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Compactness In Bitopological Spaces

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Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ....

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Abstract

In this thesis, we introduce three concepts concerning the compactness in bitopological spaces, namely, semi compactness, pairwise compactness and Birsan compactness. Also, we introduce other concepts in bitopological spaces such as Hausdorffness, continuity, regularity and normality, and study their relations with compactness in bitopological spaces. We discuss generalizations for well known theorems and results concerning compactness in single topology, such as Alexander and Tychonoff theorems for bitopological spaces.

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Introduction

In 1962, J.C. Kelly [8] has defined the concept of the bitopological space to be a non- empty set X on which two arbitrary topologies τ_1 and τ_2 are defined. This definition is denoted by the triple (X,τ_1,τ_2). Since this initiation, several authors have considered the problem of defining the concept of compactness in bitopological spaces. And in this thesis we study the definitions for compactness in bitopological spaces, which were introduced by Swart in [1], Flectcher, Holye and Patty in [2], Kim in [5-1], Kim and Naimpaly in [5-2], Cooke and Reilly in [4], and Birsan in [6].

In fact, these definitions are summarized into just three definitions only, namely, semi compactness (s-compactness), pairwise compactness (p-compactness), and Birsan compactness (conversely and B-compactness).

Semi compactness is discussed in chapter one which is divided into five sections. Section one begins with Swart's definition for bitopological compactness. Also, we define quasi-open (closed) subsets of (X,τ_1,τ_2), quasi-closure and some relations and results concerning them. In section two, we define four different types of continuous functions in the bitopological spaces, and deduce some useful results concerning continuity and compactness in bitopological spaces in this chapter and in the following two chapters. Least upper bound topology is introduced in section three. Also, we show that semi-compactness in bitopological space is equivalent to the compactness in least upper bound topology. In section four we introduce two different and related definitions for Hausdorffness in bitopological spaces, and deduce many useful results and conclusions concerning Hausdorffness in bitopological spaces. In the last section we introduce the product topology and conclude a generalization of Tychonoff theorem in the single topological space.

In section one of chapter two, three different but dependent definitions of bitopological compactness, which we call each of them pairwise compactness because they are equivalent, are considered. Also we show that semi and pairwise compactness are dependent. In section two we define pairwise regularity and pairwise normality, and deduce related results between pairwise Hausdorffness and pairwise compactness. In section three, we define the notion of pairwise compactness in a subspace of a bitopological space and its relation with pairwise compactness in bitopological spaces. In section four, we discuss some definitions and theorems related to Datta [3-2], and prove a generalization of Alexander theorem in single topology, and show that a generalization of Tychonoff theorem in single topology fails for bitopological spaces, by giving a counter example.

In section one of chapter three, we introduce a new different concept for compactness in bitopological spaces. We show that this definition is independent from the two definitions, which are introduced in chapter one and chapter two. Also, we give two dependent definitions, one is called conversely compactness and the other is called B-compactness, and then shows that B-compactness implies conversely compactness. We introduce two different ways which are equivalent to conversely compactness, and deduce the relation between conversely (B-) compactness and compactness of τ_1 and τ_2.

In fact we show that conversely (B-) compactness implies compactness of τ_1 and τ_2. We deduce the effect of pairwise Hausdorffness in comparison of topologies. The examples at the end of this section are counter examples demonstrate the relations between conversely (B-) compactness, p-regularity and p-normality in bitopological spaces. In section two, we study the compactness in subspaces and its relations with closedness and openness. In section three, we study the effect of continuous and open functions on conversely (B-) compact bitopological spaces. Finally, in section four, we make a generalization of Alexander and Tychonoff theorems in bitopological spaces.

Chapter One

Semi compactness

1.1 Swart's definition for compactness in bitopological spaces.

1.1.1 Definition [1]:

A cover υ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$– open if $\upsilon \subset \tau_1 \cup \tau_2$.

1.1.2 Definition [1]:

A bitopological space (X, τ_1, τ_2) is called semi-compact (s- compact) if every $\tau_1\tau_2$–open cover for X has a finite subcover .

Swart in [1] consider the above definition for compactness in bitopological spaces, and uses the term compact for s-compactness in (X, τ_1, τ_2).

1.1.3 Definition [3-1]:

A subset A in a bitopological space (X,τ_1,τ_2) is said to be quasi-open if for every $x \in A$, there exists a τ_1-open nhood for x (nhood stands for neighborhood) $U_x \subset A$ or a τ_2-open nhood for x, $V_x \subset A$.

1.1.4 Theorem[3-1]:

Quasi-open sets in a bitopological space (X,τ_1,τ_2) are precisely the unions of τ_1-open and τ_2-open sets.

Proof:

Let U_1, U_2 be two τ_1 and τ_2-open sets respectively. Let $x \in U_1 \cup U_2$, then $x \in U_1$ or $x \in U_2$. Either of them is contained in $U_1 \cup U_2$, which means that $U_1 \cup U_2$ is quasi-open.

Conversely, assume A is quasi-open in (X,τ_1,τ_2), and let $x \in A$. Then there is a τ_1-open nhood U_1 s.t. $U_1 \subset A$ or τ_2-open nhood U_2 s.t. $U_2 \subset A$. Let

$A_1 = \{ x \in A \mid \exists\, U_1 \in \tau_1$ such that $x \in U_1 \subset A\}$ and

$A_2 = \{ x \in A \mid \exists\, U_2 \in \tau_2$ s.t. $x \in U_2 \subset A\}$. Then $A = A_1 \cup A_2$. Now A_1 is τ_1- open. For let $x \in A_1$, then there exist $U_1 \in \tau_1$ such that $x \in U_1 \subset A$. Let $y \in U_1$, then $y \in U_1 \subset A$, and $U_1 \in \tau_1$, so $y \in A_1$. By the Definition of A_1, we have $U_1 \subset A_1$. Now $x \in U_1 \subset A_1$, $U_1 \in \tau_1$, and x was arbitrary in A_1, so $A_1 \in \tau_1$, similarly, we can show that $A_2 \in \tau_2$.

From this theorem and the definition of quasi-open sets we conclude the following results.

1.1.5 Corollary:

In a bitopological space (X,τ_1,τ_2):

a) Every τ_1-open (τ_2-open) set is quasi-open.

b) Arbitrary unions of quasi-open sets is quasi-open.

The converse of (a) does not necessarily hold. For let $\tau_1 = \{\{a\},\{a,b\},X, \emptyset\}$, $\tau_2= \{\{c\},\{c,b\},X, \emptyset\}$, $X=\{a,b,c\}$. Then $\{a,c\}$ is quasi-open but not in τ_1 nor in τ_2 .

Proof of (a): follows from definition (1.1.3).

Proof of (b):

Let $\{ A_\alpha : \alpha \in \Delta \}$ be any collection of quasi-open sets . Then $\forall\ \alpha \in \Delta$,

$A_\alpha = A`_\alpha \bigcup A``_\alpha$ ; where $A`_\alpha \in \tau_1$ and $A``_\alpha \in \tau_2$. So,

$\bigcup_{\alpha\in A} A_\alpha = \bigcup_{\alpha\in A} (A`_\alpha \cup A``_\alpha ) = ( \bigcup_{\alpha\in A} A`_\alpha) \cup ( \bigcup_{\alpha\in A} A``_\alpha)$. But $\bigcup_{\alpha\in A} A`_\alpha \in \tau_1$

and $\bigcup_{\alpha\in A} A``_\alpha \in \tau_2$, therefore $\bigcup_{\alpha\in A} A_\alpha$ is quasi-open.

Finite intersection of quasi-open sets need not be quasi-open as the following example shows.

1.1.6 Example:

Let $X = \Re$; the real line, let τ_1 = the topology with the base

$\{[a,b): a,b \in \Re\}$ and τ_2 = the topology with the base $\{(c,d]: c,d \in \Re\}$.

Let $a < b < c$. Then $(a,b]$ and $[b,c)$ are quasi- open sets, but

$(a,b] \cap [b,c) = \{b\}$ is not a quasi-open set.

1.1.7 Definition [3-1]:

A quasi-closed set is the complement of a quasi-open set.

From this definition and Thm. (1.1.4) and its results we conclude the following results.

1.1.8 Corollary:

In a bitopological space (X,τ_1,τ_2):

a) Every τ_1- closed (τ_2-closed) set is quasi-closed.

b) Arbitrary intersection of quasi-closed sets is quasi-closed.

c) Finite union of quasi-closed sets need not be quasi-closed.

d) Every quasi-closed set is the intersection of a τ_1- closed set and a τ_2-closed set.

Proof of (a):

Holds since every τ_1- closed (τ_2-closed) set is the complement of a τ_1-open (τ_2-open) set which is quasi-open.

Proof of (b):

Let $\boldsymbol{F} = \{F_\alpha : \alpha \in \Delta\}$ be any collection of quasi-closed sets, then $\forall\ \alpha \in \Delta$,

$F_\alpha = X \setminus U_\alpha$; U_α is quasi-open . So $\bigcap_{\alpha\in A} F_\alpha = \bigcap_{\alpha\in A} (X\setminus U_\alpha) = X\setminus(\bigcup_{\alpha\in A} U_\alpha)$. But $\bigcup_{\alpha\in A} U_\alpha$ is quasi-open, then $\bigcap_{\alpha\in A} F_\alpha$ is quasi-closed.

Proof of (c):

Consider the topologies in Ex. (1.1.6) and define $F_1 = X \setminus (a,b]$, $F_2 = X \setminus [b,c)$. F_1 and F_2 are quasi-closed, but $F = F_1 \cup F_2 = X \setminus \{b\}$ is not quasi-closed, because $\{b\}$ is not quasi-open .

Proof of (d):

Let F be quasi-closed, then $F=X \setminus U$; where U is quasi-open. But $U = A_1 \cup A_2$, where A_1 is τ_1- open and A_2 is τ_2-open. Then we have $X\setminus A_1$ is τ_1- closed and $X\setminus A_2$ is τ_2- closed, and $F= X \setminus (A_1 \cup A_2) = (X\setminus A_1) \cap (X\setminus A_2)$, which is the intersection of a τ_1-closed set $X\setminus A_1$ and a τ_2- closed set $X\setminus A_2$.

1.1.9 Definition [3-1]:

The quasi-closure of $A \subset (X,\tau_1,\tau_2)$; denoted by $q(\bar{A})$ is defined to be $Cl\tau_1(A) \cap Cl\tau_2(A)$; where $Cl\tau_i(A)$ denotes the closure of A in τ_i (i =1,2).

1.1.10 Theorem [3-1]:

If $A \subset (X,\tau_1,\tau_2)$, then $q(\bar{A})$ is the smallest quasi- closed set containing A.

Proof:

$q(\bar{A})$ is quasi-closed set containing A , as it is the intersection of two quasi-closed sets each of which contains A . Assume that there is a quasi- closed set B containing A. By Cor. (1.1.8-d), $B = B_1 \cap B_2$; where B_1 is τ_1- closed and B_2 is τ_2- closed. Since

$B \supset A$, then $B_1 \supset A$ and $B_2 \supset A$. Since $Cl\tau_1(A)$ is the smallest closed set containing A, then $Cl\tau_1(A) \subset B_1$. Similarly, $Cl\tau_2(A) \subset B_2$.Thus

$q(\bar{A}) = Cl\tau_1(A) \cap Cl\tau_2(A) \subset B_1 \cap B_2 = B$. Hence $q(\bar{A})$ is the smallest quasi- closed set containing A.

1.1.11 Lemma:

Let A be a subset of a bitopological space (X, τ_1, τ_2), and let $x \in X$. Then $x \in q(\bar{A})$ if and only if every quasi - open set containing x intersects A.

Proof:

($\rightarrow$) Let $x \in q(\bar{A})$, and let U be any quasi-open set containing x. Then there exists a τ_1- open nhood or a τ_2-open nhood W of x such that $x \in W \subset U$. Since $x \in q(\bar{A}) = Cl\tau_1(A) \cap Cl\tau_2(A)$, then $W \cap A \neq \emptyset$, which implies that $U \cap A \neq \emptyset$.

($\leftarrow$) We prove the converse by contrapositive. Assume that $x \notin q(\bar{A})$, then the set $U = X \backslash q(\bar{A})$ is a quasi-open set containing x that does not intersect A.

1.2 Continuity in bitopological spaces.

1.2.1 Definition [3-1]:

Let f: $(X, \tau_1, \tau_2) \rightarrow (X`, \tau_1`, \tau_2`)$ be a function, then:

a) f is said to be quasi-continuous if the inverse image of every quasi-open set is quasi - open .

b) If f is continuous as a function from (X,τ_1) into $(X`,\tau_1`)$ or f is continuous as a function from (X,τ_2) into $(X`,\tau_2`)$, then f is called semi continuous (denoted by s – continuous).

c) If for each $U \in \tau_1` \cup \tau_2`$; the inverse image of U is in $\tau_1 \cup \tau_2$ (i.e $f^{-1}(U) \in \tau_1 \cup \tau_2$ for each $U \in \tau_1` \cup \tau_2`$;), then f is called pairwise–continuous function (denoted by p-continuous).

d) If f is continuous (resp. open) as a function from (X,τ_1) into $(X`,\tau_1`)$ and f is continuous (resp. open) as a function from (X,τ_2) into $(X`,\tau_2`)$, then f is called continuous (resp. open) function.

Here are some results concerning the above definitions.

1.2.2 Theorem:

Every continuous function f: $(X,\tau_1,\tau_2) \rightarrow (X`,\tau_1`,\tau_2`)$ is p-continuous.

Proof:

Let $U \in \tau_1` \cup \tau_2`$. Then $U \in \tau_1`$ or $U \in \tau_2`$, therefore $f^{-1}(U) \in \tau_1$ or $f^{-1}(U) \in \tau_2$ as f is continuous ;Hence $f^{-1}(U) \in \tau_1 \cup \tau_2$.

The following example shows that the converse of the above theorem is not true.

1.2.3 Example:

Let f: $(\Re, \tau_1,\tau_2) \rightarrow (\Re, \tau_1,\tau_2)$; where τ_1 is the left ray topology and τ_2 is the right ray topology. Consider $f(x) = -x$. Then $f^{-1}(a,\infty) = (-\infty, -a) \in \tau_1$ and $f^{-1}(-\infty, a) = (-a, \infty) \in \tau_2$, for all $a \in \Re$. Also $f^{-1}(\Re) = \Re$ and $f^{-1}(\emptyset) = \emptyset$. Thus f is p- continuous. Observe that $f^{-1}(a,\infty) \notin \tau_2$ and so f is not continuous from $(\Re,\tau_2)$ into $(\Re,\tau_2)$. Hence f is not continuous.

