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Fractional Calculus in Economics

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Fractional Calculus In Economics

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Dedication

I present this as a way of gratitude to my father and my mother whom I'm truly proud of and for them I am grateful as they stood by my side every day and moment.

To my dear wife Shireen Who supported me through all of this.

To my Kids Noor,Omar,Hoor,Eilya'a and Mahmoud.

To all people who encouraged me.

I present this to all of them.

Declaration

I certify that this thesis submitted for the degree of master is the result of my own reseach, except where otherwise acknowledge, and this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed:

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Abstract

Fractional Calculus has become used in many applications in various types of science and it has many important applications.

In this thesis, we referred to the most important definitions of the subject of fractional calculus and studied some of important applications of it.

In economic sciences in particular, many countries have become using fractional calculus in their economic calculations, such as calculating the gross domestic product and demand elasticity and supply elasticity.

In this thesis, we focused on the uses of fractional calculus in economic and financial sciences, and we studied the application of fractional calculus in calculating the gross domestic product of Spain as a case study, and analyzed and compared the results.

At the end of this thesis, we applied fractional calculus in calculating the gross domestic product of Palestine using fractional calculus.

The fractional calculus, as we have shown in this thesis, can provide results characterized by high quality and accuracy if it used in many different scientific and research fields.

0.Introduction:

Fractional calculus is a science with many applications in a wide variety of fields of engineering and sciences such as electromagnetism, viscoelasticity, fluid mechanics, electrochemistry, models of biological assembly, optics, and signal processing. It has been used to model physical and engineering processes which found to be best described by partial differential equations. Partial derivative models are used for accurate modeling of those systems where accurate damping modeling is required. In these areas, various analytical and numerical methods including their applications to new problems have been proposed in recent years.

Fractional calculus has other applications in economics such as GDP modeling and in demand elasticity, elasticity presentation, and more other topics in economics. We are now going to study some applications of fractional integrals and partial derivatives in some sciences and discuss fractional calculus in economics in chapter four and chapter five.

The outline of the thesis is as follows:

In the first chapter, some definitions of the work of fractional calculus were presented and we presented some of the basic functions used in fractional calculus such as gamma and beta functions, and we also presented some definitions of fractional calculus such as fractional differentiation and fractional integration. In addition to that we referred to the most important scientists who were concerned this science and its applications.

In the second chapter, we presented a number of the most important applications in which fractional calculus was used in various sciences such as engineering, biology and applied mathematics.

In the third chapter of this thesis, we pointed out that the science of fractional calculus has wide and many applications in the field of financial and economic sciences. Emphasis was placed on the use of fractional calculus in the field of calculating price elasticity of demand and price elasticity of supply and the characteristics that characterize the use of fractional calculus in this field. We also presented a number of important economic concepts in which fractional calculus is used.

In the fourth chapter, we presented the use of fractional calculus in analyzing economic growth, and the Spanish case was presented and how fractional calculus was used in calculating GDP and comparing the results obtained using fractional calculus with the results obtained by the traditional method in calculating GDP.

Finally, in the fifth chapter, we tried to apply fractional calculus in calculating the gross domestic product in Palestine and comparing the results we obtained with calculating the gross domestic product in the traditional way.

At the end of the thesis, we presented the most important results and recommendations that we were able to obtain through our study.

0.1 Problem of Studying

This study attempts to answer the following question: Does the applications of fractional calculus in economics increase the efficiency and accuracy of results, and thus make important decisions based on that?

0.2 Objectives of studying

First: Learn about fractional calculus and its definitions.

Second: Identify the applications of fractional calculus in various types of sciences.

Third: Identify the applications of fractional calculus in economics.

Fourth: A study of the use of fractional calculus in calculating the gross domestic product GDP.

Fifth: Use the fractional calculus in calculating the gross domestic product in Palestine.

0.3 The importance of studying

Fractional calculus becomes used in many applications and scientific, engineering and industrial fields. Therefore, it is necessary to study this science in order to be able to use its many applications in areas that serve human life and its development.

This study sheds light on the use of fractional calculus in economics, because economics is one of the most important sciences through which administrative and economic decisions are made.

This study indicates the importance of using fractional calculus in calculating the gross domestic product of many countries and compares the efficiency of the fractional order model and other models.

Chapter One

1.Fractional Calculus

1.1 Introduction

Fractional Calculus is the field of mathematical study which deals with the investigations and applications of integrals and derivatives of arbitrary order. The birth of fractional calculus occurred in a letter from G.F.A de L'Hospital to G.W Leibniz in 1695 posing a possible question " what if the order of derivative $(d^n f/dt^n)$ such that $n = 1/2$ " in his reply he wrote it will lead to paradox , from which one day useful consequences will be drawn ". [7]

There are many types of fractional integral and differential operators that are proposed by Riemann , Liouville , Grunwald , Weyl , Riesz , Caputo and other scientists . Fractional Calculus has wide applications in applied mathematics and other sciences such as differential equations, physics, engineering, economics and many other sciences.

1.2 Historical Notes of fractional Calculus

The first mention of fractional calculus apparently by S. E Lacroix around 1819. He started by stating the common n^{th} derivative of the power function $y = x^m$ in term of the Gamma function.

$$y^{(n)} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \tag{1}$$

where $m \geq n$

He then let n be any real number to arrive at the first definition of the fractional derivative of a power function. After this period, a list of mathematicians who have provided important contributions up to the middle of the 20^{th} century ,included Fourier, N H Abel, J. Liouville, Riemann, Grunwald, Riesz, W. Feller and others.

In recent years considerable interest in fractional calculus has been stimulated by the applications that it found in different fields of sciences, including numerical analysis, economics and finance, engineering, physics and other sciences.

1.3 Special Functions

This section deals with some basic concepts of special functions that were adapted in this thesis, namely:

1.3.1 Gamma Function

Definition 1.3.1.1 The gamma function is defined as follows

$$\Gamma(n) = \lim_{M\to\infty} \int_0^M x^{n-1}\, e^{-x} dx \qquad n > 0\,,\;\; x \in \mathbb{R} \tag{2}$$

Example 1.3.1.2 Find $\Gamma(2)$.

Solution :

$$\Gamma(2) = \lim_{M\to\infty} \int_0^M x^{2-1}\, e^{-x} dx$$

$$= \lim_{M\to\infty} \int_0^M xe^{-x}\, dx$$

$$= \lim_{M\to\infty} [-xe^{-x} - e^{-x}]_0^M$$

$$= \lim_{M\to\infty} (-M/e^M - \frac{1}{e^M} + 0 + e^0)$$

$$= 1$$

The main rules for gamma function:

$$\Gamma(n+1) = n\Gamma(n) \qquad \forall n \neq 0$$

$$\Gamma(n+1) = n! \qquad for\; n \in \mathbb{N}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

1.3.2 Beta Function

Definition1.3.2.1 The beta function is defined as follows

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx\;\;, \qquad n > 0\,, m > 0 \tag{3}$$

Example 1.3.2.2 Find $\beta(2{,}3)$

Solution:

$$\beta(2{,}3) = \int_0^1 x^{2-1}(1-x)^{3-1}\, dx$$

$$= \int_0^1 x\,(1-x)^2 dx$$

$$= \int_0^1 x\,(1-2x+x^2) dx$$

$$= \int_0^1 (x - 2x^2 + x^3) dx$$

$$= \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \;\;\Big|_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

Example 1.3.2.3 Show that $\beta(m,n) = \beta(n,m)$.

Solution:

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^1 u^{n-1} (1-u)^{m-1} du$$

$where\, u = 1 - x, \quad du = -dx$

$\beta(m,n) = \beta(n,m)$

Theorem 1.3.2.4 The gamma and beta functions are related by the relation

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{4}$$

Example 1.3.2.5 Solve example 1.3.4 in using (4).

$$\beta(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)}$$

$$= \frac{1!\,2!}{4!} = \frac{1 \times 2}{4 \times 3 \times 2 \times 1}$$

$$= \frac{2}{24} = \frac{1}{12}$$

Example 1.3.2.6 The following integrals are found by using gamma and beta functions:

$$a)\ \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n) \tag{5}$$

$$b)\ \int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \qquad 0 < p < 1. \tag{6}$$

1.4 Preliminaries for fractional calculus

In this section, we illustrate how fractional calculus can be utilized in understanding the topics of several sciences and there applications.

To recognize the applications of fractional calculus, traditional integer differentiation is first examined for a power function:

Example1.4.1 Let $(x) = x^m$. Show that $f^{(n)}(x) = \frac{\Gamma(m+1)x^{m-n}}{\Gamma(m-n+1)}$

Solution:

$$f'(x) = mx^{(m-1)}$$
$$f''(x) = m(m-1)x^{(m-2)}$$
$$\vdots$$
$$f^{(n)}(x) = \frac{m!\, x^{(m-n)}}{(m-n)!} = \frac{\Gamma(m+1)x^{m-n}}{\Gamma(m-n+1)} \tag{7}$$

In a likewise manner the n^{th} repetitive integer integration process can also be examined for a power function.

Example 1.4.2 Let $f(x) = x^n$, Show that $\int \cdots \int f(x)\, dx \cdots dx = \frac{\Gamma(n)x^{m+n}}{\Gamma(m+n+1)}$

Solution:

$$\int f(x)dx = \frac{x^{n+1}}{n+1}$$

$$\iint f(x)dxdx = \frac{x^{n+2}}{(n+1)(n+2)}$$

$$\iiint f(x)\, dxdxdx = \frac{x^{n+3}}{(n+1)(n+2)(n+3)}$$

$$\vdots$$

$$\int \cdots \int f(x)\, dx \cdots dx = \frac{n! x^{n+m}}{(n+m)!} = \frac{\Gamma(n)x^{m+n}}{\Gamma(m+n+1)} \tag{8}$$

The equation (7) can be modified to utilize the gamma function:

$$\frac{d^n f}{dx^n} = \frac{\Gamma(m+1)x^{m-n}}{\Gamma(m-n+1)} \tag{9}$$

The equation (8) can be modified to utilize the gamma function:

$$\int \cdots \int f(x)\, dx \cdots dx = \frac{\Gamma(n)x^{m+n}}{\Gamma(m+n+1)} \tag{10}$$

1.5 Fractional integrals

In most calculus books, derivatives are usually explained before integration. But in our study of fractional calculus, we'll show the fractional integrals before we cover the fractional derivatives. Because fractional derivatives are defined in terms of fractional integrals.

We will see for a lot of functions the definitions for both will agree and so one formula can be defined for both fractional integrals and derivatives of these functions. [23]

Definition 1.5.1

a) **Riemann Integral**

$$ {}_cD_x^{-v}f(x) = \frac{1}{\Gamma(v)}\int_c^x (x-t)^{v-1} f(t)dt \tag{11}$$

$where\ f(x) \in \mathbb{R}$

b) **Liouville's Integral:**

$$ {}_{-\infty}D_x^{-v}f(x) = \frac{1}{\Gamma(v)}\int_{-\infty}^x (x-t)^{v-1} f(t)dt \tag{12}$$

$where\ f(x) \in \mathcal{L}$

c) **Riemann-Liouvill's Integral:**

$$ {}_0D_x^{-v}f(x) = \frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1} f(t)dt \tag{13}$$

$where\ f(x) \in \mathbb{R}$ Or $f(x) \in \mathcal{L}$

d) **Weyl's Integral:**

$$ {}_xD_\infty^{-v}f(x) = \frac{1}{\Gamma(v)}\int_x^\infty (t-x)^{v-1} f(t)dt \tag{14}$$

We note that the only difference between the four integrals is the limit of integration, and the most used integrals is the Riemann-Liuville's integral.

There are many examples of fractional integrals, let's now look at some examples that will establish fractional integrals of general functions involving exponential, sine, cosine, logarithmic, and $f(t)$ where $f(t)$ is a function of class C.

Example 1.5.2 Let $f(t) = e^{at}$ where a is any constant. Since $f(t)$ is continuous on $[0, \infty)$ and integrable on any finite subinterval of $(0, \infty)$, $f \in C$, and $v > 0$ we have

$$D^{-v}e^{at} = \frac{1}{\Gamma(v)}\int_0^t (t-\xi)^{v-1} e^{a\xi} d\xi$$

$$= \frac{1}{\Gamma(v)}\int_t^0 x^{v-1} e^{a(t-x)} (-dx)$$

$$= \frac{e^{at}}{\Gamma(v)}\int_0^t x^{v-1} e^{-ax}\, dx.$$

Example 1.5.3 Let $f(t) = \sin(at)$ $where\ a$ is any constant. We have $f(t) \in C$ and

$$D^{-v}\sin(at) = \frac{1}{\Gamma(v)}\int_0^t \xi^{v-1}\sin\big(a(t-\xi)\big)\,d\xi \quad = S_t\,(v,a)$$

Example 1.5.4 Let $f(t) = \cos^2(at)$ $where$ a is any constant. Using identity

$\cos(2at) = 2\cos^2(at) - 1$

We have

$$D^{-v}(\cos^2 at) = \frac{1}{2}\big(C_t(v,2a) + \frac{1}{\Gamma(v+1)}\ t^v\big)$$

Similarly using $\cos(2at) = 1 - 2\sin^2(at)$ we have for $f(t) = \sin^2(at)$

$$D^{-v}(\sin^2 at) = \frac{1}{2}\big(\frac{1}{\Gamma(v+1)}\ t^v - C_t(v,2a)\big)$$

1.6 Fractional Derivatives

There are many different definitions of fractional derivatives, the most popular ones are Riemann-Liouville and Caputo derivative.

Definition 1.6.1 Riemann-Liouville derivative:

$$D_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\ \frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1} f(t)dt \tag{15}$$

Guy Jamarie proposed a simple alternative definition to Reimann-Liouville derivative

$$D_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\ \frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}\{f(t) - f(0)\}dt \tag{16}$$

Definition 1.6.2 Caputo derivative:

$${}_0^cD_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^nf(t)}{dt^n}\ dt \tag{17}$$

$where\ \ n-1 < \alpha \le n$

Definition 1.6.3 Grunwald-Letnikov two sides derivatives are

a) Left sided derivative:

$$D_{a+}^\alpha[f(x)] = \lim_{h\to 0^+}\frac{1}{h^\alpha}\sum_{k=0}^{[n]}(-1)^k\frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \tag{18}$$

$;\ nh = x - a$

b) Right sided derivative:

$$D_{b-}^{\alpha}[f(x)] = \lim_{h\to 0^-} \frac{1}{h^{\alpha}} \sum_{k=0}^{[n]} (-1)^k \frac{\Gamma(\alpha+1) f(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \qquad (19)$$

$; nh = b - x$

Definition 1.6.4 Riesz derivative:

$$D_x^{\alpha} = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)} \frac{1}{\Gamma(\alpha)} \frac{d^n}{dx^n} \left\{ \int_{-\infty}^{x} (x-\xi)^{n-\alpha-1} f(\xi) d\xi + \int_{x}^{\infty} (\xi - x)^{n-\alpha-1} f(\xi) d\xi \right\} \qquad (20)$$

There are many examples of fractional derivatives , let's now look at some example.

Example 1.6.5 Let $(x) = x$, $find$ $\frac{f^{\frac{1}{2}}(x)}{dx^{1/2}}$.

Solution: $f^{\alpha}(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$

$$\frac{f^{\frac{1}{2}}(x)}{dx^{1/2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-1/2}$$

$$= \frac{1\Gamma(1)}{\Gamma(\frac{1}{2}+1)} x^{1/2} = \frac{1\Gamma(1)}{\frac{1}{2}\Gamma(\frac{1}{2})} \sqrt{x} = \frac{1}{\frac{\sqrt{\pi}}{2}} \sqrt{x} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

Example 1.6.6 Solve the following differential equation.

$$\frac{d^{1/2}y}{dx^{1/2}} = x^2 \ , \qquad y(0) = 0$$

Solution:

$$\frac{d^{1/2}}{dx^{1/2}} \frac{d^{1/2}y}{dx^{1/2}} = \frac{d^{1/2}}{dx^{1/2}} x^2$$

$$\frac{dy}{dx} = \frac{d^{1/2}}{dx^{1/2}} x^2$$

$$\frac{d^{1/2}}{dx^{1/2}} x^2 = \frac{\Gamma(2+1)}{\Gamma(2-\frac{1}{2}+1)} x^{2-\frac{1}{2}} = \frac{2}{\Gamma(\frac{5}{2})} x^{3/2} = \frac{2}{\frac{3}{4}\sqrt{\pi}} x^{3/2} = \frac{8}{3\sqrt{\pi}} x^{3/2}$$

$$\frac{dy}{dx} = \frac{8}{3\sqrt{\pi}} x^{3/2}$$

$$dy = \frac{8}{3\sqrt{\pi}}\, x^{\frac{3}{2}}\, dx$$

$$\int dy = \int \frac{8}{3\sqrt{\pi}}\, x^{\frac{3}{2}}\, dx$$

$$y = \frac{8}{3\sqrt{\pi}}\frac{2}{5}\, x^{\frac{5}{2}} + c$$

$$y = \frac{16}{15\sqrt{\pi}}\, x^{\frac{5}{2}} + c$$

$$y = \frac{16}{15\sqrt{\pi}}\, x^{\frac{5}{2}} + 0$$

$$y = \frac{16}{15\sqrt{\pi}}\, x^{\frac{5}{2}}$$

1.7 Terms and Concepts in Economics

1.7.1 Economics: The science of making decisions in the presence of scarce resources.

1.7.2 Managerial Economics: The study of how to direct scarce resources in the way that most efficiently achieves a managerial goal.

