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Noneffective Weights in Variable Lebesgue Spaces

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Dedication

To my parents, to my brothers, to my teachers, to my friends.

Walaa Jawabreh

Declaration

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledge, and that this thesis (or any part of the same) has not been submitted for the higher degree to any other university or institution.

Signed:.............................

Walaa Mohammed Abd Al-Rahman Jawabreh

Date:

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Abstract

Many of the concepts of mathematics can be generalized. In this thesis we introduce the generalized classical Lebesgue spaces, these generalized spaces are called Variable Exponent Lebesgue Spaces, denoted by $\mathrm{L}^{p(.)}$. Other generalized spaces of Lebesgue spaces are called Weighted Variable Exponent Lebesgue Spaces, denoted by $\mathrm{L}_{\omega}^{p(.)}$.

A noneffective weights in Variable Lebesgue Spaces for any weight function (positive locally integrable) is the subject of this thesis. Here the definition will depend on exponent function $p(.)$, and weight function $\omega(.)$. This definition used for equivalent of two Banach spaces without calculating their norms. The result we have obtained through a theorem that gives us the conditions of weight function to be noneffective. We proved that the weight function is noneffective (i.e. $\mathrm{L}_{\omega}^{p(.)} = \mathrm{L}^{p(.)}$) if and only if $\omega(x)^{\frac{1}{p(x)}}$ is constant almost everywhere in the set where $p(.) < \infty$, and $\omega(x)$ is constant almost everywhere in the set where $p(.) = \infty$. This theorem is used as another definition of noneffective weights in Variable Lebesgue Spaces.

الأوزان غير المؤثرة في فضاءات لبج ذو القوة المتغيرة

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إشراف: د. جميل جمال

ملخص:

العديد من مفاهيم الرياضيات يمكن تعميمها. في هذه الرسالة نقدم فضاء لبج المعمم، ويسمى هذا الفضاء المعمم بفضاء لبج ذو القوة المتغيرة ويرمز له بالرمز $L^{p(\cdot)}$. وهناك تعميم آخر لفضاء لبج يسمى فضاء لبج الموزون ذو القوة المتغيرة ويرمز له بالرمز $L_{\omega}^{p(\cdot)}$. موضوع هذه الأطروحة هو الأوزان غير المؤثرة في فضاء لبج $L^{p(\cdot)}$ لاي اقتران وزن موجب قابل للتكامل محليا .هذا الموضوع يعتمد على الاقتران الأسي$p(\cdot)$ واقتران الوزن $\omega(\cdot)$. يتطلب تعريف الأوزان غير المؤثرة في فضاءات لبج المعممة أن يكون $L^{p(.)}$ و$L_{\omega}^{p(.)}$ متكافئين. لقد حصلنا في هذه الأطروحة على نتيجة تعطينا شروط اقتران الوزن ليكون غير مؤثر, حيث أثبتنا أن اقتران الوزن هو غير مؤثر (أي أن $L_{\omega}^{p(.)} = L^{p(.)}$) إذا وفقط إذا كان الاقتران $\omega(x)^{\frac{1}{p(x)}}$ ثابت في كل مكان تقريبا في مجموعة ما حيث $p(.)<\infty$ ، و $\omega(.)$ هو اقتران ثابت في كل مكان تقريبا في مجموعة ما حيث $p(.) = \infty$. يتم استخدام هذه النظرية بمثابة تعريف آخر للأوزان غير المؤثرة في $L^{p(\cdot)}$.

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Introduction

The concept of equivalent of two Banach spaces depends on their norms . In this thesis we focus our eyes on equivalent of two specific Banach spaces $L_{\omega}^{p(.)}$ and $L^{p(.)}$ without finding their norms but by knowing if the weight function $\omega(.)$ power of $\frac{1}{p(x)}$ is constant almost everywhere in the set where $p(.)$ is finite measurable function or if weight function $\omega(.)$ is constant almost everywhere in the set where $p(.)$ is infinity measurable function . If this equivalent holds then we call the weight function $\omega(.)$ is noneffective in Variable Lebesgue Spaces .

In chapter one , we described the basic ideas of Lebesgue measure μ and the Lebesgue integral .After we have given measure space ,measurable space, measurable functions , and Lebesgue integral concepts ,we moved in sect. 1.5 to convergence in measure and gave important theorems like Fatou's Lemma , Monotone Convergence Theorem, and Lebesgue Convergence Theorem .All these theorems talk about how the integral of sequence of real –valued functions can be convergent.

In chapter two , we displayed some basic function spaces .For the investigation of (weighted)(variable) Lebesgue spaces it is enough to stay in the framework of Banach spaces which we reviewed in sect. 2.1 .In sect. 2.2 a space defined called (semi)modular spaces which then induces a norm and so is normed space . We defined the appropriate to Φ-function for variable exponent spaces in sect. 2.3 and studies its properties. Sect 2.4 deals with modular space which corresponding Φ-function called Musielak–Orlicz space.

