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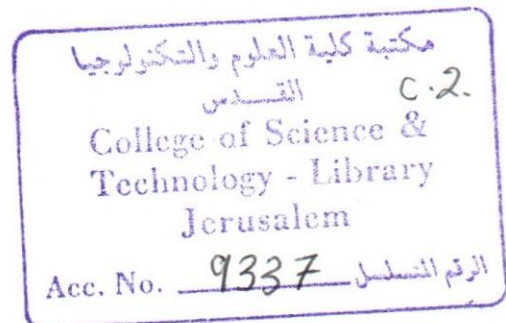
Al-Quds University

SEMI-CONNECTEDNESS IN
BITOPOLOGICAL
SPACES

By

Hani Mahmoud Sallah El-Din

M. Sc. Thesis



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SEMI-CONNECTEDNESS IN BITOPOLOGICAL
SPACES

By

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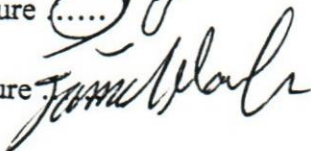
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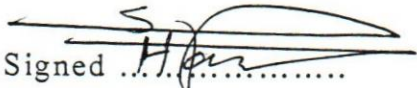
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Declaration :

I Certify that this thesis submitted for the degree of Master is the result of my own research except where otherwise acknowledged , and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution .


Signed

Hani Mahmoud Ahmad Sallah El-Din

Date : 4\2\2002

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ABSTRACT

In this thesis , we study some properties of pairwise connected and pairwise semi-connectedness in bitopological spaces , proffering some properties of semi-open sets in topological space .

Also , we study the totally disconnectedness and the semi-components in bitopological spaces .

Finally , we study connectedness and semi-connectedness between sets in bitopological spaces .

In most of these , we make generalization of similar concepts in topological spaces , and have generalization for results in topological spaces .

DEDICATION

To my mother

To my wife

To my brothers and sisters

To my children

To free people in Palestine and Afghanistan

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Introduction

The triple (X, T_1, T_2) , where X is a set and $T_i, i = 1, 2$ are arbitrary topologies on X , is called a bitopological space. In 1963, Kelly [5] introduced the notion of bitopological spaces. Several authors have studied this notion and other related concepts .e .g Weston, and Kim , Pervin [13] .

Pervin [13] defined connectedness and studied its properties for bitopological spaces. J.Swart [4] defined total disconnectedness in bitopological spaces. And Fora and Hdeib defined pairwise connectedness and semi pairwise connectedness for bitopological spaces.

This thesis consists of 4 chapters. The first chapter deals with pairwise connectedness and some types of continuous functions between bitopological spaces and their effects on some types of connected spaces.

The properties of semi-open sets, the definition of pairwise semi-connected and semi-continuous function and some related results in bitopological spaces are studied in the second chapter .

The third chapter concerns with the study of the totally disconnected Bitopological spaces and the components in bitopological spaces.

The study of pairwise connectedness between sets and pairwise semi-connectedness between sets and some relations between

them in bitopological spaces are covered in the fourth chapter.

We use p- and s- to denote pairwise and semi, respectively. e.g. p- connected and s-connected stand for pairwise connected and semi connected, respectively.

By saying that the bitopological space (X, T_1, T_2) has a certain topological property, we mean that both (X, T_1) and (X, T_2) have this property. By saying that a function $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ satisfies a property that is used for functions between topological spaces, we mean that both $f : (X, T_1) \rightarrow (Y, T^*_1)$ and $f : (X, T_2) \rightarrow (Y, T^*_2)$ satisfy this property. T_S , T_L , T_R , and T_D will denote the standard topology, the left ray topology, the right ray topology, and the discrete topology, respectively.

Chapter .1

Pairwise Connected Spaces

1.1 Pairwise Separation .

1.1.1 Definition : [13]

A bitopological space (X, T_1, T_2) is called p -connected if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap \text{cl}_1(B)) \cup (\text{cl}_2(A) \cap B) = \emptyset$, where $\text{cl}_1(B)$ and $\text{cl}_2(A)$ denote the closures of B and A with respect to T_1 and T_2 , respectively.

When X can be expressed as the union of such A and B , we write $X = A|B$ and call it p -disconnection of X , or pairwise separation, and denoted by p -separation. (This is called a separation of X according to William [13]).

1.1.2 Definition :

A bitopological space (X, T_1, T_2) which is not p -connected is called p -disconnected .

1.1.3 Theorem : [13]

For any bitopological space (X, T_1, T_2) , the following conditions are equivalent :

- 1- (X, T_1, T_2) is p -connected .
- 2- X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is T_1 open and B is T_2 open .
- 3- X contains no nonempty proper subset which is both T_1 open and T_2 closed .

Proof:

1 \Rightarrow 2) Suppose $X = A \cup B$, where A and B are nonempty T_1 open and T_2 open, respectively , such that $A \cap B = \emptyset$. A is T_1 open, then B is T_1 closed and this implies that $cl_1(B) = B$. Then $A \cap cl_1(B) = \emptyset$. And since B is T_2 open , then A is T_2 closed and so $cl_2(A) = A$, then $cl_2(A) \cap B = \emptyset$. So $[A \cap cl_1(B)] \cup [cl_2(A) \cap B] = \emptyset$. Then X is not p -connected which contradicts the assumption .

2 \Rightarrow 1) Suppose that X is not p -connected . Then $X = A \cup B$ is a p -disconnection of X , for some nonempty subsets A and B of X . Then $[A \cap cl_1(B)] = \emptyset$ and $[cl_2(A) \cap B] = \emptyset$. Now since $[A \cap cl_1(B)] = \emptyset$ and $B = X \setminus A$, then $[A \cap cl_1(X \setminus A)] = \emptyset$.

But $cl_1(X \setminus A) = X \setminus int_1(A)$, so $[A \cap (X \setminus int_1(A))] = \emptyset$. So $int_1(A) \subset A$, and so $A = int_1(A)$.i.e. A is T_1 open set. Similarly,

we show that B is a T_2 open set. Therefore $X = A \cup B$, and A is T_1 open and B is T_2 open and both of them are nonempty sets. This contradicts our assumption. Hence X is p -connected.

$2 \Rightarrow 3$) Suppose that there exists a nonempty proper subset A of X which is both T_1 open and T_2 -closed. Then $X = A \cup (X \setminus A)$, where A is T_1 open and $X \setminus A$ is T_2 open, which contradicts our assumption of X , as A and $X \setminus A$ are nonempty disjoint subsets of X .

$3 \Rightarrow 2$) Suppose $X = A \cup B$, A is T_1 open and B is T_2 open and A, B are nonempty disjoint sets. Since $B = X \setminus A$ then A is T_2 closed. This contradicts the assumption. This completes the proof of the theorem.

The following corollary results from the equivalence of 1 and 3 in the previous theorem.

1.1.4 Corollary : [1]

The space (X, T_1, T_2) is p -connected if and only if X and \emptyset are the only subsets of the space X which are both open in T_i and closed in T_j ; ($i, j = 1, 2$, and $i \neq j$).

1.1.5 Remark : [13]

For any topology T , (X, T, T) is p -connected if and only if (X, T) is connected.

Proof:

\Leftarrow) Suppose that (X, T, T) is p -disconnected. Then there exist two nonempty disjoint open sets in T , A and B such that $X = A \cup B$. So (X, T) is disconnected which contradicts the assumption.

\Rightarrow) Suppose (X, T) is disconnected. Then there exist two nonempty disjoint open sets A and B such that $X = A \cup B$. Then A and B are closed sets, so $A = \text{cl}(A)$ and $B = \text{cl}(B)$.

Hence $A \cap \text{cl}(B) = \emptyset = \text{cl}(A) \cap B$.

1.1.6 Note : [13]

If the topological spaces (X, T_1) , (X, T_2) are both connected, then (X, T_1, T_2) may fail to be p -connected. Also, E. Berven [13] showed that if (X, T_1, T_2) is p -connected, then (X, T_1) and (X, T_2) need not be connected.

The following examples show these and all other possibilities.

1.1.7 Example :

Let $X = \{1, 2, 3, 4, 5\}$, $T_1 = \{X, \emptyset, \{2\}, \{3\}, \{2, 3\}\}$ and $T_2 = \{X, \emptyset, \{1, 3, 4, 5\}\}$.

It is clear that (X, T_1) and (X, T_2) are connected. But (X, T_1, T_2) is not p-connected. Because there exist two nonempty disjoint sets $A = \{2\}$ which is open in T_1 and $B = \{1, 3, 4, 5\}$ which is open in T_2 , and $X = A \cup B$.

1.1.8 Example :

Let $X = \{1, 2, 3, 4, 5\}$, $T_1 = \{X, \emptyset, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}, \{2, 3, 4, 5\}\}$ and $T_2 = \{X, \emptyset, \{2, 3, 4\}, \{1, 5\}\}$. It is clear that (X, T_1) and (X, T_2) are not connected, and that (X, T_1, T_2) is p-connected as the complement of every nonempty proper T_1 open subset of X is not T_2 open.

1.1.9 Example :

Let $X = \{1, 2, 3, 4, 5\}$. Then $T_1 = \{X, \emptyset, \{1, 2, 3\}, \{4, 5\}, \{5\}, \{5, 3, 2, 1\}\}$ is not connected, and $T_2 = \{X, \emptyset, \{4\}, \{\{1, 2, 3, 5\}, \{2\}, \{2, 4\}\}\}$ is not connected and (X, T_1, T_2) is not p-connected, as X has the p-disconnection $\{1, 2, 3, 5\} | \{4\}$.

1.1.10 Example :

Let $X = \{1, 2, 3, 4, 5\}$. Then $T_1 = \{X, \emptyset, \{1, 2, 3\}, \{2\}, \{1\}, \{1, 2\}\}$ is connected, $T_2 = \{X, \emptyset, \{4\}, \{4, 3\}, \{3\}\}$ is connected, and (X, T_1, T_2) is p-connected, as the complement of every T_1 open set is not T_2 open.

1.1.11 Example :

Let $X = \{1,2,3,4,5\}$. Then $T_1 = \{X, \emptyset, \{4,3,2\}, \{1,5\}\}$ is not connected, $T_2 = \{X, \emptyset, \{2\}, \{1,2\}, \{2,5\}, \{1,2,5\}\}$ is connected, and (X, T_1, T_2) is p-connected, as the complement of every T_1 open set is not T_2 open.

1.1.12 Example :

Let $X = \{1,2,3,4,5\}$. Then $T_1 = \{X, \emptyset, \{4,3,2\}, \{1,5\}\}$ is not connected, $T_2 = \{X, \emptyset, \{1,5\}, \{5\}, \{1\}\}$ is connected, and (X, T_1, T_2) is not p-connected, as X has the p-disconnection $\{2,3,4\} | \{1,5\}$.

1.1.13 Definition : [13]

A set E of a bitopological space (X, T_1, T_2) will be called p-connected if the space (E, T_{1E}, T_{2E}) is p-connected.

1.1.14 Theorem : [13]

If C is a p-connected subset of a bitopological space (X, T_1, T_2) which has a p-separation $X = A | B$, then either C is a subset of A or C is a subset of B .

Proof :

$X = A \cup B$, where A and B are nonempty disjoint sets such that A is T_1 open and B is T_2 open.

$$C = C \cap X = C \cap (A \cup B) = (C \cap A) \cup (C \cap B).$$

Since $C \cap A \subset A$, then $cl_i(C \cap A) \subset cl_i(A)$. Similarly

$cl_i(C \cap B) \subset cl_i(B)$.

Since $X=A|B$, then $[A \cap cl_1(B)] \cup [cl_2(A) \cap B] = \emptyset$. But

$[(C \cap A) \cap cl_1(C \cap B)] \cup [cl_2(C \cap A) \cap (C \cap B)]$ is subset of $[(A \cap cl_1(B))] \cup [cl_2(A) \cap B]$, so

$[(C \cap A) \cap (cl_1(C \cap B))] \cup [cl_2(C \cap A) \cap (C \cap B)] = \emptyset$.

But $cl_{1c}(C \cap B) \subset cl_1(C \cap B)$, and $cl_{2c}(C \cap A) \subset cl_2(C \cap A)$,

so $[(C \cap A) \cap cl_{1c}(C \cap B)] \cup [cl_{2c}(C \cap A) \cap (C \cap B)] = \emptyset$.

Suppose that $C \cap A$ and $C \cap B$ are nonempty, then (C, T_{1c}, T_{2c}) is not p-connected, i.e. C is not a p-connected subset of (X, T_1, T_2) , which contradicts the hypothesis.

So $C \cap A = \emptyset$ or $C \cap B = \emptyset$. i.e. $C \subset A$ or $C \subset B$.

1.1.15 Note :

If (X, T_1, T_2) is a bitopological space, $A \subset X$ and $E \subset A$, then the subspaces that E inherits from T_{1A} and T_{2A} are the same subspaces E inherits for T_1 and T_2 . Hence we have :

1.1.16 Remark :

If (X, T_1, T_2) is a bitopological space, and $E \subset A \subset X$, then E is a p-connected subset of (A, T_{1A}, T_{2A}) if and only if E is a p-connected subset of (X, T_1, T_2) .

1.1.17 Corollary : [13]

If every two points of a set E are contained in some p -connected subset of E , then E is a p -connected set .

Proof :

Suppose that E is not a p -connected set .Then $E = A|B$ is a p -disconnection of E for some subsets A and B of E . Let $a \in A$ and $b \in B$. From the hypothesis , we know that a and b must be contained in some p -connected subset C contained in E . By (Th1.1.14) C is a subset of A or C is a subset of B .