1.7.3 Elasticity: A measure of responsiveness of one variable to changes in another variable; the percentage change in one variable that arises due to a given percentage change in another variable.

1.7.4 Own Price Elasticity: A measure of the responsiveness of the quantity demanded of a good to a change in the price of that good; the percentage change in quantity demanded divided by the percentage change in the price of the good.

1.7.5 Economic Growth: Economic growth is the increase in the ability of an economy to produce goods and services over time, in other word economic growth is the rise in aggregate productivity of an economy.

Chapter two

2.Applications of fractional calculus

The subject of fractional calculus has applications in diverse and widespread fields of engineering and science, such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics and signal processing. It has been used to model physical and engineering processes that are found to be the best described by fractional differential equations.

The fractional derivative models are used for accurate modeling for those systems that require accurate modeling of damping. In these fields, various analytical and numerical methods including their applications to new problems have been proposed in recent years.

Fractional calculus has other applications in economic sciences such as modeling GDP and elasticity demand and elasticity supply and more other subjects in economic sciences. We now discuss some of applications for fractional calculus for some sciences and will discuss the fractional calculus in economics in chapters four and five.

2.1 Tautochronous problem

Mehdi Dalir [15]

It may be important to point out that the first application of fractional calculus was made by Abel in the solution of an integral equation that arises in the formulation of the tautochronous problem. The problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass m can fall in a time that is independent of the starting position.

If the sliding time is constant, then the Abel integral equation is

$$\sqrt{2g}T = \int_0^{\eta} (\eta - y)^{-1/2} f'(y)dy \tag{21}$$

Where g is the acceleration due the gravity, (ξ, η) is the initial position and $s = f(y)$ is the equation of the sliding curve. It turns out that this equation is equivalent to the fractional integral equation

$$T\sqrt{2g} = \Gamma\left(\frac{1}{2}\right) {}_0D_{\eta}^{-\frac{1}{2}} f'(\eta) \tag{22}$$

Indeed, Heaviside [15] gave the interpretation of $\sqrt{p} = D^{1/2}$ so that ${}_0D_t^{1/2}\, 1 = 1/\sqrt{\pi t}$.

2.2 Electric transition lines

Mehdi Dalir [15]

During the last decades of the nineteenth century, Heaviside successfully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transition lines. Based on the symbolic operator from solution of heat equation due to Gregory, Heaviside introduced the letter p for the differential operator $\frac{d}{dt}$ and gave the solution of the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 p \tag{23}$$

For the temperature distribution $u(x,t)$ in the symbolic form

$$u(x,t) = A \exp\left(ax\sqrt{p}\right) + B \exp(-ax\sqrt{p}) \tag{24}$$

In which $p \equiv d/dx$ was treated as constant, where a, A, B are also constants.

2.3 Ultrasonic wave propagation in human cancellous bone

Mehdi Dalir [15]

Fractional calculus is used to describe the viscous interactions between fluids and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellous bone in the elastic frame using Blot's theory.

Experimental results are compared with theoretical predictions for slow and fast waves transmitted through human cancellous bone samples.

2.4 Modelling the cardiac tissue electrode interface using fractional calculus

Mehdi Dalir [15]

The tissue electrode interface is common to all forms of biopotential recording (e.g ECG, EMG, EEG) and functional electrical stimulation (e.g pacemaker, cochlear implant, deep brain stimulation). Conventional lumped element circuit models of electrodes can be extended by generalization of the order of differentiation through modification of the defining current-voltage relationships.

Such fractional order models provide an improved description of observed bioelectrode behavior. But recent experimental studies of cardiac tissue suggest that additional mathematical tools may be needed to describe this complex system.

2.5 Application of fractional calculus in the theory of viscoelasticity

Mehdi Dalir [15]

The advantage of the method of fractional derivatives in theory of viscoelasticity is that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. Also the fractional derivative method has been used in studies of the complex moduli and impedances for various models of viscoelastic substances.

2.6 Application of fractional calculus in fluid mechanics

Mehdi Dalir [15]

Applications of fractional calculus to the solution of time-dependent, viscous diffusion fluid mechanics problems are presented. Together with the Laplace transform method, the applications of fractional calculus to the classical transient viscous-diffusion equation in a semi-infinite space is shown to yield explicit analytical (fractional) solutions for the shear-stress and fluid speed anywhere in the domain.

Comparing the fractional results for boundary shear-stress and fluid speed to the existing analytical results to the first and second stokes problems, the fractional methodology is validated and shown to be much simpler and more powerful than existing techniques.

2.7 Discovering Heat Flux at the Boundary of a Semi-Infinite Rod

Adam Loverro [1]

2.7.1 Introduction

Assume you have a semi-infinite bar of unknown radius which extends in length from $x = 0 \; as \; x \to \infty$. The Temperature in the bar, give by the function $u(x,t)$ can be expressed by the following partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{25}$$

or

$$u_t - u_{xx} = 0 \tag{26}$$

For this application, we suppose $u(x,0) = 0 \;\; for \;\; 0 < x < \infty$

$x = 0$ Corresponds to the boundary across which heat is flowing into/out of the rod. Furthermore, let us assume that change in temperature with respect to x at the boundary is given by the heat influx function $P(t)$. We may consider this an extension of fourier's law

$$q = -k\frac{\partial u}{\partial x} \Longrightarrow \frac{\partial u}{\partial x} = u_x(x,t) = P(t) \tag{27}$$

Finally, we suppose that the temperature and the change in temperature with respect to x go to 0 at $x \rightarrow \infty$.

i.e. $\lim_{x\to\infty} u_x(x,t) = \lim_{x\to\infty} u_x(x,t) = 0$ (28)

2.7.2 Cosine Transform of Partial Differential Equation

Fractional Calculus is useful in this problem after an appropriate transform is completed on the differential equation. The effect for this problem is to change the style of the problem from its initial partial differential equation form to a form resembling that of Abel's Integral Equation of the First Kind. Provides a detailed execution of this transformation which will be briefly related here.

Definition 2.7.2.1: Fourier Cosine Transform on $u(x,t)$

$$u_{\boldsymbol{c}}(s,t) = \sqrt{2/\pi}\ \int_0^\infty u(x,t)\cos sxdx \tag{29}$$

$of\ \ u_{xx}(x,t)$ is

$$u_{xx_c}(s,t) = -s^2u_c(s,t) - \sqrt{\frac{2}{\pi}}\,u_x(0,t) \tag{30}$$

Given these transformations, we can rewrite the partial differential equation for the temperature in the rod as

$$\frac{\partial u_c(s,t)}{\partial t} = -\sqrt{\frac{2}{\pi}} *\ P(t) - s^2u_c(s,t) \tag{31}$$

The solution to this first-order non-linear partial differential equation is given by

$$u_c(s,t) = -\sqrt{\frac{2}{\pi}}\int_0^t P(\tau)e^{-s^2(t-\tau)}d\tau \tag{32}$$

$u_c(s,t)$ is inverted back to the x-domain.

$$u(x,t) = \sqrt{2/\pi}\ \int_0^\infty u_c\,(s,t)\cos sx\ \ ds \tag{33}$$

$$u(x,t) = -\frac{2}{\pi}\int_0^t P(\tau)d\tau\int_0^\infty e^{-s^2(t-\tau)}\cos sx\ \ ds \tag{34}$$

By a relation to Green's Function, the transformed solution becomes [1]

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{P(\tau)}{\sqrt{t-\tau}}\, e^{\frac{-s^2}{[4(t-\tau)]}}\, d\tau \tag{35}$$

2.7.3 Implementation of the Fractional Integral Equation

The above equation (35) gives the temperature distribution as a function of time and location for the semi-infinite bar given a known heat flux at the boundary $x = 0$ given by $P(t)$. However, through the introduction of Fractional Calculus, it is possible to obtain, given temperature measurements at the boundary, to arrive at a solution for an unknown heat influx (or efflux, as the case may be).

At the boundary $x = 0+$, the temperature can be written as $u(0+, t) = \emptyset(t)$. Thus

$$\emptyset(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{P(\tau)}{\sqrt{t-\tau}}\, d\tau \tag{36}$$

This equation corresponds to the form of the Abel Integral Equation of the First Kind, and can be re-written with J operator as a fractional integral of order $\frac{1}{2}$. It can then be solved by differentiating by order ½.

$P(t)$ can then be written directly in terms of $\emptyset(t)$.

$$J^{1/2}P(t) = \emptyset(t) \tag{37}$$

$$P(t) = D^{1/2}\emptyset(t) \tag{38}$$

$$P(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\emptyset(\tau)}{\sqrt{t-\tau}}\, d\tau \tag{39}$$

Although the one dimensionality in this particular example makes its usefulness limited, the fractional-order relationship that appears demonstrates a potential for use of fractional calculus in similar problems. This example shows that for a conductive bar of a length significantly greater than its width, it is possible to determine from temperature measurements along its length to calculate the heat flux at its boundary. By using fractional calculus to express this relationship, it is possible to actually reduce the complexity of turning these temperature measurements into usable results. See [1]

2.8 Electrical circuits with fractance

Classical electrical circuits consist of resistors and capacitors and are described by integer-order models. However, circuits may have the so-called fractance which represents an electrical element with fractional-order impedance as suggested by Le Mehaute and Crepy.[14]

A fractance is fractional order models , there are two kinds of fractances**:**

2.8.1 Tree fractance: which consists of a finite self-similar circuit with resistors of resistance R and capacitors of capacitance C where ω is a frequency .

The impedance of the fractance is given by

$$Z(i\omega) = \sqrt{\frac{R}{C}}\, \omega^{-1} exp\, \left(-\frac{\pi i}{4}\right) \tag{40}$$

The associated fractional-order transfer function of this tree fractance is

$$Z(s) = \sqrt{\frac{R}{C}}\, s^{-1/2} \tag{41}$$

2.8.2 Chain fractance: which consists of N pairs of resistor-capacitor connected in a chain. We have shown that the transfer function is approximately given by

$$G(s) \approx \sqrt{\frac{R}{C}\frac{1}{\sqrt{s}}} \tag{42}$$

It can be shown that this chain fractance behaves as a fractional-order integrator of order 1/2 in the time domain $6RC \le t < \left(\frac{1}{6}\right) N^2 RC$.

Due to microscopic electrochemical processes at the electrode-electrolyte interface, electric batteries produce a limited amount of current. At metal electrolyte interfaces the impedance function $Z(\omega)$ does not show the desired capacitive features for frequencies ω. Indeed, as $\omega \to 0$,

$$Z(\omega) \approx (i\omega)^{-\eta}, \quad 0<\eta< 1. \tag{43}$$

Or, equivalently, in the Laplace transform space, the impedance function is

$$Z(s) \approx s^{-\eta} \tag{44}$$

This illustrates the fact that the electrode-electrolyte interface is an example of a fractional-order process. The value of η is closely associated with the smoothness of the interface as the surface is infinitely smooth as $\eta \to 1$. Kaplan proposed a physical model by the self-affine

Cantor block with N-stage electrical circuit of fractance type. Under suitable assumptions, they found the importance of the fractance circuit in the form

$$Z(\omega) = k(i\omega)^{-\eta}, \tag{45}$$

where $\eta = 2 - log(N^2)/\log a$, K and a are constants, and $N^2 > a$ implies $0 < \eta < 1$. This shows that the model of Kaplan. Is also an example of the fractional-order electric circuit. In a resistive-capacitive transmission line model, the inter conductor potential $\emptyset(x,t)$ or inter conductor current $i(x,t)$ satisfies the classical diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \qquad t < 0, \tag{46}$$

Where the diffusion constant k is replaced by $(RC)^{-1}$, R and C denote the resistance and capacitance per unit length of the transmission line, and $u(x,t) = \emptyset(x,t)\ or\ i(x,t)$.

Using the initial and boundary conditions

$$\emptyset(x,0) = 0\ \forall x \in (0,\infty), \qquad \emptyset(x,t) \to 0\ as\ x \to \infty. \tag{47}$$

It turns out that

$$i(0,t) = -\frac{1}{R}\frac{d}{dt}\,\emptyset(0,t) = \left(\frac{C}{R}\right)^{\frac{1}{2}}\ {}_0D_t^{\frac{1}{2}}\emptyset(0,t) \tag{48}$$

This confirms that the current field in the transmission line of infinite length is expressed in terms of the fractional derivative of order $1/2$ of the potential $\emptyset(0,t)$. This is another example of the involvement of fractional-order derivative in the electric transmission line.[14]

2.9 Generalized voltage divider

Westerlund observed that both the tree fractance and chain fractance consist not only of resistors and capacitors properties, but also they exhibit electrical properties with non-integer-order impedance. He generalized the classical voltage divider in which the fractional order impedances F_1 and F_2 represent impedances not only on Westerlund's capacitors, classical resistors, and induction coils, but also impedances of tree fractance and chain fractance. The transfer function of Westerlund's voltage divider circuit is given by

$$H(s) = \frac{k}{s^{\alpha}+k}, \tag{49}$$

Where $-2 < \alpha < 2$ and k is a constant that depends on the elements of the voltage divider. The negative values of α correspond to a highpass filter, while the positive values of α correspond to a lowpass filter. Westerlund considered some special cases of the transfer function (49) for voltage dividers that consist of different combinations of resistors (R), capacitors (C), and induction coils (L). If $U_{in}(s)$ is the Laplace transform of the unit-step input signal $U_{in}(t)$, then the Laplace transform of the output signal $U_{out}(s)$ is given by

$$U_{out}(s) = \frac{ks^{\alpha-1}}{s^{\alpha}+k} \tag{50}$$

Example 2.9.1: The inverse Laplace transform of (50) is obtained from Laplace transform

$$\mathcal{L}\left\{t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}(\pm\alpha t^{\alpha})\right\} = (m!\,s^{\alpha-\beta})/(s^{\alpha}\mp a)^{m+1} \tag{51}$$

To obtain the output signal

$$U_{out}(t) = kt^{\alpha}E_{\alpha,\alpha+1}(-kt^{\alpha}) = k\mathcal{E}_0(t,-k;\alpha,\alpha+1) \tag{52}$$

Although the exact solution for the output signal is obtained, this cannot describe physical properties of the signal. Some physically interesting properties of the output signal can be described for various values of α by evaluating the inverse Laplace transform in the complex s-plane. For $1 < |\alpha| < 2$, the output signal exhibits oscillations. [14]

Chapter Three

3.Fractional calculus in economics

Fractional calculus has wide applications in dynamical systems theory since it allows us to describe systems and media that are characterized by power-law non-locality and long-term memory. A variety of models, which are based on applications of the fractional order derivatives and integrals, have been proposed to describe behavior of financial and economical processes from different points of view.

Mathematical economics is a theoretical and applied science; whose purpose is a mathematically formalized description of economic objects, processes and phenomena. Most of the economic theories are presented in terms of economic models. In mathematical economics, the properties of these models are studied based on formalizations of economics concepts and notions.

In reality, the most important purpose is to formulate economic notions and concepts in mathematical forms, which will be mathematically adequate and self- consistent, and then, on their basis to construct mathematical models of economic processes and phenomena. Moreover, it is not enough to prove the existence of a solution and find it an analytic or numerical form, but it is necessary to give an economic interpretation of these obtained mathematical results.

3.1 History of fractional mathematical economics

Mathematical economics began in the 19^{th} century with the use of differential (and integral) calculus to describe and explain economic behavior. The emergence of modern economic theory occurred almost simultaneously with the appearance of new economic concepts, which were actively used in various economic models.

In modern mathematics, derivatives and integrals of arbitrary order are well known. The derivative (or integral) order of which is a real or complex number and not just an integer, is called fractional derivative and integral.

Fractional mathematical economics is a theory of fractional dynamic models of economic processes, phenomena and effects. In this framework of mathematical economics, the fractional calculus methods are being developed for applications to problems of economics and finance. The field of fractional mathematical economics is the applications of fractional calculus to solve problems in economics (and finance) and for the development of fractional calculus for such applications.