1.2.4 Theorem [3-1]:

Let f: $(X,\tau_1,\tau_2) \rightarrow (X',\tau_1',\tau_2')$ be quasi-continuous. Let $G \subset X$. Then $f(q(\overline{G})) \subset q\,\overline{[f(G)]}$.

Proof:

Let $x \in q(\overline{G}) = Cl\tau_1(G) \cap Cl\tau_2(G)$. Let U be any τ_1'-open set of $f(x)$ in X'. Since f is quasi-continuous, $f^{-1}(U)$ is quasi-open in X and $x \in f^{-1}(U)$. Therefore there exists either τ_1–open nhood or τ_2–open nhood W of x such that $W \subset f^{-1}(U)$. W meets G (i.e $W \cap G \neq \emptyset$) by lemma (1.1.11). Hence $f^{-1}(U) \cap G \neq \emptyset$, and so $U \cap f(G) \neq \emptyset$. Hence $f(x) \in Cl\tau_1'(f(G))$. Similarily $f(x) \in Cl\tau_2'(f(G))$.

Thus $f(q(\overline{G})) \subset q(\overline{f(G)})$.

1.3 l. u. b. topology.

1.3.1 Definition of l.u.b topology [12]:

Let τ_1 and τ_2 be two topologies on X. Then $\tau_1 \cup \tau_2$ forms a subbase for some topology on X; this topology is called the least upper bound topology and is denoted by $(X,<\tau_1,\tau_2>)$. Each basic open set B in $(X,<\tau_1,\tau_2>)$ has the form $B = \bigcap_{i=1}^{n} B_i$; where $B_i \in \tau_1$ or $B_i \in \tau_2$ for all $i = 1,2,\ldots,n$. The intersection of the B_i's which are in τ_1, is in τ_1 and the intersection of the B_i's which are in τ_2 is in τ_2 . So $B = U \cap V$ where $U \in \tau_1$ and $V \in \tau_2$.

1.3.2 Definition [3-1]:

A subset A of (X,τ_1,τ_2) is said to be s-open if it is open in the l.u.b. topological space $(X,<\tau_1,\tau_2>)$.

1.3.3 Theorem:

Every quasi-open set in a bitopological space is an s - open set.

Proof:

By Thm. (1.1.4), if A is quasi-open in (X,τ_1,τ_2), then $A = A_1 \cup A_2$; $A_1 \in \tau_1$ and $A_2 \in \tau_2$. $A_1 \in \tau_1 \cup \tau_2 \subset <\tau_1,\tau_2>$, and $A_2 \in \tau_1 \cup \tau_2 \subset <\tau_1,\tau_2>$, thus $A = A_1 \cup A_2 \in <\tau_1,\tau_2>$.

1.3.4 Definition [3-1]:

A subset A of (X,τ_1,τ_2) is said to be s-closed if it is the complement of an s-open set.

1.3.5 Theorem [12]:

A bitopological space (X,τ_1,τ_2) is s-compact if and only if $(X,<\tau_1,\tau_2>)$ is compact.

Proof:

Let (X,τ_1,τ_2) be an s-compact space, and let υ be any $<\tau_1,\tau_2>$ - open cover for X consisting of subbasic open sets. We will use the fact that [X is compact if and only if every subbasic open cover for X has a finite subcover] [11;page 129]. Then $\upsilon \subset \tau_1 \cup \tau_2$, so υ is $\tau_1\tau_2$ –open cover for X. Thus υ has a finite subcover for X. which means that $(X, <\tau_1,\tau_2>)$ is compact.

Conversely, let $(X, <\tau_1,\tau_2>)$ be compact, and let υ be $\tau_1\tau_2$ – open cover for X. Then $\upsilon \subset \tau_1 \cup \tau_2 \subset <\tau_1,\tau_2>$, and so υ has a finite subcover for X. Hence (X,τ_1,τ_2) is s-compact.

In fact, Datta in [3-1] uses the following definition for semi compactness of a set of a bitopological space (X,τ_1,τ_2).

1.3.6 Definition [3-1]:

A subset $A \subset (X,\tau_1,\tau_2)$ is said to be semi compact if it is compact in the l.u.b. topology of τ_1 and τ_2. In other words, A is semi compact if and only if, given any covering of A by semi -open subsets of X there exists a finite subcovering.

1.3.7 Theorem:

Every s - closed (and therefore every τ_1-closed, τ_2 - closed and quasi-closed) subset of s-compact bitopological space is s-compact.

Proof:

Let A be s-closed subset of the s-compact bitopological space (X,τ_1,τ_2). Let υ be any $\tau_1\tau_2$ – open (s-open) cover for A. Since A is s-closed, then $X\backslash A$ is s-open and so the collection $\{X\backslash A\} \cup \{U: U \in \upsilon\}$ forms an s-open cover for X. Since X is s-compact then X has a finite subcover; say $\{X\backslash A\} \cup \{U_i : U_i \in \upsilon\ \forall\ i = 1,2,\ldots,n \}$. (We add $X\backslash A$ to the cover if necessary). Therefore $\{U_i : i = 1,2,\ldots,n$ and $U_i \in \upsilon\}$ forms a finite subcover of υ for A. Hence A is s-compact.

1.3.8 Theorem [3-1]:

The quasi-continuous image of s-compact bitopological space is s-compact.

Proof:

Let $f: (X,\tau_1,\tau_2) \rightarrow (X`,\tau_1`,\tau_2`)$ be an onto quasi-continuous function. Let $(X,\tau_1,\tau_2)$ be s-compact, and $(X`,\tau_1`,\tau_2`)$ be an arbitrary bitopological space. We shall show that $(X`,\tau_1`,\tau_2`)$ is s-compact,(by showing that $(X`,<\tau_1`,\tau_2`>)$ is compact using Thm(1.3.5).

To do so, let $\mathbf{V} = \{U_\alpha : \alpha \in A\}$ be a covering of $(X`,<\tau_1`,\tau_2`>)$; where each $U_\alpha$ is s-open and so of the form $U_\alpha = \bigcup_{i,j} (V_{\alpha_i} \cap W_{\alpha_j})$ ; where $V_{\alpha_i}$ is $\tau_1`$-open and W_{α_j} is $\tau_2`$-open. Then $f^{-1}(U_\alpha) = f^{-1}(\bigcup_{i,j}(V_{\alpha_i} \cap W_{\alpha_j}) = \bigcup_{i,j}(f^{-1}(V_{\alpha_i}) \cap f^{-1}(W_{\alpha_j}))$. Now, V_{α_i} is quasi-open, so $f^{-1}(V_{\alpha_i})$ is quasi - open and hence s- open . Similarily, $f^{-1}(W_{\alpha_j})$ is s-open . Hence, $f^{-1}(U_\alpha)$ is s-open. Therefore the collection

$\mathbf{U} = \{ f^{-1}(U_\alpha): \alpha \in A\}$ is s-open covering of $(X, <\tau_1,\tau_2>)$. Since $(X, <\tau_1,\tau_2>)$ is s-compact, there exists a finite subcovering of $\mathbf{U}$; say $\{ f^{-1}(U_{\alpha_k}) : \alpha_k \in A$ and $k=1,2,..,n \}$, for some $n \in \mathcal{N}$, covers X, which implies that $\{ U_{\alpha_k} ; \alpha_k \in A; k=1,2,..,n\}$, for some $n \in \mathcal{N}$, covers X`, which means that $(X`,<\tau_1`,\tau_2`>)$ is s- compact.

1.3.9 Theorem [12]:

If f is a p-continuous function from (X,τ_1,τ_2) into $(X`,\tau_1`,\tau_2`)$, then $f$ is continuous as a function from $(X,<\tau_1,\tau_2>)$ into $(X`,<\tau_1`,\tau_2`>)$.

Proof:

Let U be any subbasic open set in $(X`,<\tau_1`,\tau_2`>)$. Then $U \in \tau_1` \cup \tau_2`$, and so, $f^{-1}(U) \in \tau_1 \cup \tau_2$. But $\tau_1 \cup \tau_2 \subset <\tau_1,\tau_2>$, therefore $f$ is a continuous function from $(X,<\tau_1,\tau_2>)$ into $(X`,<\tau_1`,\tau_2`>)$.

The following example shows that the converse of the above theorem is not true:

1.3.10 Example:

Let f: $(\Re, \tau_1,\tau_2) \rightarrow (\Re, \tau_1,\tau_2)$, where $f(x) = | x |$, τ_1 be the left ray topology, and τ_2 be the right ray topology. Then f is continuous as a function from $(\Re,<\tau_1,\tau_2>)$ to itself because $<\tau_1,\tau_2>$ is the usual topology on $\Re$. Since $(1, \infty) \in \tau_2$ and $f^{-1}((1,\infty)) = (1, \infty) \cup (-\infty,-1) \notin \tau_1 \cup \tau_2$, therefore f is not p-continuous.

1.3.11 Theorem [12]:

The p-continuous image of s-compact bitopological space is s-compact.

Proof:

Assume that (X,τ_1,τ_2) is s-compact and the function f: $(X,\tau_1,\tau_2) \rightarrow (X`,\tau_1`,\tau_2`)$ is an onto p-continuous. Then by Thm. (1.3.5), $(X,<\tau_1,\tau_2>)$ is compact and by Thm. (1.3.9). $f$ is continuous as a function from $(X,<\tau_1,\tau_2>)$ onto $(X`,<\tau_1`,\tau_2`>)$. So, $(X`,<\tau_1`,\tau_2`>)$ is compact which implies that $(X`,\tau_1`,\tau_2`)$ is s-compact.

From the above theorem and Thm. (1.2.2), we see that the continuous image of s-compact bitopological space is s-compact.

1.4 Hausdorffness in bitopological spaces.

1.4.1 Definition [3-1]:

The bitopological space (X,τ_1,τ_2) is said to be quasi -Hausdorff if given $x_1 \neq x_2$, there exist quasi-open sets U_1 and U_2 s.t. $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

1.4.2 Definition [9]:

The bitopological space (X,τ_1,τ_2) is called pairwise Hausdorff (p-Hausdorff) if given $x_1 \neq x_2$ there exist a τ_1-open set U_1 and a τ_2-open set U_2 containing x_1 and x_2 respectively such that $U_1 \cap U_2 = \emptyset$.

From the above definitions we can conclude the following:

1.4.3 Theorem:

Every p-Hausdorff bitopological space is also quasi- Hausdorff.

Proof:

This conclusion holds since each τ_1, τ_2 - open sets are quasi-open.

The converse of the above theorem need not be true as the following example shows.

1.4.4 Example:

Consider the bitopological space ($\mathfrak{R}$, τ_1, τ_2), where $\mathfrak{R}$ is the set of real numbers, τ_1 is the left ray topology and τ_2 is the right ray topology. Now, let $x, y \in \mathfrak{R}$ and $x \neq y$. Define $z = (x+y)/2$. If $x < y$, then $x \in (-\infty, z) \in \tau_1$, $y \in (z, \infty) \in \tau_2$.

$(-\infty, z) \cap (z, \infty) = \emptyset$, and both $(-\infty, z)$, (z, ∞) are quasi-open. A similar argument can be done if $x > y$, *so*, ($\mathfrak{R}$, τ_1, τ_2) is quasi-Hausdorff. But ($\mathfrak{R}$, τ_1, τ_2) is not p-Hausdorff, because if $x > y$, we cannot find two disjoint open sets U_1 and U_2 such that

$x \in U_1 \in \tau_1$, $y \in U_2 \in \tau_2$ and $U_1 \cap U_2 = \emptyset$.

1.4.5 Theorem [12]:

If the bitopological space (X,τ_1,τ_2) is p-Hausdorff, then $(X,<\tau_1,\tau_2>)$ is Hausdorff.

Proof:

Let (X, τ_1,τ_2) be p-Hausdorff, and let $x, y \in X$ s.t. $x \neq y$. Then there exists $U \in \tau_1$ and $V \in \tau_2$ s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $\tau_1 \cup \tau_2 \subset <\tau_1,\tau_2>$, therefore

$U \in <\tau_1,\tau_2>$ and $V \in <\tau_1,\tau_2>$. Thus $(X,<\tau_1,\tau_2>)$ is Hausdorff.

The converse of the above theorem is not true, because as in Ex. (1.4.4), the usual topology ($\mathfrak{R}$, $<\tau_1, \tau_2>$) is Hausdorff but ($\mathfrak{R}$, τ_1, τ_2) is not p- Hausdorff.

1.4.6 Theorem:

An s-compact subset A of a p-Hausdorff bitopological space (X,τ_1,τ_2) is s-closed (i.e closed in $(X,<\tau_1,\tau_2>)$).

Proof:

Let $p \in X \setminus A$. Then $\forall\, x \in A$, there exists $U_x \in \tau_1$ and $V_x \in \tau_2$ s.t. $x \in U_x$, $p \in V_x$ and $U_x \cap V_x = \emptyset$. Now, let $\mathbf{U} = \{U_x : x \in A\}$. Then $\mathbf{U}$ is a $\tau_1\tau_2$-open cover for A. Since A is s-compact, then $\mathbf{U}$ has a finite subcover, say $\mathbf{U}` = \{U_{x_i} : i=1,2,\ldots,n\}$.

Let $W = \bigcap_{i=1}^{n} V_{x_i}$. Then $p \in W \in \tau_2$, so W is quasi-open which implies that W is s-open.

Let $U = \bigcup_{i=1}^{n} U_{x_i}$, then $U \in \tau_1$ and this implies that U is s-open. Also, $W \cap U = \emptyset$. Hence $W \cap A = \emptyset$. Consequently, $X \setminus A$ is an s-open set , and thus A is s-closed.

1.4.7 Theorem [3-1]:

Let (X,τ_1,τ_2) be quasi-Hausdorff bitopological space. Let U_1 and U_2 be quasi-open sets such that $U_1 \cap U_2 = \emptyset$. Then $q(\overline{U_1}) \cap U_2 = \emptyset$.

Proof:

Let U_1 and U_2 be quasi-open sets such that $U_1 \cap U_2 = \emptyset$. We shall show that $q(\overline{U_1}) \cap U_2 = \emptyset$. Suppose that $q(\overline{U_1}) \cap U_2 \neq \emptyset$, then $\exists\, y \in q(\overline{U_1}) \cap U_2$.