Since A and B are disjoint then this is a contradiction, because a belongs to A and b belongs to B , and a, b belong to C . Hence E must be p - connected.

1.1.18 Corollary : [13]

The union of any family of p -connected sets having a nonempty intersection is a p -connected set.

Proof :

Let $E = \{C_i : i \in \Omega\}$ be a family of a nonempty intersect p -connected sets. If $E = \bigcup C_i$ is not p -connected , it must have a p -separation $E = A|B$. By hypothesis, we may choose a point x belongs to $\bigcap C_i$. The point x must belong to either A or B . Suppose x belong to A .

Since x belongs to C_i for each $i \in \Omega$, then $C_i \cap A \neq \emptyset$,
 $\forall i \in \Omega$. Since C_i is p -connected and is a subset of $E = A|B$, then
 C_i must be either a subset of A or a subset of B , $\forall i \in \Omega$. Since
 $C_i \cap A \neq \emptyset$ and $A \cap B = \emptyset$, then $C_i \subset A$, for each $i \in \Omega$, which
implies that $E = \bigcup C_i \subset A$. So $E \subset A$, and so $B = \emptyset$, which is a
contradiction. Hence E is p -connected.

1.1.19 Corollary : [13]

If C is a p -connected set and $C \subset E \subset [cl_1(C) \cap cl_2(C)]$, then E
is p -connected.

Proof:

Suppose E is not p -connected, then $E=A|B$ is a p -disconnection,
for some subsets A and B of E . Since C is a p -connected subset of E
and $E = A|B$ is a p -disconnection, then C is a subset of A or C is a
subset of B . W.L.O.G. let C be a subset of A . Then $cl_i(C) \subset cl_i(A)$;
 $i = 1, 2$.

Since $C \subset E \subset [cl_1(C) \cap cl_2(C)]$ and $E = A \cup B$, then
 $C \subset A \cup B \subset [cl_1(C) \cap cl_2(C)]$, so $B \subset [cl_1(C) \cap cl_2(C)]$ which
implies that $B \cap cl_1(C) \neq \emptyset$ and $B \cap cl_2(C) \neq \emptyset$, as $B \neq \emptyset$. But
 $cl_i(C)$ is subset of $cl_i(A)$ $i = 1, 2$, so $B \cap cl_i(A) \neq \emptyset$, $i = 1, 2$ which is
a contradiction because $E = A|B$ is a p -disconnection. Hence E is p -
connected.

1.2 Continuous functions in bitopological spaces

1.2.1 Definition : [13]

Let (X, T_1, T_2) and (Y, T^*_1, T^*_2) be bitopological spaces. A function $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ is called continuous if and only if $f : (X, T_1) \rightarrow (Y, T^*_1)$ and $f : (X, T_2) \rightarrow (Y, T^*_2)$ are continuous.

1.2.2 Definition : [13]

Let (X, T_1, T_2) and (Y, T^*_1, T^*_2) be bitopological spaces. A function $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ is called h-continuous if and only if $f : (X, T_1) \rightarrow (Y, T^*_1)$ is continuous or $f : (X, T_2) \rightarrow (Y, T^*_2)$ is continuous.

1.2.3 Definition : [13]

Let (X, T_1, T_2) and (Y, T^*_1, T^*_2) be bitopological spaces.

A function $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ is called p-continuous if and only if, for every $U \in (T^*_1 \cup T^*_2)$, the inverse image of U belongs to $T_1 \cup T_2$.

1.2.4 Definition :

A bitopological space (X, T_1, T_2) is called h-disconnected if and only if there exists a proper nonempty subset U of X such that U belongs to $(T_1 \cup T_2)$ and $(X \setminus U)$ belongs to $(T_1 \cup T_2)$.

1.2.5 Definition :

A bitopological space (X, T_1, T_2) is called h-connected if and only if the space is not h-disconnected.

The following Theorem follows directly from the definitions.

1.2.6 Theorem :

In a bitopological space (X, T_1, T_2) , every continuous function is p-continuous, and h-continuous.

The following examples show that the converse of this theorem is not necessarily true, and that h-continuity and p-continuity are independent.

1.2.7 Example :

Let f be the function from the bitopological space (\mathbb{R}, T_L, T_R) into itself, defined by $f(x) = -x$. Clearly f is p-continuous. $(-\infty, 0) \in T_L$,



but $f^{-1}(-\infty, 0) = (0, \infty) \notin T_L$. So $f: (R, T_L) \rightarrow (R, T_L)$ is not continuous, and so $f: (R, T_L, T_R) \rightarrow (R, T_L, T_R)$ is not continuous. Also $(0, \infty) \in T_R$, but $f^{-1}(0, \infty) = (-\infty, 0) \notin T_R$. So $f: (R, T_R) \rightarrow (R, T_R)$ is not continuous. Hence $(R, T_L, T_R) \rightarrow (R, T_L, T_R)$ is not h-continuous.

1.2.8 Example :

Let $f: (R, T_D, T_L) \rightarrow (R, T_D, T_L)$ be defined by $f(x) = -x$. Then $f: (R, T_D) \rightarrow (R, T_D)$ is continuous, but $f: (R, T_L) \rightarrow (R, T_L)$ is not continuous. So f is h-continuous, but not continuous. Also, $T_L \subset T_D$, so f is p-continuous.

1.2.9 Example :

Let $f: (R, T_L, T_R) \rightarrow (R, T_D, T_D)$ be defined by $f(x) = |x|$. Then f is neither p-continuous nor h-continuous.

1.2.10 Example :

Let $f: (R, T_L, T_R) \rightarrow (R, T_L, T_D)$ be defined by $f(x) = x$. Then f is h-continuous but not p-continuous.

The following lemma follows directly from the definition.

1.2.11 Lemma :

Every p-disconnected bitopological space is h-disconnected.

The following example shows that the converse of the previous lemma is not necessarily true.

1.2.12 Example :

Let $X = \{1, 2, 3, 4, 5\}$, $T_1 = \{X, \emptyset, \{4, 3, 2\}, \{1, 5\}\}$ and $T_2 = \{X, \emptyset, \{2\}, \{1, 2\}, \{2, 5\}, \{1, 2, 5\}\}$. Then the bitopological space (X, T_1, T_2) is h-disconnected but not p-disconnected .

1.2.13 Theorem :

Every h-connected bitopological space is p-connected.

Proof:

The proof follows directly from lemma 1.2.11 .

The converse of the last Theorem is not necessarily true as can be seen from the example 1.2.12 .

1.2.14 Theorem : [13]

The continuous image of a p-connected bitopological space is p-connected.

Proof :

Let f be a continuous mapping from (X, T_1, T_2) onto (Y, T_1^*, T_2^*) . Suppose (Y, T_1^*, T_2^*) is not p-connected . Then

$Y = A|B$ is a p -separation of Y , for some nonempty subsets A and B of Y . By continuity of f , $f^{-1}(A) \in T_1$, $f^{-1}(B) \in T_2$, and $f^{-1}(A), f^{-1}(B)$ are nonempty disjoint subsets of X , and $f^{-1}(A) \cup f^{-1}(B) = X$. Thus $f^{-1}(A)|f^{-1}(B)$ is a p -separation of X . i. e. X is not p -connected, and this contradicts the assumption.

Hence (Y, T_1^*, T_2^*) is p -connected.

1.2.15 Theorem : [1]

A bitopological space (X, T_1, T_2) is p -connected if and only if every continuous mapping of (X, T_1, T_2) into (D, T_L, T_R) is constant; where

$D = \{0, 1\}$, $T_L = \{D, \emptyset, \{0\}\}$, and $T_R = \{D, \emptyset, \{1\}\}$.

Proof:

\Rightarrow) Let $f : (X, T_1, T_2) \rightarrow (D, T_L, T_R)$ be a continuous function.

Want to show that f is constant. Suppose that f is nonconstant.

It's clear that $X = f^{-1}(0) \cup f^{-1}(1)$, $f^{-1}(0) \cap f^{-1}(1) = \emptyset$, $f^{-1}(0) \in T_1$, $f^{-1}(1) \in T_2$, and $f^{-1}(0)$ and $f^{-1}(1)$ are nonempty proper subsets of X .

Then $X = f^{-1}(0) | f^{-1}(1)$ is a p -disconnection of X . But X is p -connected, so this is a contradiction. Therefore the function f is constant.

\Leftarrow) Suppose that X is not p -connected, then $X = A|B$ is p -disconnection for some nonempty subsets A and B of X . We can define a continuous function $f : (X, T_1, T_2) \rightarrow (D, T_L, T_R)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then $f^{-1}(0) \in T_1$ and $f^{-1}(1) \in T_2$. So f is a nonconstant

continuous function from (X, T_1, T_2) into (D, T_L, T_R) , which contradicts the assumption . So (X, T_1, T_2) is p-connected.

1.2.16 Theorem : [13]

The h-connectedness property is preserved under p-continuous function.

Proof:

Let (X, T_1, T_2) be any h-connected bitopological space , and let $f : (X, T_1, T_2) \rightarrow (Y, T_1^*, T_2^*)$ be a p-continuous function. Suppose that (Y, T_1^*, T_2^*) is h-disconnected. Then there exists a nonempty proper subset $U \in (T_1^* \cup T_2^*)$ such that $(Y \setminus U) \in (T_1^* \cup T_2^*)$. By p-continuity, $f^{-1}(U) \in (T_1 \cup T_2)$ and $f^{-1}(Y \setminus U) \in (T_1 \cup T_2)$. Also $f^{-1}(U) \cap f^{-1}(Y \setminus U) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(Y \setminus U) = X$.

Hence (X, T_1, T_2) is h-disconnected , and this is a contradiction with the hypothesis. So (Y, T_1^*, T_2^*) is h-connected.

The following theorem follows from the fact that every continuous function is p-continuous and the previous theorem.

1.2.17 Theorem :

h-connectedness is preserved under continuous functions.

1.2.18 Theorem :

A bitopological space (X, T_1, T_2) is h-connected if and only if it is connected and p-connected.

Proof:

\Rightarrow) Suppose that X is disconnected or p-disconnected . In either case there exist $U \in (T_1 \cup T_2)$ and $V \in (T_1 \cup T_2)$ such that U and V are nonempty sets , $U \cap V = \emptyset$, and $U \cup V = X$. So (X, T_1, T_2) is h-disconnected , which contradicts the hypothesis .

\Leftarrow) Suppose that (X, T_1, T_2) is h-disconnected . Then there exists a nonempty proper set A such that A and $X \setminus A$ belong to $(T_1 \cup T_2)$. Now there are two cases :

1) If A and $X \setminus A \in T_i$ for some $i = 1, 2$, then (X, T_i) is disconnected : and this contradicts the hypothesis that (X, T_1, T_2) is connected.

2) If $A \in T_i$ and $X \setminus A \in T_j$; $i \neq j$, then (X, T_1, T_2) is p-disconnected and this contradicts the hypothesis that (X, T_1, T_2) is p-connected.

1.2.19 Theorem :

If $\text{fr}_i(A) = \emptyset$, $i = 1, 2$ for some nonempty proper subset A of a bitopological space (X, T_1, T_2) , then (X, T_1, T_2) is not p-connected .

Proof:

$fr_1(A) = fr_2(A) = \emptyset$. But $fr_1(A) = cl_1(A) \setminus int_1(A)$, therefore $cl_1(A) \setminus int_1(A) = \emptyset$. So $cl_1(A) = A = int_1(A)$, and so $A, X \setminus A$ are T_1 open.

Similarly, A and $X \setminus A$ are T_2 open. Since $A \in T_1$, and $X \setminus A \in T_2$, then X is not p -connected.

1.2.20 Theorem :

Let (X, T_1, T_2) be a p -connected bitopological space, A be a nonempty proper subset of X , and $A \in (T_1 \cap T_2)$, then

$fr_i(A) \neq \emptyset; i = 1, 2$.

Proof:

Since $A \in (T_1 \cap T_2)$, then $A \in T_1$ and $A \in T_2$. But (X, T_1, T_2) is p -connected, so $X \setminus A \notin T_i; i = 1, 2$ i.e. A is not closed in T_i which implies that $A \neq cl_i(A); i = 1, 2$. Since $A \in T_1$ and $A \in T_2$ then $int_i(A) = A, i = 1, 2$.

So $fr_i(A) = cl_i(A) \setminus int_i(A) = cl_i(A) \setminus A \neq \emptyset; i = 1, 2$.

1.2.21 Remark :

The following example shows that the converse of the above theorem is not necessarily true.

1.2.22 Example :

Let $X = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $T_1 = \{X, \emptyset, \{1, 2\}, \{4\}, \{1, 2, 4\}\}$, and $T_2 = \{X, \emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$. Since $\text{int}_1(A) = A$ and $\text{cl}_1(A) = \{1, 2, 3\}$, therefore $\text{fr}_1(A) = \text{cl}_1(A) - A = \{3\} \neq \emptyset$. Also, since $\text{int}_2(A) = A$ and $\text{cl}_2(A) = \{1, 2, 4\}$, therefore $\text{fr}_2(A) = \text{cl}_2(A) - A = \{4\} \neq \emptyset$.

Hence $\text{fr}_i(A) \neq \emptyset$; $i = 1, 2$. But X is not p -connected; as $X = \{4\} \mid \{1, 2, 3\}$ is a p -disconnection of (X, T_1, T_2) .

Chapter 2

Pairwise Semi-Connected Spaces

2.1 Semi-open and semi-closed sets .

2.1.1 Definition : [12]

Let (X,T) be a topological space . A subset A of X is said to be semi-open if and only if there exists an open set U such that $U \subset A \subset cl(U)$.