Fractional mathematical economics can be considered as a branch of applied mathematics that deals with economic problems. However, this point of view is obviously a narrowing of the field of research, goals and objectives of this area. An important part of fractional mathematical economics is the use of fractional calculus to formulate new economic concepts, notions, effects and phenomena.

This is especially important due to the fact that the fractional mathematical economics is now only being formed as an independent science. Moreover, the development of the fractional calculus itself and its generalizations will largely be determined precisely by such goals and objectives in economics, physics and other sciences.

The history of application of fractional calculus in economics can be divided into the following stages of development:

3.1.1 ARFIMA Stage [22]**:** This stage is characterized by models with discrete time and applications of the Grunwald-Letnikov fractional differences. Granger and Joyeux and Hosking proposed the fractional generalization of ARFIMA models that improved the statistical methods for researching of processes with memory. As the main mathematical tool for describing memory, fractional differencing and integrating were proposed for discrete time case. The suggested generalization of the ARFIMA model is realized be considering non-integer order d instead of positive integer values of d. The Granger-Joyeux-Hosking operators were proposed and used without relationship with the fractional calculus.

3.1.2 Fractional Brownian motion Stage [22]**:** This stage is characterized by financial models and the applications of stochastic calculus methods and stochastic differential equations. Andrey N. Kolmogorov, who is one of the founders of modern probability theory, was the first who considered in 1940 the continuous Gaussian processes with stationary increments and with the self-similarity property A.N. Kolmogorov called such Gaussian processes “Wiener Spirals”. Its modern name is the fractional Brownian motion that can be considered as a continuous self-similar zero-mean Gaussian process and with the stationary increments.

3.1.3 Econophysics Stage [22]**:** This stage is characterized by financial models and the applications of physical methods and equations. Twenty years ago, a new branch of the econophysics, which is connected with the application of fractional calculus, has appeared. In fact, this branch, which can be called fractional econophysics, was porn in 2000 and it can be primarily associated with the works of Francesco Mainardi, Rudolf Gorenflo, Enrico Scalas, and Marco Roberto on the continuous-time finance. In fractional econophysics, the fractional diffusion models are used in finance, where price jumps replace the particle jumps in the physical diffusion models. The corresponding stochastic models are called continuous time random walks, which are random walks that also incorporate a random waiting time between jumps. In finance, the waiting times measure delay between transactions. These two random variables (price change and waiting time) are used to describe the long-time behavior in financial markets.

3.1.4 Deterministic Chaos Stage [22]**:** This stage is characterized by financial and economic models and applications of methods of nonlinear dynamics. Strictly speaking, this stage should be attributed to the econophysics stage. Nonlinear dynamics models are useful to explain irregular and chaotic behavior of complex economic and financial processes. The complex behaviors of nonlinear economic processes restrict the use of analytical methods to study nonlinear economic models. In 2008, for the first time, Wie-Ching Chen proposed in a fractional generalization of a financial model with deterministic chaos. Chen studied the fractional-dynamic behaviors and describes fixed points, periodic motions, chaotic motions, and identified period doubling and intermittency routes to chaos in the financial process that is described by a system of three equations with the Caputo fractional derivatives. He demonstrates by numerical simulations that chaos exists when orders of derivatives is less than three and that the lowest order at which chaos exists was 2.35. The work studied the chaos control method of such a kind of system by feedback control, respectively.

3.1.5 Mathematical Economics Stage [22]**:** This stage is characterized by macro- economic and microeconomic models with continuous time and generalization of basic economic concepts and notions. The fractional calculus approach has been used to describe the concepts of memory itself for economic processes. This stage began with a proposal of generalizations of the basic economic concepts and notions at the beginning of 2016, when the concepts of elasticity for economic processes with memory was proposed. Then in 2016, the concepts of the marginal values with memory, the concept of accelerator and multiplier with memory and others were suggested. These concepts are used in fractional generalizations of some standard economic models with the continuous times, which were proposed in 2016 and subsequent years. These dynamic models describe fractional dynamics of economic processes with memory.

3.2 Some of concepts and definitions of economics

Before we start studying some case studies that use fractional calculus, we must define some economic concepts, which we will mention in this thesis.

3.2.1 Law of demand: Price and quantity demanded are inversely related. That is, as the price of a good rises(falls) and all other things remain constant, the quantity demanded of the good falls(rises).

3.2.2 Law of supply: As the price of a good rises(falls) and other things remain constant, the quantity supplied of the good rises(falls).

3.2.3 Price elasticity of demand : A measure of how much the quantity demanded of a good responds to a change in the price of that good, computed as the percentage change in quantity demanded divided by the percentage change in price.

3.2.4 Price elasticity of supply: A measure of how much the quantity supplied of a good responds to a change in the price of that good, computed as the percentage change in quantity supplied divided by the percentage change in price.

3.2.5 The Coefficient of determination R^2: The simplest commonly used measure of fit is R^2 or the coefficient of determination. R^2 is the ratio of the explained sum of squares to the total sum of squares.

The higher R^2 is, the closer the estimated regression equation fits the sample data. Measure of this type are called " goodness of fit " measures.

R^2must lie in the interval $0 \leq R^2 \leq 1$, a value of R^2 close to one shows an excellent overall fit, whereas a value near zero shows a failure of estimated regression equation to explain the values.

3.2.6 GDP Formula for Calculation: The following formula can be used to calculate GDP.

$$GDP = C + G + I + NX \tag{53}$$

Where
C = consumer spending.
G = Government spending.
I = Investment.
NX = Net Export (Exports – Imports).

3.3 Elasticity for Economic Processes with Memory

One of the most important areas of applications of differential operators is description of economic dynamics by using the concept of elasticity. Elasticity shows a relative change of an economic indicator under influence of change of an economic factor on which it depends at constant remaining factors acting on it. Usually effects of memory are ignored in the concept of elasticity. For example, the definition of the standard point-price elasticity of demand at time point t_0 , is expressed by the equation

$$E(Q(t); p(t); t_0) = \left(\frac{p(t)}{Q(t)} \frac{\frac{dQ}{dt}}{\frac{dp}{dt}} \right)_{t=t_0} = (\frac{p}{Q} \frac{dQ}{dp})_{t=t_0} \quad , \tag{54}$$

Where Q is the quantity demanded and p is the price of a good. Equation (54) assumes that the elasticity depends only on the current price at $t = t_0$ a price at infinitesimal neighborhood of point t_0 . In general, we should take into account that demand can depend on all changes of prices during a finite interval of time, since behavior of buyers can be determined by the presence of a memory of previous price changes.

3.4 Fractional elasticities of Y with respect to X

The most important area of application of differential calculus is to describe the economics with the help of the concept of elasticity. The elasticity shows a relative change of an economic indicator under influence of change of economic factors on which it depends at constant remaining factors affecting it. In this section, we define generalizations of point elasticity of Y with respect to X by taking into account a long-term memory and a finite-interval memory of changes of economic factor X and indicator. We will consider the following forms of memory. [20]

In the general case, the economic indicator and economic factors can depend on time, i.e. Y and X are functions of time $t \in [t_i ; t_f]$. The absence of memory (amnesia) means that the value of $Y(t)$ is determined only by the values of $X(t)$ at the point $t = t_0 \in [t_i ; t_f]$ and in infinitely small neighborhood of this point. The presence of a memory means that the value of $Y(t)$ depends on values of $X(t)$ at all points t of the finite interval $\left[t_i ; t_f \right]$.

The presence of a memory can also mean that the values of $Y(X_0)$ actually depends not only on X_0 but it also depends on X from the intervals $[X_i ; X_0)$ and $\left(X_0; X_f \right]$. In general, the time parameter cannot be excluded to have an explicit dependence of Y on X in the form of a function. In a rigorous mathematical description of processes with memory, we should apply integro-differential equations. For simplification, we will assume that we have a solution of this equation in the form $Y = Y(X)$.

Definition 3.4.1 The fractional T- elasticity $E_\alpha(Y(t); X(t); [t_i , t_0])$ of order α at $t = t_0$ of $Y = Y(t)$ with respect to $X = X(t)$ is defined by the equation

$$E_\alpha(Y(t); X(t); [t_i , t_0]) := \frac{X(t_0)}{Y(t_0)} \frac{{}_{t_i}^{C}D_{t_0}^{\alpha} Y(t)}{{}_{t_i}^{C}D_{t_0}^{\alpha} X(t)} \tag{55}$$

Where $t \in [t_i; t_0]$.

The fractional T- elasticity $E_\alpha(Y(t); X(t); [t_i , t_0])$ describes an elasticity for the economic processes with a memory of the changes of economic factors and indicator. This type of memory describes the dependence of the economic indicator Y not only on $X(t_0)$ at the current time t_0 but also the economic factor $X(t)$ at all $t \in [t_i; t_0]$. The order α is the parameter that characterizes the degree of damping memory over time. In general, we can consider fractional elasticity with two different parameters α and β to describe fading memory of $Y(t)$ and $X(t)$ respectively.

Definition 3.4.2 Let $Y = Y(X)$ be an economic indicator where X is a function of economic factor $X \in [X_i ; X_f]$. The left-sided and right-sided fractional X-

elasticities $E_{\alpha,l}(Y(X);[X_i\,,X_0])\ and\ E_{\alpha,r}(Y(X);[X_0,X_f\ ])$ of order α at $X_0\ \in\ [X_i\,;X_f]$ of Y with respect to X are defined by the equation

$$E_{\alpha,l}(Y(X);[X_i\,,X_0]) \coloneqq \frac{(X_0)^{\alpha}}{Y(X_0)}\ \ {}_{X_i}^{\ C}D_{X_0}^{\alpha}[X]Y(X)\,, \tag{56}$$

$$E_{\alpha,r}\big(Y(X);\big[X_0\,,X_f\big]\big) \coloneqq \frac{(X_0)^{\alpha}}{Y(X_0)}\ \ {}_{X_0}^{\ C}D_{f}^{\alpha}[X]Y(X)\,, \tag{57}$$

Where $X_i\ =\ X_{min}$ and $X_f\ =\ X_{max}$ are initial and final points of the investigated interval of the economic factor $X\ \in\ [X_i\,;X_f]$. Here ${}_{X_i}^{\ C}D_{X_0}^{\alpha}$ is the left-sided Caputo derivative and ${}_{X_0}^{\ C}D_{f}^{\alpha}$ is the right-sided Caputo derivative of order $\alpha\ >\ 0$.

Using that the standard (point) elasticity of Y with respect to X can be represented as a derivative of $f(t)\ =\ ln(Y(t))$ by $g(t)\ =\ ln(X(t))$ in the form

$$E(p,t_0) \coloneqq \left(\frac{df(t)}{dg(t)}\right)_{t=t_0} \coloneqq \left(\frac{d\ln\big(Y(t)\big)}{d\ln X(t))}\right)_{t=t_0}, \tag{58}$$

We can also define the corresponding fractional generalization by using the fractional derivative of function $f(t) = \ln(Y(t))$ by a function $g(t) = \ln(X(t))$.

Definition 3.4.3 The fractional *Log*-elasticity $E_{\alpha,log}\,(Y(t);X(t);[t_i\,,t_0])$ of order α at $t\ =\ t_0\in\ [t_i;t_f\ ]$ is defined by the equation

$$E_{\alpha,log}(Y(t);X(t);[t_i\,,t_0])\ =$$

$$\frac{1}{\Gamma(n-\alpha)}\int_{t_i}^{t_0} d\tau\ \ \frac{dg(\tau)}{d\tau}\ \frac{f(\tau)}{\big(g(t)-g(\tau)\big)^{\alpha+1-n}}\left(\frac{1}{\frac{dg(\tau)}{d\tau}}\ \frac{d}{d\tau}\right)^{n}\ f(t)\ \ ,(t_0>t_i) \tag{59}$$

$Where\ n-1\leq\alpha\leq n\,, f(t)=\ln\big(Y(t)\big)\quad and\quad g(t)=\ln\big(X(t)\big).$

Remark 3.4.4 For the $case\ \alpha\ =\ 1$, equations (55), (56) and (57) take the forms

$$E1(Y(t);X(t);[t_i\,,t_0])\ =\ \frac{X(t_0)}{Y(X_0)}\left(\frac{\frac{dY(t)}{dt}}{\frac{dX(t)}{dt}}\right)_{t=t_0} \tag{60}$$

And

$$E_{1,l}(Y(X);[X_i\,,X_0]) =\ E1,r\big(Y(X);\big[X_0,X_f\big]\big) = \frac{X_0}{Y(X_0)}\ \left(\frac{dY(X)}{dX}\right)_{X=X_0} \tag{61}$$

Where the elasticity does not depend $on\ t \neq t_0\ and\ X \neq X_0$. This means that the case $\alpha = 1$ corresponds to the economic processes without memory.

Remark 3.4.5 Using the chain rule

$$\frac{dY(X(t))}{dt} = (\frac{dY(X)}{dX})_{X=X(t)} \frac{dX(t)}{dt} \tag{62}$$

We have the equality of the fractional T -elasticity and the fractional X -elasticity for the case $\alpha = 1$ of the economic dynamics without memory,

$$E_1(Y(t); X(t); [t_i , t_0]) = E_{1,l}(Y(X); [X_i , X_0]) = E_{1,r}\left(Y(X); \left[X_0, X_f \right]\right) \tag{63}$$

This case corresponds to economic the case of total amnesia. The standard point elasticity of Y with respect to X describes economic processes, when market participants have amnesia.

3.5 Properties of fractional elasticities

Let us give main properties of the suggested fractional elasticities. For simplification, we describe these properties for $t_i = 0$ and $X_i = 0$.

The fractional elasticity is a dimensionless quantity,

$$E_\alpha(\lambda Y(t); X(t); [0, t_0]) = E_\alpha(Y(t); X(t); [0, t_0]), \tag{64}$$
$$E_\alpha(Y(t); \lambda X(t); [0, t_0]) = E_\alpha(Y(t); X(t); [0, t_0]), \tag{65}$$

These equations mean that its do not depend on units of the economic indicator Y and the economic factor X .

The fractional T -elasticity of inverse function is inverse

$$E_\alpha(X(t); Y(t); [0, t_0]) = \frac{1}{E_\alpha(Y(t); X(t); [0, t_0]),} \tag{66}$$

In general, the fractional X -elasticities of inverse functions are not inverse

$$E_{\alpha,l}(Y(X); [0, X_0]) \neq \frac{1}{E_\alpha(X(Y); [0, Y_0])} ,$$
$$E_{\alpha,r}\left(Y(X); \left[X_0, X_f\right]\right) \neq \frac{1}{E_\alpha\left(X(Y); \left[Y_0, Y_f\right]\right)} \tag{67}$$

These inequalities become equalities for $\alpha = 1$.

In general, the fractional elasticity of the product of two functions, which depend on the same argument, does not equal to the sum of elasticities unless $\alpha = 1$

$$E_\alpha(Y_1(t) \cdot Y_2(t); X(t); [0, t_0]) \neq E_\alpha(Y_1(t); X(t); [0, t_0]) + E_\alpha(Y_2(t); X(t); [0, t_0]) \quad (68)$$

For $\alpha \neq 1$.

This inequality becomes an equality for $\alpha = 1$. Inequality (68) caused by the violation of the Liebniz rule

$$ {}_a^C D_t^\alpha \ [\tau]\ (Y_1(\tau) Y_2(\tau)) \neq (\ {}_a^C D\ _t^\alpha\ [\tau] Y_1(\tau)\ )\ Y_2(\tau) + Y_1(\tau)\ ({}_a^C D\ _t^\alpha\ [\tau] Y_2(\tau)) \quad (69)$$

The fractional elasticity of the sum of two functions, which depend on the same argument, is given by the equation

$$E_\alpha(Y_1(t) + Y_2(t); X(t); [0, t_0]) = \frac{1}{Y_1 + Y_2}\big(Y_1(t) E_\alpha(Y_1(t); X(t); [0, t_0]) + Y_2(t) E_\alpha(Y_1(t); X(t); [0, t_0])\big) \quad (70)$$

The fractional elasticity of the power function is a constant

$$E_{\alpha,l}\big(X^\beta; [0, X_0]\big) = \frac{\beta}{\alpha}\,, \quad (71)$$

$where\ \beta > n - 1$ and $n - 1 < \alpha < n$ for all $n \in \mathbb{N}$.