$y \in q(\overline{U_1})$ implies that every τ_1 –open nhood and τ_2 –open nhood of y meets U_1 . But U_2 is a quasi-open set containing y and so there exists either τ_1 –open nhood or τ_2 –open nhood W of y s.t. $W \subset U_2$. But $U_1 \cap U_2 = \emptyset$, so $W \cap U_1 = \emptyset$. This contradiction establishes that $q(\overline{U_1}) \cap U_2 = \emptyset$.

In the above theorem we can omit the word quasi –Hausdorff, because we do not use it in the proof. So, the above theorem can be restated as follows:

[Let (X,τ_1,τ_2) be a bitopological space. Let U_1 and U_2 be quasi-open sets such that $U_1 \cap U_2 = \emptyset$. Then $q(\overline{U_1}) \cap U_2 = \emptyset$].

1.4.8 Theorem [3-1]:

An s-compact subset A of a quasi-Hausdorff bitopological space (X,τ_1,τ_2) is s-closed.

Proof:

We shall show that $X \setminus A$ is semi-open. Let $x \in X\setminus A$, then $x \notin A$, so $\forall$ a $\in A$, there exists disjoint quasi-open sets V (a) and U (a) containing a and x respectively,(as (X,τ_1,τ_2) is quasi - Hausdorff). The family $\boldsymbol{F} = \{V(a): a \in A\}$ forms a quasi-open cover for A, and so $\boldsymbol{F}$ is an s-open cover for A (because: every quasi-open set is s-open). Since A is s-compact, then there exists a finite subcover, say $\{V(a_i): i = 1,\ldots,n\}$ for some $n \in \mathcal{N}$ that covers A . Define $U = \bigcap_{i=1}^{n} U(a_i)$, where $U(a_i)$ is the corresponding s-open set which is disjoint from $V(a_i)$. Then:

a) U is s-open and containing x.

b) U does not intersect any of $V(a_i)$; $i=1,2,\ldots,n$, as $U \subset U(a_i)$ $\forall$ $i=1,2,\ldots,n$. So $x \in U$, $A \subset \bigcup_{i=1}^{n} V(a_i)$ and $U \cap (\bigcup_{i=1}^{n} V(a_i)) = \emptyset$. Which means that $x \in U \subset X \setminus A$. So $X \setminus A$ is s – open.

1.4.9 Theorem [12]:

The union of two s-compact subspaces of a bitopological space is s-compact.

Proof:

Let A and B be two s-compact subspaces of the bitopological space (X,τ_1,τ_2). Then, by Def. (1.3.6), A and B are compact subspaces of $(X,<\tau_1,\tau_2>)$ and thus $A \cup B$ is a compact subspaces of $(X, <\tau_1,\tau_2>)$. This implies, by the same definition, that $A \cup B$ is an s-compact subspace of (X,τ_1,τ_2).

1.5 Product topology in bitopological spaces.

In single topology we have, if $\{(X_i,\tau_i) : i\in I\}$ is a family of topological spaces, then the product topology $(\prod_{i\in I} X_i , \tau)$ is the topology generated by the collection $\{ \pi_i^{-1}(U): U\in \tau_i ; i \in I \}$ as a subbase; where π_i is the natural projection from $(\prod_{i\in I} X_i,\tau) \to (X_i,\tau_i)$. In bitopological spaces we have the following analogous definition.

1.5.1 Definition [3-1]:

Let $\{(X_i,\tau_i,\tau_i`) : i\in I\}$ be a family of bitopological spaces. We construct in a natural way two topologies on the product set $X = \prod_{i\in I} X_i$ .Let $\tau$ be the product topology on X determined by the $\tau_i$'s , and let $\tau`$ be the product topology on X determined by the

τ_i`'s.The resulting bitopological space ( X,$\tau$,$\tau$`) will be called the product bitopological space generated by the family {(X_i,τ_i,τ_i`) : i∈ I}.

1.5.2 Theorem [3-1]:

The natural projection from a product bitopological space (X,τ,τ`) on the component bitopological space ( $X_i$,$\tau_i$,$\tau_i$`); i∈ I is quasi-continuous.

Proof:

Let π_i denote the projection from X onto X_i . Let U be any quasi-open set of X_i. Then U = V ∪ W; where V is τ_i- open and W is τ_i`-open. $\pi_i^{-1}$(U)= $\pi_i^{-1}$(V)∪$\pi_i^{-1}$(W). But $\pi_i^{-1}$ (V) is $\tau$-open and $\pi_i^{-1}$ (W) is $\tau$`-open. Therefore π_i^{-1} (U) is quasi-open in (X,τ,τ`). Hence π_i is quasi-continuous.

1.5.3 Theorem [3-1]:

The product bitopological space (X,τ,τ`) of the family of bitopological spaces {( $X_i$,$\tau_i$,$\tau_i$`) : i∈ I} is s-compact iff every (X_i,τ_i,τ_i`) is s-compact ∀ i ∈ I .

Proof:

Assume that (X,τ,τ`) is s-compact. Since the natural Projection $\pi_i$: X→$X_i$ , ∀i ∈I is quasi – continuous, by Thm. (1.5.2), and since the quasi-continuous image of s-compact bitopological space is s-compact, by Thm. (1.3.2), then ( $X_i$,$\tau_i$,$\tau_i$`) is s-compact ∀i ∈ I .

Conversely, assume that $(X_i,\tau_i,\tau_i`)$ is s-compact $\forall i \in I$ .Let $(X, \delta) = \prod_{i\in I} (X_i,<\tau_i,\tau_i`>)$ be the product space , and let $(X,\tau,\tau`) = \prod_{i\in I} (X_i,\tau_i,\tau_i`)$ be the product bitopological space . Then (X, δ) is compact, since each $(X_i,<\tau_i,\tau_i`>)$ is compact . Since $(X, \delta)$ contains the l.u.b. of $(X,\tau,\tau`)$, then $(X,\tau,\tau`)$ is s-compact.

Chapter Two

Pairwise Compactness

In this chapter, we introduce a new type of compactness in bitopological spaces, and then we derive some related results. Also, we obtain some results that are generalizations of well known results in single topology.

2.1 Pairwise compactness in bitopological spaces.

We begin with Fletcher, Hoyle and Patty definition of pairwise Compactness in the bitopological space. (denoted FHP-compactness).

2.1.1 Definition [2]:

A $\tau_1\tau_2$-open cover υ of a bitopological space (X,τ_1,τ_2) is called p-open cover if υ contains at least one nonempty member of τ_1 and a nonempty member of τ_2.

A bitopological space (X,τ_1,τ_2) is called pairwise compact (denoted p-compact) provided that every p-open cover of X has a finite subcover .

The following theorem illustrates the relation between s-compactness and p-compactness .

2.1.2 Theorem [4]:

The bitopological space (X, τ_1, τ_2) is s-compact if and only if it is p-compact, τ_1-compact and τ_2 –compact.

Proof:

Assume that the bitopological space (X, τ_1, τ_2) is s-compact, and let υ be any p-open cover of the space X , then υ is $\tau_1\tau_2$ –open cover for X . Since X is s-compact, then υ has a finite subcover for X . Thus X is p-compact . Also, let υ be any τ_i – open cover of X, (i =1,2), then $\upsilon \subset \tau_1 \cup \tau_2$, which means that υ is a $\tau_1\tau_2$ – open cover of X . Since (X, τ_1, τ_2) is s-compact, then there is a finite subcover of υ of X , which implies that X is τ_i-compact (i = 1,2). Conversely, assume that (X, τ_1, τ_2) is p-compact, τ_1– compact and τ_2–compact . Let υ be any $\tau_1\tau_2$ -open cover for X, then $\upsilon \subset \tau_1 \cup \tau_2$.

Case 1:

If υ contains at least one nonempty member of τ_2, and at least one nonempty member of τ_1, then υ is p-open. Thus there is a finite subcover of υ for X (as X is p-compact). Hence X is s-compact.

Case 2:

υ is contained entirely in τ_1 or τ_2, then υ is either τ_1–open cover for X or τ_2–open cover for X . In either case, there is a finite subcover of υ for X (as X is τ_1–compact and τ_2 –compact). Hence X is s-compact.

The following example shows that:

" Not every p-compact bitopological space is s-compact ".

2.1.3 Example:

Consider the bitopological space $(\Re, \tau_1, \tau_2)$, where $\Re$ is the set of real numbers, τ_1 is the left ray topology, and τ_2 is the right ray topology. Then $(\Re, \tau_1, \tau_2)$ is p-compact, but not s- compact.

Proof:

Let $\upsilon = \{U_\alpha : \alpha \in \Lambda\}$ be a p-open cover for $\Re$. Then there exist $\beta, \gamma \in \Lambda$ such that $U_\beta \in \tau_1$, $U\gamma \in \tau_2$, $U_\beta \neq \emptyset$ and $U\gamma \neq \emptyset$. If $U_\beta = \Re$ or $U\gamma = \Re$, then υ has a finite subcover for $\Re$, namely $\{ \Re \}$. Otherwise, let $U_\beta = (-\infty, x)$ and $U\gamma = (y, \infty)$ for some x, $y \in \Re$. If $x > y$, then $\{ U_\beta , U\gamma \}$ is a finite subcover of υ for $\Re$. If $x = y$, then there is $\lambda \in \Lambda$ such that $x \in U_\lambda$, and then $\{ U_\beta , U\gamma , U_\lambda \}$ is a finite subcover of υ for $\Re$. Now, let $x < y$. Let $A = \{ z \in [x,y] :$ there is no $\alpha \in \Lambda$ such that $z \in U_\alpha \in \tau_2\}$. If $A = \emptyset$, then $x \in U_\alpha \in \tau_2$ for some $\alpha \in \Lambda$, and then $\{ U_\beta , U_\alpha \}$ is a finite subcover of υ for $\Re$. If $A \neq \emptyset$, then A is bounded above and so, by completeness axiom for $\Re$, it has a least upper bound, say t . Then $x \leq t \leq y$. Suppose that there exists $\alpha \in \Lambda$ such

That $t \in U_\alpha \in \tau_2$. If $U_\alpha = \Re$, then υ has a finite subcover for $\Re$, Namely$\{\Re\}$. Otherwise $U_\alpha = (z, \infty)$ for some $z \in \Re$. Then $z < t$, and so there exists $w \in \Re$ such that $z < w < t$. It is clear that there exists $\lambda \in \Lambda$ such that $w \in U_\lambda \in \tau_1$, and then $\{U_\alpha, U_\lambda\}$ is a finite subcover of υ for $\Re$. Suppose now that there exists no $\alpha \in \Lambda$ such that $t \in U_\alpha \in \tau_2$, then there exists $\alpha \in \Lambda$ such that $x \in U_\lambda \in \tau_1$. If $U_\alpha = \Re$, then υ has a finite subcover for $\Re$, namely $\{\Re\}$. Otherwise $U_\alpha = (-\infty, z)$ for some $z \in \Re$. Then $t < z$, and so there exists $w \in \Re$ such that $t < w < z$. By definition of A and **t**, there exists $\lambda \in \Lambda$ such that $w \in U_\lambda \in \tau_2$, and then υ has $\{U_\alpha, U_\lambda\}$ as a finite subcover for $\Re$. Hence, $(\Re, \tau_1, \tau_2)$ is p-compact.

However $(\Re, \tau_1, \tau_2)$ is not s-compact, for $(\Re, \tau_1)$ is not compact.

The following example shows that if the bitopological space (X, τ_1, τ_2) is τ_i-compact, i=1,2, then it is not necessarily that it is s-compact.

2.1.4 Example [1]:

Let $X = [0,1]$, $\tau_1 = \{\emptyset, X\} \cup \{[0,b) : b \in X\}$, $\tau_2 = \{\emptyset, X, \{1\}\}$. Every τ_1–open cover υ for X must contain X, as X is the only τ_1–open set that contains 1. So $\{X\}$ is a finite subcover of υ for X. So (X,τ_1) is compact. Also, (X,τ_2) is compact as τ_2 is finite. However, (X,τ_1,τ_2) is not s-compact: Consider the following $\tau_1\tau_2$-open cover υ for X, where $\upsilon = \{[0,b) \mid b \in X\} \cup \{\{1\}\}$. Suppose there exists a finite subfamily of υ which covers X. This is equivalent to supposing that there is a subfamily

$\{[0,b_i) \mid i=1,2,\ldots,n\}$ of $\{ [0,b) \mid b \in X\}$ that covers $[0,1)$. Now each b_i is in $[0,1)$, so $m = \max \{ b_1 , b_2 ,\ldots, b_n \}$ satisfies $0 < m < 1$, and so $m \notin \bigcup \{ [0, b_i) \mid i=1,2,..,n\}$. Thus (X,τ_1,τ_2) is not s-compact.

Thm.(2.1.2) shows that the bitopological space (X,τ_1, τ_2) in the last example is not p-compact. Hence, not every compact bitopological space (X,τ_1, τ_2) [i.e. (X,τ_1) and (X,τ_2) are compact] is p-compact .

We now obtain two alternative characterizations for pairwise compactness, which were introduced by Kim [5-1], and Kim and Naimpally [5-2]. We need first the notion of the adjoint topology, which was introduced also by Kim [5-1]. And then we show in Thm. (2.1.6) that these definitions are equivalent.

2.1.5 Lemma [5]:

If τ is a topology on X and A is a subset of X, then the collection $\tau (A)$ given by $\tau (A) =\{\emptyset\} \cup \{A \cup U: U \in \tau\}$ is a topology on X, called the adjoint topology of A.

Proof:

i. It is clear that $\emptyset, X \in \tau (A)$.

ii. Let $\mathcal{F} = \{F_i: i = 1, 2,..,n\}$ be a finite collection of members of $\tau(A)$. If one of these members is $\emptyset$, then their intersection is $\emptyset$ which is in $\tau(A)$. Otherwise each member of $\mathcal{F}$ is nonempty and has the form $A \cup U_i$, where

$U_i \in \tau$ for each i = 1,2,..,n . So $\bigcap_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (A \cup U_i) = A \cup (\bigcap_{i=1}^{n} U_i) \in \tau(A)$,

because $\bigcap_{i=1}^{n} U_i \in \tau$.

iii. Consider an arbitrary collection of members of $\tau(A)$. If each of these members is Ø, then their union is Ø which is in $\tau(A)$. Otherwise each member of this collection, after neglecting the empty sets, is of the form $A \cup U$ for some $U \in \tau$.