2.1.2 Definition : [12]

Let (X,T) be a topological space . A subset A of X is said to be semi-closed if and only if A is the complement of a semi-open set.

2.1.3 Definition : [12]

Let (X,T) be a topological space , and let A be a subset of X . The intersection of all semi-closed sets which contain A is called the semi closure of A , and denoted by $scl(A)$.

2.1.4 Theorem :[13]

If A is a subset of a topological space (X, T) , then A is semi-open if and only if $A \subset \text{cl}(\text{int}(A))$.

Proof :

\Rightarrow) Suppose that A is semi-open. Then $B \subset A \subset \text{cl}(B)$ for some open set B . Since B is open set and contained in A , then $B \subset \text{int}(A)$ and thus $\text{cl}(B) \subset \text{cl}(\text{int}(A))$. Hence $A \subset \text{cl}(B) \subset \text{cl}(\text{int}(A))$.

\Leftarrow) Let $A \subset \text{cl}(\text{int}(A))$. Then for $B = \text{int}(A)$, we have $B \subset A \subset \text{cl}(B)$.

2.1.5 Theorem :

If A is a subset of a topological space (X, T) , then A is semi-closed if and only if there exists a closed set W such that $\text{int}(W) \subset A \subset W$.

Proof :

\Rightarrow) Suppose that A is semi-closed. Then $X \setminus A$ is semi-open, so $B \subset X \setminus A \subset \text{cl}(B)$, for some open set B .

Therefore $X \setminus \text{cl}(B) \subset A \subset X \setminus B$. Since $X \setminus \text{cl}(B)$ is open set and is contained in $X \setminus B$, then $X \setminus \text{cl}(B) \subset \text{int}(X \setminus B)$.

Want to show that $\text{int}(X \setminus B) \subset X \setminus \text{cl}(B)$. Let $x \in \text{int}(X \setminus B)$. Then there exists an open set E such that $x \in E$ and $E \cap B = \emptyset$. So $x \notin \text{cl}(B)$, and so $x \in X \setminus \text{cl}(B)$. Hence $\text{int}(X \setminus B) = X \setminus \text{cl}(B)$. Hence if we let $W = X \setminus B$, then $\text{int}(W) \subset A \subset W$, where W is a closed set.

\Leftarrow) Suppose $\text{int}(W) \subset A \subset W$ for some closed set W . Want to show A is semi-closed set. $X \setminus W \subset X \setminus A \subset X \setminus \text{int}(W)$.

Since $X \setminus \text{int}(W) = \text{cl}(X \setminus W)$, then $X \setminus A$ is semi-open. Hence A is semi-closed.

2.1.6 Lemma : [11]

If A is a subset of a topological space, then A is semi-closed if and only if $\text{int}(\text{cl}(A)) \subset A$.

Proof :

\Leftarrow) Let $\text{cl}(A) = B$, then $A \subset B$ and so $\text{int}(B) \subset A \subset B$.

\Rightarrow) Since A is semi-closed, then $\text{int}(W) \subset A \subset W$, for some closed set W . $\text{cl}(A) \subset \text{cl}(W) = W$. So $\text{int}(\text{cl}(A)) \subset \text{int}(W)$. So $\text{int}(\text{cl}(A)) \subset A$ as $\text{int}(W) \subset A$.

2.1.7 Lemma :

If $\{A_i : i \in \Omega\}$ is a collection of semi-closed subsets in a topological space (X, T) , and $A = \bigcap A_i$, then A is a semi-closed set.

Proof :

Let $i \in \Omega$, then A_i is semi-closed set, so there is a closed set W_i such that $\text{int}(W_i) \subset A_i \subset W_i$. So $\bigcap (\text{int}(W_i)) \subset \bigcap A_i \subset \bigcap W_i$.

Let $W = \bigcap W_i$, then W is a closed subset of X . Also, $W \subset W_i, \forall i \in \Omega$, so $\text{int}(W) \subset \text{int}(W_i), \forall i \in \Omega$. So $\text{int}(W) \subset \bigcap (\text{int}(W_i))$. Hence

$\text{int}(W) \subset A \subset W$, and W is a closed subset of X . So A is a semi-closed set .

2.1.8 Theorem : [8]

$\text{scl}(A) = A$ if and only if A is semi-closed.

Proof :

\Rightarrow) A is an intersection of semi-closed sets , so by the previous lemma , A is a semi-closed set.

\Leftarrow) By the definition of $\text{scl}(A)$, $A \subset \text{scl}(A)$. But A is semi-closed set containing A , so by the definition of $\text{scl}(A)$ we have $\text{scl}(A) \subset A$. Hence $\text{scl}(A) = A$.

2.2 Pairwise Semi-separation .

2.2.1 Definition: [2]

Two non-empty subsets A and B of (X, T_1, T_2) are said to be pairwise semi-separated if and only if $A \cap \text{scl}_1(B) = \emptyset = \text{scl}_2(A) \cap B$.

2.2.2 Definition : [2]

Two nonempty subsets A and B are said to form a pairwise semi-separation of X , if $X = A \cup B$ and A and B are pairwise semi-separated.

2.2.3 Theorem :[2]

If A and B are pairwise semi separated subsets of (X, T_1, T_2) and $A_1 \subset A$, $B_1 \subset B$ then A_1 and B_1 are pairwise semi separated.

Proof:

Since A and B are pairwise semi-separated , then we have $A \cap scl_1(B) = \emptyset = B \cap scl_2(A)$. But $A_1 \cap scl_1(B_1) \subset A \cap scl_1(B)$ and $B_1 \cap scl_2(A_1) \subset B \cap scl_2(A)$. so $A_1 \cap scl_1(B_1) = \emptyset$ and $B_1 \cap scl_2(A_1) = \emptyset$. So A_1 and B_1 are pairwise semi- separated.

2.2.4 Theorem :

If A and B are subsets of (X, T_1, T_2) such that $A \cap B = \emptyset$, A is T_1 -semi open and B is T_2 -semi open, then A and B are pairwise semi separated .

Proof:

We have $A \cap scl_1(B) \subset A \cap scl_1(X \setminus A) = A \cap (X \setminus A) = \emptyset$. And $B \cap scl_2(A) \subset B \cap scl_2(X \setminus B) = B \cap (X \setminus B) = \emptyset$. Thus A and B are pairwise semi separated .

2.2.5 Remark :

A subset A of (X, T_1, T_2) is T_i -semi-open if and only if $X \setminus A$ is T_i -semi-closed , $i = 1, 2$. So the following theorem follows directly from theorem 4 .

2.2.6 Theorem :

If A and B are nonempty subset of (X, T_1, T_2) such that $A \cap B = \emptyset$ where A is T_2 -semi-closed and B is T_1 -semi closed then A and B are pairwise semi-separated .

2.2.7 Theorem : [2]

If A and B are pairwise semi-separated subsets of (X, T_1, T_2) such that $A \cup B$ is T_1 -semi-closed as well as T_2 -semi-closed, then A is T_2 -semi-closed and B is T_1 -semi-closed .

Proof :

As A and B are pairwise semi-separated, we get

$A \cap scl_1(B) = \emptyset = B \cap scl_2(A)$. Also as $A \cup B$ is T_2 -semi-closed ,

so $scl_2(A) \subset scl_2(A \cup B) = A \cup B$.

Now $scl_2(A) = scl_2(A) \cap scl_2(A) \subset (A \cup B) \cap scl_2(A)$

$$= (A \cap scl_2(A)) \cup (B \cap scl_2(A)) = A ,$$

as $B \cap scl_2(A) = \emptyset$ and $A \subset scl_2(A)$. Thus A is T_2 -semi-closed, by

(Th2.1.8) . Similarly we show that B is T_1 -semi-closed .

2.2.8 Theorem : [12]

If A and Y are subsets of a topological space (X, T) such that $A \subset Y \subset X$ and A is T.semi- open, then A is T_Y semi-open.

Proof:

Since A is T -semi-open, then there exists $B \in T$ such that $B \subset A \subset \text{cl}(B)$. Then $B \in T_Y$ and $B \subset A \subset \text{cl}(B) \cap Y = \text{cl}_Y(B)$, which implies that A is T_Y -semi-open.

2.2.9 Theorem : [1]

If A and Y are subsets of a topological space (X, T) such that $A \subset Y$ and Y is open, then A is T_Y -semi-open if and only if A is T semi-open.

Proof:

\Rightarrow) Suppose that A is T_Y -semi-open. Then there exists $B \in T_Y$ such that $B \subset A \subset \text{cl}_Y(B)$. Since $B \in T_Y$ and $Y \in T$, then $B \in T$ and $B \subset A \subset \text{cl}_Y(B) \subset \text{cl}(B)$, which implies that A is T -semi-open.

\Leftarrow) The converse follows from Theorem 8.

2.2.10 Lemma : [8]

Let (X, T) be a topological space, let A be an open subset of X , and let B be a semi-open subset of X . Then $A \cap B$ is a semi-open subset in (A, T_A) .

Proof:

B is a semi-open subset of X , so there exist $U \in T$ such that

$U \subset B \subset \text{cl}(U)$. Let $x \in (A \cap B) \setminus (U \cap A)$. Suppose that $x \notin \text{cl}_A(U \cap A)$. Then there exist $W \in T_A$ such that $x \in W$ and $W \cap (U \cap A) = \emptyset$. So $W \cap U = \emptyset$ and this implies that $x \notin \text{cl}(U)$, and so $x \notin B$. So $x \notin (A \cap B)$ and this is a contradiction, so $A \cap B \subset \text{cl}_A(U \cap A)$. Hence $(U \cap A) \subset (A \cap B) \subset \text{cl}_A(U \cap A)$, and $(U \cap A) \in T_A$. So $(A \cap B)$ is a semi-open set in (A, T_A) .

2.2.11 Lemma : [1]

Let Y be a subset of the topological space (X, T) , and let A be a T_Y semi-open subset of Y . Then $A = M \cap Y$ for some T semi-open set M .

Proof :

Since A is T_Y semi-open then there exist $U \in T_Y$ such that $U \subset A \subset \text{cl}_Y(U)$. But $U \in T_Y$, so $U = B \cap Y$, for some T -open set B . So $A \cup B \subset \text{cl}_Y(U) \cup B \subset \text{cl}(B)$. Since $B \subset A \cup B \subset \text{cl}(B)$, then $A \cup B$ is semi-open in (X, T) . Let $M = A \cup B$, then $M \cap Y = A$.

2.2.12 Lemma :

Let A be a subset of the topological space (X, T) , and let M be a T_A semi-closed subset of A . Then $M = A \cap F$ for some T semi-closed set F .

Proof :

Follows from the previous lemma and the fact if $A \setminus M = U \cap A$, then $M = (X \setminus U) \cap A$.

2.2.13 Note :

The converse of the last two lemmas are not necessarily true , as the following examples show .

2.2.14 Example:

Let $X = \{1,2,3,4\}$, $T = \{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}\}$, $F = \{3,4\}$, and $A = \{2,3\}$. Then $F \cap A = \{3\}$. Now , the set of all closed sets in $(X, T) = \{X, \emptyset, \{2,3,4\}, \{1,2\}, \{2\}\}$. $T_A = \{A, \emptyset, \{3\}\}$, and the set of closed sets in $(A, T_A) = \{A, \emptyset, \{2\}\}$. Let $M = \{2,3,4\}$. Then M is closed set in (X, T) and $\text{int}(M) = \{3,4\}$. So $\text{int}(M) \subset F \subset M$. And so , F is a semi-closed set in (X, T) . But the only T_A closed set containing $\{3\}$ is A , and $\text{int}_A(A) = A \not\subset \{3\}$. So $\{3\}$ is not T_A -semi-closed set . i . e $F \cap A$ is not T_A -semi-closed set in A .

2.2.15 Example :

In the last example , $\{1,2\}$ is T -semi-open , but $\{1,2\} \cap A = \{2\}$ is not T_A -semi-open .

2.2.16 Theorem : [1]

If A and Y are subsets of a topological space (X, T) such that $A \subset Y$ and Y is open , then $\text{scl}_Y(A) = Y \cap \text{scl}(A)$.

Proof:

Since $scl_Y(A)$ is T_Y semi-closed, then by (lem.2.2.12) there exists a T semi-closed B such that $scl_Y(A) = B \cap Y$. Since $A \subset B$, we have $T-scl(A) \subset B$ and hence $Y \cap T-scl(A) \subset Y \cap B = scl_Y(A)$.

Let $x \in scl_Y(A)$. Then $x \in Y$. To show that $x \in scl(A)$. Let W be any semi-open set in (X, T) containing x , as Y is open in (X, T) . Then by (Lem.2.2.10) $W \cap Y$ is a semi-open subset in (Y, T_Y) containing x . But $x \in scl_Y(A)$, so $(W \cap Y) \cap A \neq \emptyset$. Hence $W \cap A \neq \emptyset$. So $x \in scl(A)$. And so $x \in scl(A) \cap Y$.

Hence $scl_Y(A) \subset scl(A) \cap Y$. Then $scl_Y(A) = scl(A) \cap Y$.

2.2.17 Theorem: [2]

If A and B are pairwise semi-separated in a bitopological space (X, T_1, T_2) and if $A \cup B$ is both T_1 open and T_2 open, then A is T_1 -semi-open and B is T_2 -semi-open.