The fractional elasticity of the exponential function is given by the equation

$$E_{\alpha,r}\left(e^{-\lambda X}; [X_0; \infty)\right) = (\lambda X)^\alpha, \quad (72)$$

Where $\lambda > 0$

The fractional elasticity of the linear function is given by the equation

$$E_{\alpha,l}(a_0 + a_1 X; [0, X_0]) = \frac{1}{\Gamma(2 - \alpha)}\,\frac{a_1 X}{a_0 + a_1 X} \quad (73)$$

For derivatives of non-integer orders, the standard chain rule cannot be satisfied in general. For example, the chain rules for fractional derivatives of a composite function have the form that is similar to the following

$$\mathcal{D}_t^\alpha Y = \frac{t^\alpha Y(X(t))}{\Gamma(1 - \alpha)} + \sum_{K=1}^{\infty} C_t^\alpha\,\frac{k!\,t^{k-\alpha}}{\Gamma(k - \alpha + 1)} \times \sum_{m=1}^{k} (D_X^m\, Y(X))_{X=X(t)}\ \ \sum \prod_{r=1}^{k} \frac{1}{a_r!}\ \ (\frac{(D_t^r X)(t)}{r!})^{ar}\,, \quad (74)$$

Where $D_n^t = d^n/dt^n$. Therefore, we have the inequalities

$$E_\alpha(Y(t); X(t); [t_i, t_0]) \neq E_{\alpha,l}(Y(X); [X_i, X_0], \tag{75}$$

$$E_\alpha(Y(t); X(t); [t_i, t_0]) \neq E_{\alpha,r}(Y(X); [X_i, X_f], \tag{76}$$

For non-integer values of the order α. As a result, the fractional X -elasticities and the fractional T -elasticity should be considered as independent characteristics in the economic dynamics with memory.

The fractional elasticities of constant demand are equal to zero.

$$E_\alpha\,(\text{const}; X(t); [t_i, t_0]) = 0,$$
$$E_{\alpha,l}(\text{const}; [X_i\,, X_0]) = \; E_{\alpha,r}(\text{const}; [X_0\,, X_f]) \tag{77}$$

That corresponds to perfectly inelastic demand.

These properties can be directly derived from the properties of the Caputo fractional derivative and the definition of the fractional elasticities.

3.6 Fractional elasticities of demand

In this section, we define generalizations of point-price elasticity of demand to the cases of memory. In these generalizations we take into account dependence of demand not only from the current price (price at current time), but also changes of prices in some interval (prices that were before this current price). For simplification, we will assume that there is one parameter α, which characterizes a degree of damping memory over time.[20]

Definition 3.6.1 Let demand $Q = Q(t)$ and price $p = p(t)$ be functions of time variable $t \in [t_i\,; t_f\,]$. the fractional T -elasticity $E_\alpha(Q(t);\, p(t); [t_i\,, t_0])$ of order α at $t = t_0$ of demand $Q(t)$ with respect to price $p(t)$ is defined by the equation

$$E_\alpha(Q(t); p(t); [t_i\,, t_0]) \coloneqq \frac{p(t_0)}{Q(t_0)} \frac{{}_{t_i}^{C}D_{t_0}^{\alpha}\, Q(t)}{{}_{t_i}^{C}D_{t_0}^{\alpha}\, p(t)}\,, \tag{78}$$

Where $t \in [t_i\,, t_0]$, and $t_i < t_0 < t_f$. The fractional T -elasticity (78) describes an elasticity of demand for the processes in economic dynamical systems with the memory of price changes over time. This type of memory describes the dependence of demand Q not only from the price $p = p(t_0)$ at the current time t_0 but also the prices $p(t)$ that were before this price, i.e. all prices at $t \in [t_i\,; t_0]$. The order α is the parameter that characterizes the degree of damping memory over time.

Definition 3.6.2 Let us consider a demand $Q = Q(p)$ as a function of price $p \in [p_l\,;\, p_h]$. the left-sided and right-sided fractional p-elasticities $E_{\alpha,l}(Q(p); [p_l\,, p_0])$ and $E_{\alpha,r}(Q(p); [p_0, p_h])$ of order α at $p_0 \in [p_l\,;\, p_h]$ of demand $Q = Q(p)$ is defined by the equations

$$E_{\alpha,l}(Q(p);[\,p_l\,,p_0]) := \frac{(p_0)^{\alpha}}{Q(p_0)}\ {}_{p_l}^{C}D_{p_0}^{\alpha}[p]Q(p), \tag{79}$$

$$E_{\alpha,r}(Q(p);[\,p_0\,,p_h]) := \frac{(p_0)^{\alpha}}{Q(p_0)}\ {}_{p_0}^{C}D_{p_h}^{\alpha}[p]Q(p), \tag{80}$$

$where\ p_l = p_{min}$ is the lowest price and $p_h = p_{max}$ is the highest price; ${}_{X_i}^{C}D_{X_0}^{\alpha}$ and ${}_{X_0}^{C}D_{X_f}^{\alpha}$ are the left-sided and right-sided Caputo derivatives of order $\alpha > 0$.

The fractional p-elasticities (79) and (80) describe an elasticity of demand for the processes in economic dynamical systems with price memory. The elasticity (79) takes into account a "memory of low prices". The "memory of high prices" is taken into account by the elasticity (80). These types of memory describe a dependence of demand Q not only on the current price p_0 but also all prices p of the given range ($p_l \leq p \leq p_h$). The order α characterizes a degree of damping memory over time.

Analogously to generalization of the price elasticities of demand, we can generalize of other types of elasticity. For example, we can give definitions of fractional income elasticity of demand. Using the demand function $Q = Q(t)$ and income function $I = I(t)$ of time variable $t \in [t_i\,;t_0]$, the fractional income T -elasticity $E\alpha(Q(t);I(t);[t_i\,;t_0])$ of order α at $t = t_0$ can be defined by the equation

$$E_{\alpha}(Q(t);I(t);t_i\,;[t_i\,;t_0]) := \frac{I(t_0)}{Q(t_0)}\,\frac{{}_{t_i}^{C}D_{t_0}^{\alpha}Q(t)}{{}_{t_i}^{C}D_{t_0}^{\alpha}I(t)}\,, \tag{81}$$

Remark 3.6.3 In the definition of the fractional elasticities, we use the Caputo fracional derivatives instead of other types of derivatives. It is caused by that the Caputo fractional derivatives of a constant is equal to zero. This property leads us to zero values of fractional elasticities for constant demand. Contrary to it the RiemannLiouville fractional derivatives of a constant is not equal to zero

$${}_{0}^{RL}D_{p}^{\alpha}[p']Q(p') = \frac{p^{-\alpha}}{\Gamma(1-\alpha)}\,, \tag{82}$$

Therefore the fractional elasticities, which are defined by this type of derivatives, cannot be considered as a perfectly inelastic demand for the constant demand functions. For example, the corresponding left-sided fractional p-elasticity of the constant demand $Q(p) = q_0 = \text{const}$ is the constant.

$${}_{RL}E_{\alpha,l}(Q(p);[0,p]) := \frac{p^{\alpha}}{Q(p)}\ {}_{0}^{RL}D_{p}^{\alpha}[p']Q(p') = \frac{1}{\Gamma(1-\alpha)}\,, \tag{83}$$

Where ${}_{0}^{RL}D_{p}^{\alpha}$ is the left-sided Riemann-Liouville derivative.

It should be noted that the fractional p- elasticities and the fractional T - elasticity should be considered as independent indicators of the economic dynamics with memory. This fact is based on the violation of the standard chain rule for derivatives of non-integer orders.

3.7 Examples of calculations

Let us consider simple examples of calculations of fractional elasticities. For simplification, we will use the demand equation

$$Q(p) = a_0 + a_1 p + a_2 p^2 \tag{84}$$

Where p is the unit price and $Q(p)$ is the quantity demanded when the price is p. Equation (84) is considered as a demand function for a product. Point-price elasticity is the elasticity of demand, which is defined by the equation

$$E(p) = \left(\frac{p}{Q(p)}\right)\left(\frac{dQ(P)}{dp}\right)$$

To find the point elasticity of demand $E(p)$ for (84), we use

$$\frac{dQ(p)}{dp} = a_1 + 2a_2 p\,, \tag{85}$$

As a result, the standard (point-price) elasticity of demand is

$$E(p) = \frac{p}{Q(p)}\,\frac{dQ(p)}{dp} = \frac{a_1 p + 2a_2 p^2}{a_0 + a_1 p + a_2 p^2}\,, \tag{86}$$

Let us consider some examples of fractional elasticity for demand (84).

Example 3.7.1 Let us consider the demand and price functions in the form

$$Q(t) = q_0 + q_1 t + q_2 t^2, \tag{87}$$

$$p(t) = p_0 t\,, \tag{88}$$

It is obvious that the substitution of (88) into (87) gives (84) with

$$a_0 = q_0\,, \qquad a_1 = \frac{q_1}{p_0}\,, \qquad a_2 = \frac{q_2}{p_0^2}\,, \tag{89}$$

Let us consider the fractional T - elasticity (78) with $t_i = 0$ and $\alpha \in (0,1)$ from the properties of the Caputo fractional derivatives of power functions we are given,

$${}_0^C D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad (t > 0, n-1 < \alpha < n\,, \beta > n-1), \tag{90}$$

Using (90) we get

$$ {}_{0}^{C}D_{t}^{\alpha} Q(t) = q(1)\frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} + q(2)\frac{\Gamma(\beta 3+1)}{\Gamma(3-\alpha)} t^{2-\alpha}, \tag{91} $$

And

$$ {}_{0}^{C}D_{t}^{\alpha} p(t) = p_0 \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}, \tag{92} $$

Substitution of (91) and (92) into (78) gives the fractional T –elasticity

$$ \begin{aligned} E_{\alpha}(Q(t); p(t); [0\,, t_0]) &:= \frac{p(t)}{Q(t)} \frac{{}_{0}^{C}D_{t}^{\alpha}\, Q(t)}{{}_{0}^{C}D_{t}^{\alpha}\, p(t)} \\ &= \frac{p_0 t}{q_0 + q_1 t + q_2 t^2} \frac{q_1 \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + q_2 \frac{2}{\Gamma(2-\alpha)} t^{2-\alpha}}{p_0 \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}} \\ &= \frac{p_0 t}{q_0 + q_1 t + q_2 t^2} \frac{q_1 t^{1-\alpha} + q_2 \frac{2}{(2-\alpha)} t^{2-\alpha}}{p_0 t^{1-\alpha}} \end{aligned} $$

$$ = \frac{q_1 t + q_2 \frac{2}{(2-\alpha)} t^2}{q_0 + q_1 t + q_2 t^2} = \frac{a_1 p + a_2 \frac{2}{2-\alpha} p^2}{a_0 + a_1 p + a_2 p^2}\ , \tag{93} $$

Where we use $\Gamma(z+1) \ = \ z\Gamma(z)$ and equations (88) , (89).

In the general case, these fractional elasticity can be distinguished not only by a factor.

$$ E_{\alpha,l}(Q(p); [0, p]) = \frac{1}{\Gamma(2-\alpha)} E_{\alpha}(Q(t); p(t); [0, t]). \tag{94} $$

Example 3.7.2 Let us consider the demand and price functions in the following simple form

$$ Q(t) = a t^{\beta}, \tag{95} $$

$$ p(t) = b t^{\gamma}, \tag{96} $$

Substitution of (96) into (95) gives

$$ Q(t) = \frac{a}{b^{\beta/\gamma}} p^{\beta/\gamma}, \tag{97} $$

The standard (point-price) elasticity has the form

$$E(p) = \frac{p}{Q(t)} \frac{dQ(p)}{dp} = \frac{\beta}{\gamma}\,, \tag{98}$$

The fractional p-elasticity is given by the equation

$$E_{\alpha,l}(Q(p);[0,p]) = \frac{(p)^{\alpha}}{Q(p)}\, {}_{0}^{C}D_{p}^{\alpha} Q(p) = \frac{\Gamma\left(\frac{\beta}{\gamma}+1\right)}{\Gamma\left(\frac{\beta}{\gamma}+1-\alpha\right)}, \tag{99}$$

The fractional T -elasticity is written in the form

$$E_{\alpha}(Q(t);p(t);[0,t_0]) = \frac{P(t)}{Q(t)} \frac{{}_{0}^{C}D_{t}^{\alpha} Q(t)}{{}_{0}^{C}D_{t}^{\alpha} p(t)} \frac{\Gamma(\beta+1)\Gamma(\gamma-1+\alpha)}{\Gamma(\gamma+1)\Gamma(\beta-1+\alpha)}\,, \tag{100}$$

$For\ \alpha = 1$, we get

$$E_{\alpha}(Q(t);p(t);[0,t_0]) = E_{\alpha,l}(Q(p);[0,p_0]) = E(p)$$

since $\Gamma(z+1) = z\Gamma(z)$. It is easy to see that the expression of the fractional p-elasticity (99) and the fractional T -elasticity (100) are different for $\alpha \neq 1$. It is well-known that the following conditions, if $E(p) < -1$, then demand is elastic and a percent increase in price yields a larger percent decrease in demand.

$If\ -1 < E(p) < 0$, then demand is inelastic and a percent increase in price yields a smaller percent decrease in demand. It is evident that taking into account the effect of memory $(0 < \alpha < 1)$, we can get other inequalities for the price in comparison with the standard case $(\alpha = 1)$.

Chapter Four

4.Using Fractional Calculus to Calculate GDP

4.1 Economic Meaning of Derivatives

The economic meaning of the derivative of the first order describes an intensity of changes of an economic indicator regarding the investigated factor by assuming that other factors remain unchanged. First-order derivative of the function of an indicator defines the marginal value of this indicator. The marginal values shows the growth of the corresponding indicator per unit increase of the determining factor. This is analogous to the physical meaning of speed (The physical meaning of speed is the path length travelled per unit of time). In economic theory, the main marginal values of indicators are marginal product, marginal utility, marginal cost, marginal revenue, marginal propensity to save and consume, marginal tax rate, marginal demand and some others.[4]

In the study of economic processes is usually performed by calculation of the marginal and average values of indicators that are considered as functions of the determining factors.

In modern mathematics it is known the concept of derivative (integer-differentiation) of non-integer orders. This concept is used in the natural sciences to describe the processes with memory. Recently, non-integer order derivatives have been used to describe the financial processes and the economic processes. There are various types of derivatives of non-integer orders. In this thesis we will consider the Caputo fractional derivative. One of the distinguishing features of this derivative is that its action on the constant function gives zero. The use of the Caputo derivative in the economic analysis produces zero value of marginal indicator of non-integer order for the function of the corresponding indicator. There are left-handed and right-handed Caputo derivatives. We will consider only the left-sided derivatives, since the economic process at time T depends only on state changes of this process in the past, that is, for t < T, and the right-sided Caputo derivative is determined by integrating the values of t > T. Left-sided Caputo derivative of order α > 0 is defined by the formula

$$ {}_{0}^{C}D_{T}^{\alpha} f(T) \coloneqq \frac{1}{\Gamma(n-\alpha)} \int_{0}^{T} \frac{f^{(n)}(t)dt}{(T-t)^{\alpha-n+1}} , \qquad (101) $$

Where $\Gamma\,(z)$ is the gamma function, and $T > 0, n \coloneqq [\alpha] + 1$, and $f^{(n)}(t)$ is standard derivative of integer order n of the function $f(t)$ at time $t.$ For integer values of $\alpha = n \in N$, the Caputo derivative coincides with standard derivatives of order $n,$ i.e.

$$ {}_{0}^{C}D_{T}^{n} f(t) \coloneqq \frac{d^n f(T)}{dT^n} , \quad {}_{0}^{C}D_{T}^{0} f(t) \coloneqq f(T). \qquad (102) $$

The economic interpretation of non-integer order derivative $\alpha \geq 0$ directly related with the concept of T-indicator. The fractional derivatives can be interpreted as a growth of indicator Y per unit increase of the factor X at time $t = T$ in the economic process with memory of power type.

4.2 Case Study

4.2.1 Applying Fractional Calculus to Analyze Economic Growth Modelling

It is well known that EGM is one of the most important models in studying the dynamics of finance behaviour. After reviewing the classical EGM in the literature, one can see that the integer order derivatives and integrals are always used to characterize such procedure in the development of economics. However, there exist some gaps by using the classical calculus to simulate the data from the real models. Recently, the basic theory including existence theory, stability and control theory for all kinds of fractional differential equations and inclusions is studied extensively. In addition, one can see that fractional calculus is also widely used to construct economic models involving the memory effect in the evolutionary process. It has been proved that fractional models are better than integer models and provide an excellent tool for the description of memory of EGM .