In chapter three , we defined (weighted) variable exponent Lebesgue spaces ($L_{\omega}^{p(.)}$) $L^{p(.)}$ which generalize the classical Lebesgue spaces L^{p}, where the constant exponent p is replaced by a function $p(.)$ .They fall within the scope of Musielak-Orlicz spaces hence the general theory implies their basic properties .We are in a position to apply the results of general Musielak–Orlicz spaces to our case in sect 3.1 .Finally , we have collected all what we learn in previous chapters and in sect 3.1 to realize the main theorem that contains the conditions of weight function to be noneffective in sect 3.2.

Chapter One

Measure and Lebesgue Integration

In the history ,people where engaged in the problem of measuring length ,areas and volumes .In mathematical formulation the task was, for given set A how to determine its size("measure") $\mu(A)$ .It was required that the length of interval on $\mathbb{R}$ or the volume of a cube in $\mathbb{R}^n$ showed to agree with well-known formulae .It was also that this measure showed to be nonnegative and additive for disjoint collection of sets .

Through this chapter, we describe the basic ideas of Lebesgue measure μ and the Lebesgue integral; also we give some of their main properties. The full details and proofs can be found in[2].

Section 1.1 σ-$algebra$

Definition 1.1.1 (σ-$algebra$)

A σ-$algebra$ on a set X is a collection F of subset of X such that

(1) $\emptyset \in F$
(2) If A $\in F$ then $A^c \in F$
(3) $\forall i \in \mathbb{N}$, If$A_i \in F$ then $\bigcup_{i=1}^{\infty} A_i \in F$

We will give some examples of $\sigma - algebras$.
Example 1.1.1: Let X be a nonempty set .Then the collection $\{\emptyset, X\}$ and P (X)=$\{E: E \subseteq X\}$ are trivial examples of σ-$algebra\ of\ X$.

Example 1.1.2: Let X be any uncountable set and let S=$\{A \subseteq X |\ A\ orA^c\ is$ countable $\}$ then S is σ-$algebra$ of subset of X.

From De Morgan's laws we have $\bigcap_{i=1}^{\infty} A_i \in F$ whenever $A_i \in F$ $\forall i \in \mathbb{N}$. also F is closed under finite union and intersection.

Section 1.2 Measure Space

Definition 1.2.1:(Measurable Space)
A non-empty set X together with σ- algebra F defined over it is called measurable space, denoted by (X, F).

Definition 1.2.2:(Measure)
A measure of a measurable space (X, F) is a set function μ : $F \to [0,\infty]$ such that
(1) $\mu(\emptyset)=0$

(2) (countable additively) $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for pairwise disjoint sets A_i *in* F

Note that for any sequence $\{E_i\}_1^{\infty}$ of measurable sets we have

$\mu(\cup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} \mu(E_i)$

Definition 1.2.3:(Measure Space)
A measurable space (X, F) together with a measure μ is called a measure space, denoted by (X, F, μ).
There are some properties of measure space .We will give them as in following theorem.

Theorem 1.2.1[2]: Given measure space (X, F, μ).Then we have

(1) $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ for pairwise disjoint sets A_i *in* F
(2) If $A, B \in F$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$.
(3) If $A = \cup_{i=1}^{\infty} A_i$, $A_i \in F$ and $A_i \subset A_{i+1}$ then $\lim_{i\to\infty} \mu(A_i) = \mu(A)$.
(4) If $A = \cap_{i=1}^{\infty} A_i$, $A_i \in F$ and $A_{i+1} \subset A_i$. if $\mu(A_1) < \infty$
then $\lim_{i\to\infty} \mu(A_i) = \mu(A)$.

Proof: (1) By induction , want to show the equation is true for n=2 . Let A_i are pairwise disjoint sets *in* F.

$$\mu(\cup_{i=1}^{2} A_i) = \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) = \sum_{i=1}^{2} \mu(A_i)$$

Assume the equation is true for n=k then

$$\mu\left(\cup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$$

for disjoint sets A_i .Want to proof that the equation is true for n= k+1 then

$$\mu\left(\cup_{i=1}^{k+1} A_i\right) = \mu\left(A_{k+1} \cup \left(\cup_{i=1}^{k} A_i\right)\right) = \mu(A_{k+1}) + \mu\left(\cup_{i=1}^{k} A_i\right)$$
$$= \mu(A_{k+1}) + \sum_{i=1}^{k} A_i$$
$$= \sum_{i=1}^{k+1} \mu(A_i)$$

(2) Since $A \subseteq B$ then $B = (B - A) \cup A$ and since $(B - A) \cap A = \emptyset$. Therefore by (1) we have $\mu(B) = \mu(B - A) + \mu(A)$ .Since $A, B \in F$ then $(B - A) \in F$ and so

$$\mu(B - A) \ge 0$$

Hence $\mu(B) \ge \mu(A)$.