Proof:

Let $T^*_1 = T_1|_{A \cup B}$ and $T^*_2 = T_2|_{A \cup B}$. As A and B are semi-separated, $A \cap scl_1(B) = \emptyset = B \cap scl_2(A)$. And so by (Th.2.2.16), $A \cap scl_{\bullet 1}(B) = \emptyset = B \cap scl_{\bullet 2}(A)$, so we have $scl_{\bullet 1}(B) \subset B$, and $scl_{\bullet 2}(A) \subset A$.

So by (Th.2.1.8), A is T^*_2 -semi-closed and B is T^*_1 -semi-closed which again imply by (Re.2.1.2) that A is T^*_1 -semi open and B is T^*_2 -semi open. Now, as $A \cup B \in T_1$ and A is T^*_1 -semi open, then A is T_1 -

semi open by (Th.2.2.9) . Similarly B is T_2 -semi open .

2.2.18 Remark : [2]

If A and B are pairwise semi-separated and if $A \cup B$ is both T_1 -semi-open and T_2 -semi-open , then A is not necessarily T_1 -semi-open and B is not necessarily T_2 -semi-open . This is shown by the following example .

2.2.19 Example :

Let us consider the bitopological space (X, T_1, T_2) , where $X = \{a, b, c, d\}$.

$T_1 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $T_2 = \{X, \emptyset, \{d\}, \{b, c\}, \{b, c, d\}\}$.

It is easy to see that

$C(T_1) = \{X, \emptyset, \{b, c, d\}, \{a, d\}, \{d\}\}$,

$C(T_2) = \{X, \emptyset, \{a, b, c\}, \{a, d\}, \{a\}\}$,

$S.O(T_1) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}\}$,

$S.O(T_2) = \{X, \emptyset, \{d\}, \{b, c\}, \{b, c, d\}, \{a, d\}, \{a, b, c\}\}$,

$S.C(T_1) = \{X, \emptyset, \{b, c, d\}, \{a, d\}, \{d\}, \{b, c\}, \{a\}\}$,

$S.C(T_2) = \{X, \emptyset, \{a, b, c\}, \{a, d\}, \{a\}, \{b, c\}, \{d\}\}$. Let $A = \{d\}$ and $B = \{a\}$.

Then A and B are pairwise semi-separated . Although

$A \cup B = \{a, d\}$ is both T_1 -semi-open and T_2 -semi-open . But A is not T_1 -semi-open and B is not T_2 -semi-open .

2.2.20 Definition: [9]

A bitopological space (X, T_1, T_2) is called pairwise semi-connected if and only if X can not be expressed as the union of two nonempty pairwise semi-separated sets .

2.2. 21 Theorem : [9]

A bitopological space (X, T_1, T_2) is pairwise semi-connected if and only if X can not be expressed as the union of nonempty disjoint sets one of which is T_1 -semi-open and the other is T_2 -semi-open.

Proof:

\Rightarrow) Suppose that $X = A \cup B$, where A is T_1 -semi-open and B is T_2 -semi-open . Since A is T_1 -semi-open , then B is T_1 -semi-closed and so $A \cap scl_1(B) = \emptyset$. And since B is T_2 -semi-open , then A is T_2 -semi-closed and $B \cap scl_2(A) = \emptyset$. This implies that (X, T_1, T_2) is not pairwise semi-connected and this is a contradiction . Hence X cannot be expressed as the union of two non empty disjoint sets one of which is T_1 - semi-open and the other is T_2 -semi-open .

\Leftarrow) Suppose that (X, T_1, T_2) is not pairwise semi-connected .

Then $X = A \cup B$, for some pairwise semi-separated sets A and B .

Since $A \cap scl_1(B) = \emptyset$ and $A \cup B = X$, then $B = scl_1(B)$, and so B is T_1 - semi-closed and so A is T_1 -semi-open . Similarly we see that B is T_2 -semi-open and this is a contradiction with the assumption.

2.2.22 Remark :

Every pairwise semi-connected space is pairwise connected, as every T_i open set is T_i semi-open ; $i = 1,2$. But the converse is not necessarily true as shown by the following example.

2.2.23 Example :

Let us consider the bitopological space (X, T_1, T_2) , where $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$, and $T_2 = \{X, \emptyset, \{b\}, \{a, b\}\}$. It is easy to see that

$$C(T_1) = \{X, \emptyset, \{b, c\}, \{c\}\}.$$

$$C(T_2) = \{X, \emptyset, \{a, c\}, \{c\}\}.$$

$$S.O(T_1) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$$

$$S.O(T_2) = \{X, \emptyset, \{b\}, \{b, a\}, \{b, c\}\}.$$

$$S.C.(T_1) = \{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}, \text{ and}$$

$$S.C.(T_2) = \{X, \emptyset, \{a, c\}, \{c\}, \{a\}\}.$$

Clearly X is p -connected . But $X = \{a\} \cup \{b, c\}$ is a pairwise semi-separation of X , and so (X, T_1, T_2) is not pairwise semi-connected .

2.2.24 Remark :

Pairwise semi-connectedness for a bitopological space is not equivalent to the semi connectedness of the two topologies , as shown by the following Example .

2.2.25 Example :

Consider the bitopological space (X, T_1, T_2) , where $X = \{a, b\}$, $T_1 = \{X, \emptyset, \{a\}\}$, and $T_2 = \{X, \emptyset, \{b\}\}$. Both (X, T_1) and (X, T_2) are semi connected, but (X, T_1, T_2) is not pairwise semi-connected as $X = \{a\} \cup \{b\}$ is a pairwise semi-separation of X .

2.3 $T_1 T_2$ pairwise semi-connected sets

2.3.1 Definition: [9]

A subset A of (X, T_1, T_2) is said to be $T_1 T_2$ -pairwise semi-connected if and only if A can not be expressed as the union of two pairwise-semi separated sets in (X, T_1, T_2) .

2.3.2 Theorem : [9]

Let A be a $T_1 T_2$ -pairwise semi-connected. If $A \subset C \cup D$ where C and D are pairwise semi-separated sets in (X, T_1, T_2) , then either $A \subset C$ or $A \subset D$.

Proof:

Suppose $A \cap C \neq \emptyset$ and $A \cap D \neq \emptyset$. Since $A \cap C \subset C$, then $scl_2(A \cap C) \subset scl_2(C)$. Similarly $scl_1(A \cap D) \subset scl_1(D)$.

Since $C \cap scl_1(D) = \emptyset$ and $D \cap scl_2(C) = \emptyset$, and since $(A \cap C) \subset C$ and $scl_1(A \cap D) \subset scl_1(D)$, then $(A \cap C) \cap scl_1(A \cap D) = \emptyset$, and similarly, $(A \cap D) \cap scl_2(A \cap C) = \emptyset$. This implies that A is not T_1T_2 -pairwise semi-connected, and this is a contradiction with the assumption. Hence $A \cap D = \emptyset$ or $A \cap C = \emptyset$, and so $A \subset C$ or $A \subset D$.

2.3.3 Theorem: [2]

The union E of any family $\{C_i : i \in \Omega\}$ of T_1T_2 -pairwise semi-connected sets in (X, T_1, T_2) having a non-empty intersection is T_1T_2 -pairwise semi-connected.

Proof:

Let $E = \{C_i : i \in \Omega\}$ be a family of a nonempty intersect T_1T_2 -pairwise semi-connected sets. If $E = \bigcup C_i$ is not T_1T_2 -pairwise semi-connected, it must have a pairwise semi-separation $E = A|B$ in X . By hypothesis we may choose a point x belongs to $\bigcap C_i$. The point x must belong to either A or B . Suppose x belong to A . Since x belong to C_i for each $i \in \Omega$, then $(\bigcap C_i) \cap A \neq \emptyset$. Since C_i is T_1T_2 -pairwise semi-connected and is a subset of $E = A|B$ in X , then C_i must be either a subset of A or a subset of B for each $i \in \Omega$.

Since $C_i \cap A \neq \emptyset$ and $A \cap B = \emptyset$, then $C_i \subset A; \forall i \in \Omega$, which implies that $E = \bigcup C_i \subset A$. So $E \subset A$. So $B = \emptyset$, which is a contradiction. Hence E is T_1T_2 -pairwise semi-connected.

2.3.4 Theorem :

If A is a T_1T_2 -pairwise semi connected subset of (X, T_1, T_2) , and if $A \subset E \subset [scl_1(A) \cap scl_2(A)]$, then E is T_1T_2 -pairwise semi-connected.

Proof:

If E is not T_1T_2 -pairwise semi-connected, we can write $E = G \cup H$, where $G \neq \emptyset$, $H \neq \emptyset$, and $G \cap scl_1(H) = \emptyset$ and $H \cap scl_2(G) = \emptyset$. By (Th.3.2.2), either $A \subset G$ or $A \subset H$. We suppose that $A \subset G$. Then

$$\begin{aligned} H &\subset E && ; \text{ as } E = H \cup G \\ &\subset scl_2(A) && ; \text{ as } E \subset (scl_1(A) \cap scl_2(A)) \\ &\subset scl_2(G) && ; \text{ as } A \subset G \end{aligned}$$

So $H \subset scl_2(G)$. But $H \cap scl_2(G) = \emptyset$, so $H = \emptyset$.

Which is a contradiction with the fact that $H \neq \emptyset$. Hence E must be semi connected.

2.3.5 Corollary :

If E is a T_1T_2 -pairwise semi connected subset of (X, T_1, T_2) , then $scl_1(E) \cap scl_2(E)$ is T_1T_2 -pairwise semi connected.

2.3.6 Theorem : [2]

The union of two T_1T_2 -pairwise semi connected subsets A and B of a bitopological space (X, T_1, T_2) which are not pairwise semi-separated in X

is T_1T_2 -pairwise semi connected.

Proof:

If $A \cup B$ is not T_1T_2 -pairwise semi-connected, then we can write $A \cup B = G \cup H$; where $G \neq \emptyset$, $H \neq \emptyset$, $G \cap scl_1(H) = \emptyset$ and $H \cap scl_2(G) = \emptyset$. By (Th.2.3.2), $(A \subset G \text{ or } A \subset H)$ and $(B \subset G \text{ or } B \subset H)$.

Let $A \subset G$. Then, as H is not empty, $B \subset H$, and so $A \cap scl_1(B) \subset G \cap scl_1(H) = \emptyset$, and $B \cap scl_2(A) \subset H \cap scl_2(G) = \emptyset$. So A and B are pairwise semi-separated in X . The contradiction obtained proves the theorem.

2.3.7 Theorem : [2]

Let (X, T_1, T_2) be a bitopological space such that for any $A \subset X$, $cl_2(cl_1(A)) = cl_1(A)$ and $cl_1(cl_2(A)) = cl_2(A)$. Then X is pairwise semi-connected if and only if there are no two non-empty disjoint sets one of which is T_1 -semi open and the other is T_2 -semi open.

Proof:

\Rightarrow) First we suppose that (X, T_1, T_2) is pairwise semi connected. Suppose that there exist two non-empty disjoint sets U and V where U is T_1 -semi open and V is T_2 -semi open. By definition 2.1.1 we have two non-empty disjoint sets C and D such that $C \subset U \subset cl_1(C)$ and $D \subset V \subset cl_2(D)$ and where $C \in T_1$ and $D \in T_2$.

Now $X = cl_1(C) \cup (X \setminus cl_1(C))$

2.4 Pairwise semi-connected sets

2.4.1 Theorem : [9]

If a subset A of (X, T_1, T_2) is $T_1 T_2$ -pairwise semi-connected, then (A, T_{1A}, T_{2A}) is pairwise semi-connected.

Proof:

In fact if $U|V$ is a pairwise semi-separation of A in (A, T_{1A}, T_{2A}) , then it is also a pairwise semi-separation of A in X , since $scl_1(B) \cap A \subset scl_{1A}(B)$ for every subset B of A .

2.4.2 Remark :

The converse of theorem 1 need not be true as can be seen from the following example.

2.4.3 Example :

Let $X = \{a, b, c, d\}$, $T_1 = \{X, \emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$ and

$T_2 = \{X, \emptyset, \{b, d\}, \{a, c, d\}, \{d\}\}$.

Let $A = \{a, b, c\}$. Then $T_{1A} = \{A, \emptyset, \{a, b\}, \{c\}\}$ and

$T_{2A} = \{A, \emptyset, \{a, c\}, \{b\}\}$. Clearly (A, T_{1A}, T_{2A}) is a pairwise semi-connected space, but A is not $T_1 T_2$ pairwise semi-connected in X

because the sets $\{a,b\}$ and $\{c\}$ are pairwise semi-separated in X and $A = \{a,b\} \cup \{c\}$.

2.4.4 Definition: [1]

A subset A of a topological space X is said to be preopen if $A \subset \text{int}(\text{cl}(A))$.

2.4.5 Remark:

It is clear that every open set is a preopen set.

2.4.6 Definition:

A subset A of a bitopological space (X, T_1, T_2) is said to be preopen if $A \subset \text{int}_1(\text{cl}_1(A))$ and $A \subset \text{int}_2(\text{cl}_2(A))$.

2.4.7 Definition:

A subset Y of (X, T_1, T_2) is said to be pairwise s -connected if and only if (Y, T_{1Y}, T_{2Y}) is pairwise semi-connected.

2.4.8 Lemma:

Let $A \subset B \subset X$ in a topological space (X, T) . If $B = \text{int}(\text{cl}(A))$, then A is dense in (B, T_B)

Proof:

Let U be a nonempty open set in (B, T_B) , then there exists $x \in U$ and U is open set in (X, T) as $B \in T$. Also, $x \in B$, so $x \in \text{cl}(A)$ and so $U \cap A \neq \emptyset$. Hence A is dense in (B, T_B) .

