2b_\alpha \cos(\psi^\alpha - \theta^\alpha))} d\,\theta^\alpha \\
&+ \sum_{n=1}^{K} \frac{\phi_{2n}^\alpha}{2\pi} \frac{a_\alpha}{r_\alpha} \int_0^{2\pi} \cos n\theta^\alpha \frac{(b_\alpha - \cos(\psi^\alpha - \theta^\alpha)}{(1 + b_\alpha^2 - 2b_\alpha \cos(\psi^\alpha - \theta^\alpha)} d\,\theta^\alpha \\
&= \sum_{n=1}^{K} \frac{\phi_{1n}^\alpha}{2\pi} b_\alpha \int_{0+\psi^\alpha}^{2\pi+\psi^\alpha} \frac{b_\alpha - \cos\theta^\alpha}{1 + b_\alpha^2 - 2b_\alpha \cos\theta^\alpha} \sin n(\theta^\alpha + \psi^\alpha) d\,\theta^\alpha \\
&+ \sum_{n=1}^{K} \frac{\phi_{2n}^\alpha}{2\pi} b_\alpha \int_{0+\psi^\alpha}^{2\pi+\psi^\alpha} \frac{b_\alpha - \cos\theta^\alpha}{1 + b_\alpha^2 - 2b_\alpha \cos\theta^\alpha} \cos n(\theta^\alpha + \psi^\alpha) d\,\theta^\alpha \\
&= \sum_{n=1}^{K} \frac{\phi_{1n}^\alpha}{2\pi} b_\alpha \int_0^{2\pi} \frac{b_\alpha - \cos\theta^\alpha}{1 + b_\alpha^2 - 2b_\alpha \cos\theta^\alpha} (\sin n\theta^\alpha \cos n\dot{\psi}^\alpha + \cos n\theta^\alpha \sin n\psi^\alpha) d\,\theta^\alpha \\
&+ \sum_{n=1}^{K} \frac{\phi_{2n}^\alpha}{2\pi} b_\alpha \int_0^{2\pi} \frac{b_\alpha - \cos\theta^\alpha}{1 + b_\alpha^2 - 2b_\alpha \cos\theta^\alpha} (\cos n\theta^\alpha cos n\psi^\alpha - \sin n\theta^\alpha \sin n\psi^\alpha) d\,\theta^\alpha \\
&= \sum_{n=1}^{K} \frac{b_\alpha}{2\pi} (\phi_{1n}^\alpha \sin n\psi^\alpha + \phi_{2n}^\alpha \cos n\psi^\alpha) \int_0^{2\pi} \frac{b_\alpha - \cos\theta^\alpha}{1 + b_\alpha^2 - 2b_\alpha \cos\theta^\alpha} \cos n\theta^\alpha d\,\theta^\alpha \\
&+ \sum_{n=1}^{K} \frac{b_\alpha}{2\pi} (\phi_{1n}^\alpha \cos n\psi^\alpha - \phi_{2n}^\alpha \sin n\psi^\alpha) \int_0^{2\pi} \frac{b_\alpha - \cos\theta^\alpha}{1 + b_\alpha^2 - 2b_\alpha \cos\theta^\alpha} \sin n\theta^\alpha d\,\theta^\alpha \\
&= \sum_{n=1}^{K} \frac{b_\alpha}{2\pi} (\phi_{1n}^\alpha \sin n\psi^\alpha + \phi_{2n}^\alpha \cos n\psi^\alpha) \left\{ \frac{2\pi b_\alpha^{n+1} - \pi b_\alpha^{n-1} - \pi b_\alpha^{n+1}}{1 - b_\alpha^2} \right\} \\
&= \sum_{n=1}^{K} \frac{b_\alpha}{2\pi} (\phi_{1n}^\alpha \sin n\psi^\alpha + \phi_{2n}^\alpha \cos n\psi^\alpha) \frac{\pi b_\alpha^n (b_\alpha - \frac{1}{b_\alpha})}{1 - b_\alpha^2} \\
&= \sum_{n=1}^{K} \frac{b_\alpha}{2\pi} (\phi_{1n}^\alpha \sin n\psi^\alpha + \phi_{2n}^\alpha \cos n\psi^\alpha) \frac{-\pi b_\alpha^n (b_\alpha^2 - 1)}{b_\alpha (b_\alpha^2 - 1)} \\
&= -\sum_{n=1}^{K} \frac{b_\alpha^n}{2} (\phi_{1n}^\alpha \sin n\psi^\alpha + \phi_{2n}^\alpha \cos n\psi^\alpha) \qquad (1.1.5)
\end{aligned}$$

and

Conclusion and outlook

In this thesis we apply SBIEM to Dirichlet problems where the potential is taken to be constant. The agreement between the numerical solution and the exact one was very high, comparing to the results of the projection methods. This partially because of the simple geometry used, and the assumption that the potential on the circular holes is constant.

The case of nonconstant potential is an interesting one. On the other hand Neumann problems with constant and nonconstant fluxes are very important.

Another important aspect is to give a rigorous mathmatical proofs to show that the obtained linear algebric systems are well-conditioned.

These aspects can be considered as a basis for a further work in the frame of SBIEM schemes.