(3)Let $B_1 = A_1$ and $B_n = A_n - A_{n-1}$ for n=2,3,… then for all i≥ 1 we have B_i is pairwise disjoint measurable set and $A_n = \cup_{i=1}^{n} B_i$ Therefore $\mu(A_n) = \mu(\cup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \mu(B_i)$.
Let $A = \cup_{i=1}^{\infty} B_i$ so that

$$\mu(A) = \mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu(A_n)$$

(4) Let $A = \cap_{i=1}^{\infty} A_i$ and let $F_i = A_i - A_{i+1}$.Then for all i≥ 1 we have F_i is pairwise disjoint measurable set and $A_1 - A = \cup_{i=1}^{\infty} F_i$. Now since $A \subset A_1$ and $A_{i+1} \subset A_i$ then $A_1 = (A_1 - A) \cup A$ and $A_i = (A_i - A_{i+1}) \cup A_{i+1}$. Hence $\mu(A_1) = \mu(A_1 - A) + \mu(A)$ and $\mu(A_i) = \mu(A_i - A_{i+1}) + \mu(A_{i+1})$. Also since

$$\mu(A_i) < \mu(A_1) < \infty$$

Thus $\mu(A_1 - A) = \mu(A_1) - \mu(A)$ and $\mu(A_i - A_{i+1}) = \mu(A_i) - \mu(A_{i+1})$. Therefore

$$\mu(A_1) - \mu(A) = \mu(A_1 - A) = \sum_{i=1}^{\infty} \mu(F_i)$$

$$= \sum_{i=1}^{\infty} \mu(A_i - A_{i+1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i - A_{i+1})$$

$$= \lim_{n\to\infty} \big(\mu(A_1) - \mu(A_n)\big)$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

Since $\mu(A_1) < \infty$ .We have $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Example 1.2.1:Let (X, F) be a measurable space then the simplest measure is zero measure which define by μ(A)=0 , ∀ A ∈ F .

Example 1.2.2: Let X be uncountable set, (X, F) be a measurable space then

$$\mu(A) = \begin{cases} 0 & if\ A\ is\ countable \\ \infty & if\ A\ is\ uncontable \end{cases}$$

then μ is measure on (X, F).

Example 1.2.3: Let P (ℝ) be the set all subset of ℝ and (ℝ, P (ℝ)) be a measurable space and define

$$\mu(E) = \begin{cases} \infty & if\ E\ is\ infinte \\ n & if\ |E| = n \end{cases}$$

then μ is measure on P (ℝ).

We try now to construct a measure on X and on special cases when X= ℝ and when X= $\mathbb{R}^n$, which is called the Lebesgue measure.

Definition1.2.4:(Outer Measure)

Let F be a σ- algebra of a set X and μ :F → [0,∞] be a measure on F. For E ⊆ X we define

$$\mu^*(E) = inf \left\{ \sum_{i=1}^{\infty} \mu(E_i) \,\middle|\, E_i \in F\ , E \subset \bigcup_{i=1}^{\infty} E_i \right\}$$

Then the set function μ^* is called the outer measure of E.

If $X = \mathbb{R}$,we will deal with open interval $I_n \in F$ instead of E_i such that $E \subset \bigcup_{n=1}^{\infty} I_n$ then $\mu(I_n) = l(I_n)$ where $l(I_n)$ be the length of interval I_n therefore the outer measure of E will be defined as

$$\mu^*(E) = inf \left\{ \sum_{n=1}^{\infty} l(I_n) \,\middle|\, I_n \in F\ , E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

In the same way as for $\mathbb{R}$, we introduce the outer measure on X= $\mathbb{R}^n$. Recall that by an interval in $\mathbb{R}^n$ we understand an arbitrary Cartesian Product of n-dimensional intervals in $\mathbb{R}$. Any interval I of $\mathbb{R}^n$ is of the form

$$I=\prod_{i=1}^{n}(a_i,b_i) :=\{x=(x_1,x_2,\dots\dots,x_n)\in \mathbb{R}^n | x_i\in(a_i,b_i)\ \forall 1\le i\le n\}$$

We define its volume as Vol $I=\prod_{i=1}^{n}(b_i-a_i)$ which is a finite non-negative number .

Given an arbitrary set E $\subset \mathbb{R}^n$. Then we define the outer measure of E as

$$\mu^*(E)=inf\left\{\sum_{n=1}^{\infty}\text{Vol}(I_n)\middle| I_n\in F\text{ , E}\subset\bigcup_{n=1}^{\infty}I_n\right\}$$

Example 1.2.4 : Let E be a countable subset of $\mathbb{R}$ then the outer measure $\mu^*(E)$ of E is equal to zero .

Definition1.2.5 (Measurable Set)
A subset $E\subseteq X$ is called (Lebesgue)measurable set if for every $A\subseteq X$,

$$\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^c)$$

Remark 1.2.1: If (X, F, μ) is measure space then any set in σ-algebra F is measurable set .

Example 1.2.5 :Any subset E $\subset \mathbb{R}$ of outer measure $\mu^*(E)=0$ is measurable set .

Example 1.2.6: Any open (closed)interval of $\mathbb{R}$ is measurable set .

Example 1.2.7: Any Cartesian Product of intervals (a ,b)×(c, d) (which is an interval in $\mathbb{R}^2$) is measurable set.