2.4.9 Lemma:

Let Y be a dense subset of a topological space (X, T) . Let M be a semi-open subset of X , then $M \cap Y$ is a semi-open subset in (Y, T_Y) .

Proof:

Since M is semi-open subset of X , so $U \subset M \subset \text{cl}(U)$, for some open subset U of X . So $Y \cap U \subset Y \cap M$ and $Y \cap U \in T_Y$. Now, let $x \in Y \cap M$. To show that $x \in \text{cl}_Y(Y \cap U)$. Let $x \in W \in T_Y$. Then $W = V \cap Y$ for some $V \in T$. $x \in M \subset \text{cl}(U)$, so $x \in \text{cl}(U)$. But $x \in V \in T$, so $U \cap V \neq \emptyset$. But Y is dense in X and $U \cap V$ is a nonempty open set in (X, T) , so $(U \cap V) \cap Y \neq \emptyset$. Hence $(V \cap Y) \cap (U \cap Y) \neq \emptyset$. i.e. $W \cap (U \cap Y) \neq \emptyset$.

Hence $x \in \text{cl}_Y(U \cap Y)$. So $U \cap Y \subset Y \cap M \subset \text{cl}_Y(U \cap Y)$, and so $Y \cap M$ is semi-open in (Y, T_Y) .

2.4.10 Theorem:

Let Y be a preopen subset of a topological space (X, T) , and let B be a semi-open subset of X , then $Y \cap B$ is semi-open in Y .

Proof :

Follows directly from last two lemmas .

2.4.11 Lemma: [10]

Let Y be preopen subset of a topological space (X,T) , and let B be a semi-closed subset of X , then $Y \cap B$ is semi-closed in Y .

Proof :

Since $Y \subset \text{int}(\text{cl}Y)$ and $X \setminus B$ is semi-open, then by the last lemma, $Y \cap (X \setminus B)$ is semi-open in Y . So $A \subset [Y \cap (X \setminus B)] \subset \text{cl}_Y(A)$. Since $Y \setminus (Y \cap B) = Y \cap (X \setminus B)$ then $A \subset Y \setminus (Y \cap B) \subset \text{cl}_Y(A)$. So $Y \cap B$ semi-closed .

2.4.12 Theorem : [1]

If A and Y are subsets of a topological space (X,T) such that $A \subset Y$ and Y is preopen , then $\text{scl}_Y(A) = Y \cap \text{scl}(A)$.

Proof :

Since $\text{scl}_Y(A)$ is T_Y semi-closed , then by (lem.2.2.12) there exists a T semi-closed B such that $\text{scl}_Y(A) = B \cap Y$. Since $A \subset B$, we have $T\text{-scl}(A) \subset B$ and hence $Y \cap T\text{-scl}(A) \subset Y \cap B = \text{scl}_Y(A)$. Since Y is preopen set and $T\text{-scl}(A)$ is T -semi closed then by (lem.2.4.11) $Y \cap T\text{-scl}(A)$ is T_Y semi-closed . Since $A \subset Y \cap T\text{-scl}(A)$, then $\text{scl}_Y(A) \subset Y \cap (T\text{-scl}(A))$.

2.4.13 Theorem : [9]

If Y is a preopen subset of (X, T_1, T_2) , then Y is $T_1 T_2$ -pairwise semi-connected if and only if (Y, T_{1Y}, T_{2Y}) is pairwise semi-connected.

Proof :

\Rightarrow) Direct by (Th.2.4.1) .

\Leftarrow) Suppose that Y is not $T_1 T_2$ -pairwise semi-connected, then $Y = U \cup V$, $U \cap (\text{scl}_1(V)) = \emptyset = (\text{scl}_2(U)) \cap V$. Since $U \subset Y$ and Y is preopen then, by (Th.2.4.12) $\text{scl}_{2Y}(U) = Y \cap \text{scl}_2(U)$ and $\text{scl}_{1Y}(V) = Y \cap \text{scl}_1(V)$. So $\text{scl}_{2Y}(U) \subset \text{scl}_2(U)$ and $\text{scl}_{1Y}(V) \subset \text{scl}_1(V)$. Thus $U \cap \text{scl}_{1Y}(V) = \emptyset = \text{scl}_{2Y}(U) \cap V$. So U and V are pairwise semi separation of (Y, T_{1Y}, T_{2Y}) and this is a contradiction .

2.4.14 Lemma :

Let A be a preopen subset of (X, T_1, T_2) . Then A is $T_1 T_2$ -pairwise semi-connected if and only if A is pairwise s-connected.

Proof :

\Rightarrow) Direct by (Th.2.4.13) and (Def. 2.4.7), so A is pairwise semi-connected.

\Leftarrow) Since A is pairwise s-connected then by (Def.2.4.7), (A, T_{1A}, T_{2A}) is pairwise semi-connected, and also by (Th.2.4.13) A is $T_1 T_2$ -pairwise semi-connected .

2.4.15 Theorem : [11]

If A is a pairwise s -connected preopen set in (X, T_1, T_2) , and if $A \subset C \cup D$, where C and D are pairwise semi-separated sets in X , then $A \subset C$ or $C \subset D$.

Proof:

The proof is direct by using (Th.2.4.13) and (Th.2.3.2).

2.4.16 Definition : [11]

A function $f : X \rightarrow Y$ is said to be irresolute if $f^{-1}(A)$ is semi-open for every semi-open subset A of Y .

2.4.17 Definition : [9]

A function $f : X \rightarrow Y$ is said to be semi continuous if $f^{-1}(A)$ is semi-open for every open subset A of Y .

2.4.18 Theorem : [9]

Let $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ be a semi-continuous mapping from X onto Y . Then (Y, T^*_1, T^*_2) is pairwise connected if (X, T_1, T_2) is pairwise semi-connected.

Proof:

Suppose that Y is not pairwise connected, then $Y = A \cup B$ is

pairwise separation of Y for some subset A and B of Y . A and B are non-empty subset of Y such that $Y = A \cup B$ where $A \in T^*_1$ and $B \in T^*_2$. By pairwise semi-continuity of f , $f^{-1}(A)$ is T_1 -semi-open and $f^{-1}(B)$ is T_2 -semi-open. Also $X = f^{-1}(A) \cup f^{-1}(B)$, $f^{-1}(A) \neq \emptyset$, $f^{-1}(B) \neq \emptyset$, and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus X is not pairwise semi-connected, which is a contradiction.

2.4.19 Lemma : [8]

Let (X, T) and (Y, T^*) be topological spaces. If $f : (X, T) \rightarrow (Y, T^*)$ is continuous and open, then $f^{-1}(\text{cl}(A)) = \text{cl}(f^{-1}(A))$, for every subset A of Y .

Proof:

Let A be any subset of (Y, T) , then $A \subset \text{cl}(A)$ and $f^{-1}(A) \subset f^{-1}(\text{cl}(A))$. So $\text{cl}(f^{-1}(A)) \subset \text{cl}(f^{-1}(\text{cl}(A))) = f^{-1}(\text{cl}(A))$, by the continuity of f . Hence $\text{cl}(f^{-1}(A)) \subset f^{-1}(\text{cl}(A))$.

Let $x \in f^{-1}(\text{cl}(A))$, and let U be arbitrary open set in X containing x . Then $f(x) \in \text{cl}(A)$, and $f(U) \in T^*$, as f is open function. Now, $f(x) \in f(U) \in T^*$ and $f(x) \in \text{cl}(A)$, so $f(U) \cap A \neq \emptyset$, and so $U \cap f^{-1}(A) \neq \emptyset$. So $x \in \text{cl}(f^{-1}(A))$. Hence $f^{-1}(\text{cl}(A)) \subset \text{cl}(f^{-1}(A))$.

This completes the proof of the lemma.

2.4.20 Theorem:

If $f : (X, T) \rightarrow (Y, T^*)$ is continuous and open where X and Y are topological spaces, then f is irresolute.

Proof:

Let A is a semi-open subset of Y , then there is an open set B in Y such that $B \subset A \subset \text{cl}(B)$. By the above lemma, $f^{-1}(\text{cl}(B)) = \text{cl}(f^{-1}(B))$. Also, $f^{-1}(B) \subset f^{-1}(A) \subset f^{-1}(\text{cl}(B)) = \text{cl}(f^{-1}(B))$. And since f is continuous, $f^{-1}(B)$ is open. Thus $f^{-1}(A)$ is a semi-open subset of X .

2.4.21 Example :

A continuous, irresolute function need not be open. Let $X = \{a, b, c\}$, and consider the topologies $T = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $T^* = \{X, \emptyset, \{a\}, \{a, b\}\}$.

Consider the function $f: (X, T) \rightarrow (X, T^*)$ defined by $f(x) = x$ for all $x \in X$.

$$C(T) = \{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}.$$

$$S.O(T) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}.$$

$$S.C(T) = \{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}.$$

$$C(T^*) = \{X, \emptyset, \{b, c\}, \{c\}\}.$$

$$S.O(T^*) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}.$$

It is clear that f is continuous and irresolute function, but the function f is not open, as $f(\{a, c\})$ is not open in (X, T^*) .

2.4.22 Theorem : [9]

If f is open , semi continuous and onto mapping from (X,T) to (Y,T^*) , then f is irresolute .

Proof:

Let B be an arbitrary semi-open set in Y , then there exist an open set V in Y such that $V \subset B \subset \text{cl}(V)$. Since f is semi-continuous and V is open, $f^{-1}(V)$ is semi-open . Thus we obtain that $f^{-1}(V)$ is semi-open and $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(\text{cl}(V)) \subset \text{cl}(f^{-1}(V))$ as f is open , so that $f^{-1}(B)$ is a semi-open set in X .

2.4.23 Theorem : [9]

The irresolute image of a pairwise semi-connected bitopological space is pairwise semi-connected.

Proof:

Let $f : (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ be an irresolute from the pairwise semi-connected bitopological space (X, T_1, T_2) onto the bitopological space (Y, T^*_1, T^*_2) . Suppose that Y is not pairwise semi-connected . Then $Y = A|B$ is pairwise semi separation of Y for some subsets A and B of Y . Then A and B are non- empty disjoint subsets of Y such that $Y = A \cup B$, where A is T^*_1 semi-open and B is T^*_2 semi-open . But f is irresolute, so $f^{-1}(A)$ is T_1 -semi-open and $f^{-1}(B)$ is T_2 -semi-open . Also $X = f^{-1}(A) \cup f^{-1}(B)$, $f^{-1}(A) \neq \emptyset$, $f^{-1}(B) \neq \emptyset$, and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus (X, T_1, T_2) is not pairwise semi-connected , which is a contradiction.

The following theorem follows directly from theorem (2.4.22) and theorem (2.4.23) .

2.4.24 Theorem : [9]

If f is an open , semi continuous and onto mapping from (X, T_1, T_2) to (Y, T^*_1, T^*_2) , and if (X, T_1, T_2) is pairwise semi connected , then (Y, T^*_1, T^*_2) is also pairwise semi connected .

2.4.25 Corollary :

Let $f: (X, T) \rightarrow (Y, T^*)$ be an open , and an irresolute function, then $f|_A: (A, T_A) \rightarrow (f(A), T^*_{f(A)})$ is irresolute , if A is an open set in X .

Proof :

Let V be a semi-open set in $(f(A), T^*_{f(A)})$. Then by (Lem.2.2.11) $V = U \cap f(A)$ for some T^* semi-open subset U of Y . f is an irresolute , so $f^{-1}(U)$ is T semi-open subset of X . Now $A \in T$ and $f^{-1}(U)$ is T semi-open subset of X , so by (Lem. 2.2.10) , $A \cap f^{-1}(U)$ is semi-open in

$$(A, T_A). \text{ But } (f|_A)^{-1}(V) = A \cap f^{-1}(V) = A \cap f^{-1}(U \cap f(A))$$

$$= A \cap f^{-1}(U) \cap f^{-1}(f(A)) = A \cap f^{-1}(U) , \text{ as}$$

$A \subset f^{-1}(f(A))$. Hence $(f|_A)^{-1}(V)$ is semi-open set in (A, T_A) . This shows that $f : (A, T_A) \rightarrow (f(A), T^*_{f(A)})$ is an irresolute .

2.4.26 Theorem : [9]

Let (X, T_1, T_2) and (Y, T^*_1, T^*_2) be two bitopological spaces .

If $f: (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ is open, irresolute surjection, and if A is an open, $T_1 T_2$ pairwise semi-connected subset of (X, T_1, T_2) , then $f(A)$ is $T_1 T_2$ pairwise semi-connected subset of (Y, T^*_1, T^*_2) .

Proof:

Since A is $T_1 T_2$ pairwise semi-connected subset of (X, T_1, T_2) , then (A, T_{1A}, T_{2A}) is pairwise semi-connected by (Th.2.4.1). Now since A is open then $f|_A : (A, T_{1A}, T_{2A}) \rightarrow (f(A), T^*_{1A}, T^*_{2A})$ is a irresolute surjection and so by (Th.2.4.23), $(f(A), T^*_{1A}, T^*_{2A})$ is a pairwise semi-connected space. Since f is open, $f(A)$ is open in (Y, T^*_1, T^*_2) and so by (Th.2.4.13), $f(A)$ is $T_1 T_2$ pairwise semi-connected in (Y, T^*_1, T^*_2) .

Chapter 3

Totally disconnectedness in bitopological spaces

3.1 Pairwise separation axioms .

3.1.1 Definition: [4]

A bitopological space (X, T_1, T_2) is called pairwise totally disconnected space if for every two distinct points x and y in X , there exists a p -disconnection $X=A|B$ for X , with x belongs to A and y belongs to B .