Let this collection be $\mathcal{F} = \{A \cup U_\alpha : \alpha \in \Lambda\}$, where $U_\alpha \in \tau$, for all $\alpha \in \Lambda$. Then

$\cup \mathcal{F} = \bigcup_{\alpha\in\Lambda} (A \cup U_\alpha) = A \cup (\bigcup_{\alpha\in\Lambda} U_\alpha)$ which is in $\tau(A)$, because τ is a topology,

and so $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$. Hence $\tau(A)$ is a topology.

2.1.6 Theorem [4]:

For the bitopological space (X, τ_1, τ_2), the following are equivalent:

a) (X, τ_1, τ_2) is p-compact.

b) For each non-empty set V in τ_1, the topology $\tau_2(V)$ is compact, and for each non -empty set V in τ_2, the topology $\tau_1(V)$ is compact. (Kim definition for compactness in bitopological spaces).

c) Each τ_1–closed proper subset of X is τ_2– compact, and each τ_2 – closed proper subset of X is τ_1–compact. (Kim and Naimpally definition for compactness in bitopological spaces).

Proof:

(a) → (b) :

Let V be any non-empty τ_1- open set, and let υ be a τ_2 (V) open cover of X. So υ = $\{V \cup U_\alpha : \alpha \in \Lambda\}$ where $U_\alpha \in \tau_2$ for each $\alpha \in \Lambda$. We show that $\{V\} \cup \{U_\alpha : \alpha \in \Lambda\}$ is p-open cover of X. V is a nonempty τ_1- open member of υ. If V = X, then V is also a nonempty τ_2 –open member of υ. If $V \neq X$, then $U_\alpha \neq \emptyset$ for some $\alpha \in \Lambda$, and so U_α is a nonempty τ_2 –open member of υ. And so $\{V\} \cup \{U_\alpha : \alpha \in \Lambda\}$ has a finite subcover for X which we denote by $\{V\} \cup \{U_{\alpha_i} : i = 1,2\ldots,n\}$.We add {V} to the subcover if necessary . Then $\{V \cup U_{\alpha_i} : i = 1,2\ldots,n\}$ is the desired finite subcover of υ for X. So τ_2 (V) is compact. Similarly, we show that τ_1(V) is compact for each non-empty set V in τ_2.

(b) → (c):

Let K be any proper τ_1. closed subset of X, then V = X\K is a nonempty τ_1-open set. Let $\{U_\alpha : \alpha \in \Lambda\}$ be τ_2-open cover of K . Then $\{V \cup U_\alpha : \alpha \in \Lambda\}$ is a τ_2 (V) open cover of X , so we have $X = V \cup [\cup\{U_{\alpha_i} : i = 1,2\ldots,n\}]$ for some integer n, and $\alpha_i \in \Lambda$ for each i=1,2,..,n. . Then $K \subset \cup\{U_{\alpha_i} : i = 1,2\ldots,n\}$ as $K \cap V = \emptyset$. So K is τ_2 - compact. Similarly, we show that each τ_2- closed proper subset of X is τ_1- compact.

(c)→ (a):

Let υ be p-open cover of X. Let the τ_1. open sets in υ be $\{U_\beta : \beta \in \Lambda_1\}$, and let the τ_2 - open sets in υ be $\{V_\alpha : \alpha \in \Lambda_2\}$. Then two cases arise:

(i) $\cup\{V_\alpha : \alpha \in \Lambda_2\} = X$. Choose $\beta_0 \in \Lambda_1$ such that $U_{\beta_0} \neq \emptyset$. Then $\{V_\alpha : \alpha \in \Lambda_2\}$ is a τ_2 - open cover of the τ_1- closed proper subset $X \setminus U_{\beta_0}$. So there is a finite subcover $\{V_{\alpha_i} : i = 1,2\ldots,n\}$, such that

$X \setminus U_{\beta_0} \subset \bigcup \{ V_{\alpha_i} : i = 1,2\ldots,n\}$. Then $\{ U_{\beta_0}, V_{\alpha_i}, \ldots, V_{\alpha_n} \}$ is a finite subcover of υ for X.

(ii) $\bigcup \{ V_\alpha : \alpha \in \Lambda_2\} \neq X$. Then $K = X \setminus \bigcup \{ V_\alpha : \alpha \in \Lambda_2\}$ is a proper τ_2 –closed subset of X , and $K \subset \bigcup \{U_\beta : \beta \in \Lambda_1\}$. Hence there is a finite subcover $\{ U_{\beta_i} : i = 1,2,.., m \}$ of $\{ U_\beta : \beta \in \Lambda_1 \}$ such that $K \subset \bigcup \{ U_{\beta_i} : i = 1,2,.., m\}$, and $\beta_i \in \Lambda_1 \ \forall \ i = 1,2,\ldots,m$.

If $\bigcup \{ U_{\beta_i} : i = 1,2,.., m \} = X$, then there is nothing to prove.

If $\bigcup \{ U_{\beta_i} : i = 1,2,.., m\} \neq X$, then $X \setminus \bigcup \{ U_{\beta_i} : i = 1,2,.., m \}$ is a proper τ_1-closed subset of X contained in $\bigcup \{ V_\alpha : \alpha \in \Lambda_2\}$. By hypothesis, there is a finite subcover $\{ V_{\alpha_j} : j = 1,2\ldots,p\}$ with $[X \setminus \bigcup \{ U_{\beta_i} : i = 1,2,.., m \}] \subset \cup\{ V_{\alpha_j} : j = 1,2\ldots,p\}$. Then $\{ U_{\beta_i} : i = 1,2,.., m \} \cup \{ V_{\alpha_j} : j = 1,2\ldots,p\}$ is the required finite subcover of υ for X. Hence, (X, τ_1, τ_2) is p-compact.

2.1.7 Theorem [2]:

If (X,τ_1,τ_2) is Hausdorff bitopological space, [i.e. (X,τ_1) and (X,τ_2) are Hausdorff spaces], and (X,τ_1,τ_2) is p-compact, then $\tau_1 = \tau_2$.

Proof:

Let A be a proper τ_1-closed subset of X. By Thm. (2.1.6), A is τ_2-compact, therefore A is τ_2-closed as (X,τ_2) is Hausdroff. But X is τ_2-closed also, so every τ_1-closed subset of X is τ_2-closed. Similarly, we show that every τ_2-closed subset of X is τ_1-closed. Hence $\tau_1 = \tau_2$.

2.1.8 Theorem [2]:

If (X,τ_1,τ_2) is p-Hausdorff and (X,τ_1,τ_2) is compact (i.e (X,τ_1) and (X, τ_2) are compact), then $\tau_1 = \tau_2$.

Proof:

Let A be τ_1-open set. Want to show that A is τ_2-open. Now, X\A is τ_1-closed and so τ_1-compact. Let a be an arbitrary element in A. Since (X,τ_1,τ_2) is p-Hausdorff, $\forall x \in X\backslash A$, there is a τ_1-open set U_x and a τ_2-open set V_x s.t. $x \in U_x$, $a \in V_x$, and $U_x \cap V_x = \emptyset$ .Then the collection $\upsilon = \{ U_x \mid x \in X\backslash A\}$ is a τ_1-open cover for X\A . Therefore υ has a finite subcover $\{ U_{x_i} : i =1,2,..,n\}$ for X\A , for some positive integer n . Then there is a corresponding V_{x_i} which is τ_2-open and $U_{x_i} \cap V_{x_i} = \emptyset$, $\forall$ i =1,2,..,n . Define $V = \bigcap_{i=1}^{n} V_{x_i}$, then V is τ_2-open set and $a \in V$. Also $V \cap (X\backslash A) = \emptyset$, as $V \cap (\bigcup_{i=1}^{n} U_{x_i}) = \emptyset$. So, $a \in V \subset A$, this implies that A is τ_2-open .In a similar argument , we can show that every τ_2-open set is τ_1-open. Hence $\tau_1 = \tau_2$.

The following theorem represents the disadvantage of semi compactness, that is, when the bitopological space is s- compact and p-Hausdorff, then we are talking about the same topology.

2.1.9 Theorem [12]:

If the bitopological space (X,τ_1,τ_2) is p- Hausdorff and s-compact, then $\tau_1 = \tau_2$.

Proof:

Since s-compactness of (X,τ_1,τ_2) implies that (X,τ_1) and (X,τ_2) are compact by Thm. (2.1.2), then the result holds by Thm. (2.1.8).

We now give an example of a p-compact, p-Hausdorff bitopological space (X,τ_1,τ_2) such that $\tau_1 \neq \tau_2$.

2.1.10 Example:

Let X be the nonnegative reals, let τ_1 be the usual topology on X, and let

$\tau_2 = \{\emptyset\} \cup \{U \cup (x,\infty) \mid U \in \tau_1 \text{ and } x \in X\}$. We show first that τ_2 is a topology.

Proof:

I. $\emptyset \in \tau_2$. Since $X \in \tau_1$ and for each $x \in X$, $X \cup (x, \infty) = X$, then $X \in \tau_2$.

II. Let $\mathcal{F} = \{ F_i : i=1,2,..,n\}$ be a finite collection of elements of τ_2 . If one of them is $\emptyset$, then their intersection is $\emptyset$ which is in τ_2 . Otherwise, each member of $\mathcal{F}$ is nonempty and thus has the form $F_i = U_i \cup (x_i, , \infty)$; $U_i \in \tau_1$ and $x_i \in X$, $\forall\, i =1,2,..,n$.

So $\bigcap_{i=1}^{n} F_i = \bigcap_{i=1}^{n} [U_i \cup (x_i , \infty)] = (\bigcap_{i=1}^{n} U_i) \cup [\bigcap_{i=1}^{n}(x_i, , \infty)]$. Now,

$\bigcap_{i=1}^{n} U_i \in \tau_1$ as τ_1 is a topology. Also $\bigcap_{i=1}^{n} (x_i, , \infty) = (y, \infty)$, where

$y = \max \{ x_1,\dots, x_n \} \in X$. Then $\bigcap_{i=1}^{n} F_i \in \tau_2$.

III. Consider an arbitrary collection of members of τ_2. If each of these members is $\emptyset$, then their union is $\emptyset$ which is in τ_2. Otherwise after neglecting the empty members, each member of this collection has the form $U \cup (x, \infty)$; where

$U \in \tau_1$ and $x \in \Re$. So the collection is $\mathcal{F} = \{ U_\alpha \cup (x_\alpha, \infty) : \alpha \in \Delta \}$ for some Δ, $U_\alpha \in \tau_1$, and $x_\alpha \in X \ \forall \alpha \in \Delta$. Then:

$$\bigcup \mathcal{F} = \Big(\bigcup_{\alpha\in\Delta} U_\alpha\Big) \cup \Big(\bigcup_{\alpha\in\Delta} (x_\alpha, \infty)\Big) = \Big(\bigcup_{\alpha\in\Delta} U_\alpha\Big) \cup (y, \infty); \quad \text{where}$$

$y = \inf\{ x_\alpha : \alpha \in \Delta \}$. Since τ_1 is a topology, then $\big(\bigcup_{\alpha\in\Delta} U_\alpha\big) \in \tau_1$. So $B \in \tau_2$.

We have $\tau_1 \neq \tau_2$, in fact, τ_1 contains τ_2 properly, as $[0,1)$ is in τ_1 but not in τ_2. To show that (X,τ_1,τ_2) is p-Hausdorff, let $x, y \in X$ such that $x \neq y$.

Let $z = (x + y)/2$. If $x < y$, then $x \in [0, z) \in \tau_1$, $y \in (z, \infty) \in \tau_2$ and $[0,z) \cap (z, \infty) = \emptyset$. If $x > y$, then $x \in (z, x+1) \in \tau_1$, $y \in [0, z) \cup (x+1, \infty) \in \tau_2$ and $(z, x+1) \cap ([0, z) \cup (x+1, \infty)) = \emptyset$. Hence (X,τ_1,τ_2) is p-Hausdorff.

Let υ be p-open cover for X, and let V be a nonempty member of υ such that V is τ_2 – open. Let $a \in X$ such that $(a, \infty) \subset V$. Since $\tau_2 \subset \tau_1$, υ is τ_1-open cover of $[0,a]$, which is τ_1 compact. So υ has a finite subcover υ_1 for $[0,a]$. Therefore, the finite subcollection $\upsilon_1 \cup \{V\}$ of υ covers X, and hence (X,τ_1,τ_2) is p-compact.

2.2 Regularity and normality in bitopological spaces.

The following definition is introduced by Kelly [8].

2.2.1 Definition of p-regularity [8]:

In a bitopological space (X,τ_1,τ_2), τ_1 is said to be regular with respect to τ_2 if for each point $x \in X$ and each τ_1-closed set C such that $x \notin C$, there exist a τ_1-open set U_1 and a τ_2-open set U_2 such that $x \in U_1$, $C \subset U_2$ and $U_1 \cap U_2 = \emptyset$. We say that (X,τ_1,τ_2) is p-regular if τ_1 is regular with respect to τ_2 and vice versa.

2.2.2 Theorem [2]:

If (X,τ_1,τ_2) is a p-Hausdorff and p-compact in the bitopological space, then (X,τ_1,τ_2) is p-regular.

Proof:

Let C be a τ_1-closed subset of X and let $p \in X \setminus C$. Since (X,τ_1,τ_2) is p-Hausdorff , then for each $x \in C$, there is a τ_1-open set U_x and a τ_2-open set V_x s.t. $p\in U_x$, $x\in V_x$, and $U_x \cap V_x = \emptyset$ .Now $\upsilon = \{ V_x | x\in C \}$ is a τ_2-open cover of C. Thus, by Thm. (2.1.6) and p- compactness of (X,τ_1,τ_2) , there is a finite subcollection $\{ V_{x_1}$. $V_{x_2}, \dots, V_{x_n} \}$ of υ which covers C . If $U = \bigcap_{i=1}^{n} U_{x_i}$ and $V= \bigcup_{i=1}^{n} V_{x_i}$ then U is τ_1-open, V is τ_2-open, $p \in U$, $C \subset V$, and $V\cap U = \emptyset$. Therefore τ_1 is regular with respect to τ_2 .

A similar argument shows that τ_2 is regular with respect to τ_1. Hence (X,τ_1,τ_2) is p-regular.

The following definition is introduced by Kelly [8].

2.2.3 Definition of p- normality [8]:

A bitopological space (X,τ_1,τ_2) is said to be pairwise normal (denoted p-normal) if, given a τ_1-closed set A_1 and a τ_2-closed set A_2 with $A_1 \cap A_2 = \emptyset$, there exist a τ_2-open set U_2 and a τ_1-open set U_1 such that $A_1 \subset U_2$, $A_2 \subset U_1$, and $U_1 \cap U_2 = \emptyset$.