Definition1.2.6: (Lebesgue Measure)
If E is measurable set then we define the Lebesgue measure of E to be the outer measure of E that is $\mu(E)=\mu^*(E)$.

Example 1.2.8 : The examples1.2.1, 1.2.2 of measure is also Lebesgue measure.

Note that all properties of arbitrary measure that we took about it is also satisfied for Lebesgue measure .

Example 1.2.9: Let I be an arbitrary interval in $\mathbb{R}$ then the Lebesgue measure of I is the length of I .

Example 1.2.10 : Any Cartesian Product of intervals (a,b)×(c,d) has Lebesgue measure (b-a)(d-c).

Definition1.2.7: (Finite And σ-Finite Measure)
Let (X, F, μ) be measure space . A measure μ is called finite if $\mu(X)<\infty$ and is called σ-finite if there exist a sequence $\{X_n\}$ of measurable sets in F such that

$$X=\bigcup_{n=1}^{\infty}X_n \qquad \text{and} \qquad \mu(X_n)<\infty$$

Example 1.2.11 : The Lebesgue measure on $[0,1]$ is finite and the Lebesgue measure on $\mathbb{R}$ is σ-finite measure.

Definition1.2.8: (Complete Measure Space)
A measure space (X, F, μ) is said to be complete if the σ-algebra F contains all subset of sets of measure zero .

Example 1.2.12: Lebesgue measure space $(\mathbb{R}, P(\mathbb{R}), \mu)$ is complete measure space.

Definition1.2.9: (Absolutely Continuous Measure)
A measure γ is called absolutely continuous with respect to the measure μ if γ(A)=0 whenever μ(A)=0 for each set A. In this case we write $\ll \mu$.

Section 1.3 Measurable Function

Definition1.3.1 (Measurable Function)
Let f:D→ $[-\infty,\infty]$ where D is measurable set then f is said to be (Lebesgue) measurable function on D if it satisfies one of the following statements;

(1) For each real number α the set $\{x: f(x) > \alpha\}$ is measurable set
(2) For each real number α the set $\{x: f(x) \geq \alpha\}$ is measurable set
(3) For each real number α the set $\{x: f(x) < \alpha\}$ is measurable set
(4) For each real number α the set $\{x: f(x) \leq \alpha\}$ is measurable set

These statements imply

For each extended real number α the set $\{x: f(x) = \alpha\}$ is measurable set .

Theorem 1.3.1[2]: Let c be a constant ,f and g are two measurable real valued functions define on the same domain .Then the functions
(1) $f + c$
(2) cf
(3) $f + g$
(4) $g - f$
(5) fg
are also measurable functions.

Proof: (1) Let α be any real number .Then the set

$$\{x: f(x) + c < \alpha\} = \{x: f(x) < \alpha - c\}$$

is measurable set since f is measurable function . So that $f + c$ is measurable function.

(2) Let α be any real number .If c=0 its obvious. Assume $c > 0$ then the set

$$\{x: f(x)c < \alpha\} = \left\{x: f(x) < \frac{\alpha}{c}\right\}$$

is measurable set since f is measurable function . So that cf is measurable function.

(3) Let α be any real number .If $f(x) + g(x) < \alpha$ then $f(x) < \alpha - g(x)$ and there exist a rational number r such that

$$f(x) < r < \alpha - g(x)$$

Hence

$$\{x: f(x) + g(x) < \alpha\} = \cup_r(\{x: f(x) < r\} \cap \{x: g(x) < \alpha - r\})$$

Since f and g are two measurable functions and the intersection and union of measurable sets is measurable set then $\{x: f(x) + g(x) < \alpha\}$ is measurable set and so $f + g$ is measurable function.

(4) Since $-g = (-1)\, g$ is measurable function when g is measurable function and since by (3) $f + g$ is measurable function then $f - g$ is measurable function.

(5) If $\alpha \geq 0$ .Then the set

$$\{x: f^2(x) > \alpha\} = \{x: f(x) > \sqrt{\alpha}\} \cup \{x: f(x) < -\sqrt{\alpha}\}$$

is measurable set since f is measurable function . And if $\alpha < 0$ then the set

$$\{x: f^2(x) > \alpha\} = D$$

is measurable set ,where D is the domain of f . So that f^2 is measurable function. Now since $(f + g)^2 = f^2 + 2fg + g^2$ so that

$$fg = \frac{1}{2}[(f + g)^2 - f^2 - g^2]$$

is measurable function .

Example1.3.1 : Let A be any set ,we define the characteristic function of the set A to be

$$\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Then χ_A is measurable function if and only if A is measurable set .

Definition1.3.2 : (Simple Function)

Let E be measurable set .A function $\varphi: E \to \mathbb{R}$ is called simple function if there exist $\alpha_i \in \mathbb{R}$ and E_i are measurable subset of E for all $1 \leq i \leq n$ such that

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

where E_i =$\{x \in E |\, \varphi(x) = \alpha_i\}$.