3.1.2 Definition : [4]

A bitopological space (X, T_1, T_2) is called pairwise weakly totally disconnected if for each two distinct points x and y in X , there exists a p -disconnection $X=A|B$ for X such that $x \in A$ and $y \in B$ or $y \in A$ and $x \in B$.

3.1.3 Definition : [4]

A bitopological space (X, T_1, T_2) is called pairwise Hausdorff space if and only if for every two distinct points x and y in X there are T_1

neighbourhood U of x and T_2 neighbourhood V of y such that $U \cap V = \emptyset$.

3.1.4 Definition : [4]

A space (X, T_1, T_2) is said to be pairwise weakly Hausdorff, if for each two distinct points x and y in X , at least one has a T_1 neighbourhood U and the other has a T_2 neighbourhood V satisfying $U \cap V = \emptyset$.

The following results follow directly from the above definitions.

3.1.5 Theorem: [4]

Every pairwise totally disconnected bitopological space is pairwise Hausdorff space.

3.1.6 Theorem: [4]

Every pairwise weakly totally disconnected bitopological space is pairwise weakly Hausdorff space.

3.1.7 Theorem: [4]

If (X, T_1, T_2) is pairwise weakly totally disconnected or pairwise totally disconnected space containing more than one point then the space is p -disconnected.

3.1.8 Corollary : [4]

Every pairwise totally disconnected space is pairwise weakly totally disconnected .

The following example shows that the converse of the above corollary is not necessarily true.

3.1.9 Example: [4]

Consider the bitopological space (Q, Q_L, Q_R) . Let x and z be any two distinct points in Q , and suppose $x > z$. Then there exists an irrational number y such that $x > y > z$.

Thus $Q = Q \cap (-\infty, y) \cup Q \cap (y, \infty)$ is a p -disconnection of Q which separates z and x , as $Q \cap (-\infty, y)$ is open in Q_L topology which contains z and does not contain x , and $Q \cap (y, \infty)$ is open in Q_R topology which contains x and does not contain z . But any nonempty open set in Q_L topology that contains x must contain z . Hence the bitopological space (Q, Q_L, Q_R) is clearly pairwise weakly totally disconnected and not pairwise totally disconnected .

3.2 Pairwise semi separation axioms .

3.2.1 Definition : [9]

A bitopological space (X, T_1, T_2) is called pairwise totally semi disconnected , if for each two distinct points x and y in X , there exists a pairwise semi-separation $X=A \mid B$ such that x belongs to A and y belongs to B .

3.2.2 Definition : [9]

A space (X, T_1, T_2) is called pairwise weakly totally semi-disconnected if for each two distinct points x and y of X , there exists a pairwise semi- separation $X=A \mid B$ such that either x belongs to A and y belongs to B or y belongs to A and x belongs to B .

3.2.3 Fact :

Every pairwise separated sets in a bitopological space (X, T_1, T_2) is pairwise semi separated ; as $scl(A) \subset cl(A)$ for every subset A of X .

3.2.4 Theorem : [9]

Every pairwise weakly totally disconnected bitopological space is pairwise weakly totally semi-disconnected .

Proof:

This is obvious from the fact that any pairwise separated sets in a

bitopological space are pairwise semi-separated .

3.2.5 Remark :

The converse of the last theorem need not to be true , as can be seen in the following example .

3.2.6 Example:

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}\}$, and $T_2 = \{X, \emptyset, \{c\}, \{b, c\}\}$.

$$C(T_1) = \{X, \emptyset, \{b, c\}\}.$$

$$S.O(T_1) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}.$$

$$S.C(T_1) = \{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}.$$

$$C(T_2) = \{X, \emptyset, \{a, b\}, \{a\}\}.$$

$$S.O(T_2) = \{X, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}.$$

$$S.C(T_2) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Since for each $x \neq y$ of X , there exists a pairwise semi-separation $X = A|B$ such that A contains x or y and B contains the other, then (X, T_1, T_2) is pairwise weakly totally semi-disconnected. But not pairwise weakly totally disconnected, because for b and c , there exists no pairwise separation $X = A|B$ such that A contains b or c , and B contains the other.

3.2.7 Corollary : [4]

Every pairwise totally disconnected space is pairwise totally semi-disconnected .

Proof:

This is obvious , and can be seen from the fact that every open set is semi-open .

3.2.8 Definition : [9]

A space (X, T_1, T_2) is said to be pairwise weakly semi-Hausdorff if for every pair of distinct points of X , at least one belongs to a T_1 -semi-open set U and the other belongs to a T_2 semi-open set V satisfying $U \cap V = \emptyset$. If the roles of the points are interchangeable , then (X, T_1, T_2) is called a pairwise semi Hausdorff .

3.2.9 Theorem : [13]

Every pairwise (weakly) totally semi-disconnected bitopological space (X, T_1, T_2) is pairwise (weakly) semi Hausdorff.

Proof:

Since X is pairwise totally semi disconnected , then for each $x \neq y$ in X there exists a pairwise semi separation $X = A|B$ such that $x \in A$ and $y \in B$. So, A is T_1 semi-open , B is T_2 semi-open , $x \in A$, $y \in B$, and $A \cap B = \emptyset$. Hence (X, T_1, T_2) is pairwise semi Hausdorff . Similar argument works for pairwise weakly totally semi- disconnectedness.

3.2.10 Theorem : [9]

The preopen , pairwise semi connected subsets of a pairwise weakly totally semi-disconnected space are its point .

Proof:

Let (X, T_1, T_2) be a pairwise weakly totally semi-disconnected bitopological space, and let Y be a preopen subset of X which contains more than one point. Let $x, y \in Y$ with $x \neq y$, and let $A|B$ be a pairwise semi separation of X such that $(x \in A \text{ and } y \in B)$, or $(x \in B \text{ and } y \in A)$. Then $Y = (Y \cap A) \cup (Y \cap B)$, $(Y \cap A) \neq \emptyset$, $(Y \cap B) \neq \emptyset$ and $(Y \cap A) \cap (Y \cap B) = \emptyset$. But Y is preopen, so $(Y \cap A)$ and $(Y \cap B)$ are pairwise semi-separated in A . Thus Y is not pairwise semi-connected, which contradicts the assumption. This proves the theorem.

3.3 The Components in Bitopological Spaces

3.3.1 Definition: [13]

Let (X, T_1, T_2) be a bitopological space, and $x \in X$. The component of x , denoted by $C(x)$, is defined to be the union of all pairwise connected subsets of X containing x .

3.3.2 Remark:

If (X, T_1, T_2) is a bitopological space, then $x \in C(x)$ for

each $x \in X$; as $\{x\}$ is a pairwise connected subset of X containing x .

3.3.3 Theorem: [13]

The components of a weakly totally disconnected bitobological space are its points .

Proof:

Let (X, T_1, T_2) be a weakly totally disconnected bitopological space , and let Y be a component of X which contains more than one point . Let x and y be two elements of Y with $x \neq y$, and let $X=A|B$ be p -disconnection of X with $(x \in A$ and $y \in B)$, or $(x \in B$ and $y \in A)$. This implies that $Y = (Y \cap A)|(Y \cap B)$ is a p -disconnection of Y , and so Y is pairwise disconnected . This is a contradiction , because Y is a component and so it is pairwise connected . So Y does not contain more than one point . But Y is component , so it contains at least one point. Hence Y contains exactly one point .

3.3.4 Theorem : [13]

In a bitopological space (X, T_1, T_2) :

- (i) Each component $C.(x)$ is a maximal pairwise connected set in X .
- (ii) The set of all distinct components of X form a partition of X .
- (iii) Each component $C.(x)$ is the maximum pairwise connected set in X containing x .

Proof (i):

Since $C.(x)$ is the union of pairwise connected sets in X which has a nonempty intersection , so it is pairwise connected. If A is a pairwise

connected set such that $A \supset C.(x)$, then A is a pairwise connected set containing x , and so $A \subset C.(x)$. Hence $A = C.(x)$.

Proof (ii):

Let x and y be any two distinct points of X , and let $C.(x)$, $C.(y)$ be their pairwise components, respectively. If $C.(x) \cap C.(y) \neq \emptyset$, then by (Th.1.1.19) $C.(x) \cup C.(y)$ is a pairwise connected set containing each of $C.(x)$ and $C.(y)$. But $C.(x)$ and $C.(y)$ are maximal pairwise connected sets in X , so $C.(x) = C.(x) \cup C.(y)$ and $C.(y) = C.(x) \cup C.(y)$. Hence $C.(x) = C.(y)$. So distinct pairwise components in X are disjoint.

Also $\forall x \in X, x \in C.(x)$. So the union of all distinct pairwise components of X equals X . So the set of all distinct pairwise components of X forms a partition of X .

Proof (iii) :

$C.(x)$ is a pairwise connected set in X containing x . Let A be any pairwise connected set in X containing x , then $C.(x)$ is the union of a family of sets including A . Hence $A \subset C.(x)$. So $C.(x)$ is the maximum pairwise connected set in X containing x .

3.3.5 Lemma :

In any bitopological space (X, T_1, T_2) and for each $x, y \in X$, we have $C.(x) = C.(y)$ if and only if x and y belong to a pairwise connected subset of X .

Proof:

\Rightarrow) Clear, as both x and y belong to $C.(x)$ which is a pairwise connected subset of X .

\Leftarrow) Suppose that $x \in A$ and $y \in A$, where A is a pairwise connected subset of X . Then $C.(x) \cup A$ is a pairwise connected subset of X containing both x and y . Then by the previous theorem, $C.(x) \cup A \subset C.(y)$; as $C.(x) \cup A$ is a pairwise connected subset of X containing y . So $C.(x) \subset C.(y)$. Similarly, we show that $C.(y) \subset C.(x)$. Hence $C.(x) = C.(y)$.

3.3.6 Theorem: [13]

Any component C of a bitopological space (X, T_1, T_2) satisfies the equation $C = cl_1(C) \cap cl_2(C)$.

Proof:

$C \subset cl_1(C) \cap cl_2(C)$. By (Cor1.1.20), $cl_1(C) \cap cl_2(C)$ is pairwise connected. But it contains C , So by the maximality of C ,
 $C = cl_1(C) \cap cl_2(C)$.

The following corollary follows directly from the maximality of each component.

3.3.7 Corollary :

The pairwise connected bitopological space (X, T_1, T_2) has exactly one component, namely X .

3.4 Semi-components in bitopological spaces .

3.4.1 Definition : [9]

Let (X, T_1, T_2) be a bitopological space and $x \in X$. The semi-component of x , denoted by $S.C.(x)$ is defined to be the union of all $T_1 T_2$ -pairwise semi-connected subsets of X containing x .

3.4.2 Theorem : [9]

In a bitopological space (X, T_1, T_2) :

- (i) Each semi-component $S.C.(x)$ is a maximal $T_1 T_2$ -pairwise semi-connected set in X .
- (ii) The set of all distinct semi-components of X forms a partition of X .
- (iii) Each $S.C.(x)$ satisfies the equation
$$S.C.(x) = scl_1(S.C.(x)) \cap scl_2(S.C.(x)) .$$
- (iv) Each semi-component $S.C.(x)$ is the maximum $T_1 T_2$ -pairwise semi-connected set in X containing x .

Proof (i):

Since $S.C.(x)$ is a union of T_1T_2 - pairwise semi-connected sets in X which has a nonempty intersection set so it is T_1T_2 -pairwise semi-connected set in X . If A is a T_1T_2 -pairwise semi-connected set such that $A \supset S.C.(x)$, then A is T_1T_2 pairwise semi-connected set containing x , and so $A \subset S.C.(x)$. Hence $A = S.C.(x)$.

Proof (ii):

Let x and y be any two distinct points of X , and let $S.C.(x)$, $S.C.(y)$ be their semi-components . If $S.C.(x) \cap S.C.(y) \neq \emptyset$ then by (Th.2.3.3) $S.C.(x) \cup S.C.(y)$ is a T_1T_2 -pairwise semi-connected set containing each of $S.C.(x)$ and $S.C.(y)$. But $S.C.(x)$ and $S.C.(y)$ are maximal T_1T_2 -pairwise semi-connected sets in X , so $S.C.(x) = S.C.(x) \cup S.C.(y)$ and $S.C.(y) = S.C.(x) \cup S.C.(y)$. Hence $S.C.(x) = S.C.(y)$.

So distinct semi component in X are disjoint .

Also $\forall x \in X$, $x \in S.C.(x)$. So the union of all distinct semi components of X equals X . So the set of all distinct semi-components of X forms a partition of X .

Proof (iii):

Let x be any point in X , and let $S.C.(x)$ be its semi-component . Then $scl_1(S.C.(x)) \cap scl_2(S.C.(x))$ is T_1T_2 -pairwise semi-connected set containing $S.C.(x)$, by (Th.2.3.5) . But , by (i), $S.C.(x)$ is a maximal T_1T_2

-pairwise semi connected set in X , so

$$S.C.(x) = scl_1(S.C.(x)) \cap scl_2(S.C.(x)) .$$

Proof (iv) :

$S.C.(x)$ is a T_1T_2 -pairwise semi-connected set in X containing x . Let A be any T_1T_2 - pairwise semi connected set in X containing x , then $S.C.(x)$ is the union of a family of sets including A .

Hence $A \subset S.C.(x)$. So $S.C.(x)$ is the maximum T_1T_2 -pairwise semi-connected set in X containing x .

The following corollary follows directly from the last theorem .

3.4.3 Corollary :

The pairwise semi-connected bitopological space (X, T_1, T_2) has exactly one semi-component , namely X .