In this thesis we start with study GDP growth for the Spanish and Portuguese cases by applying Grunwald-Letnnikov fractional EGM via data between 1960 and 2012. By setting the mean absolute deviation as performance index and using Nelder-Mead's simplex search method, the coefficients and orders proposed in the fractional EGM are obtained. By comparing the coefficients of fractional EGM and integer EGM, a new hybrid model involving integer calculus and fractional calculus is established to remove low influence variables in the models. It is shown that fractional models have a better performance than the classical models. In this thesis, we go on the study of GDP growth for the Spanish case to improve fractional EGM by using different computational methods. More precisely, we use four different EGMs, namely Grunwald-Letnnikov integer/fractional type and Caputo integer/fractional type models. Moreover, Nelder-Mead's simplex search method is replaced by genetic algorithm to give orders in the current work. The method of least squares is used to give the estimation of the coefficients. In spite of software of Matlab, SPSS and R are also used in linear regression analysis. We note that the Spanish case is used in this thesis only for possibility to compare our achievement and proposed models with previous ones.[4]

4.2.2 Integer and fractional EGMS

Through out of this thesis, we denote land area by LA (km^2), arable land area by AL (km^2), population by P, school attendance by SA, gross capital formation by GCF, exports of goods and services by EGS, general government final consumption expenditure by GGFCE, money and quasi money by MQM, number of variables of the model by NVM and number of parameters of the model by NPM. We remark that all the data used here are taken from 1960 to 2012. We also denote the mean square error by MSE, the mean absolute deviation by MAD, the coefficient of determination by R^2, Akaike Information Criterion by AIC and the weight of AIC for the i-th model by ωi .[4]

Consider the following general formulation of $EGM: z(t) = f(x_1, x_2, \cdots)$ where f is a given function. For simplify, we introduce the following notations:

x_1 is LA, x_2 is AL, x_3 is P, x_4 is SA, x_5 is GCF, x_6 is EGS, x_7 is GGFCE, and x_8 is MQM

Where LA: land area, AL: arable land, P: population, SA: school attendance, GCF: gross capital formation, EGS: exports of goods and services, GGFCE: general government final consumption expenditure, MQM: money and quasi money.

And

z is GDP and t is year.

Where the output model z is the GDP (in 2012 euros) and x_k are the variables on which the output depends. The inputs considered are the following:

$\boldsymbol{x_1}$**:** Land area ((km^2);

$\boldsymbol{x_2}$**:** Arable land(km^2);

$\boldsymbol{x_3}$**:** Population;

$\boldsymbol{x_4}$**:** School attendance (years);

$\boldsymbol{x_5}$**:** Gross capital formation (GCF) (in 2012 euros);

$\boldsymbol{x_6}$**:** Exports of goods and services (in 2012 euros);

$\boldsymbol{x_7}$**:** General government final consumption expenditure (GGFCE) (in 2012 euros);

$\boldsymbol{x_8}$**:** Money and quasi money (MQM) (in 2012 euros).

The rationale behind this choice of variables is the following:

Natural resources are represented by x_1 , and their quality by x_2;

Human resources are represented by x_3 , and their quality by x_4;

Manufactured resources are represented by x_5;

External impacts in the economy are represented by x_6;

Internal impacts in the economy are represented by x_7 (budgetary impacts) , x_8 (monetary impacts) and also by x_5 (investment). Rather than having x_5 play two roles, we will rather use another variable $x_9 \equiv x_5$ to represent the impact of investment in the economy, bringing the number of inputs up to 9.

Define

$$\text{MSE} = \frac{\sum_{i=1}^{n}(z_i - \tilde{z})^2}{n}, \tag{103}$$

$$\text{MAD} = \frac{\sum_{i=1}^{n}|z_i - \tilde{z}_i|}{n}, \tag{104}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(z_i - \tilde{z})^2}{\sum_{i=1}^{n}(z_i - \bar{z})^2}, \tag{105}$$

Where $z \ is \ GDP, \bar{z} \ is \ the \ mean \ of \ GDP \ , \tilde{z} \ is \ the \ expected \ of \ the \ GDP$.

Next, we recall the following standard integer order model (IOM)

$$z(t) = \sum_{i=1,2,3,4,6,7} c_i x_i(t) + c_5\left(I_{t_0,t}^{1} x_5\right)(t) + \sum_{i=8,9} c_i x'_i(t) \tag{106}$$

We also need the following modified models:

IOM1 (Grunwald-Letnnikov integer type)

$$z(t) = \sum_{i=1,2,3,4,6,7} c_i({}^{GL}D_{t_0,t}^{0}\, x_i)(t) + c_5\left({}^{GL}D_{t_0,t}^{-1} x_5\right)(t) + \sum_{i=8,9} c_i({}^{GL}D_{t_0,t}^{1} x_i)(t) \tag{107}$$

IOM2 (Caputo integer type)

$$z(t) = \sum_{i=1,2,3,4,6,7} c_i({}^{C}D_{t_0,t}^{0}\, x_i)(t) + c_5\left(I_{t_0,t}^{1} x_5\right)(t) + \sum_{i=8,9} c_i({}^{C}D_{t_0,t}^{1} x_i)(t) \tag{108}$$

FOM1 (Grunwald-Letnnikov fractional type)

$$z(t) = \sum_{i=1}^{9} c_i({}^{GL}D_{t_0,t}^{\alpha_k}\, x_i)(t) \tag{109}$$

FOM2 (Caputo fractional type)

$$z(t) = \sum_{i=1}^{9} c_i({}^{C}D_{t_0,t}^{\alpha_i} x_i)(t) \tag{110}$$

where t_0 denotes the first year and the fractional calculus is given by

$$\left(I_{a,t}^{\alpha} u\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(\tau)}{(1-\tau)^{1-\alpha}} d\tau, \qquad 0 < \alpha \leq 1, \tag{111}$$

and the Grunwald-Letnnikov (GL) derivative

$${}^{GL}D_{a,t}^{\alpha} u(t) = \lim_{h \to 0} \frac{\sum_{j=0}^{[(t-a)/\,h]}}{h^{\alpha}} (-1)^j\, C_\alpha^j u(t - jh), \tag{112}$$

$$C_\alpha^j = \frac{(-1)^j \Gamma(\alpha - j)}{\Gamma(j+1)\Gamma(-\alpha - j + 1)}\ , \qquad 0 < \alpha \leq 1, \tag{113}$$

$$C_\alpha^j = \frac{\Gamma(\alpha + 1)}{\Gamma(j+1)\Gamma(\alpha - j + 1)}\ \ , \qquad -1 < \alpha \leq 0\,, \tag{114}$$

$$C_\alpha^j = 1, \qquad \alpha = 0$$

and the Caputo derivative

$${}^{C}D_{a,t}^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(s)}{(t-s)^{\alpha}} ds, t > a, \qquad 0 < \alpha \leq 1 \tag{115}$$

4.2.3 Main Results

4.2.3.1 Economic data for Spanish economy by using the Spanish data from 1960 to 2012, we apply Matlab to obtain the following figures (4.1),(4.2),(4.3),(4.4),(4.5),(4.6),(4.7),(4.8)

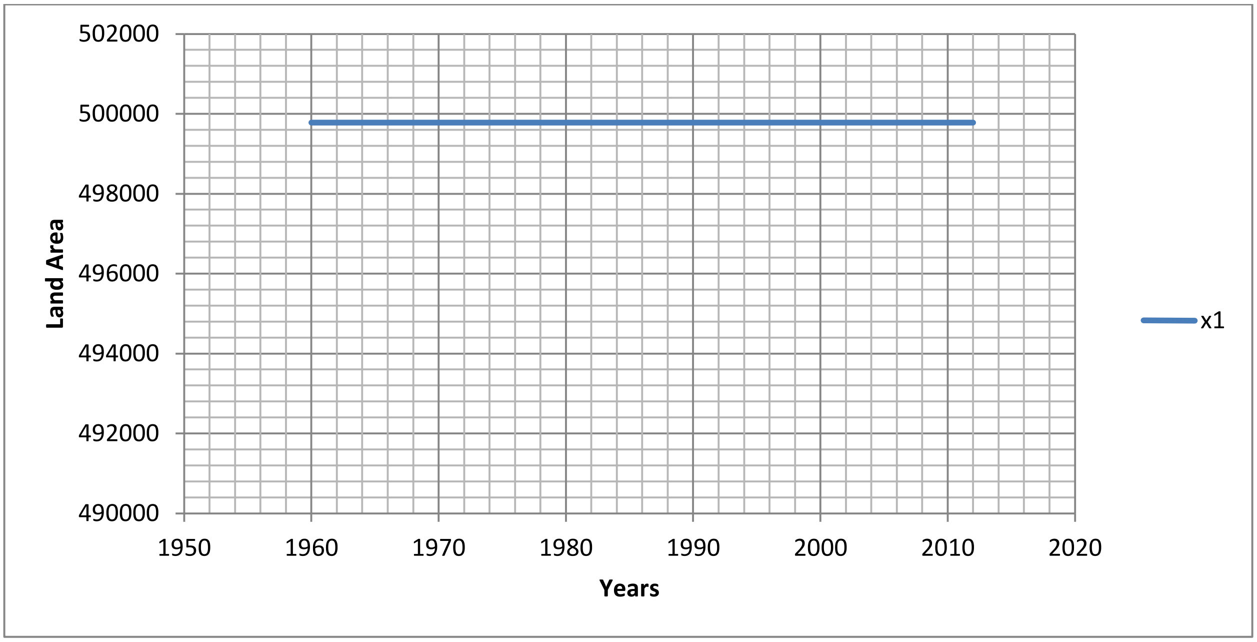

Figure 4.1: x_1 Land Area Curve.

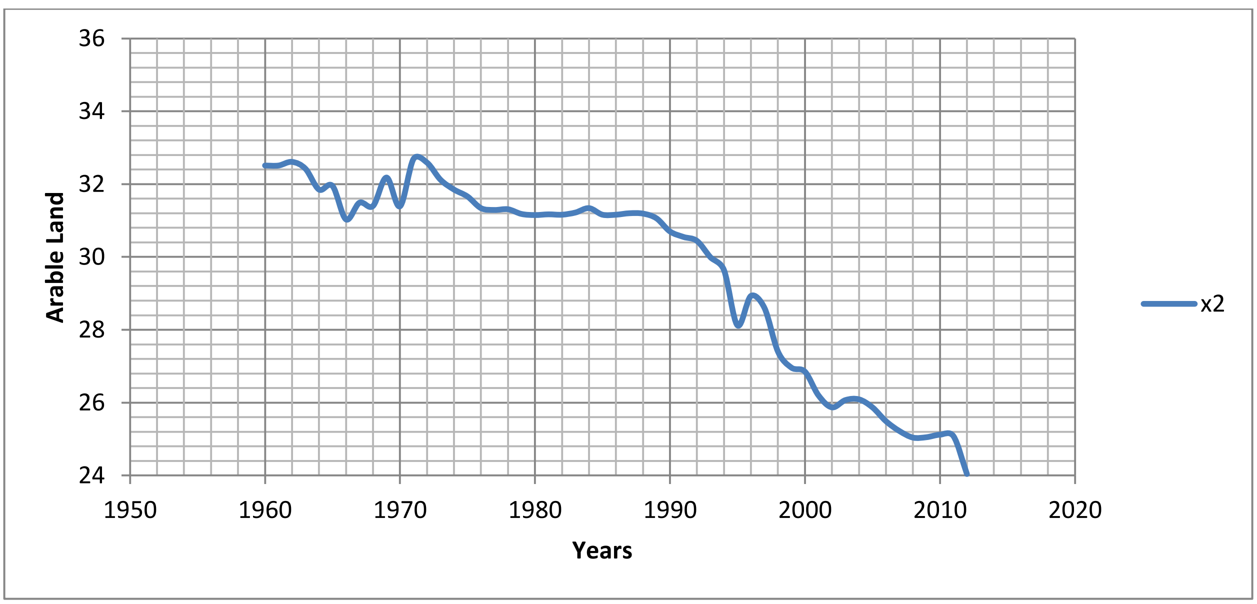

Figure 4.2: x_2 Arable Land Curve.

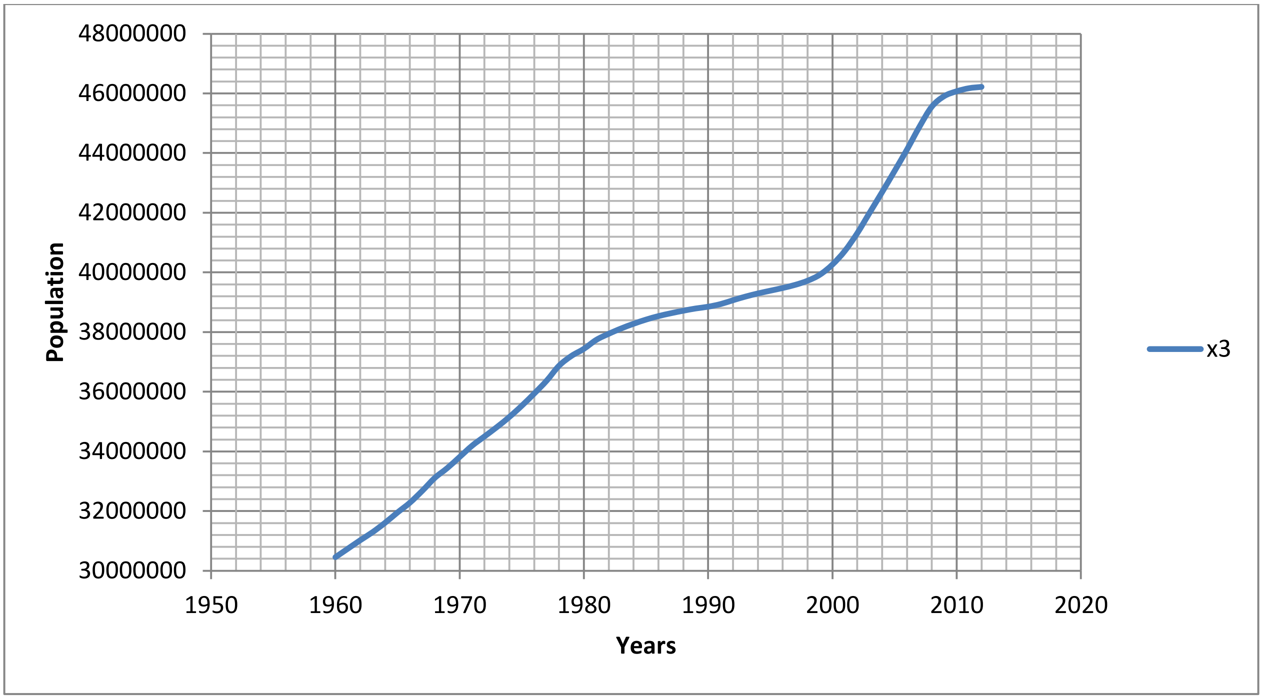

Figure 4.3: x_3 Population Curve.

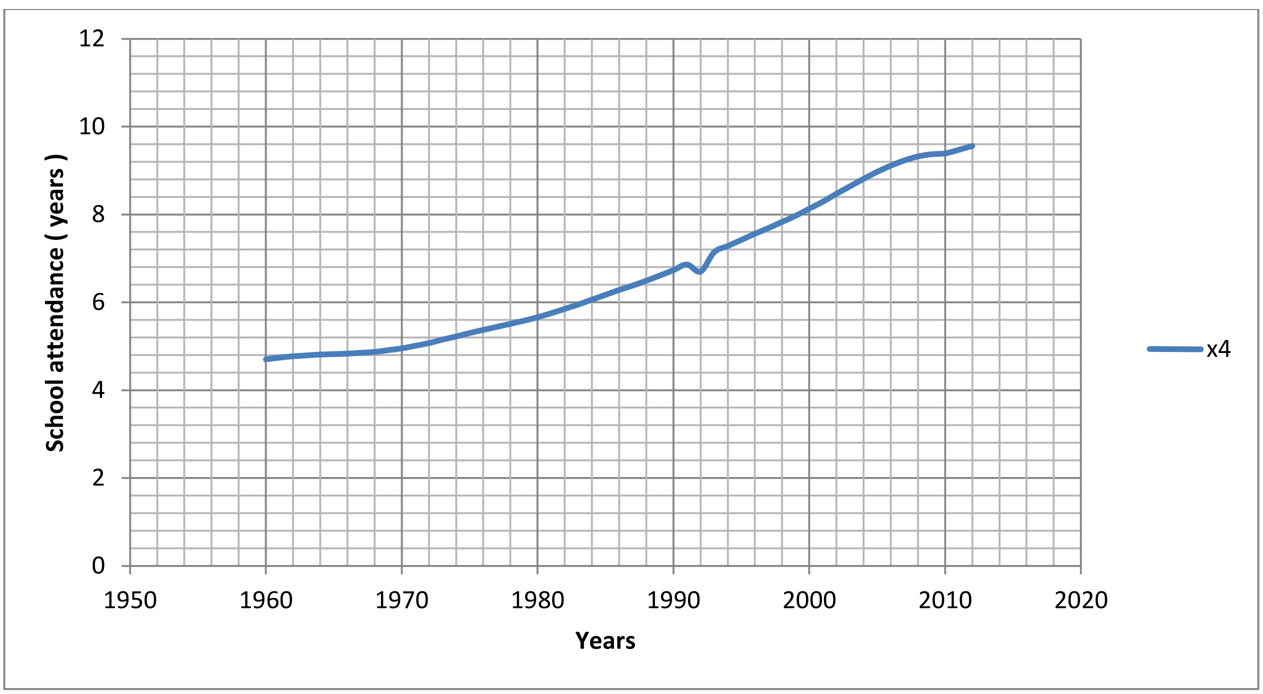

Figure 4.4: x_4 School Attendance Curve.