2.2.4 Theorem [2]:

If (X,τ_1,τ_2) is p-compact and either τ_1 is regular with respect to τ_2 or τ_2 is regular with respect to τ_1, then (X,τ_1,τ_2) is p-normal.

Proof:

Suppose that τ_1 is regular with respect to τ_2, and let H be a τ_1-closed set and K be a τ_2-closed set such that $H \cap K = \emptyset$. For each $x \in K$ ($x \notin H$) and so, there are a τ_1-open set U_x and a τ_2-open set V_x such that $x \in U_x$, $H \subset V_x$, and $U_x \cap V_x = \emptyset$. Then the collection $\upsilon = \{U_x | x \in K\}$ is a τ_1-open cover of K and so, by Thm. (2.1.6), there is a finite subcollection $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ of υ which covers K. If $U = \bigcup_{i=1}^{n} U_{x_i}$ and $V = \bigcap_{i=1}^{n} V_{x_i}$, then U is τ_1-open, V is τ_2-open, $H \subset V$, $K \subset U$ and $U \cap V = \emptyset$. Hence,

(X,τ_1,τ_2) is p-normal. A similar argument proves the theorem if τ_2 is regular with respect to τ_1.

The following result follows directly from Thm (2.2.2) and Thm. (2.2.4).

2.2.5 Corollary:

If (X,τ_1,τ_2) is a p-Hausdorff and a p-compact, then the bitopological space (X,τ_1,τ_2) is p-normal .

2.3 Pairwise compactness of subspaces of bitopological spaces.

2.3.1 Definition:

Let (X,τ_1,τ_2) be a bitopological space. Then a subset A of X is called p-compact, if the bitopological space $(A, \tau_{1A}, \tau_{2A})$ is p-compact, where

$\tau_{iA} = \{ U_i \cap A : U_i \in \tau_i \}, i = 1,2.$

2.3.2 Definition:

Let A be a subset of the bitopological space (X,τ_1,τ_2), then a $\tau_1\tau_2$- open cover $\upsilon = \{U\alpha: \alpha \in\Lambda\}$ of A is called p-open cover of A in (X,τ_1,τ_2), if $A \subset \bigcup_{\alpha\in\Lambda} U_\alpha$, and if there exists $\beta, \gamma \in \Lambda$ such that $U_\gamma \in \tau_1$, $U_\beta \in \tau_2$, $U_\gamma \cap A \neq \emptyset$ and $U_\beta \cap A \neq \emptyset$.

As in the single topology, we have:

2.3.3 Theorem:

If A is a subset of a bitopological space (X,τ_1,τ_2), then A is p-compact if and only if every p-open cover of A in (X,τ_1,τ_2) has a finite subcover for A.

Proof:

($\rightarrow$) Let A be a p-compact subset of the bitopological space (X,τ_1,τ_2), and let $\upsilon = \{U_\alpha: \alpha \in\Lambda \}$ be a p-open cover of A in (X,τ_1,τ_2). Then the collection $\upsilon` = \{ U_\alpha \cap A : \alpha \in\Lambda \}$ is a p-open cover of A in $( A, \tau_{1A}, \tau_{2A})$. But A is a p-compact subset of $(X,\tau_1,\tau_2)$ , so $( A, \tau_{1A}, \tau_{2A})$ is p-compact, and so $\upsilon`$ has a finite subcover , say $\{ U_{\alpha_i} \cap A :i=1,2,..,n \}$ for some integer $n \in \mathcal{N}$, for A. Then $\{U_{\alpha i} : i=1,2,..,n \}$ is the required finite subcover of υ for A in X.

($\leftarrow$) Want to show that $(A, \tau_{1A}, \tau_{2A})$ is p-compact. Let $\upsilon = \{U_\alpha: \alpha \in\Lambda\}$ be a p-open cover in $(A, \tau_{1A}, \tau_{2A})$ for A . Then there exist $\beta , \gamma \in \Lambda$ such that $U_\gamma \in \tau_{1A}$, $U_\beta \in \tau_{2A}$, $U_\gamma \neq \emptyset$ and $U_\beta \neq \emptyset$. So, there exist $V_\gamma \in \tau_1$ and $V_\beta \in \tau_2$ such that

$U_\gamma = V_\gamma \cap A$ and $U_\beta = V_\beta \cap A$. For each other $\alpha \in \Lambda$, there exists $V_\alpha \in \tau_1$ or $V_\alpha \in \tau_2$ such that $U_\alpha = V_\alpha \cap A$. Now, $\{V_\alpha : \alpha \in \Lambda\}$ is a p-open cover for A in (X,τ_1,τ_2), as $V_\gamma \cap A \neq \emptyset$, $V_\beta \cap A \neq \emptyset$, $V_\gamma \in \tau_1$ and $V_\beta \in \tau_2$. Hence, the collection $\{V_\alpha : \alpha \in \Lambda\}$ has a finite subcover for A, say $\{V_{\alpha_i} : i=1,2,..,n\}$ for some $n \in \mathcal{N}$. Then

$\{U_{\alpha_i} : i=1,2,..,n\}$ is a finite subcover of υ for A.

Hence $(A, \tau_{1A}, \tau_{2A})$ is p-compact, and so A is a p-compact subset of (X,τ_1,τ_2).

2.3.4 Theorem [12]:

Let (X,τ_1,τ_2) be a p-Hausdorff bitopological space, and let A be a p-compact subset of X. Then A is a closed subset of $(X,<\tau_1,\tau_2>)$ (i.e A is s-closed).

Proof:

If $A = \emptyset$, then it is a closed subset of $(X,<\tau_1,\tau_2>)$. Let A be nonempty. If $p \in X\backslash A$, then $\forall x \in A$, there exist $U_x \in \tau_1$ and $V_x \in \tau_2$ such that $x \in U_x$, $p \in V_x$, and $U_x \cap V_x = \emptyset$. Let $y \in A$ be any element, then there exist $U \in \tau_2$ and $V \in \tau_1$ such that $y \in U$, $p \in V$ and $U \cap V = \emptyset$. Now, let $\upsilon = \{U_x : x \in A\} \cup \{U\}$. Then υ is a p-open cover for A in (X,τ_1,τ_2) and hence υ has a finite subcover, say

$\upsilon' = \{U_{x_1}, \dots, U_{x_n}, U\}$, as A is p-compact, by Thm. (2.3.3).

Let $W = \cap\{V_{x_1}, \dots, V_{x_n}, V\}$. Then $W \in (X,<\tau_1,\tau_2>)$, $p \in W$

and $W \cap [U \cup \bigcup_{i=1}^{n} U_{x_i}] = \emptyset$. Hence $W \cap A = \emptyset$. Consequently, $X\backslash A$ is an s-open set, and thus A is s-closed.

In the following example, we show that if A is s-compact and B is p-compact, then $A \cup B$ need not be p-compact.

2.3.5 Example:

Consider the bitopological space $(\mathfrak{R}, \tau_1, \tau_2)$, where τ_1 is the left ray topology and τ_2 is the right ray topology . The subset $A = [-1,1]$ is s-compact, and the subset $B = (2, \infty)$ is p-compact. But $A \cup B$ is not p-compact, because the cover $\{[-1,1]\} \cup \{(2 + 1/n, \infty): n \in \mathcal{N}\}$ is p-open cover for $A \cup B$ in $(\mathfrak{R}, \tau_1, \tau_2)$, which does not have any finite subcover .The proof that $B = (2, \infty)$ is p-compact is similar to the proof that $(\mathfrak{R}, \tau_1, \tau_2)$ is p-compact.

It is clear that the union of a compact space with an s-compact space is compact. The following example shows that the union of a compact space and an s-compact space is not necessarily s-compact, even it is not necessarily p-compact.

2.3.6 Example:

In the bitopological space in Ex. (2.3.5), the subspace $A = [-1,1]$ is s-compact and the subspace $B = [1,3) \cup (3,4]$ is compact. However $A \cup B$ is not p-compact because the cover $\{[-1,3)\} \cup \{(3 + 1/n, 4]: n \in \mathcal{N}, n \neq 1\}$ is p-open cover for $A \cup B$ which does not have any finite subcover, so $A \cup B$ is not p- compact.

2.4 p-compactness of the product topology in bitopological spaces.

2.4.1 Theorem [1]:

Let $\{(X,\tau_i,\tau`_i) : i \in I\}$ be an arbitrary family of nonempty bitopological spaces. Then for each fixed i, the natural projection map, $\pi_i$: $(X,\tau,\tau`) \rightarrow (X_i,\tau_i,\tau`_i)$ is continuous.

Proof:

Let $U \in \tau_i$, then π_i^{-1} (U) is a subbasic open set in (X,τ) , and this implies that π_i: $(X,\tau) \rightarrow (X_i,\tau_i)$ is continuous. Simillary, we see that π_i: $(X,\tau`) \rightarrow (X_i,\tau`_i)$ is continuous. Hence π_i: $(X,\tau,\tau`) \rightarrow (X_i,\tau_i,\tau`_i)$ is continuous.

2.4.2 Theorem [12]:

The continuous image of p-compact bitopological space is p-compact .

Proof:

Suppose that (X,τ_1,τ_2) is p-compact, let $(X`,\tau`_1,\tau`_2)$ be a bitopological space, and let $f$: $(X,\tau_1,\tau_2) \rightarrow (X`,\tau`_1,\tau`_2)$ be a continuous function. Let $\{H_\alpha : \alpha \in \Delta\}$ be p-open cover of $f(X)$ in $(X`,\tau`_1,\tau`_2)$ . Then there exist $\gamma$ , $\beta \in \Delta$ such that $H_\gamma \in \tau`_1$, $H_\beta \in \tau`_2$, $H_\gamma \cap f(X) \neq \emptyset$, and $H_\beta \cap f(X) \neq \emptyset$. So, $f^{-1}(H_\gamma)$ and $f^{-1}(H_\beta)$ are nonempty τ_1-open and τ_2-open sets respectively. So, the family $\{ f^{-1}(H_\alpha): \alpha \in \Delta\}$ is p-open cover of (X,τ_1,τ_2), and so by the p-compactness of (X,τ_1,τ_2), it has a finite subcover of X , say $\{ f^{-1}(H_{\alpha_1}), f^{-1}(H_{\alpha_2}) , \ldots, f^{-1}(H_{\alpha_n}) \}$ for some positive integer n, and $\alpha_i \in \Delta$ for each i =1,2,..,n .The corresponding family { H_{α_1}, H_{α_2} . ....., H_{α_n} } forms the required finite subcover of { $H_\alpha : \alpha \in \Delta$} for $f(X)$ in $(X`,\tau`_1,\tau`_2)$. Hence $f(X)$ is p-compact subset of $(X`,\tau`_1,\tau`_2)$.

2.4.3 Definition:

A family $\mathcal{F}$ of a bitopological space (X,τ_1,τ_2) is said to be p-closed if each member of $\mathcal{F}$ is τ_1-closed or τ_2- closed set, if at least one member of $\mathcal{F}$ is a proper τ_1-closed set, and if at least one member of $\mathcal{F}$ is a proper τ_2- closed set.

2.4.4 Definition [10]:

The family $\mathcal{F}$ = { $F_\alpha : \alpha \in \Delta$ } is said to have the finite intersection property (F.I.P) if and only if every finite nonempty subcollection { F_{α_1} , F_{α_2} , ..., F_{α_n} }of $\mathcal{F}$ has the property that $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \emptyset$.

2.4.5 Theorem [3-2]:

A bitopological space (X,τ_1,τ_2) is p-compact if and only if each family of p-closed subsets of X with F.I.P has nonempty intersection .

Proof:

($\rightarrow$) Let $\mathcal{F}$ = { $F_\alpha : \alpha \in \Delta$ } be a family of p-closed subsets of X with F.I.P. We are to show that $\bigcap_{\alpha\in\Delta} F_\alpha \neq \emptyset$. Suppose not; i.e $\bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$. Then the family

$\upsilon = \{ X \setminus F_\alpha : \alpha \in \Delta \}$ is a family of p-open sets and

$\bigcup_{\alpha\in\Delta} (X \setminus F_\alpha) = X \setminus \bigcap_{\alpha\in\Delta} F_\alpha = X \setminus \emptyset = X$. By p-compactness of (X,τ_1,τ_2), υ has a finite subcover; say $\upsilon_1 = \{ X \setminus F_{\alpha_i} : i=1,2,..,n\}$; $\alpha_i \in \Delta$ $\forall$ $i=1,2,.., n$, for X, and consequently, $\bigcap_{i=1}^{n} F_{\alpha_i} = \bigcap_{i=1}^{n} X \setminus (X \setminus F_{\alpha_i}) = X \setminus (\bigcup_{i=1}^{n} X \setminus F_{\alpha_i}) = X \setminus X = \emptyset$. This contradicts the fact that $\mathcal{F}$ has F.I.P . Hence $\mathcal{F}$ has non- empty intersection .

($\leftarrow$) Let $\upsilon = \{U_\alpha : \alpha \in \Delta\}$ be any p-open cover for X. Then, $\mathcal{F} = \{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of p- closed sets such that $\bigcap_{\alpha\in\Delta} (X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in\Delta} (U_\alpha) = X \setminus X = \emptyset$. Consequently , our hypothesis implies that $\mathcal{F}$ does not have F.I.P . Therefore, there is some finite subcollection of $\mathcal{F}$; say $X \setminus U_{\alpha_1}, X \setminus U_{\alpha_2}, .., X \setminus U_{\alpha_n}$ such that

$\bigcap_{i=1}^{n} (X \setminus U_{\alpha_i}) = \emptyset$, and this implies that

$$\bigcup_{i=1}^{n} U_{\alpha_i} = \bigcup_{i=1}^{n} X \setminus (X \setminus U_{\alpha_i}) = X \setminus (\bigcap_{i=1}^{n} (X \setminus U_{\alpha_i})) = X \setminus \emptyset = X.$$

Thus, $U_{\alpha_1}, U_{\alpha_2}, .., U_{\alpha_n}$ is a finite subcover of υ for X , implying that (X,τ_1,τ_2) is p-compact .

2.4.6 Definition [3-2]:

A family $\mathcal{F}$ of subsets of a bitopological space (X,τ_1,τ_2) is called inadequate if and only if the family does not cover X . The family is called finitely inadequate if and only if no finite subfamily of $\mathcal{F}$ covers X.