This representation for φ is called the canonical representation ,and it is characterized by the fact that the E_i are disjoint and α_i are distinct and nonzero .

Remark 1.3. 1: The sum ,the product, and difference of two simple functions are simple.

Example 1.3.2 : The characteristic function of the set of rational number $\mathbb{Q}$

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & if\ x \in \mathbb{Q} \\ 0 & if\ x \notin \mathbb{Q} \end{cases}$$, is simple function .

Section 1.4 Lebesgue Integral

Definition1.4.1 : Let (X, F, μ) be a measure space and $\varphi: X \to [0, \infty)$ is a simple function . We define the integral of φ by

$$\int \varphi(x)dx = \sum_{i=1}^{n} \alpha_i \, \mu(E_i)$$

When φ has the canonical representation ,

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

Definition 1.4.2 : (Lebesgue Integral)
Let f : $X \to [0, \infty]$ be nonnegative measurable function on a measure space (X, F, μ) .We define the Lebesgue integral of f over a measurable set E by

$$\int_E f d\mu = sup \int_E \varphi \, d\mu$$

For all simple function $\varphi \leq f$

Definition1.4.3 :(Almost Everywhere)
A property is said to hold almost everywhere (write $a.e$) ,if the set of points where it fails to hold is a set of measure zero

Example1.4.1 : We say that $f = g \; a.e$ if $f \; and \; g$ have the same domain and $\mu\{x | f(x) \neq g(x)\} = 0$

Theorem1.4.1[2]: Let (X, F, μ) be measure space. If f and g are nonnegative measurable functions on (X, F, μ) and E be measurable set then

1) $\int_E c\, f = c \int_E f \quad , c > 0$
2) $\int_E f + g \geq \int_E f + \int_E g$
3) If $\mu(E) = 0$ then $\int_E f = 0$
4) If $\int_E f = 0$ then f = 0 $a.e$ on E

Proof :
(1) Let φ be a simple function and $E_i = \{x | \, \varphi(x) = \alpha_i\}$ then for c > 0 we have c $\varphi = c \sum_{i=1}^{n} \alpha_i \chi_{E_i} = \sum_{i=1}^{n} (c\alpha_i) \, \chi_{E_i}$ is also simple function thus

$$\int c \, \varphi = \sum_{i=1}^{n} (c\alpha_i) \, \mu(E_i) \; = c \sum_{i=1}^{n} \alpha_i \, \mu(E_i) = c \int \varphi$$

It follows that ,

$$\int_E c\, f = \sup_{\varphi\le f}\int_E c\,\varphi = \sup_{\varphi\le f} c\int_E \varphi = c\sup_{\varphi\le f}\int_E \varphi = c\int_E f$$

(2) Let φ_1, φ_2 be two simple functions such that $\varphi_1 \le f$ and $\varphi_2 \le g$. Let $A_i = \{x|\varphi_1(x) = a_i\}, B_i = \{x|\varphi_2(x) = b_i\}$, $A_0 = \{x|\varphi_1(x) = 0\}$ and $B_0 = \{x|\varphi_2(x) = 0\}$, then $\mu(A_0) < \infty$ and $\mu(B_0) < \infty$. Set $E_k = A_i \cap B_j$ then E_k is finite disjoint measurable sets ,and we may write

$$\varphi_1 = \sum_{k=1}^{n} a_k \chi_{E_k}$$

and

$$\varphi_2 = \sum_{k=1}^{n} b_k \chi_{E_k}$$

And so ,

$$\varphi_1 + \varphi_2 = \sum_{k=1}^{n} a_k \chi_{E_k} + \sum_{k=1}^{n} b_k \chi_{E_k} = \sum_{k=1}^{n} (a_k + b_k)\, \chi_{E_k}$$

is a simple function such that $\varphi_1 + \varphi_2 \le f + g$ whence

$$\int (\varphi_1 + \varphi_2) = \sum_{k=1}^{n} (a_k + b_k)\mu(E_k)$$

$$= \sum_{k=1}^{n} a_k\, \mu(E_k) + \sum_{i=1}^{n} b_k\, \mu(E_k)$$

$$= \int \varphi_1 + \int \varphi_2$$

So that

$$\int_E \varphi_1 + \int_E \varphi_2 = \int_E (\varphi_1 + \varphi_2) \le \int_E (f + g)$$

Taking supremum for both sides as $\varphi_1 + \varphi_2 \le f + g$ then we have ,

$$\sup_{\varphi_1\le f}\int_E \varphi_1 + \sup_{\varphi_2\le g}\int_E \varphi_2 \le \sup_{(\varphi_1+\varphi_2)\le(f+g)}\int_E (f + g)$$