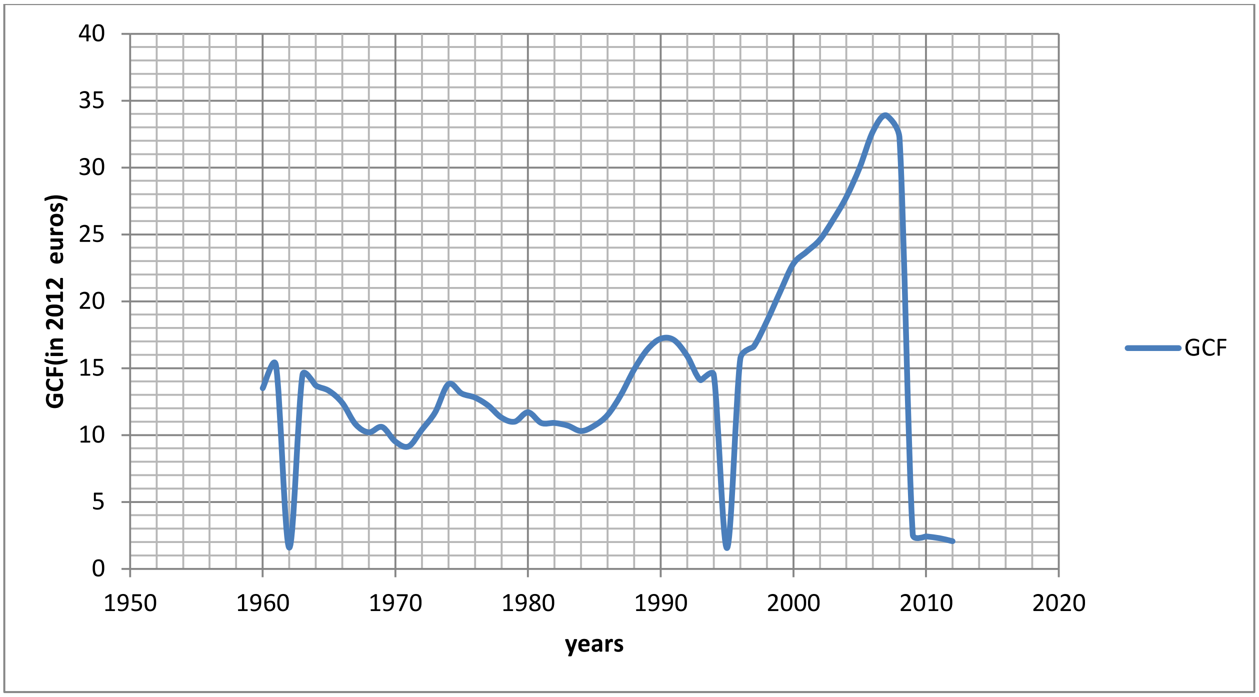

Figure 4.5: $x_5 \equiv x_9$ Gross Capital Formation (GCF) Curve.

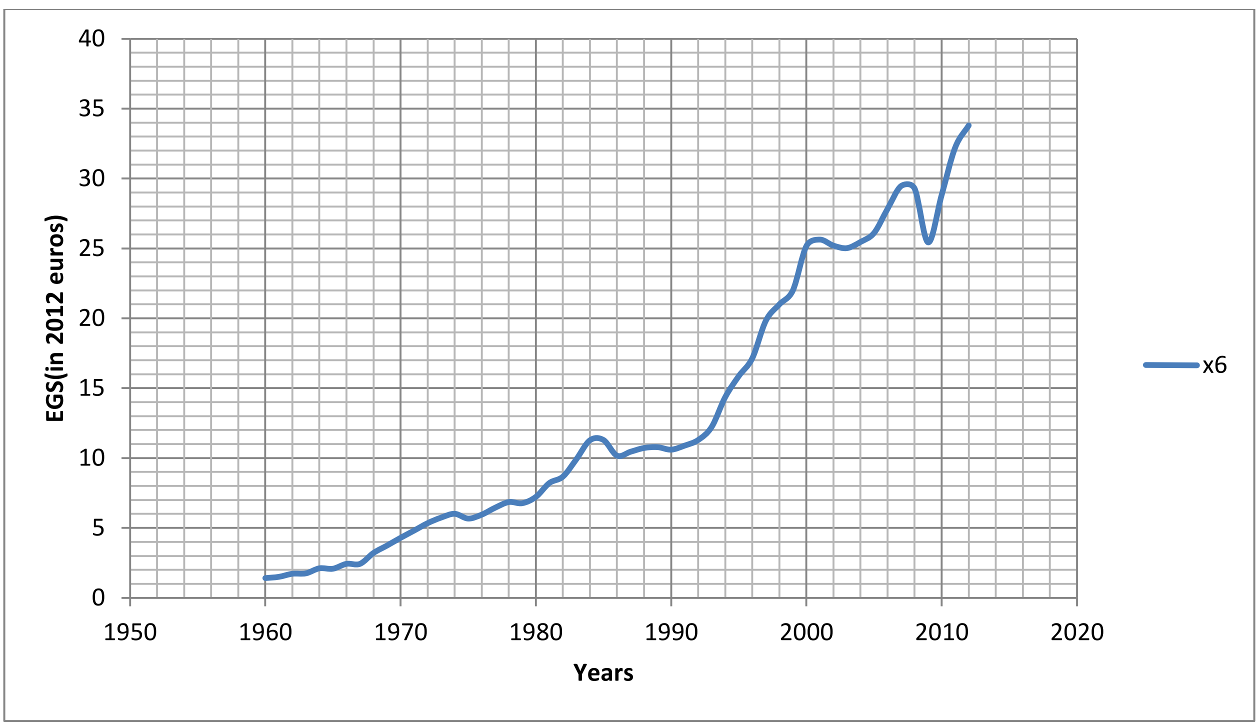

Figure 4.6: x_6 Exports of Goods and Services (EGS) Curve.

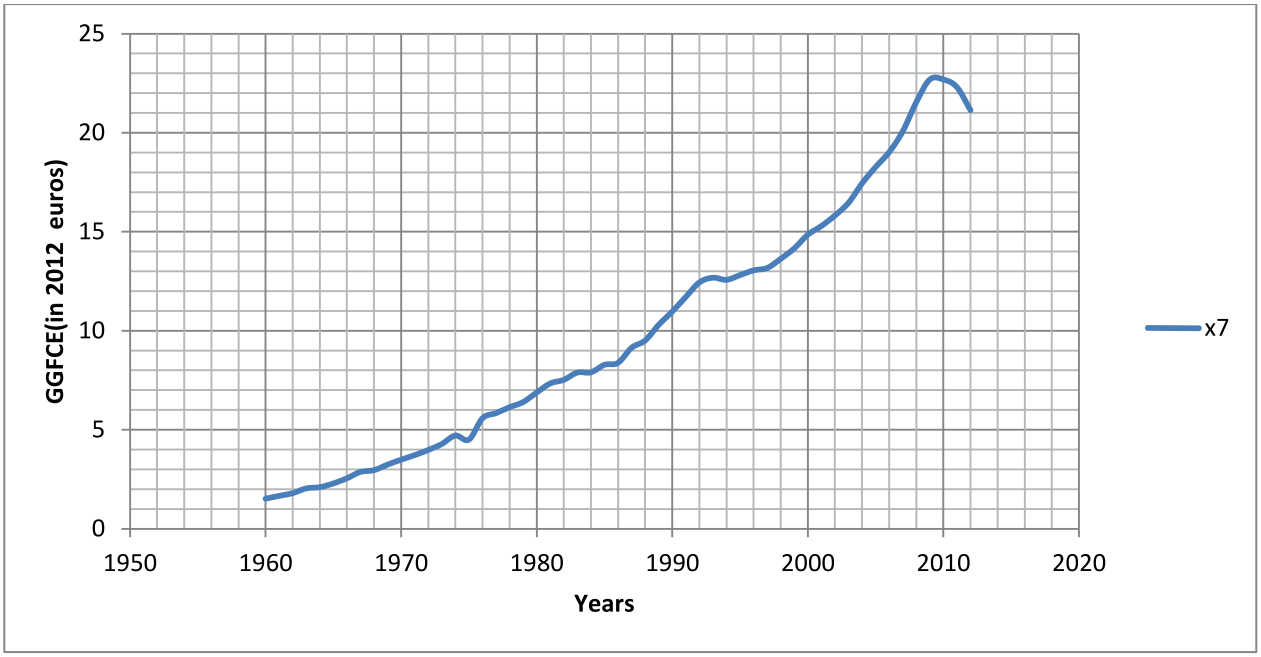

Figure 4.7: x_7 General Government Final consumption Expedenture (GGFCE) Curve.

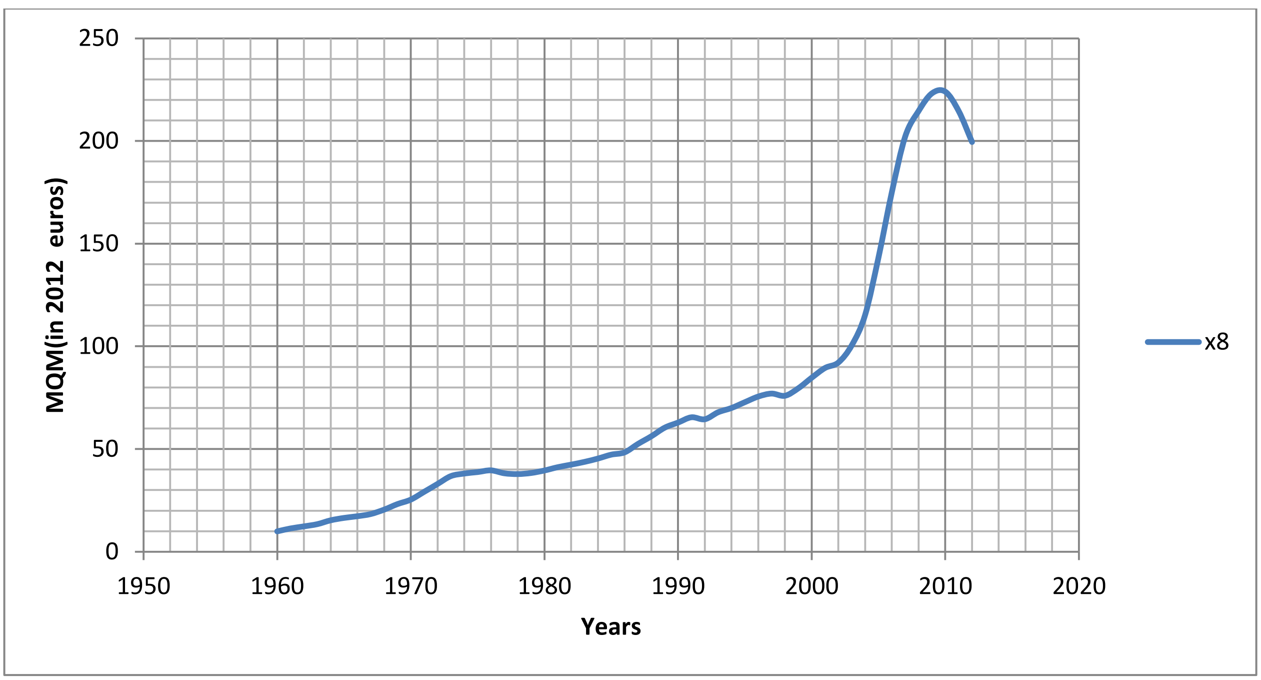

Figure 4.8: x_8 Money and Quasi Money (MQM) Curve.

4.2.3.2 The coefficients and orders by using genetic algorithm in the Matlab, we obtain the following data (see Tables:4.2 and 4.3). Here we remark that the coefficients are estimated by using the method of least squares.

Table 4.1.a: Spanish economic data for years 1960 –2012. GDP, x_5 , x_6 , x_7 and , x_8 in 2012 euros , x_1 in km^2 , x_2 in % of x_1 , x_3 in people and , x_4 in years.

Year	GDP ($\times 10^{11}$)	x_1	x_2	x_3	x_4	GCF ($\times 10^{10}$)	x_6 ($\times 10^{10}$)	x_7 ($\times 10^{10}$)	x_8 ($\times 10^{10}$)
1960	1.69	499780	32.51	30455000	4.70	13.5	1.41	1.52	9.91
1961	1.89	499780	32.51	30739250	4.74	15.2	1.50	1.66	11.33
1962	2.07	499780	32.61	31023366	4.77	1.58	1.72	1.80	12.34
1963	2.27	499780	32.42	31296651	4.79	14.5	1.75	2.04	13.42
1964	2.39	499780	31.85	31609195	4.81	13.7	2.11	2.10	15.28
1965	2.54	499780	31.95	31954292	4.82	13.3	2.08	2.29	16.46
1966	2.73	499780	31.03	32283194	4.83	12.4	2.43	2.55	17.27
1967	2.85	499780	31.49	32682947	4.85	10.8	2.43	2.87	18.34
1968	3.03	499780	31.40	33113134	4.87	10.2	3.21	2.96	20.48
1969	3.30	499780	32.18	33441054	4.91	10.6	3.74	3.24	23.18
1970	3.45	499780	31.39	33814531	4.95	9.51	4.29	3.49	25.37
1971	3.61	499780	32.69	34191678	5.01	9.15	4.81	3.72	29.16
1972	3.90	499780	32.59	34502705	5.07	10.4	5.34	3.98	32.99
1973	4.20	499780	32.12	34817071	5.15	11.7	5.74	4.28	36.82
1974	4.44	499780	31.85	35154338	5.22	13.8	6.01	4.71	38.07
1975	4.47	499780	31.66	35530725	5.3	13.1	5.67	4.50	38.78
1976	4.61	499780	31.34	35939437	5.37	12.8	5.95	5.58	39.61
1977	4.74	499780	31.29	36370050	5.44	12.2	6.45	5.84	38.18
1978	4.81	499780	31.31	36872506	5.51	11.3	6.85	6.14	37.81
1979	4.82	499780	31.18	37201123	5.58	11.0	6.77	6.40	38.33
1980	4.92	499780	31.15	37439035	5.66	11.7	7.22	6.88	39.55
1981	4.92	499780	31.17	37740556	5.75	10.9	8.21	7.34	41.16
1982	4.98	499780	31.16	37942805	5.85	10.9	8.67	7.52	42.40
1983	5.07	499780	31.22	38122429	5.95	10.7	9.92	7.89	43.74
1984	5.16	499780	31.34	38278575	6.06	10.3	11.27	7.90	45.36
1985	5.28	499780	31.16	38418817	6.17	10.7	11.29	8.28	47.26
1986	5.45	499780	31.16	38535617	6.28	11.5	10.17	8.38	48.38
1987	5.75	499780	31.20	38630820	6.38	13.0	10.45	9.14	52.48
1988	6.04	499780	31.19	38715849	6.49	14.9	10.72	9.51	56.20
1989	6.33	499780	31.06	38791473	6.61	16.4	10.78	10.30	60.39
1990	6.57	499780	30.70	38850435	6.73	17.2	10.60	10.97	62.88
1991	6.74	499780	30.55	38939049	6.86	17.1	10.89	11.71	65.46
1992	6.80	499780	30.44	39067745	6.7	15.9	11.29	12.44	64.49
1993	6.73	499780	29.99	39189400	7.14	14.1	12.23	12.68	67.90
1994	6.89	499780	29.64	39294967	7.28	14.5	14.36	12.57	69.98
1995	7.08	499780	28.12	39387017	7.42	1.55	15.86	12.81	72.81
1996	7.25	499780	28.93	39478186	7.56	15.7	17.14	13.05	75.56
1997	7.53	499780	28.60	39582413	7.69	16.6	19.82	13.17	77.00
1998	7.87	499780	27.40	39721108	7.83	18.5	20.99	13.63	75.94
1999	8.24	499780	26.96	39926268	7.97	20.7	21.99	14.16	79.58
2000	8.66	499780	26.85	40263216	8.13	22.8	25.17	14.85	84.71

Table 4.1.b: Spanish economic data for years 1960 –2012. GDP, x_5 , x_6 , x_7 and , x_8 in 2012 euros , x_1 in km^2 , x_2 in % of x_1 , x_3 in people and , x_4 in years.