2.4.7 Definition [3-2]:

A family $\zeta = \{ C_\alpha : \alpha \in \Delta \}$ of subsets of a bitopological space (X,τ_1,τ_2) is called common if $\bigcap_{\alpha\in\Delta} C_\alpha \neq \emptyset$. ζ is called finitely common if and only if each finite subcollection of ζ has nonempty intersection.

2.4.8 Theorem [3-2]:

Let (X,τ_1,τ_2) be a bitopological space. Then the following are equivalent:

(a) Each finitely inadequate family of p-open sets in X is inadequate.

(b) Each finitely common family of p-closed sets of X is common.

Proof:

Suppose (a), and let $\mathcal{A} = \{ A_\alpha : \alpha \in \Delta \}$ be a finitely common family of p-closed sets in X. Let $\mathcal{A}` = \{ A_{\alpha_i} : i = 1,2,.., n\}$ for some positive integer n, where $\alpha_i \in \Delta$, $\forall$ i =1,2,.., n, be an arbitrary finite subfamily of $\mathcal{A}$. Then

$\bigcap_{i=1}^{n} A_{\alpha_i} \neq \emptyset$. So, $X \setminus \bigcap_{i=1}^{n} A_{\alpha_i} \neq X$ which implies that $\bigcup_{i=1}^{n} (X\setminus A_{\alpha_i}) \neq X$. But $\forall$ i=1,2,.., n , $X\setminus A_{\alpha_i}$ is p-open and $\mathcal{A}`$ is an arbitrary finite subfamily of $\mathcal{A}$, therefore, the collection $\{ X\setminus A_\alpha : \alpha \in \Delta\}$ is finitely inadequate family of p-open sets in X . By the hypothesis, the collection $\{X\setminus A_\alpha : \alpha \in \Delta\}$ is inadequate, implying that the collection $\{A_\alpha : \alpha \in \Delta\}$ has nonempty intersection (i.e. common).

Conversely, suppose (b), and let $\mathcal{U} = \{ U_\alpha : \alpha \in \Delta\}$ be a finitely inadequate family of p-open sets in X. Let $\mathcal{U}` = \{ U_{\alpha_i} : =1,2,.., n\}$ for some positive integer n, where

$\alpha_i \in \Delta, \forall$ i =1,2,.., n, be an arbitrary finite subfamily of $\mathcal{V}$. Then $\bigcup_{i=1}^{n} U_{\alpha_i} \neq X$, so

$X \setminus \bigcup_{i=1}^{n} U_{\alpha_i} \neq \emptyset$, and this implies that $\bigcap_{i=1}^{n} (X \setminus U_{\alpha_i}) \neq \emptyset$. But

$\forall$ i=1,2,.., n , $X \setminus U_{\alpha_i}$ is p-closed and $\mathcal{V}`$ is arbitrary finite subfamily of $\mathcal{V}$, therefore, the collection $\{ X \setminus U_\alpha : \alpha \in \Delta\}$ is common , this means that $\bigcap_{\alpha\in\Delta} X \setminus U_\alpha \neq \emptyset$ implying that $\bigcup_{\alpha\in\Delta} U_\alpha \neq X$.

Since the second condition in Thm. (2.4.5) is equivalent to the condition (b) in the above above theorem, then each of the conditions (a) and (b) in the above theorem is equivalent to p-compactness in (X,τ_1,τ_2).

2.4.9 Lemma [3-2]:

Let $\mathcal{F}$ be a finitely inadequate family of p-open subsets of (X,τ_1,τ_2) . Then there is a maximal finitely inadequate family $\mathcal{D}$ of p-open subsets of (X,τ_1,τ_2) such that $\mathcal{F} \subset \mathcal{D}$.

Proof:

Let ζ be the collection of all finitely inadequate families of p-open sets . Let ζ be ordered by inclusion ; i.e if C_1 , $C_2 \in \zeta$ then $C_1 \leq C_2$ if and only if $C_1 \subset C_2$. Now $\mathcal{F} \in \zeta$. So, by Hausdorff maximal principle, there is a maximal linearly ordered subcollection $\mathcal{A}$ of ζ such that $\mathcal{F} \in \mathcal{A}$. Let $\mathcal{D} = \bigcup \{C : C \in \mathcal{A}\}$. $\mathcal{D}$ is a family of p-open sets, since each member of $\mathcal{A}$ is such a family. We shall show:

(i) $\mathcal{D}$ is finitely inadequate .

(ii) $\mathcal{D}$ is a maximal finitely inadequate family of p-open subsets of (X,τ_1,τ_2) containing $\mathcal{F}$.

(i) Let $D_1, D_2, \ldots, D_k \in \mathcal{D}$. Then for each i , there is some $C_i \in \mathcal{A}$ such that $D_i \in C_i$. Since $\mathcal{A}$ is linearly ordered one of these C_i's , say C_j, contains each of the other C_i's . So , $D_i \in C_j$ for each i=1,2,..,k . Thus $\bigcup_{i=1}^{k} D_i \neq X$, as C_j is finitely inadequate . Hence $\mathcal{D}$ is finitely inadequate .

(ii) By (i) $\mathcal{D}$ is finitely inadequate family. Also, its clear that $\mathcal{D}$ contains $\mathcal{F}$. Suppose that $\mathcal{D}$ is not maximal ; i.e, suppose that there were some open (either τ_1-open or τ_2-open) set $G \notin \mathcal{D}$ such that $\mathcal{D} \cup \{G\}$ is still finitely inadequate. Then $\mathcal{A} \cup \{ \mathcal{D} \cup \{G\} \}$ would by linearly ordered and would properly contain $\mathcal{A}$, contradicting the maximality of $\mathcal{A}$. Thus (ii) must be true, and $\mathcal{D}$ is the required maximal finitely inadequate family containing $\mathcal{F}$.

2.4.10 Lemma [3-2]:

Let (X,τ_1,τ_2) be a bitopological space. Let $\mathcal{D}$ be a maximal finitely inadequate family of p-open sets. If some member of $\mathcal{D}$ contains $\bigcap_{i=1}^{n} G_i$, where each G_i is τ_1-open (τ_2-open) , then $G_k \in \mathcal{D}$ for some k in $\{1,2,\ldots, n\}$.

Proof:

First suppose that n =2. Suppose that $G_1 \notin \mathcal{D}$ and $G_2 \notin \mathcal{D}$. Then by maximality of $\mathcal{D}$, there must be members $A_1, \ldots, A_m$ of $\mathcal{D}$ such that $G_1 \cup A_1 \cup \ldots \cup A_m = X$. Also, there are members $B_1, \ldots, B_n$ of $\mathcal{D}$ such that $G_2 \cup B_1 \cup \ldots \cup B_n = X$.

Claim: $(G_1 \cap G_2) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_n = X$.

Proof of the claim:

Let $x \in X$. If $x \in A_i$ for some i=1,2,..,m , or $x \in B_i$ for some i =1,2,..,n , then $x \in (G_1 \cap G_2) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_n$. Otherwise, $x \in G_1$ and $x \in G_2$, which means that

$x \in (G_1 \cap G_2) \cup A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_n$. This proves the claim.

Now, if $(G_1 \cap G_2)$ is contained in some member of $\mathcal{D}$, then $\mathcal{D}$ is not finitely inadequate which contradicts the assumption . Hence, no member of $\mathcal{D}$ can contain $(G_1 \cap G_2)$. Thus we have proved the result for n = 2. By mathematical induction, we conclude the lemma for any positive integer n.

The following is the generalization of Alexander's Thm. [13].

2.4.11 Theorem [3-2]:

Let $\mathcal{S}(\tau_1)$ be a subbase for τ_1 and $\mathcal{S}(\tau_2)$ be a subbase for τ_2, where (X,τ_1,τ_2) is a bitopological space . If each p-open cover of X consisting sets from $\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)$ has a finite subcover, then X is p-compact.

Proof:

Assume the hypothesis of the theorem, and let $\mathcal{B}$ be a finitely inadequate family of p-open subsets in the bitopological space (X,τ_1,τ_2). Then by Lem. (2.4.9), there is a maximal finitely inadequate family $\mathcal{A}$ of p-open sets of (X,τ_1,τ_2), containing $\mathcal{B}$.

If we show that $\mathcal{A}$ is inadequate then $\mathcal{B}$ will be inadequate and then (X,τ_1,τ_2) will be p-compact , by Thm. (2.4.8) and its remark. The family $(\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A}$ of all members of $\mathcal{A}$ which belong to $(\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2))$ is finitely inadequate. We show now, that $(\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A}$ is p-open . $\mathcal{A}$ is p-open, so $\mathcal{A}$ has a nonempty τ_1-open set, say B. Then some finite intersection of members of $\mathcal{S}(\tau_1)$ is contained in B, and so, by Lem. (2.4.10), one of this finite family belongs to $\mathcal{A}$. Similarly, we show that $\mathcal{A}$ contains a nonempty member of $\mathcal{S}(\tau_2)$ which is τ_2-open. So,

$(\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A}$ is p-open family. Hence, by the hypothesis, the family

$(S(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A}$ is inadequate .To show that $\mathcal{A}$ is inadequate we will show that

$\bigcup\{ A : A \in \mathcal{A} \} = \bigcup\{ A: A \in (\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A} \}$.

Now, $\bigcup\{A : A \in \mathcal{A} \} \supset \bigcup\{ A: A \in (\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A} \}$, because

$(\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A}) \subset \mathcal{A}$. Conversely, since $\mathcal{S}(\tau_1)$, $\mathcal{S}(\tau_2)$ are subbases for τ_1 and τ_2 respectively , each point x of a member A of $\mathcal{A}$ belongs to some finite intersection of members of $\mathcal{S}(\tau_i)$ for some i = 1,2 which is contained in $A \in \mathcal{A}$. By lemma (2.4.10), one of this finite family belongs to $\mathcal{A}$, so,

$x \in \bigcup\{ A : A \in (\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A} \}$. So,

$\bigcup\{A: A \in \mathcal{A} \} \subset \bigcup\{ A: A \in (\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A} \}$, and so

$\bigcup\{ A : A \in \mathcal{A} \} = \bigcup\{ A: A \in (\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A} \}$. But $(\mathcal{S}(\tau_1) \cup \mathcal{S}(\tau_2)) \cap \mathcal{A}$ is inadequate , so is $\mathcal{A}$. This completes the proof of the theorem.

Then $\upsilon' = \{(-\infty,n_k) \times (-\infty,n_k) : k= 1,2,..,m \} \cup \{ (1, \infty) \times (1, \infty) \}$ for some positive integer m . Let $t = \max \{ n_k : k = 1,2,..,m \}$, then $(-1, t+1) \notin \cup \upsilon'$. So, υ' is not subcover of υ for $\Re \times \Re$. Hence υ has no finite subcover for $\Re \times \Re$. i.e

$(\Re\times \Re , \tau_1 \times \tau_1 , \tau_2 \times \tau_2)$ is not p-compact.

Chapter Three

Birsan Compactness

In this chapter we introduce a new definition of compactness in bitopological spaces which is independent from the previous definitions. Also, we deduce some related results and generalizations of some theorems in single topology.

3.1 Conversely compactness and B- compactness in bitopological spaces.

3.1.1 Definition [6]:

We say that $\upsilon_1 = \{V_i : i \in I\}$ is finer than $\upsilon = \{ U_\alpha : \alpha \in A\}$ if for each $i \in I$, there exists $\alpha \in A$ s.t. $V_i \subset U_\alpha$.

The following definition of bitopological compactness is due to Birsan [6].

3.1.2 Definition [6]:

A bitopological space (X,τ_1,τ_2) space is called τ_1-compact with respect to τ_2 if for each τ_1-open cover υ for X , there is a finite family of τ_2-open sets finer than υ and covers X.

The space is called conversely compact if it is τ_1-compact with respect to τ_2 and is τ_2 -compact with respect to τ_1.

3.1.3 Definition:

A bitopological space (X,τ_1,τ_2) space is called τ_1-compact within τ_2 if each τ_1-open cover υ for X has a finite subcover of τ_2-open sets for X. The space is called B-compact if it is τ_1-compact within τ_2 and τ_2 – compact within τ_1.

Ian E. Cook and Ivan E. Reilly in [4], call the τ_1-compact within τ_2,τ_1-compact with respect to τ_2, and refer this definition to Birsan.

In fact, τ_1-compactness of (X,τ_1,τ_2) within τ_2 implies τ_1-compactness of (X,τ_1,τ_2) with respect to τ_2, but the converse need not be true, as the following example shows.

3.1.4 Example [6]:

Let $X = [0,1]$, let

$\tau_1 = \{A \subset X: 0 \in A \text{ and } X \setminus A \text{ is finite}\} \cup \{A \subset (0,1):(0,1)\setminus A \text{ is finite}\} \cup \{\emptyset\}$, and let

$\tau_2 = \{ A \subset X: 1 \in A \text{ and } X\setminus A \text{ is finite} \} \cup \{ A \subset (0,1) : (0,1) \setminus A \text{ is finite} \} \cup \{\emptyset\}$.

Then (X,τ_1,τ_2) is a bitopological space which is τ_1-compact with respect to τ_2, but not τ_1-compact within τ_2, because $\{[0,1] \setminus \{1/2\}, [0,1)\}$ is τ_1-open covering for X but has no finite τ_2-open subcovering.

B-compactness is independent of s-compactness and p-compactness, because any finite bitopological space is s-compact and p-compact but may not be B-compact as Ex. (3.1.5) shows. Also we can find a bitopological space which is B-compact but neither s-compact nor p-compact as Ex. (3.1.6) shows.

3.1.5 Example [6]:

Let $X=\{a, b, c\}$, $\tau_1 = \{\emptyset, X,\{a,b\},\{c\}\}$ and $\tau_2 =\{\emptyset, X, \{a\},\{b,c\}\}$. Then, (X,τ_1,τ_2) is s-compact and p-compact, but it is not τ_2- compact within τ_1, as $\{\{a\},\{b,c\}\}$ is a τ_2-open cover of X which has no τ_1-open subcover. Also $\{\{a\}, \{b,c\}\}$ is a τ_2- open cover of X which has no finite family of τ_1-open cover which is finer than this cover.

Hence, (X,τ_1,τ_2) is neither B-compact, nor conversely compact.

3.1.6 Example [6]:

Let $X = [0,1]$, $\tau_1 = \{ \emptyset, X, \{0\}\} \cup \{ [0,a) : a \in X \}$ and $\tau_2 = \{\emptyset, X, \{1\}\} \cup \{(a,1] : a \in X\}$. Then (X,τ_1,τ_2) is B-compact, for any τ_1-open cover of X or any τ_2–open cover of X must contain X as a member . However (X,τ_1,τ_2) is neither p-compact nor s-compact, for the p-open cover $\{\{0\}\} \cup \{(a, 1]: a \in X, a \neq 0\}$ of X has no finite subcover.