Therefore

$$\int_E f + \int_E g \le \int_E (f + g) \qquad (1)$$

(3)Let

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

be a simple function. Since $\mu(E) = 0$ then $\mu(E_i \cap E) = 0$ .Now

$$\varphi\chi_E = \sum_{i=1}^{n} \alpha_i \chi_{E_i \cap E}$$

Then

$$\int_E \varphi = \int \varphi\chi_E = \sum_{i=1}^{n} \alpha_i \, \mu(E_i \cap E) = \sum_{i=1}^{n} \alpha_i \, 0 = 0$$

it follows that

$$\int_E f = \sup_{\varphi \le f} \int_E \varphi = \sup_{\varphi \le f} 0 = 0$$

(4) Let f be nonnegative measurable function and $\int_E f = 0$. Let $B = \{x \in E | f(x) > 0\}$ and $B_n = \{x \in E \mid f(x) \ge 1/n\}$ so that $B = \bigcup_{n=1}^{\infty} B_n$ and $B_n \subset B_{n+1}$. Since $B_n \subset E$, then

$$0 = \int_E f = \int_{B_n} f \ge \int_{B_n} \frac{1}{n} = \frac{1}{n} \int_{B_n} 1 = \frac{1}{n} \mu(B_n) \ge 0$$

Therefore $\mu(B_n) = 0$ and by Theorem 1.2.1 (3) we have

$$\mu(B) = \lim_{n \to \infty} \mu(B_n) = 0$$

Whence $\mu\{x \in E \,| f(x) \ne 0\} = \mu(B) = 0$ thus, we conclude that $f = 0 \; a.e$ on .

Section 1.5 Convergence in Measure

Definition 1.5.1 : (Convergence Almost Everywhere)
Let (X, F, μ) be a measure space and let $\{f_n\}$ be a sequence of measurable functions. We say that f_n converge to measurable function f almost everywhere if $\mu\{x | f_n(x) \nrightarrow f(x)\} = 0$. In this case we write $f_n \to f \; a.e$

Definition 1.5.2 : (Converge in Measure)
Let (X, F, μ) be measure space and let $\{f_n\}$ be a sequence of measurable functions. We say that f_n converge in measure to measurable function f if for every $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mu\{x: |f_n(x) - f(x)| \ge \varepsilon\} = 0$$

That is , for every $\varepsilon > 0$ and every $\varepsilon' > 0$ there exist integer N such that for all $n \ge N$ we have $\mu\{x: |f_n(x) - f(x)| \ge \varepsilon\} < \varepsilon'$. In this case we write $f_n \underset{\mu}{\to} f$

Example 1.5.1: Let $f_n: [0, \infty) \to \mathbb{R}$ define by

$$f_n(x) = \begin{cases} \dfrac{nx}{n+1} & if\ x \notin \mathbb{N} \\ n & if\ x \in \mathbb{N} \end{cases}$$

Define $f: [0, \infty) \to \mathbb{R}$ by $f(x) = x$ then with respect to Lebesgue measure μ, $f_n \to f\ a.e\ [0, \infty)$.

Theorem 1.5.1[6] : Let (X, F, μ) be measure space .If $f_n \underset{\mu}{\to} f$ then there exist a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f\ a.e$ on X .

Proof: Since $f_n \underset{\mu}{\to} f$ then there exist an increasing sequence of integer $\{n_k\}$ such that for all $n \geq n_k$ we have $\mu\{x: |f_n(x) - f(x)| \geq 2^{-k}\} < 2^{-k}$. Now let ,
$E_k = \{x: |f_{n_k}(x) - f(x)| \geq 2^{-k}\}\ and\ A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} E_k$ Thus
$A^c = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} {E_k}^c = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} \{x: |f_{n_k}(x) - f(x)| < 2^{-k}\}$ suppose$x \in A^c$ want to show that $f_{n_k}(x) \to f(x)\ \forall\ x \in A^c$ and $\mu(A) = 0$. Since $x \in A^c$ then there is some $i \geq 1$ such that $x \in \bigcap_{k=i}^{\infty} \{x: |f_{n_k}(x) - f(x)| < 2^{-k}\}$ so for $k \geq i$ large enough we have $2^{-k} < \varepsilon$. Whence for $k \geq K$,$K \in \mathbb{N}$ we get $|f_{n_k}(x) - f(x)| < 2^{-k} < \varepsilon$ thus $f_{n_k}(x) \to f(x)$ $\forall\ x \in A^c$ .Moreover,for each $i \geq 1$ we have
$\mu(A) = \mu(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} E_k) \leq \mu(\bigcup_{k=i}^{\infty} E_k) \leq \sum_{k=i}^{\infty} \mu(E_k) \leq \sum_{k=i}^{\infty} 2^{-k} = 2^{1-i}$
so that $0 \leq \mu(A) \leq 2^{1-i}\ \forall i \geq 1$ thus $\mu(A) = 0$ .We conclude that $f_{n_k} \to f\ a.e$ on X.

Definition1.5.3 : (Cauchy in Measure)
A sequence of measurable functions $\{f_n\}$ is called Cauchy in measure if
For every $\varepsilon > 0$ we have

$$\lim_{m,n \to \infty} \mu\{x: |f_n(x) - f_m(x)| \geq \varepsilon\} = 0$$

That is, for every $\varepsilon > 0$ and every $\varepsilon' > 0$ there exist integer N such that for all n$, m \geq N$ we have $\mu\{x: |f_n(x) - f_m(x)| \geq \varepsilon\} < \varepsilon'$.