Year	GDP ($\times 10^{11}$)	x_1	x_2	x_3	x_4	GCF ($\times 10^{10}$)	x_6 ($\times 10^{10}$)	x_7 ($\times 10^{10}$)	x_8 ($\times 10^{10}$)
2001	8.98	499780	26.20	40720484	8.29	23.7	25.63	15.29	89.47
2002	9.22	499780	25.87	41313973	8.47	24.6	25.20	15.82	92.14
2003	9.51	499780	26.07	42004522	8.64	26.1	25.02	16.46	100.5
2004	9.82	499780	26.09	42691689	8.81	27.8	25.46	17.44	115.3
2005	10.17	499780	25.87	43398143	8.97	30.0	26.10	18.27	143.3
2006	10.58	499780	25.49	44116441	9.11	32.7	27.83	19.02	175.1
2007	10.95	499780	25.22	44878945	9.23	33.9	29.46	20.07	202.2
2008	11.05	499780	25.04	45555716	9.32	32.2	29.28	21.54	214.6
2009	10.63	499780	25.05	45908594	9.37	2.55	25.43	22.69	223.2
2010	10.60	499780	25.12	46070971	9.39	2.42	28.82	22.69	224.0
2011	10.65	499780	25.08	46174601	9.47	2.29	32.22	22.30	214.6
2012	10.49	499780	25.04	46217961	9.56	2.06	33.80	21.14	199.5

Table 4.2: The orders of the fractional operators

α	IOM1	FOM1	IOM2	FOM2
α_1	0	0.31072	0	0.26735
α_2	0	-0.75424	0	-0.69281
α_3	0	-0.73633	0	-0.70304
α_4	0	-0.99999	0	-0.67233
α_5	-1	-1	-1	-0.60068
α_6	0	-0.83616	0	-0.95969
α_7	0	0.31073	0	0.43180
α_8	1	-0.13985	1	0.31072
α_9	1	-0.34465	1	-0.94727

Table 4.3: The coefficients of the fractional operators

c	IOM1	FOM1	IOM2	FOM2
$c_1(\times 10^{05})$	9.903	1.954	10.393	227.211
$c_2(\times 10^{09})$	7.531	11.371	-0.872	9.732
$c_3(\times 10^{04})$	-1.416	-1.150	-0.851	-1.009
$c_4(\times 10^{10})$	-2.455	0.141	-0.207	1.576
$c_5(\times 10^{-1})$	2.887	0.296	1.658	1.963
$c_6(\times 10^{-1})$	-2.123	4.707	-0.269	2.646
$c_7(\times 10^{00})$	-3.845	1.513	-1.16	1.072
$c_8(\times 10^{-2})$	9.582	4.236	16.556	5.358
$c_9(\times 10^{-2})$	1.759	9.542	12.459	-6.060

Now we are ready to give analysis by virtue of estimated value from Matlab via true value.

4.2.3.3 Fitting results : we are using data in (table 4.1a),(table 4.1b) and (table 4.2),(table 4.3) to calculate GDP, IOM1,IOM2,FOM1and FOM2. After that we are ready to give the fitting results for IOM,FOM1 and FOM2

(a) From the figure of data fitting in IOM1 and FOM1, one can see that the simulation result of FOM1 is better than the simulation result of IOM1.(See figure 4.9)

(b) From the figure of data fitting in IOM2 and FOM2, you can see that the simulation results of IOM2 and FOM2 are very close to original data. However, R^2 of FOM2 is closer to 1 than $R^2$of IOM2. Thus, FOM2 is better than IOM2.(See figure 4.10)

(c) From the figure of data fitting in IOM1 and IOM2 , one can see that the simulation result of IOM2 is better than the simulation result of IOM1.(See figure 4.11)

(d) From the figure of data fitting in FOM1 and FOM2, one can see that FOM2 is closer to original data than FOM1 although the value of $R^2$for both FOM1 and FOM2 tend to 1.(See figure 4.12)

From above, one can deduce that FOM2 is the most suitable model for this case.

$\times 10^{11}$

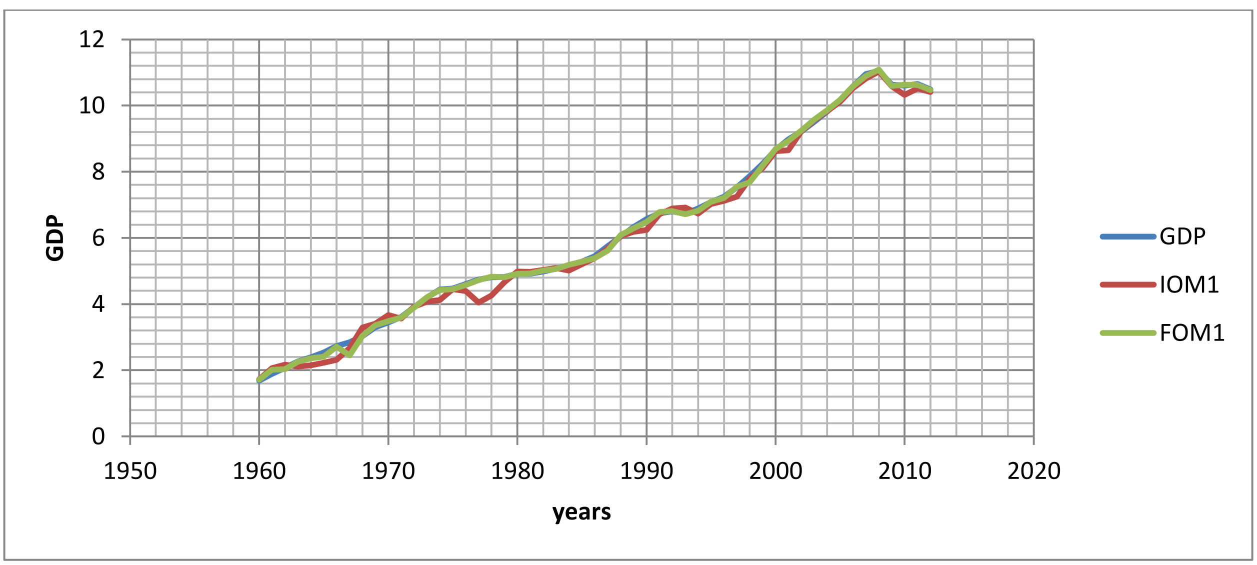

Figure 4.9: Data fitting in IOM1 and FOM1

IOM1($R^2 = 0.9947$)

FOM1($R^2 = 0.9992$)

$\times 10^{11}$

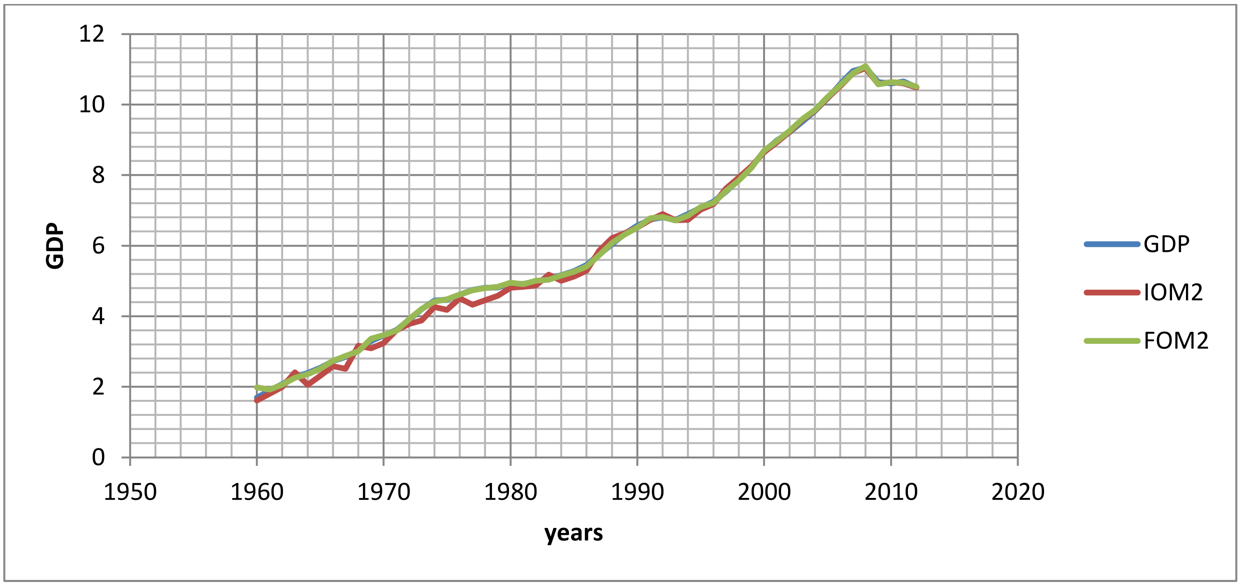

Figure 4.10: Data fitting in IOM2 and FOM2

IOM2($R^2 = 0.9968$)

FOM2($R^2 = 0.9997$)

$\times 10^{11}$

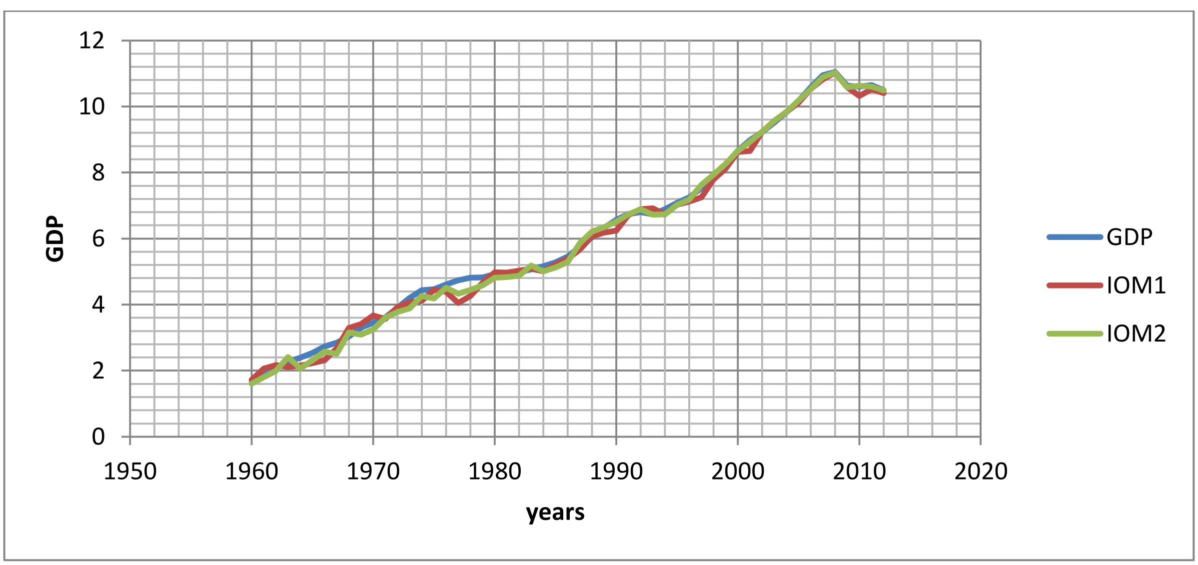

Figure 4.11: Data fitting in IOM1 and IOM2

IOM1($R^2 = 0.9947$)

IOM2($R^2 = 0.9968$)

$\times 10^{11}$

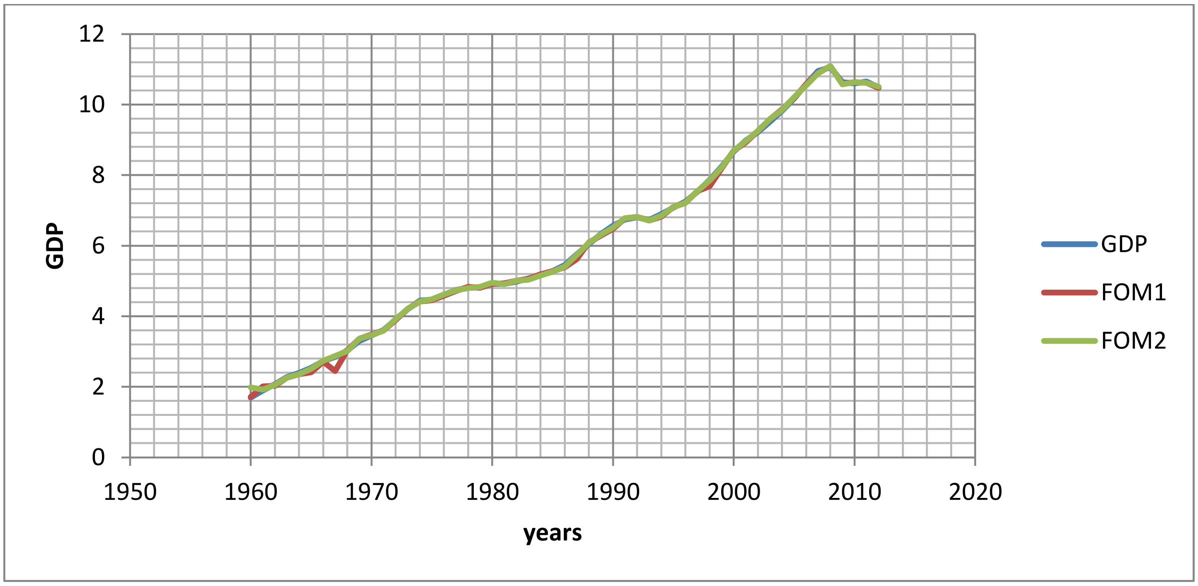

Figure 4.12: Data fitting in FOM1 and FOM2

FOM1($R^2 = 0.9992$)

FOM2($R^2 = 0.9997$)

4.2.4 Conclusions

This thesis studies a class of economic growth modelling for the Spanish case. Based on our results, four models of fractional calculus (IOM1, IOM2, FOM1 and FOM2) are proposed. It is shown that the date of GDP raised from the Caputo derivative are better than the Grunwald-Letnnikov derivative.

There are many study cases similar to the Spanish case that was explained in the previous pages, and among these study cases is the Portuguese case, which is very similar in its results with the Spanish case, where the results for this study case indicate that when we use the integer order model the coefficient of determination is ($R^2 = 0.9931$) but when using the fractional order model, the coefficient of determination is ($R^2 = 0.9985$) and this indicates that fractional models have a better performance than the other alternatives considered and proposed in the literature.

Chapter Five

5.Palestinian GDP

For more study cases that show the efficiency of fractional order model, we have studied the gross domestic product of Palestine from the year 2004 to the year 2018, through the information available through the Palestinian Statistics Center.

Unfortunately, it was found that the Palestinian Statistics Center does not use the fractional order model in calculating the GDP.

For this reason, we will present the data obtained through the Palestinian Statistics Center as contained in the published data, and we recommend the need to use the fractional order model in calculating the GDP, because it ensures a higher efficiency than the currently used method.

5.1 Terms and Indicators

5.1.1 Gross Domestic Product or GDP (Indicator)

This measures the total value added of all economic activities, which consists of the output of goods and services for final use produced by both residents and non-residents (local factors of production), and regardless of the distribution of this production, locally or externally, during a specific period of time. It does not include deductions for depreciation of fixed capital or deterioration of natural resources.

5.1.2 Final Consumption

This is the amount of expenditure on consumption of goods and services by households, government and non-profit institutions serving households (NPISHs).

5.1.3 Household Final Consumption

This consists of the expenditure, including expenditure whose value must be estimated indirectly, incurred by resident households on the individual consumption of goods and services, including those sold at prices that are not economically significant and including consumption goods and services acquired abroad.

5.1.4 Government Final Consumption

This consists of expenditure, including expenditure whose value must be estimated indirectly, incurred by general government for both individual consumption of goods and services and collective consumption of services.

5.1.5 Final Consumption Expenditure of Non-profit Institutions Serving Households

This consists of the expenditure, including expenditure whose value must be estimated indirectly, incurred by resident NPISHs on individual consumption of goods and services, and possibly on collective consumption services.

5.1.6 Gross Capital Formation

This is the acquisition minus the disposal of produced assets for the purpose of fixed capital formation, inventories or valuables.

5.1.7 Gross Fixed Capital Formation

Gross fixed capital formation consists of the value of producers' acquisitions of new and existing products of produced assets minus the value of the disposal of fixed assets of the same type.

5.1.8 Change in Inventory

This is measured by the value of entries into inventories minus the value of withdrawals and less the value of any recurrent losses of goods held in inventories during the accounting period.

5.1.9 Exports

It refers to whole commodities (goods and services) that are exported or re-exported outside the country, conditioned with ownership transcription to another economy or to free customs regions as a discount from the notional economy which results from transaction with a non-resident economy.

5.1.10 Imports

It refers to whole commodities (goods and services) entering the country by air, land and sea that are used in consumption, for conversion in the manufacturing sector and for re-exportation.

5.2 Palestinian GDP Calculations

In order to apply fractional calculus in calculating the gross domestic product for Palestine, we obtained the following data in (Table 5.1), from the Palestinian Central Bureau of Statistics.