3.1.7 Theorem [6]:

If the bitopological space (X,τ_1, τ_2) is conversely compact then (X,τ_1) and (X,τ_2) are compact.

Proof:

Let $\upsilon = \{V_\alpha : \alpha \in \Delta\}$ be any τ_1-open cover for X. Since (X,τ_1,τ_2) is conversely compact, there is a finite τ_2-open cover $\upsilon_1 = \{U_i : i = 1,..,n \}$ for X, such that υ_1 is finer than υ . So, for each i= 1,..,n , there exists $\alpha_i \in \Delta$ s.t. $U_i \subset V_{\alpha_i}$. Consider the τ_1-open collection $\upsilon_2 = \{ V_{\alpha_i} : i=1,..,n\}$, then υ_2 covers X because $U_i \subset V_{\alpha_i}$ for each i =1,2,..,n and υ_1 covers X. Since $\forall$ i = 1,..,n , $V_{\alpha_i} \in \upsilon$, then υ_2 is the desired finite subfamily of υ that covers X. which means that (X,τ_1) is compact. Hence, we show that if the bitopological space (X,τ_1,τ_2) is τ_1-compact with respect to τ_2, then (X,τ_1) is compact. Similarly, we show that (X,τ_2) is compact, if (X,τ_1,τ_2) is τ_2-compact with respect to τ_1.

We can replace conversely compact by B-compact in the above theorem because every B-compact space is conversely compact.

In Ex. (3.1.5), (X,τ_1) and (X,τ_2) are compact, but the bitopological space, (X,τ_1,τ_2) is neither B-compact, nor conversely compact, so the converse of Thm. (3.1.7) is not true.

3.1.8 Theorem [6]:

Let (X,τ_1,τ_2) be a bitopological space, then the following are equivalent:

a) (X,τ_1,τ_2) is τ_1-compact with respect to τ_2.

b) For any family $\{F_\alpha : \alpha \in \Delta\}$ of τ_1-closed sets which has empty intersection, there exists a finite family $\{G_j : j = 1,..,n\}$ of τ_2- closed sets with empty intersection and satisfies the condition that $\forall\, j = 1,2,.., n$, $\exists\, \alpha_j \in \Delta$ such that $G_j \supset F_{\alpha j}$.

c) For any family $\upsilon = \{F_\alpha : \alpha \in \Delta\}$ of τ_1-closed sets with the property that every finite family $\{G_j : j = 1,..,n\}$ of τ_2-closed sets which statisfies the condition that $\forall\, j = 1, 2, \ldots, n$, $\exists\, \alpha_j \in \Delta$ s.t. $G_j \supset F_{\alpha j}$ has nonempty intersection, it results that υ has nonempty intersection.

Proof: (a) → (b)

Assume (a) and let $\{F_\alpha : \alpha \in \Delta\}$ be any family of τ_1-closed sets with empty intersection, then the family $\upsilon = \{ U_\alpha : U_\alpha = X \setminus F_\alpha , \alpha \in \Delta\}$ is a family of τ_1-open sets which covers X because $\bigcup_{\alpha\in\Delta} U_\alpha = \bigcup_{\alpha\in\Delta} X \setminus F_\alpha = X \setminus \bigcap_{\alpha\in\Delta} F_\alpha = X \setminus \emptyset = X$.

By the hypotheses of (a) , there is a finite family $\upsilon_1=\{V_j : j = 1,..,n\}$ of τ_2-open sets which covers X s.t. $\forall j, \exists \alpha_j$ with $V_j \subset U_{\alpha_j}$. Define $G_j = X \setminus V_j$, then for each j, G_j is τ_2 - closed and $G_j = X \setminus V_j \supset X \setminus U_{\alpha_j} = F_{\alpha_j}$.

Also, $\bigcap_{i=1}^{n} G_j = \bigcap_{i=1}^{n} X \setminus V_j = X \setminus \bigcup_{i=1}^{n} V_j = X \setminus X = \emptyset$.

(b) → (a) :

Assume (b), and let $\upsilon = \{U_\alpha : \alpha \in \Delta\}$ be any τ_1-open cover for X. Then the family $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of τ_1-closed sets such that

$\bigcap_{\alpha \in \Delta} (X \setminus U_\alpha) = X \setminus (\bigcup_{\alpha \in \Delta} U_\alpha) = X \setminus X = \emptyset$, i.e has empty intersection . Consequently, the hypothesis in (b) implies that there is a finite family $\{G_j : j = 1,\ldots, n\}$ of τ_2-closed sets s.t. $\forall j, \exists \alpha_j \in \Delta$ with $G_j \supset X \setminus U_{\alpha_j}$ and $\bigcap_{i=1}^{n} G_j = \emptyset$. Consider $V_j = X \setminus G_j$, then $\forall j$, V_j is τ_2-open and $\bigcup_{i=1}^{n} V_j = \bigcup_{i=1}^{n} X \setminus G_j = X \setminus \bigcap_{i=1}^{n} G_j = X \setminus \emptyset = X$.

Since $\forall j$, $V_j = X \setminus G_j \subset X \setminus (X \setminus U_{\alpha_j}) = U_{\alpha_j}$, then the finite family $\{V_j : j = 1,..,n\}$ of τ_2-open sets covers X and satisfies the desired condition. Hence (X,τ_1,τ_2) is τ_1-compact with respect to τ_2.

(b) → (c):

Assume (b), and let $\upsilon = \{F_\alpha : \alpha \in \Delta\}$ of τ_1-closed sets with the property stated in (c). Suppose that $\bigcap_{\alpha \in \Delta} F_\alpha = \emptyset$. By the hypothesis in (b), there is a finite family $\{G_j : j=1,\ldots, n\}$ of τ_2-closed sets with empty intersection s.t. $\forall j, \exists \alpha_j \in \Delta$ with $G_j \supset F_{\alpha_j}$, and this contradicts the property of the family υ.Hence $\bigcap_{\alpha \in \Delta} F_\alpha \neq \emptyset$.

(c) $\to$ (b):

Assume (c), and let $\{F_\alpha : \alpha \in \Delta\}$ be a family of τ_1-closed sets which has empty intersection . Suppose that there exists no finite family of the form $\{G_j , j =1,\dots, n\}$ of τ_2-closed sets with empty intersection and satisfies the condition that $\forall$ j, $\exists$ $\alpha_j \in \Delta$ with $G_j \supset F_{\alpha_j}$. This means that every finite family of $\{G_j : j =1,\dots, n\}$ of τ_2-closed sets which satisfies the condition $\forall$ j, $\exists$ $\alpha_j \in \Delta$ with $G_j \supset F_{\alpha_j}$, has nonempty intersection. By (c), $\{F_\alpha : \alpha \in \Delta\}$ has nonempty intersection, and this contradicts the assumption.

3.1.9 Theorem [6]:

Let (X,τ_1,τ_2) be a p-Hausdorff bitopological space and let (X,τ_1) be a compact topological space . Then $\tau_1 \subset \tau_2$.

Proof:

To prove this, it is sufficient to show that every τ_1-closed set is τ_2-closed set. Let A be τ_1-closed, then A is τ_1-compact. Let $x \notin A$. Since (X,τ_1,τ_2) is p-Hausforff, then for each $a \in A$, there exists a τ_1-open set V(a) and a τ_2-open set U(a) such that

$a \in V(a)$, $x \in U(a)$, and $V(a) \cap U(a) = \emptyset$.

The family $\{V(a): a \in A\}$ forms a τ_1-open cover of A, and so, by compactness of A, we can find a finite subcover $\{V(a_1), V(a_2), \dots, V(a_n)\}$ of $\{V(a): a \in A\}$ for A.

For each $V(a_k)$, k =1,2,…,n , there is a corresponding τ_2-open sets $U(a_k)$, and hence $B=\bigcap_{k=1}^{n} U(a_k)$ is τ_2-open set containing x. Now $B \cap V(a_k) = \emptyset$ for each k = 1,2,…,n , for if this is not true , then $B \cap V(a_i) \neq \emptyset$ for some i =1,…,n, and then $V(a_i) \cap U(a_i) \neq \emptyset$ as $B \subset U(a_k)$ for each k=1,2,…,n and this is contrary to the way $V(a_k)$ and $U(a_k)$ were chosen. Define $C = \bigcup_{i=1}^{n} V(a_k)$ which is τ_1-open, then we have $B \cap C = \emptyset$ and this implies that $B \cap A = \emptyset$. Therefore $x \in B \subset X \setminus A$ which means that A is τ_2-closed.

3.1.10 Corollary [6]:

Let the bitopological space (X,τ_1,τ_2) be pairwise Hausforff.

(a) If the topologies τ_1 and τ_2 are compact, then $\tau_1 = \tau_2$.

(b) If (X,τ_1,τ_2) is τ_1-compact with respect to τ_2, then $\tau_1 \subset \tau_2$.

(c) If (X,τ_1,τ_2) is conversely compact, then $\tau_1 = \tau_2$.

(d) If (X,τ_1,τ_2) is B-compact, then $\tau_1 = \tau_2$.

Proof:

(a) Follows from theorem (3.1.9). [An analogue statement for Thm. (2.1.7)].

(b) By Thm. (3.1.7), τ_1 is compact, and so, by Thm. (3.1.9) we get $\tau_1 \subset \tau_2$.

(c) Follows from (b).

(d) Follows from (c) and the fact that every B-compact bitopological space is conversely compact.

3.1.11 Example [6]:

Let X=[0,1]. Let τ_1 be the topology induced on X by the usual topology on $\Re$, and τ_2 be the discrete topology on X. Then (X,τ_1,τ_2) is p-Hausdorff bitopological space, and τ_1 is compact with respect to τ_2. But the topology τ_2 is not compact with respect to τ_1, and so τ_2 is not compact within τ_1. Consequently (X,τ_1,τ_2) is neither B-compact, nor conversely compact.

3.1.12 Example [6]:

Let $X = [0,\infty)$, let τ_1 be the discrete topology, and τ_2 be the co- countable topology . (X,τ_1,τ_2) is pairwise Hausedorff, and p- normal. The topologies τ_1 and τ_2 are not compact and consequently (X,τ_1,τ_2) is not conversely compact nor B-compact.

Proof:

It is clear that τ_1 is not compact . To see that τ_2 is not compact consider the τ_2 - open covering $\{(X \setminus \mathcal{N}) \cup \{ i \}: i \in \mathcal{N}\}$ for X which has no finite subcovering for X.

3.1.13 Example [6]:

Let $X= [0,1]$, τ_1 = the topology induced on X by the usual topology on $\Re$, and τ_2 = the topology generated by the union of families of τ_1 and the families of sets whose complements are countable. The bitopological space (X,τ_1,τ_2) is p -Hausedorff and τ_1-compact with respect to τ_2, (it is even τ_1-compact within τ_2), but it is not p-normal .

3.1.14 Example [6]:

Let $X= \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, c\}, \{b, c\}, \{c\}, X\}$,$\tau_2 = \{ \emptyset, \{b\}, \{b,c\}, \{a,b\}, \{a\}, X\}$. Therefore in (X,τ_1,τ_2), $\tau_1 \neq \tau_2$. (X,τ_1,τ_2) is p-regular, p-normal and conversely compact. But its not p - Hausedorff, as τ_1 and τ_2 are finite and so, they are compact . Since $\tau_1 \neq \tau_2$, then by Cor. (3.1.10-a) it is not p- Hausdorff.

3.1.15 Example [6]:

Let $X=\{a,b,c\}$, $\tau_1 = \{\emptyset,\{a\}, X\}$, and $\tau_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$.

The bitopological space (X,τ_1,τ_2) is p-normal and B-compact but not p-regular.

Proof: The bitopological space (X,τ_1,τ_2) is

(1) p-normal , because $\{b,c\}$ is the only nonempty proper τ_1-closed subset.

And, the only nonempty proper τ_2-closed subset of X that is disjoint from $\{b,c\}$ is $\{a\}$, and $\{b,c\}$ is τ_2 -open , and $\{a\}$ is τ_1 -open .

(2) B-compact, because each τ_1 -open or τ_2 -open cover for X must contain X as a member .

(3) Not p-regular, because $\{a\}$ is τ_2 -closed and $b \notin \{a\}$, the only τ_1 -open set that contains b is X, and the only τ_2 -open set which contains $\{a\}$ is X. So, τ_2 is not regular with respect to τ_1.

3.2 Coversely compactness of sets in bitopological spaces.

3.2.1 Definition [6]:

Let (X,τ_1,τ_2) be a bitopological space, and let $A\subset X$. We say that the set A is τ_1-compact with respect to τ_2 [resp. τ_2- compact with respect to τ_2, conversely compact], if the bitopological subspace (A, τ_{1A}, τ_{2A}) is τ_{1A} compact with respect to τ_{2A} [resp. τ_{2A} -compact with respect to τ_{1A} conversely compact]; where $\tau_{1A} = \{U \cap A: U \in\tau_1 \}$ and $\tau_{2A} = \{V \cap A: V\in\tau_2 \}$.

3.2.2 Theorem [6]:

Let A be a set in a bitopological space (X,τ_1,τ_2). Then:

(a) A sufficient condition for the set A to be τ_1-compact with respect to τ_2 is:

for every τ_1-open cover υ of A, there is a finite τ_2-open cover υ_1 of A finer than υ.

(b) If the set A is τ_2-open, then a nescessary condition for A to be τ_1- compact with respect to τ_2 is : for every τ_1- open cover υ of A , there is a finite τ_2- open cover υ_1 of A finer than υ.

Proof (a):

Let $\upsilon = \{ U_j \cap A : j\in J\}$, where $U_j \in\tau_1$ for each $j\in J$, be a τ_{1A}-open cover for A. Then, $\bigcup\{(U_j \cap A): j\in J \} = A$. So, $\bigcup\{ U_j : j\in J \} \cap A = A$, and so, $\bigcup\{ U_j : j\in J \} \supset A$. i.e. $\upsilon` =\{U_j : j\in J\}$ is $\tau_1$-open cover for A. By the hypothesis, there is a finite $\tau_2$-open cover for A ; say $\upsilon_1` = \{V_i: i=1,2,\ldots,n\}$ finer than $\upsilon`$. This means that $\forall i= 1,2,..,n$, there is $j\in J$ s.t. $V_i \subset U_j$. This implies that $\forall i= 1,2,..,n$, $\exists$ $j\in J$ s.t. $(V_i \cap A) \subset (U_j \cap A)$. Hence, the collection $\upsilon_1 = \{V_i \cap A: i= 1,2,..,n\}$ is the desired finite τ_{2A}-open cover for A which is finer than υ.