3.4.4 Remarek :

In a bitopological space a pairwise semi-component need not be a pairwise component , and a pairwise component is not necessarily a pairwise semi-component , as can be seen from the following example .

3.4.5 Example :

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{c\}, \{b, c\}\}$, and $T_2 = \{X, \emptyset, \{b\}\}$. Then X is a pairwise component because (X, T_1, T_2) is pairwise connected , but X

is not semi- component since (X, T_1, T_2) is not pairwise semi-connected . Also $\{a, c\} \subset X$ is a semi-componant , as $\{a, c\}$ is pairwise semi-connected . But X is not pairwise semi-connected . So $\{a, c\}$ is a maximal pairwise semi connected set in X . So $\{a, c\}$ is a semi-component. But $\{a, c\}$ is not a maximal pairwise connected in X , as X is a pairwise connected set . So $\{a, c\}$ is not a component .

3.4.6 Theorem : [9]

A semi-continuous image of a pairwise semi-connected bitopological space is pairwise connected .

Proof:

Let $f: (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ be a semi-continuous mapping from the pairwise semi-connected bitopological space (X, T_1, T_2) onto the bitopological space (Y, T^*_1, T^*_2) . Suppose that Y is not pairwise connected , then $Y = A|B$ is pairwise disconnection of Y , for some nonempty subset A and B of Y . Then $Y = A \cup B$, where $A \in T^*_1$, $B \in T^*_2$ and $A \cap B = \emptyset$. By semi-continuity of f , $f^{-1}(A)$ is T_1 -semi-open and $f^{-1}(B)$ is T_2 -semi-open . Also $X = f^{-1}(A) \cup f^{-1}(B)$, $f^{-1}(A) \neq \emptyset$, $f^{-1}(B) \neq \emptyset$, and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus X is not pairwise semi-connected , which contradicts the assumption . Hence (Y, T^*_1, T^*_2) is pairwise connected .

3.4.7 Theorem :[9]

Let $f: (X, T_1, T_2) \rightarrow (Y, T^*_1, T^*_2)$ be irresolute and open surjection.

Then the image of an open semi-component of X lies in a semi-component of Y .

Proof :

Let $S.C.(x)$ be the semi-component of x in X . Then it is T_1T_2 -pairwise semi-connected set, and by (Th.2.4.26), $f(S.C.(x))$ is T_1T_2 -pairwise semi-connected set in Y , and $f(x) \in f(S.C.(x))$. Therefore $f(S.C.(x)) \subset S.C.(f(x))$.

Chapter 4

Pairwise set semi-connected spaces

4.1 Pairwise connected between sets .

4.1.1 Definition : [7]

An ij -coset in any bitopological space (X, T_1, T_2) is a T_i -closed and T_j -open set in X ; $i, j = 1, 2$, and $i \neq j$.

4.1.2 Definition :

A bitopological space (X, T_1, T_2) is said to be ij -disconnected between A and B (where A and B are non-empty subsets of X) , if and only if there is an ij -coset F such that $A \subset F \subset X \setminus B$; $i, j = 1, 2$, and $i \neq j$.

4.1.3 Definition : [7]

A bitopological space (X, T_1, T_2) is said to be ij -connected between

A and B , (where A and B are non-empty subsets of X) , if and only if it is not ij-disconnected between A and B .

4.1.4 Definition : [7]

A bitopological space (X, T_1, T_2) is said to be pairwise-connected between A and B , (where A and B are non-empty subsets of X) , if and only if X is 12-connected as well as 21- connected between A and B.

The following remarks follow directly from the definitions, as a subset F is ij-coset if and only if $X \setminus F$ is ji-coset , $i, j = 1, 2$, and $i \neq j$.

4.1.5 Remark :

A bitopological space (X, T_1, T_2) is 12-connected between A and B if, and only if it is 21-connected between B and A .

4.1.6 Remark :

A bitopological space (X, T_1, T_2) is pairwise connected between A and B if and only if it is pairwise connected between B and A , for any nonempty subsets A and B of X.

4.1.7 Rmark : [7]

If X is ij-connected between A and B , $A \subset C$, and $B \subset D$, then X is ij-connected between C and D.

4.1.8 Lemma : [7]

Let (X, T_1, T_2) be a bitopological space , and let A and B be two subsets of X . Then if a subspace M of X , containing the sets A and B , is ij -connected between A and B , then so is the whole space ; $i, j = 1, 2$, and $i \neq j$.

Proof:

Suppose that (X, T_1, T_2) is not ij connected between A and B , then There is an ij -coset F in X such that $A \subset F \subset X \setminus B$. Then $F \cap M$ is an ij -coset in M with $A \subset (F \cap M) \subset (M \setminus B)$. And this contradicts the assumption .

4.1.9 Lemma : [9]

If a bitopological space (X, T_1, T_2) is an ij -connected between two subsets A and B of M , where M is a subset of X , which is ij -coset in X , then (M, T_{1M}, T_{2M}) is ij -connected between A and B ; $i, j = 1, 2$, and $i \neq j$.

Proof:

Suppose that (M, T_{1M}, T_{2M}) is not ij -connected between A and B . Then there exists an ij -coset F in (M, T_{1M}, T_{2M}) such that $A \subset F \subset M \setminus B$. But M is an ij -coset also , so F is an ij -coset in (X, T_1, T_2) , $A \subset F \subset X \setminus B$. Hence (X, T_1, T_2) is not ij -connected between A and B , which contradicts the assumption . So (M, T_{1M}, T_{2M}) is ij -connected between A and B .

From the last two lemmas , we have the following :

4.1.10 Theorem : [7]

If M is an ij -coset of a bitopological space (X, T_1, T_2) , and A, B are nonempty subsets of M , then (M, T_{1M}, T_{2M}) is ij -connected between A and B if and only if (X, T_1, T_2) is ij -connected between A and B ,
 $i, j = 1, 2$, and $i \neq j$.

4.2 Pairwise semi-connectedness between sets

4.2.1 Definition : [10]

A subset F of a bitopological space (X, T_1, T_2) is called (i, j) -s-coset if and only if F is both T_i -semi closed in X and T_j -semi open in X for some $i, j = 1, 2$, and $i \neq j$.

4.2.2 Definition : [10]

A bitopological space (X, T_1, T_2) is said to be (i, j) -s-connected between its nonempty subsets A and B if and only if there exists no (i, j) -s-coset F such that $A \subset F$ and $F \cap B = \emptyset$ for $i, j = 1, 2$, and $i \neq j$.

4.2.3 Definition : [10]

A bitopological space (X, T_1, T_2) is said to be pairwise s -connected between its nonempty subsets A and B if it is $(1,2)$ - s -connected as well as $(2,1)$ - s -connected between A and B .

4.2.4 Theorem :

If (X, T_1, T_2) is $(1,2)$ s -connected between A and B then the space (X, T_1, T_2) is $(2,1)$ s -connected between B and A .

Proof:

Suppose that (X, T_1, T_2) is not $(2,1)$ s -connected between B and A . Then there exists a $(2,1)$ s -coset F such that $B \subset F$ and $A \cap F = \emptyset$. So $X \setminus F$ is $(1,2)$ - s -coset, $A \subset X \setminus F$ and $B \cap (X \setminus F) = \emptyset$. So (X, T_1, T_2) is not $(1,2)$ - s -connected, which contradicts the assumption. So (X, T_1, T_2) is $(2,1)$ - s -connected between B and A .

The following theorem follows directly from the last theorem.

4.2.5 Theorem :

If (X, T_1, T_2) is pairwise s -connected between A and B , then it is pairwise s -connected between B and A .

4.2.6 Theorem : [10]

If (X, T_1, T_2) is pairwise s -connected between its subsets A and B , then it is pairwise connected between A and B .

Proof:

Suppose that the space is not ij -connected for some $i, j = 1, 2$, and $i \neq j$. Then there exists an ij -coset F such that $A \subset F$ and $B \cap F = \emptyset$. Since every closed set is semi closed and every open set is semi open, then F is (i,j) -s-coset, so the space X is not (i,j) -s-connected between A and B .

So X is not pairwise s -connected between A and B , which contradicts our assumption. Hence (X, T_1, T_2) is pairwise connected between A and B .

4.2.7 Remark :

The converse of the above theorem need not be true, as can be seen from the following example.

4.2.8 Example :

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}\}$, and $T_2 = \{X, \emptyset, \{b\}\}$.

$$C.(T_1) = \{X, \emptyset, \{b, c\}\}.$$

$$S.O.(T_1) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}.$$

$$S.C.(T_1) = \{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}.$$

$$C.(T_2) = \{X, \emptyset, \{a, c\}\}.$$

$$S.O.(T_2) = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}.$$

$$S.C.(T_2) = \{X, \emptyset, \{a, c\}, \{c\}, \{a\}\}.$$

The set of 12 -cosets = $\{X, \emptyset\}$, the set of 21 -cosets = $\{X, \emptyset\}$,

the set of $(1,2)$ -s-cosets = $\{X, \emptyset, \{b, c\}, \{b\}\}$, and the set of $(2,1)$ -s-

cosets = $\{X, \emptyset, \{a,c\}, \{a\}\}$. Since no ij -coset F such that $\{a\} \subset F$ and $F \cap \{b\} = \emptyset$; where $i,j = 1,2$, and $i \neq j$, then the bitopological space is pairwise connected between $\{a\}$ and $\{b\}$. But it is not pairwise s -connected between $\{a\}$ and $\{b\}$. For $\{a,c\}$ is a $(2,1)$ - s -coset containing $\{a\}$ and disjoint from $\{b\}$ and so X is not $(2,1)$ - s -connected between $\{a\}$ and $\{b\}$.

4.2.9 Definition : [6]

In a topological space (X,T) , a set A is said to be s mei-clopen if and only if A is semi-open and semi closed.

4.2.10 Definition : [6]

A topological space (X,T) is said to be s -connected between A and B if and only if there exists no semi clopen subset F of X such that $A \subset F$ and $F \cap B = \emptyset$.

4.2.11 Note :

The following examples show that if (X,T_1,T_2) is pairwise s -connected between its non-empty subsets A and B , then it is not necessarily that (X,T_1) or (X,T_2) must be s -connected between A and B . Likewise, (X,T_1) and (X,T_2) can be both s -connected between A and B and still (X,T_1,T_2) may fail to be pairwise s -connected between A and B .

4.2.12 Example :

Let $X = \{a,b,c\}$, $T_1 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$, and

$T_2 = \{X, \emptyset, \{a,c\}\}$. Then :

$C.(T_1) = \{X, \emptyset, \{b,c\}, \{a,c\}, \{c\}\}$.

$S.O.(T_1) = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.

$S.C.(T_1) = \{X, \emptyset, \{b,c\}, \{a,c\}, \{c\}, \{b\}, \{a\}\}$.

$C.(T_2) = \{X, \emptyset, \{b\}\}$.

$S.O.(T_2) = \{X, \emptyset, \{a,c\}\}$.

$S.C.(T_2) = \{X, \emptyset, \{b\}\}$.

Semi clopen sets in T_1 are $X, \emptyset, \{a\}, \{b\}, \{a,c\}$, and $\{b,c\}$.

Then clearly, (X, T_1, T_2) is pairwise s-connected between $\{a\}$ and $\{c\}$, but (X, T_1) is not s-connected between $\{a\}$ and $\{c\}$, because there exists a T_1 -semi clopen set $\{a\}$ such that $\{a\} \subset \{a\}$ but $\{c\} \cap \{a\} = \emptyset$.

4.2.13 Example :

Let $X = \{a,b,c\}$, $T_1 = \{X, \emptyset, \{a\}\}$ and $T_2 = \{X, \emptyset, \{b,c\}\}$. Then clearly, (X, T_1) and (X, T_2) are both s-connected between $\{b\}$ and $\{a\}$. But (X, T_1, T_2) is not pairwise s-connected between $\{b\}$ and $\{a\}$, as it is not (1,2)-s-connected between $\{b\}$ and $\{a\}$.

4.2.14 Theorem :

If (X, T_1, T_2) is pairwise s-connected between A and B , $A \subset A_1$ and $B \subset B_1$, then it is also pairwise s-connected between A_1 and B_1 .

Proof:

Clear, as every set contains A_1 , and does not intersect B_1 , contains A and does not intersect B .

The following theorem follows directly.

4.2.15 Theorem : [6]

If (X, T_1, T_2) is pairwise s -connected between A and B then it is pairwise s -connected between $cl_i(A)$ and $cl_j(B)$; $i, j = 1, 2, i \neq j$.

4.2.16 Remark :

The converse of the above theorem is not necessarily true, as can be seen from the following example .

4.2.17 Example :

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}\}$ and $T_2 = \{X, \emptyset, \{b\}\}$. Then :

$$C(T_1) = \{X, \emptyset, \{b, c\}\},$$

$$C(T_2) = \{X, \emptyset, \{a, c\}\},$$

$$S.O.(T_1) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\},$$

$$S.O.(T_2) = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\},$$

$$S.C.(T_1) = \{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}, \text{ and}$$

$$S.C.(T_2) = \{X, \emptyset, \{a, c\}, \{c\}, \{a\}\},$$

$$cl_1 \{a\} = X, \quad cl_2 \{a\} = \{a, c\}, \quad cl_1 \{b\} = \{b, c\}, \text{ and } cl_2 \{b\} = X.$$

Clearly (X, T_1, T_2) is not pairwise s -connected between $\{a\}$ and $\{b\}$ as $\{a\}$ is a $(2, 1)$ - s -coset . Since there exists no (i, j) - s -coset F such that $\{a, c\} \subset F$ and $\{b, c\} \cap F = \emptyset$, as $\{a, c\} \cap \{b, c\} \neq \emptyset$, hence (X, T_1, T_2) is pairwise s -connected between $\{a, c\}$ and $\{b, c\}$. It is also clear that (X, T_1, T_2) is pairwise connected between $cl_1 \{a\}$ and $cl_2 \{b\}$, as $cl_1 \{a\} \cap cl_2 \{b\} \neq \emptyset$.