Table 5.1: Data of Palestinian GDP(2004-2018)

Year(t)	Final Consumption	Gross Capital Formation	Net Exports of Goods And Services	GDP
2004	6,597.9	1136.3	-2,789.6	4,944.6
2005	7370.5	1176.8	-3,278.5	5,268.8
2006	7384.5	994.7	-2,632.2	5,747.0
2007	8246.1	1172.3	-3,111.3	6,307.1
2008	8843.6	1729.4	-3,357.7	7,215.3
2009	9650.9	2142.6	-3,894.2	7,899.3
2010	9932.1	1795.1	-3,374.7	8,352.5
2011	10501.8	1879.1	-3,204.3	9,176.6
2012	11364.3	2280.9	-3,945.7	9,699.5
2013	11420.3	2539	-3,874.1	10,085.2
2014	11703.2	2765.8	-3,811.4	10,657.6
2015	12538.4	3048.2	-4,834.3	10,752.3
2016	13725.4	3251.1	-5,755.2	11,221.3
2017	13270.5	3606.6	-5,771.4	11,105.7
2018	13538.0	3843.9	-5,261.6	12,120.3

$x_1(t) = Final\ Consumption = FC$

$x_2(t) = Gross\ Capital\ Formation = GCF$

$x_3(t) = Net\ Exports\ of\ Goods\ and\ Services = NEGS$

$GDP = Gross\ Domestic\ Product.$

$$GDP = y(t) = FC + GCF + NEGS \tag{116}$$

$Integer\ Order\ Model = IOM$

$$y_1(t) = C_1 x_1(t) + C_2 \int_{t_0}^{t} x_2(t) + C_3 x_3\,(t) \tag{117}$$

$Fractional\ Order\ Model = FOM$

$$FOM = y_2(t) = C_1 x_1(t) + C_2 D^{\alpha_2} x_2(t) + C_3 x_3(t) \tag{118}$$

To find GDP we use (116).

To find Integer order model we use (117).

To find Fractional order model we use (118).

Where

$$x_2(t) = 202.52t - 404042 \tag{119}$$

$t_0 : the\ base\ year.$

$t : the\ current\ year.$

We obtained the coefficients of the variables by using the SAS program and they were as follows:

$C_1 = 0.84301$

$C_2 = 79679$

$C_3 = 0.38599$

The orders of fractional operators can be obtained by using genetic algorithm in the matlab, but we chose $\alpha = 0.5$ as an example for the fractional order.

5.2.1. Calculating GDP.

First we find GDP by using (116)

$GDP = y(t) = FC + GCF + NEGS$

We using data in (table 5.1) to calculate GDP and the results of our calculation appears in (Table 5.2) and in (figure 5.1).

Table 5.2: Palestine GDP from (2004 – 2018)

Year	X_1	X_2	X_3	GDP
2004	6,597.9	1136.3	-2,789.6	4,944.6
2005	7370.5	1176.8	-3,278.5	5,268.8
2006	7384.5	994.7	-2,632.2	5,747.0
2007	8246.1	1172.3	-3,111.3	6,307.1
2008	8843.6	1729.4	-3,357.7	7,215.3
2009	9650.9	2142.6	-3,894.2	7,899.3
2010	9932.1	1795.1	-3,374.7	8,352.5
2011	10501.8	1879.1	-3,204.3	9,176.6
2012	11364.3	2280.9	-3,945.7	9,699.5
2013	11420.3	2539	-3,874.1	10,085.2
2014	11703.2	2765.8	-3,811.4	10,657.6
2015	12538.4	3048.2	-4,834.3	10,752.3
2016	13725.4	3251.1	-5,755.2	11,221.3
2017	13270.5	3606.6	-5,771.4	11,105.7
2018	13538.0	3843.9	-5,261.6	12,120.3

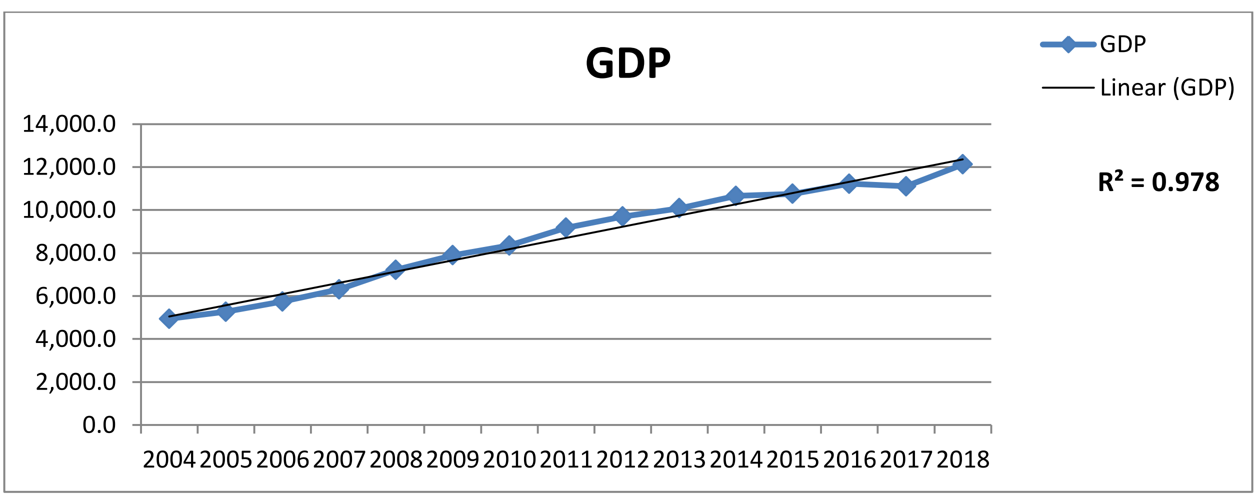

Figure 5.1: R^2 of Palestine GDP

After calculating R^2 we see that the R^2 $of\ GDP\ is$ 0.978

5.2.2. Calculating IOM.

Second we find Integer Order Model by using (117)

$$y_1(t) = C_1 x_1(t) + C_2 \int_{t_0}^{t} x_2(t) + C_3 x_3\,(t)$$

From data in (table 5.1) and by using equation (117) we obtained the results of our calculation which is appears in (Table 5.3) and in (figure 5.2)

Table 5.3: Palestine GDP with IOM

Year	X_1	X_2	X_3	IOM
2004	6,597.9	1136.3	-2,789.6	4558.12
2005	7370.5	1176.8	-3,278.5	5672.49
2006	7384.5	994.7	-2,632.2	6095.123
2007	8246.1	1172.3	-3,111.3	6797.899
2008	8843.6	1729.4	-3,357.7	7367.855
2009	9650.9	2142.6	-3,894.2	8002.7
2010	9932.1	1795.1	-3,374.7	8601.642
2011	10501.8	1879.1	-3,204.3	9309.043
2012	11364.3	2280.9	-3,945.7	9911.332
2013	11420.3	2539	-3,874.1	10147.54
2014	11703.2	2765.8	-3,811.4	10571.6
2015	12538.4	3048.2	-4,834.3	11142.65
2016	13725.4	3251.1	-5,755.2	12241.1
2017	13270.5	3606.6	-5,771.4	12127.36
2018	13538.0	3843.9	-5,261.6	12435.46

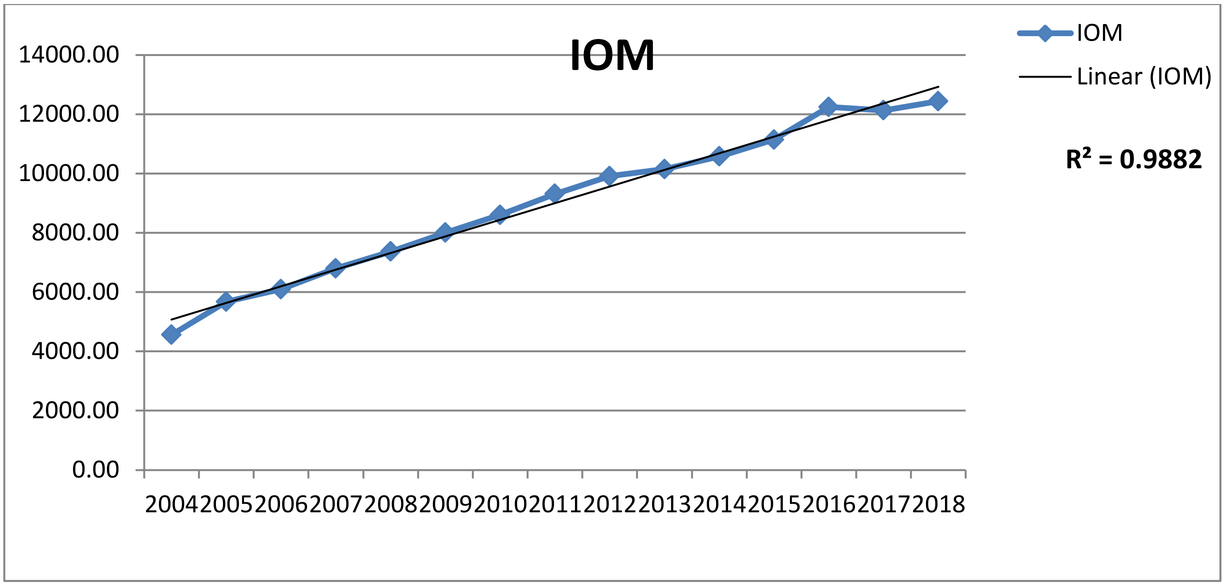

Figure 5.2: R^2 of Palestine GDP with IOM

After calculating R^2 we see that the R^2 $of\ IOM\ is$ 0.9882

From results we obtained from our calculation above, we see that the integer order model is better than GDP model since R^2 of integer order model is 0.9882 while R^2 of GDP model is 0.978 although the value of R^2 of both GDP and integer order model tend to 1.

5.2.3. Calculating FOM.

Third we find the Fractional order model by using (118)

$$y_2(t) = C_1 x_1(t) + C_2 D^{\alpha_2} x_2(t) + C_3 x_3(t)$$

From data in (table 5.1) and by using equation (118) we obtained the results of our calculation which is appears in (Table 5.4) and in (figure 5.3).

Table 5.4: Palestine GDP with FOM

Year	X_1	X_2	X_3	FOM
2004	6,597.9	1136.3	-2,789.6	4485.3
2005	7370.5	1176.8	-3,278.5	5270.67
2006	7384.5	994.7	-2,632.2	5665.62
2007	8246.1	1172.3	-3,111.3	6309.60
2008	8843.6	1729.4	-3,357.7	6804.67
2009	9650.9	2142.6	-3,894.2	7354.33
2010	9932.1	1795.1	-3,374.7	7860.79
2011	10501.8	1879.1	-3,204.3	8241.30
2012	11364.3	2280.9	-3,945.7	8520.30
2013	11420.3	2539	-3,874.1	9100.26
2014	11703.2	2765.8	-3,811.4	9415.32
2015	12538.4	3048.2	-4,834.3	9874.82
2016	13725.4	3251.1	-5,755.2	10240.30
2017	13270.5	3606.6	-5,771.4	10630.05
2018	13538.0	3843.9	-5,261.6	11220.30

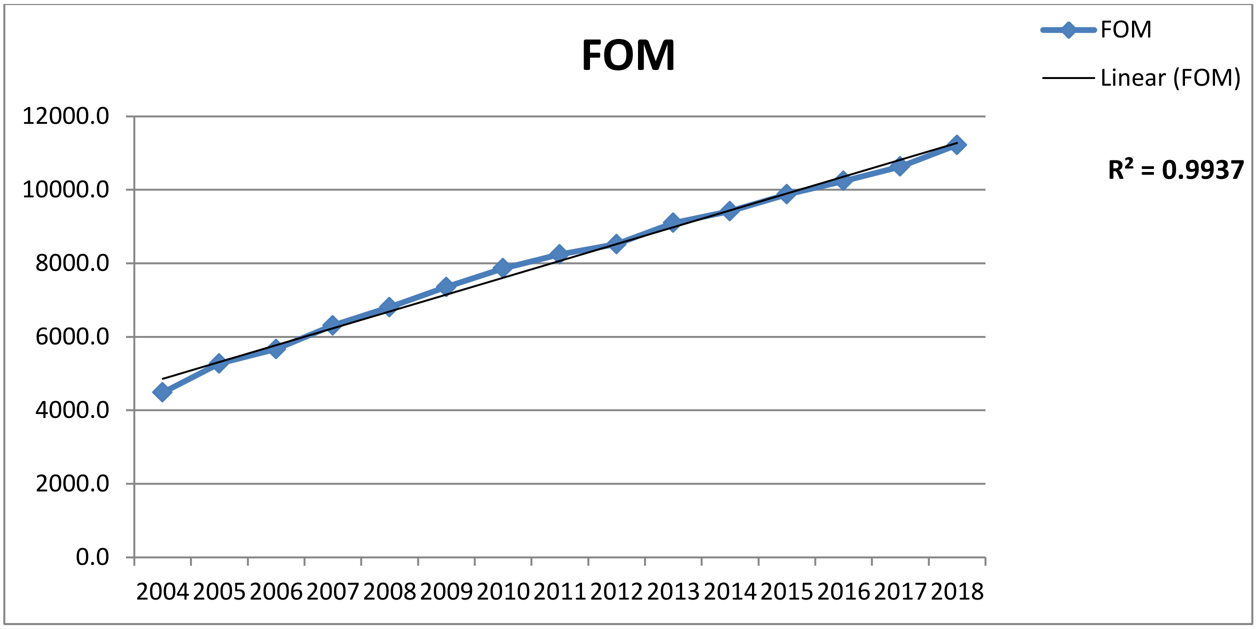

Figure 5.3: R^2 of Palestine GDP with FOM

After calculating R^2 we see that the $R^2\ of\ FOM\ is$ 0.9937

From results we obtained from our calculation above, when comparing R^2 of fractional order model with R^2 of integer order model and GDP model, we see that the fractional order model is better than integer model and GDP model although the value of R^2 of them tend to 1.

5.3 Main results for Palestinian GDP

The first result we obtained from the figure of data fitting in GDP model and IOM, we see that IOM is closer to original data than GDP model although the value of R^2 for both GDP model and IOM tend to 1.

The second result we obtained from the figure of data fitting in FOM and IOM, we see that the simulation result of FOM is better than the simulation result of IOM.

The Third result we obtained from the figure of data fitting in FOM and GDP model, we see that the simulation result of FOM is better than the simulation result of GDP model.

6.Results and Recommendations

Through our study of fractional calculus, we see that this science has many important uses in various types of science, and it has many useful and important applications.

Fractional calculus has many applications in economic sciences, which cannot be ignored because of its high accuracy in economic results.

Fractional calculus has become widely used in calculating GDP in many European countries and major industrialized countries because it gives high-quality and accurate results.

In our study of the Palestinian case, we calculated the gross domestic product using the calculation of the gross domestic product in the traditional way, and then we calculated the GDP through the Integer order model , and then we calculated it through fractional order model , and we found that using fractional calculus gives more accurate results than other methods.

In conclusion, we recommend using fractional calculus, especially in applications that require very accurate results, because of this science's ability to obtain accurate and high-quality results that can be relied upon in making important decisions.

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التفاضل والتكامل الكسري في الإقتصاد

اعداد : عائد محمد عبد الله الفواقة

إشراف : الدكتور إبراهيم الغروز

الملخص

يعتبر حساب التفاضل والتكامل الكسري من العلوم التي تم استخدامها مؤخرا في العديد من التطبيقات العلمية في مختلف المجالات الطبية والهندسية والرياضية والاقتصادية وغيرها من العلوم ، وقد كان لاستخدام حساب التفاضل والتكامل الكسري العديد من المزايا التي ساهمت في زيادة دقة التطبيقات العلمية وكفاءتها.

إن استخدام حساب التفاضل والتكامل في العلوم الاقتصادية له أثر كبير في تحسين الحسابات المتعلقة بالاقتصاد في العديد من دول العالم ، وقد تميزت الحسابات الخاصة بالناتج المحلي الإجمالي للدول بالدقة والكفاءة العالية مقارنة بالحسابات التقليدية .

في هذه الرسالة بحثنا استخدامات التفاضل والتكامل الكسري في العديد من العلوم ، وقد سلطنا الضوء على استخدامات التفاضل والتكامل الكسري في مجال الاقتصاد وخصوصا في حساب الناتج المحلي الاجمالي للدول ومقارنة الحسابات التي تم استخدام التفاضل والتكامل الكسري فيها بالحسابات التقليدية ، ومن خلال النتائج يمكننا القول بأن استخدام التفاضل والتكامل الكسري في الحسابات الاقتصادية سيؤدي الى زيادة الدقة والجودة في تلك الحسابات، كما أن الباب ما زال مفتوحا لاستخدامات أخرى لحساب التفاضل والتكامل الكسري في مجالات أخرى في العلوم الاقتصادية وغيرها من العلوم .