Proof (b):

Let A be τ_2 –open, and let the collection $\upsilon = \{U_i: i\in I\}$ be a τ_1–open cover for A. Then $\upsilon_1 = \{ U_i \cap A: i\in I \}$ is a τ_{1A}-open cover for A, so, by the hypothesis, there is a finite family υ_2 of τ_{2A}-open sets finer than υ_1 that covers A, say

$\upsilon_2 = \{V_j \cap A: j = 1,2,..,n\}$ where $V_j \in \tau_2$ $\forall j = 1,2,.., n$. Since A is τ_2 –open then for each $j = 1,2,..,n$, $V_j \cap A$ is τ_2 –open, and so, $\{ V_j \cap A: j = 1, 2, \ldots, n\}$is the desired finite family of τ_2 –open sets which is finer than υ and covers A.

The following example shows that the converse of Thm. (3.2.2-a) is not necessarily true if A is not τ_2 –open.

3.2.3 Example [6]:

Let $X= \{a, b, c\}$, $\tau_1 = \{\phi,\{a\}, \{a, c\}, \{b, c\},\{c\}, X \}$, and $\tau_2 =\{ \emptyset ,\{b\}, \{a, b\}, \{a\}, X\}$. Let $A = \{c\}$, and consider the τ_1- open cover $\{\{b, c\}\}$for A, then there is no τ_2-open cover for A finer than $\{\{b, c\}\}$. So A does not satisfy the condition in Thm. (3.2.2-a), even though $(A, \tau_{1A}, \tau_{2A})$ is τ_1-compact with respect to τ_2.

3.2.4 Theorem [6]:

Let A and B be τ_2-open sets, each of which is τ_1-compact with respect to τ_2, then their union $(A \cup B)$ is τ_1-compact with respect to τ_2.

Proof:

Let $\upsilon = \{U_j : j \in J\}$ be τ_1-open cover for $A \cup B$, then υ is τ_1-open cover for A and for B. By our hypothesis of A and B, and according to Thm. (3.2.2), there are two finite τ_2-open covers for A and B; say S_1 and S_2 respectively s.t. each of S_1 and S_2 is finer than υ. Therefore $S_1 \cup S_2$ is a finite τ_2-open cover for $A \cup B$, and $S_1 \cup S_2$ is finer than υ. It follows that $A \cup B$ is τ_1-compact with respect to τ_2, by Thm. (3.2.2).

The following example shows that the condition that A and B are τ_2-open is essential.

3.2.5 Example [6]:

Let $X=\{a, b\}$, $\tau_1= \{ \emptyset, \{a\}, \{b\}, X\}$, and $\tau_2= \{ \emptyset, X\}$. The sets $\{a\}$, $\{b\}$ are τ_1-compact with respect to τ_2, but $\{a\} \cup \{b\} = X$ is not τ_1- compact with respect to τ_2. Note that $\{a\}$ and $\{b\}$ are not τ_2-open.

3.2.6 Theorem [6]:

Let the bitopological space (X,τ_1,τ_2) be τ_1-compact with respect to τ_2 [resp. conversely compact], and let the subset A of X be τ_1-closed [resp. τ_1-closed and

τ_2-closed]. Then A is τ_1-compact with respect to τ_2 [resp. conversely compact].

Proof:

Assume that A is τ_1-closed and that (X,τ_1,τ_2) is τ_1-compact with respect to τ_2. Want to show that the subspace $(A, \tau_{1A}, \tau_{2A})$ is τ_{1A}- compact with respect to τ_{2A}.

Let $\upsilon = \{U_\alpha : \alpha \in \Delta\}$ be any τ_{1A}-open cover of A, then for each $\alpha \in \Delta$, $U_\alpha = V_\alpha \cap A$; where $V_\alpha \in \tau_1$. Since A is τ_1-closed, then $X \backslash A$ is τ_1-open, and so the collection

$\upsilon_1 = \{V_\alpha : \alpha \in A\} \cup \{X \backslash A\}$ is a τ_1-open cover of X. By τ_1- compactness of A with respect to τ_2, there is a finite τ_2- open cover for X, say υ_2 such that υ_2 is finer than υ_1. Let the collection υ_3 be the set of all elements of υ_2 which are not subsets of $X \backslash A$. Then $\upsilon_3 = \{W_i : i = 1,2,\ldots,n\}$ is a family of τ_2–open sets which is finer than υ_1 and covers A . Consequently the collection $\upsilon_4 = \{W_i \cap A : i = 1, 2, \ldots, n\}$ is the desired τ_{2A}–open cover for A which is finite and finer than υ. Which means that A is τ_1-compact with respect to τ_2. We use the same argument to complete the proof of the theorem.

3.3 Continuous (open) functions and conversely compactness in bitopological spaces.

3.3.1 Theorem [6]:

If the bitopological space (X,τ_1,τ_2) is τ_1-compact with respect to τ_2, if the function $f : (X,\tau_1) \to (X`,\tau_1`)$ is continuous and if the function $f : (X,\tau_2) \to (X`,\tau_2`)$ is open, then f(X) is $\tau_1`$- compact with respect to $\tau_2`$.

Proof:

Let $\upsilon' = \{U'_\alpha : \alpha \in \Lambda\}$ be τ_1'-open cover for $f(X)$ in (X', τ_1', τ_2'). Because $f: (X, \tau_1) \to (X', \tau_1')$ is continuous, then the collection $\upsilon = \{f^{-1}(U'_\alpha) : \alpha \in \Lambda\}$ is τ_1-open cover for X, and therefore there exists a finite τ_2-open cover say $\{V_i : i = 1,2,\ldots,n\}$ of X finer than υ. That is to say that $\forall i$, $\exists\, \alpha_i \in \Lambda$, s.t. $V_i \subset f^{-1}(U'_{\alpha_i})$. Since the function $f: (X, \tau_2) \to (X', \tau_2')$ is open, then the collection $\{f(V_i) : i=1,2,..,n\}$ is τ_2'-open cover of $f(X)$ which is finite and finer than υ' because $\forall\, i = 1,2,..,n$, $\exists\, \alpha_i \in \Lambda$ s.t. $f(V_i) \subset U'_{\alpha_i}$. This implies that $f(X)$ is τ_1'-compact with respect to τ_2', by Thm. (3.2.2-a).

The following corollary follows directly from Thm. (3.3.1).

3.3.2 Corollary [6]:

If the bitopological space (X, τ_1, τ_2) is conversely compact, and the function $f:(X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is continuous and open, then $f(X)$ is a conversely compact subset of the space (X', τ_1', τ_2').

3.3.3 Corollary [6]:

If we add to the hypothesis of corollary (3.3.2), the hypothesis that (X', τ_1', τ_2') is p-Hausdorf, then $\tau_1' = \tau_2'$ and $(f(X), \tau_1' = \tau_2')$ is a compact topological space.

Proof:

By corollary (3.3.2), (f (X), τ_1`,$\tau_2$`) is conversely compact. Then by Thm. (3.1.10-c) τ_1`= $\tau_2$`. Since (f (X), τ_1`,$\tau_1$`) is conversely compact then, f (X) is τ_1`-compact with respect to $\tau_1$`, i.e((f (X), τ_1`) is a compact topological space, according to Thm. (3.1.10) .

3.3.4 Corollary [6]:

In the bitopological space (X,τ_1,τ_2), the image of the τ_2-open (resp. τ_1 and τ_2-open) subset A of X which is τ_1-compact with respect to τ_2 (resp. conversely compact) by a function f : (X,τ_1,τ_2) $\rightarrow$ (X`,$\tau_1$`,τ_2`) with $f$ : (X,$\tau_1$)$\rightarrow$ (X`,τ_1`) is continuous and $f$: (X,$\tau_2$) $\rightarrow$ (X`,τ_2`) is open (resp. $f$ is continuous and open) is $\tau_1$`-compact with respect to τ_2` (resp. conversely compact).

Proof:

The proof is similar to the proof of Thm. (3.3.1), using Thm. (3.2.2).

The following example proves that it is not sufficient to suppose that f is only continuous in Thm. (3.3.1).

3.3.5 Example [6]:

Let X = {a, b, c} , τ_1=τ_2 ≡ the discrete topology. Let X` = {1,2,3}, $\tau_1$` = {Ø, {1}, {2,3}, X`}, and $\tau_2$`= {Ø, {1,2}, {3}, X`}. Define the function f by

$f(a) = 1$, $f(b) = 2$, $f(c) = 3$. We observe that:

1) (X,τ_1,τ_2) is conversely compact (there is exactly one compact topological space) .

2) f is a continuous function .

3) $(X`,\tau_1`,\tau_2`)$ is neither $\tau_1`$-compact with respect to $\tau_2`$, nor $\tau_2`$-compact with respect to $\tau_1`$.

Proof:

The proof of (i) and (ii) are direct. To prove (iii) we notice that $\upsilon_1 = \{ \{1\} , \{2 ,3 \}\}$ is $\tau_1`$-open cover for X , but there is no $\tau_2`$-open cover for X that is finer than υ_1 . Also, $\upsilon_2 = \{ \{1,2\}, \{3\} \}$ is $\tau_2`$-open cover for X , but there is no $\tau_1`$-open cover for X that is finer than υ_2 .

3.4 Alexander's, Tychonoff's theorems and conversely compactness in bitopological spaces.

3.4.1 Definition:

A family $\mathcal{F}$ of τ_i-open sets in the bitopological space (X,τ_1,τ_2) is called τ_i-inadequate in (X,τ_1,τ_2) , i=1,2, if and only if it fails to cover X. The family $\mathcal{F}$ of τ_1-open sets is called finitely τ_1- inadequate with respect to τ_2 in X if and only if no finite family of τ_2- open sets which is finer than $\mathcal{F}$ covers X .We can easily see that the bitopological space (X,τ_1,τ_2) is τ_1 - compact with respect to τ_2 if and only if each finitely τ_1-inadequate family with respect to τ_2 in X, is τ_1- inadequate.

The proofs of lemma (3.4.2), lemma (3.4.3) and theorem (3.4.4) are seemed to be similar to the proofs of Lemma (2.4.9), lemma (2.4.10) and theorem (2.4.11), respectively.

3.4.2 Lemma:

Let $\mathcal{F}$ be a finitely τ_1- inadequate family with respect to τ_2 in the bitopological space (X,τ_1,τ_2). Then there is a maximal finitely τ_1-inadequate family with respect to τ_2 in (X,τ_1,τ_2), say $\mathcal{D}$, such that $\mathcal{F} \subset \mathcal{D}$.

3.4.3 Lemma:

Let (X,τ_1,τ_2) be a bitopological space. Let $\mathcal{D}$ be a maximal finitely τ_1-inadequate family with respect to τ_2. If some member of $\mathcal{D}$ contains $\bigcap_{i=1}^{n} G_i$, where each G_i is τ_1-open (τ_2-open), then $G_k \in \mathcal{D}$ for some k in $\{1,2,\ldots, n\}$.

3.4.4 Theorem (Alexander):

Let (X,τ_1,τ_2) be a bitopological space, and assume that $\mathcal{S}$ is a subbase of the topology τ_1 such that, for each τ_1 – open cover υ for X by members of $\mathcal{S}$, there is a finite family of τ_2 –open sets finer than υ that covers X, then (X,τ_1,τ_2) is τ_1- compact with respect to τ_2

3.4.5 Theorem (Tychonoff):

Let the bitopological space (X,τ,τ') be the product bitopological space of the family of bitopological spaces $\{ (X_i,\tau_i,\tau_i') : i\in I \}$. Then

(i) If (X,τ,τ') is τ-compact with respect to τ'(conversely compact), then each factor space (X_i,τ_i,τ_i') is τ_i-compact with respect to τ_i' (conversely compact).

(ii) If for every $i \in I$, the bitopological space (X_i,τ_i,τ_i') is τ_i-compact with respect to τ_i'(conversely compact), then the product bitopological space (X,τ,τ') is τ-compact with respect to τ' (conversely compact).

The natural projections are continuous and open, therefore we can apply Thm.(3.3.1) and Cor. (3.3.2) to prove (i).

Conversely, let $\mathcal{S} = \{ \pi_i^{-1}(U_i) : U_i \in \tau_i , i \in I\}$, where π_i is the natural projection into the i-th coordinate space X_i, then $\mathcal{S}$ is a subbase for the tobology τ. In veiw of Thm.(3.3.10), the product bitopological space (X,τ,τ') will be τ-compact with respect to τ' if each subfamily $\mathcal{A}$ of $\mathcal{S}$ which is finitely τ-inadequate with respect to τ' in (X,τ,τ') is τ-inadequate . For each index $i \in I$, let $\mathcal{B}_i$ be the family of all sets $U_i \in \tau_i$ such that $\pi_i^{-1}(U_i) \in \mathcal{A}$. Then $\mathcal{B}_i$ is finitely τ_i-inadequate with respect to τ_i' in (X_i,τ_i,τ_i') . Since (X_i,τ_i,τ_i') is τ_i-compact with respect to τ_i', then $\mathcal{B}_i$ is τ_i-inadequate in (X_i,τ_i,τ_i'). So , there is $x_i \in X_i \setminus U_i$ for each $U_i \in \mathcal{B}_i$.

Consider the point $x \in X$ whose i-th coordinate is x_i, then x belongs to no member of $\mathcal{A}$, and consequentely, $\mathcal{A}$ is τ -inadequate, in (X,τ,τ').

Hence the product bitopological space (X,τ,τ') is τ- compact with respect to τ'.

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الملخص

في هذه الرسالة، نقدم ثلاثة مفاهيم تتعلق بالفضاءات التبولوجية الثنائية المتراصة، وهي التراص البسيط، التراص المزدوج وتراص بيرسان. ونقدم بعض مفاهيم الفضاءات الثنائية مثل التباعد، الاتصال، الانتظام والاعتدال حيث سندرس علاقتها بمفاهيم التراص في الفضاءات التبولوجية الثنائية . ونناقش كذلك التعميم لبعض النظريات المهمة في الفضاءات التبولوجية الاحادية ذات العلاقة بالتراص مثل نظرية الكسندر ونظرية تيخينوف للتراص بالفضاءات التبولوجية الثنائية.