4.2.18 Theorem : [6]

A space (X, T_1, T_2) is $(1,2)$ -s-connected between A and B if and only if it is $(1,2)$ -s-connected between $scl_1 A$ and $scl_2 (B)$.

Proof:

\Rightarrow) Direct , as $A \subset scl_1(A)$ and $B \subset scl_2(B)$.

\Leftarrow) Suppose that X is not $(1,2)$ -s- connected between A and B , then there is a $(1,2)$ -s- coset F such that $A \subset F$ and $F \cap B = \emptyset$. F is T_1 semi closed containing A so $scl_1(A) \subset F$. Also $scl_2(B) \cap F = \emptyset$. Since F is T_2 semi open , so $X \setminus F$ is T_2 semi-closed containing B , so $scl_2 (B) \subset X \setminus F$. Clearly , $scl_1 A$ and $scl_2 B$ are nonempty subsets of X . Since F is $(1,2)$ -s-coset , $scl_1(A) \subset F$, and $F \cap scl_2(B) = \emptyset$, then X is not $(1,2)$ s-connected between $scl_1(A)$ and $scl_2(B)$, which is a contradiction. Hence the space X is $(1,2)$ - s-connected between $scl_1(A)$ and $scl_2(B)$.

4.2.19 Theorem : [10]

Let A and B be two nonempty subsets of (X, T_1, T_2) such that $A \cap B \neq \emptyset$. Then (X, T_1, T_2) is pairwise s-connected between A and B .

Proof:

Follows directly from the definition .

4.2.20 Remark :

The converse of the above theorem is not necessarily true , as can be seen from the following example .

4.2.21 Example :

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $T_2 = \{X, \emptyset, \{a, c\}\}$.
Then Clearly (X, T_1, T_2) is pairwise s-connected between $\{a\}$ and $\{c\}$ But
 $\{a\} \cap \{c\} = \emptyset$.

4.2.22 Theorem : [10]

A bitopological space (X, T_1, T_2) is pairwise semi connected if and only if X contains no nonempty proper subset which is (1,2)-s-coset or (2,1)-s-coset .

Proof:

\Rightarrow) Suppose that (X, T_1, T_2) is pairwise semi-connected , and suppose that X has a nonempty proper subset F which is (1,2)-s-coset. Then F is nonempty semi closed in T_1 , and $X \setminus F$ is nonempty semi-closed in T_2 . So $F \cap scl_2(X \setminus F) = \emptyset = scl_1(F) \cap (X \setminus F)$. And since $X = (X \setminus F) \cup F$, then X is not pairwise semi connected . And this contradicts the assumption . So X has no nonempty proper subset which is (1,2)-s-coset . Similarly , we see that X has no nonempty proper subset which is (2,1)-s-coset .

\Leftarrow) If (X, T_1, T_2) is not pairwise semi-connected , then $X = A \cup B$, for some nonempty disjoint subsets A and B of X , where A is T_1 semi-open , and B is T_2 semi open . So $X \setminus B = A$ is T_2 semi-closed . So X has a nonempty proper subset which is (2,1)-s-coset . Similarly we show that

X has a nonempty proper subset which is $(1,2)$ -s-coset . And this contradicts the assumption .

4.2.23 Theorem : [10]

A bitopological space (X, T_1, T_2) is pairwise semi-connected if and only if it is pairwise s-connected between every pair of its nonempty subsets .

Proof:

\Rightarrow) Since (X, T_1, T_2) is pairwise semi-connected , then by the last theorem , X contains no nonempty proper subset which is $(1,2)$ -s-coset or $(2,1)$ -s-coset . So X is pairwise s- connected between every pair of its nonempty subsets .

\Leftarrow) Since (X, T_1, T_2) is pairwise s-connected between every pair of its nonempty proper subsets , so there is no nonempty proper subset which is $(1,2)$ -s-coset or $(2,1)$ -s-coset . Then , also by the last theorem , the space (X, T_1, T_2) is pairwise s-connected .

4.2.24 Theorem : [10]

If a bitopological space (X, T_1, T_2) is not $(1,2)$ -s-connected between its subsets A and B , then B and A are pairwise semi separated in X .

Proof:

Suppose that (X, T_1, T_2) is not $(1,2)$ -s-connected between A and B . Then there exists $(1,2)$ -s-coset F such that $A \subset F$ and $B \cap F = \emptyset$. Want to -

show that $A \cap scl_2(B) = \emptyset$ and $B \cap scl_1(A) = \emptyset$. F is T_2 -semi-open, and $F \cap B = \emptyset$. So $X \setminus F$ is T_2 -semi-closed and $B \subset X \setminus F$, so $scl_2(B) \subset X \setminus F$. And so $scl_2(B) \subset X \setminus A$, as $A \subset F$. Hence $A \cap scl_2(B) = \emptyset$.

Also, F is T_1 -semi-closed set, and $A \subset F$, so $scl_1(A) \subset F$. But $B \cap F = \emptyset$, so $B \cap scl_1(A) = \emptyset$. So B and A are pairwise semi-separated sets in X .

4.2.25 Theorem : [6]

If a bitopological space (X, T_1, T_2) is neither 12-connected between A and B_1 nor (1,2)-s connected between A and B_2 , then it is not (1,2)-s-connected between A and $(B_1 \cup B_2)$.

Proof:

Since the space is not 12-connected between A and B_1 , then there exists a 12-coset F_1 such that $A \subset F_1$ and $F_1 \cap B_1 = \emptyset$. Also, since X is not (1,2)-s-connected between A and B_2 , there exists a (1,2)-s-coset F_2 such that $A \subset F_2$ and $F_2 \cap B_2 = \emptyset$. Define $F = F_1 \cap F_2$. Since any intersection of T_1 -semi closed sets is T_1 -semi closed, so F is T_1 semi closed. Also, F is T_2 -semi open as it is the intersection of a T_1 -semi open set F_1 and an open set T_2 -open set F_2 . Hence F is a (1,2)-s-coset such that $A \subset F$ and $(B_1 \cup B_2) \cap F = \emptyset$. Thus X is not (1,2)-s-connected between A and $(B_1 \cup B_2)$.

4.2.26 Remark :

If the space (X, T_1, T_2) is not (1,2)-s-connected neither between A and

B_1 , nor between A and B_2 , then it is not necessarily that X is not (1,2)-
s-connected between A and $(B_1 \cup B_2)$. We show this by the following
example .

4.2.27 Example :

Let $X = \{a, b, c\}$, $T_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, and

$T_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then :

$$C(T_1) = \{X, \emptyset, \{b, c\}, \{a, c\}, \{c\}, \{b\}\},$$

$$C(T_2) = \{X, \emptyset, \{b, c\}, \{a, c\}, \{c\}\},$$

$$S.O.(T_1) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\},$$

$$S.O.(T_2) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\},$$

$$S.C.(T_1) = \{X, \emptyset, \{b, c\}, \{a, c\}, \{c\}, \{b\}\}, \text{ and}$$

$$S.C.(T_2) = \{X, \emptyset, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}.$$

The space (X, T_1, T_2) is clearly, not (1,2)-s-connected neither between
 $\{c\}$ and $\{a\}$ nor between $\{c\}$ and $\{b\}$. But it is (1,2)-s-connected
between $\{c\}$ and $\{a, b\}$.

4.3 Subspaces of pairwise semi-connected spaces between sets .

4.3.1 Lemma : [11]

Let Y be an open subset of a topological space (X, T) . If A is semi closed in X then $Y \cap A$ is semi closed in Y .

Proof:

Since A is semi-closed in X , so $X \setminus A$ is semi-open in X . By (Lem.2.2.10), $Y \cap (X \setminus A)$ is semi-open in (Y, T_Y) , and so , $Y \setminus (Y \cap (X \setminus A))$ is semi-closed in (Y, T_Y) . But $Y \setminus (Y \cap (X \setminus A)) = Y \cap A$, so $Y \cap A$ is semi-closed in (Y, T_Y) .

4.3.2 Lemma : [11]

Let Y be a closed subset of a topological space (X, T) and B be semi closed, in (X, T) , then $Y \cup B$ is semi closed in X .

Proof:

$X \setminus (Y \cup B) = (X \setminus Y) \cap (X \setminus B)$. But $X \setminus Y$ is open and $X \setminus B$ is semi-open in (X, T) , so by (Lem.2.2.10) , $X \setminus (Y \cup B)$ is semi-open . Hence $Y \cup B$ is semi closed .

4.3.3 Lemma :

Let A be semi-open in the topological space X and suppose that

$A \subset B \subset \text{cl}(A)$. Then B is semi-open .

Proof:

Since A is semi-open then there exists an open set V such that $V \subset A \subset \text{cl}(V)$. But $\text{cl}(A) \subset \text{cl}(V)$ and thus $B \subset \text{cl}(V)$.

Hence $V \subset B \subset \text{cl}(V)$ and B is semi-open .

4.3.4 Lemma : [11]

Let Y be a semi-open subset of a topological space (X, T) , then a subset A of Y is semi-open subset of (X, T) if and only if A is semi-open in (Y, T_Y) .

Proof:

\Rightarrow) Direct by (Th. 2.2.8)

\Leftarrow) Since A is semi-open in Y, then $U \subset A \subset \text{cl}_Y(U)$,for some $U \in T_Y$. So $U = V \cap Y$, for some $V \in T$. So $V \cap Y \subset A \subset \text{cl}_Y(V \cap Y) \subset \text{cl}(V \cap Y)$. Since $V \cap Y$ is semi-open in (X, T) , then by the above lemma, A is semi-open in (X, T) .

4.3.5 Definition :

A subset Y of (X, T_1, T_2) is said to be pairwise s-connected between two non-empty subsets A and B of Y if the space (Y, T_{1Y}, T_{2Y}) is pairwise -s- connected between A and B .

4.3.6 Theorem :

If a preopen subspace Y of a bitopological space (X, T_1, T_2) is pairwise s -connected between A and B of Y , then so is X .

Proof:

Suppose that (X, T_1, T_2) is not pairwise s -connected between A and B , then there exists a $(1,2)$ - s -coset or a $(2,1)$ - s -coset F such that $A \subset F$ and $F \cap B = \emptyset$.

Then by (Lem.2.4.10) and (Lem.2.4.11), $Y \cap F$ is a $(1,2)$ - s -coset or $(2,1)$ - s -coset in (Y, T_{1Y}, T_{2Y}) . But $A \subset F \cap Y$ and $(F \cap Y) \cap B = Y \cap (F \cap B) = \emptyset$. Hence Y is not pairwise s -connected between A and B which is a contradiction. So, (X, T_1, T_2) is pairwise s -connected between A and B .

4.3.7 Theorem : [6]

A $(1,2)$ - s -coset Y in a bitopological space (X, T_1, T_2) is $(1,2)$ - s -connected between its subsets A and B if (X, T_1, T_2) is $(1,2)$ - s -connected between A and B .

Proof:

Suppose that Y is not $(1,2)$ - s -connected between A and B . Then there is a $(1,2)$ - s -coset F in (Y, T_{1Y}, T_{2Y}) such that $A \subset F$ and $F \cap B = \emptyset$. Since Y is $(1,2)$ - s -coset in (X, T_1, T_2) , therefore F is $(1,2)$ - s -coset in (X, T_1, T_2) . But $A \subset F$ and $F \cap B = \emptyset$ also, hence (X, T_1, T_2) is not $(1,2)$ -

s-connected between A and B which contradicts the assumption . So Y is (1,2)-s-connected between A and B .

4.3.8 Theorem : [10]

Let (X, T_1, T_2) be a bitopological space .If X is not (1,2)-s-connected between A and B, then no preopen subset Y of X can be (1,2)-s-connected between $A \cap Y$ and $B \cap Y$.

Proof:

Let Y be a preopen subset of (X, T_1, T_2) . If $A \cap Y = \emptyset$ or $B \cap Y = \emptyset$, then by the definition , Y is not (1,2)-s-connected between A and B . Let $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$. Since X is not (1,2)-s-connected between A and B, then there exists a (1,2)-s-coset F such that $A \subset F$ and $F \cap B = \emptyset$. Thus $A \cap Y \subset F \cap Y$ and $(F \cap Y) \cap (B \cap Y) = \emptyset$. By (Lem.2.4.10) and by (Lem.2.4.11), $F \cap Y$ is (1,2)-s-coset in (Y, T_{1Y}, T_{2Y}) . Hence Y is not (1,2)-s-connected between $A \cap Y$ and $B \cap Y$.

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المخلص

في هذه الرسالة ندرس بعض خواص الفضاءات المترابطة و شبه المترابطة في الفضاءات التبولوجية الثنائية , مستعرضين بعض خواص المجموعات شبه المفتوحة في الفضاءات التبولوجية .

كذلك , ندرس الترابط الكلي و شبه المركبات في الفضاءات التبولوجية الثنائية .
أخيرا ندرس الترابط و شبه الترابط بين المجموعات في الفضاءات التبولوجية الثنائية .
في معظم دراستنا لهذه المواضيع , نقوم بتعميم مفاهيم معروفة في الفضاءات التبولوجية , والحصول على تعميم لنتائج معروفة في الفضاءات التبولوجية .

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