Example 1.5.2: Show that every sequence $\{f_n\}$ which converge in measure , is Cauchy in measure .

Solution : Let $\varepsilon > 0$ and $\varepsilon' > 0$ be given , since $\{f_n\}$ converge in measure then there is measurable function f and an integer N such that for all n$\geq N$ we have $\mu\left\{x: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon'}{2}$, then for all $n, m \geq N$ we have
$\left\{x: |f_n(x) - f(x)| < \frac{\varepsilon}{2}\right\} \cap \left\{x: |f(x) - f_m(x)| < \frac{\varepsilon}{2}\right\}$ is subset of $\{x: |f_n(x) - f_m(x)| < \varepsilon\}$.
Thus by taking complement,

$$\{x: |f_n(x) - f_m(x)| \geq \varepsilon\} \subseteq \left\{x: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x: |f(x) - f_m(x)| \geq \frac{\varepsilon}{2}\right\}$$

so for $n, m \geq N$ we have $\mu\{x: |f_n(x) - f_m(x)| \geq \varepsilon\}$

$$\leq \mu\left\{x: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} + \mu\left\{x: |f(x) - f_m(x)| \geq \frac{\varepsilon}{2}\right\}$$
$$\leq \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'$$

so that $\{f_n\}$ is Cauchy in measure.

Theorem1.5.2[2]: (Fatou's Lemma)
If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n \to f\ a.e$ on a set E, then

$$\int_E f \le \underline{lim} \int_E f_n$$

Proof: Without loss of generality we may assume $f_n \to f$ everywhere , since integral over sets of measure zero are zero .Let h be a bounded measurable function which vanishes outside a set E', $\mu(E') < \infty$ and $h \le f$.Define a function h_n by setting ,

$$h_n(x) = min\{h(x), f_n(x)\}$$

Then h_n is bounded by bound for h and vanishes outside E' so that $h_n(x) \to h(x)\ \ \forall x \in E'$.Thus by Bounded Convergence Theorem ,we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \le \underline{lim} \int_{E'} f_n = \underline{lim} \int_E f_n$$

Taking the supremum over h, we get

$$\int_E f = \sup_{h \le f} \int_E h \le \underline{lim} \int_E f_n$$

Theorem 1.5.3 [2]:(Monotone Convergence Theorem)
Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f_n \to f\ a.e$ .Then

$$\int f = lim \int f_n$$

Proof : If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n \to f\ a.e$ then by Fatou's Lemma(Theorem1.5.2) we have

$$\int f \le \underline{lim} \int f_n$$

But $\{f_n\}$ is an increasing sequence then for each n we have $f_n \le f$, and so $\int f_n \le \int f$.Taking $\overline{lim}$ for both side we get

$$\overline{lim} \int f_n \le \int f$$

Whence

$$\overline{lim} \int f_n \le \underline{lim} \int f_n$$

On other hand we know that

$$\underline{lim} \int f_n \le \overline{lim} \int f_n$$

Therefore

$$lim \int f_n = \underline{lim} \int f_n = \int f = \overline{lim} \int f_n$$

Hence

$$\int f = lim \int f_n$$

We see in Theorem 1.4.1 that $\int_E f + g \geq \int_E f + \int_E g$ .Now by using Monotone Convergence Theorem we will see in following theorem that both sides are realized .

Theorem 1.5.4 [7]: Let (X, F, μ) be measure space. If f and g are nonnegative measurable functions on (X, F, μ) then

$$\int f + g = \int f + \int g$$

Proof: Let $E \subset X$,$\{f_n, n \in \mathbb{N}\}$ and $\{g_n, n \in \mathbb{N}\}$ be increasing sequence of nonnegative simple functions such that $f_n \to f$ a .e on E and $g_n \to g$ a .e E as $n \to \infty$ .Then $f_n + g_n$ is increasing sequence of nonnegative simple function such that $f_n + g_n \to f + g$ $a.e$ on E as $n \to \infty$
because

$$| f_n + g_n - (f + g)| = | f_n - f + g_n - g| \leq | f_n - f| + |g_n - g| \to 0$$

as $n \to \infty$. By Monotone Convergence Theorem we have

$$\lim_{n\to\infty} \int f_n = \int f$$

And

$$\lim_{n\to\infty} \int g_n = \int g$$

Hence

$$\int (f + g)d\mu = \lim_{n\to\infty} \int (f_n + g_n)d\mu$$

$$= \lim_{n\to\infty} \left(\int f_n d\mu + \int g_n \, d\mu\right)$$

$$= \lim_{n\to\infty} \left(\int f_n d\mu\right) + \lim_{n\to\infty} \left(\int g_n \, d\mu\right)$$

$$= \int f \, d\mu + \int g d\mu$$

So that

$$\int f + g = \int f + \int g$$

Theorem1.5.5 [2] :(Lebesgue Convergence Theorem)
Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and $f_n \to f\ a.e\ on\ E$ then

$$\int_E f = lim \int_E f_n$$

Proof : Since $|f_n| \leq g$ then $g - f_n \geq 0$ and $\lim_{n\to\infty}(g - f_n) = g - f\ a.e\ on\ E$ then by Fatou's Lemma (Theorem 1.5.2) we have

$$\int_E (g-f) \leq \underline{lim} \int_E (g - f_n)$$

Since $f \leq |f| = \lim_{n\to\infty}|f_n| \leq \lim_{n\to\infty} g = g$,and g is integrable then f is integrable so that

$$\int_E g - \int_E f = \int_E (g-f) \leq \underline{lim} \int_E (g - f_n) = \int_E g - \overline{lim} \int_E f_n$$

Hence

$$\overline{lim} \int_E f_n \leq \int_E f$$

Similarly, considering $g + f_n$,we get

$$\underline{lim} \int_E f_n \geq \int_E f$$

Thus we conclude

$$\int_E f = lim \int_E f_n$$

Chapter Two

Basic Function Spaces

In present Chapter we study many important spaces that help us to achieve our goal. In this thesis we study modular spaces an Musielak–Orlicz spaces which provide the framework for different function spaces, including (weighted) Lebesgue spaces , Orlicz spaces and variable exponent Lebesgue spaces. We will study also the relationships between these spaces .We shall review normed space which include definition these spaces. Many of the results in this chapter will be used in next chapter. In this chapter we will consider $\Omega \subset \mathbb{R}^n$ and μ is Lebesgue measure.

There is no big difference in the definition of real valued and complex valued spaces. To avoid a double definition we let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We will denote the set of all measurable functions from Ω *to* $\mathbb{R}$ by $L^0(\Omega)$.

Section 2.1 Normed Space

Before we give the definition of normed space we give four definitions that we will use in several sections later.

Definition 2.1.1 :(Lebesgue Space L^p)
Let (Ω, F, μ) be Lebesgue measure space and $1 \leq p < \infty$.Then we define Lebesgue Space $L^p(\Omega, \mu)$ by

$$L^p(\Omega, \mu) = \left\{ f \in L^0(\Omega) \,\middle|\, \int_\Omega |f(x)|^p d\mu < \infty \right\}$$

To simplify , we write $L^p(\Omega)$, or L^p when the measure space has been specified . When $p = 1$ the space $L^1(\Omega)$ consists of all integrable functions on Ω.

Definition 2.1.2:(Lebesgue Space $L^p, p = \infty$)
Let(Ω, F, μ) be Lebesgue measure space .Then we define $L^\infty(\Omega, \mu)$ by

$$L^\infty(\Omega, \mu) = \{ f \in L^0(\Omega) \mid f \textit{ is essentially bounded on } \Omega \}$$

We mean f is essentially bounded on Ω, If there exist $0 < M < \infty$ such that

$$|f(x)| \leq M \quad a.e \text{ on } \Omega$$

As in case $1 \leq p < \infty$.We write $L^\infty(\Omega)$ or L^∞

Definition 2.1.3:(locally integrable)
Let Ω be open set in $\mathbb{R}^n$ and $f(x) : \Omega \to \mathbb{C}$ is a Lebesgue measurable function. If the Lebesgue integral

$$\int_K |f(x)|\, dx < \infty$$

for all compact subset K in Ω , then f is locally integrable. The set of all such functions is denoted by $L^1_{loc}(\Omega)$.

Example 2.1.1 : The function $f(x)$=1 is locally integrable but not integrable on $\mathbb{R}$.

Definition 2.1.4: (Weight Function)
A function $\omega(x) : \Omega \to (0, \infty)$ such that $\omega(x) \in \mathrm{L}^1_{loc}(\Omega)$ called weight function **.**

Definition 2.1.5:Weighted Lebesgue Space($\mathbf{L}^p_\omega(\Omega)$)
Let (Ω, F, μ) be Lebesgue measure space, ω is weight function and$1 \leq p < \infty$ .Then we define weighted Lebesgue Space $\mathrm{L}^p_\omega(\Omega, \mu)$ by

$$\mathrm{L}^p_\omega(\Omega, \mu) = \left\{ f \in \mathrm{L}^0(\Omega) \middle| \int_\Omega |f(x)|^p \omega(x) d\mu < \infty \right\}$$

To simplify , we write $\mathrm{L}^p_\omega(\Omega)$, or L^p_ω when the measure space has been specified**.**

Definition 2.1.6: (Norm)
Let X be a vector space . A norm on X over a filed $\mathbb{K}$ is a function $\|.\|: X \to [0, \infty)$ such that for every $x, y \in X$ and $\alpha \in \mathbb{K}$ we have

N1) $\|x\| \geqq 0$

N2) $\|x\| = 0 \; iff \; x = 0$

N3) $\|\alpha x\| = |\alpha| \|x\|$

N4) $\|x + y\| \leqq \|x\| + \|y\|$ (triangle inequality)

Definition 2.1.7: (Normed Space)
A normed space is a pair $(X, \|.\|)$ where X is vector space and $\|.\|$ is a norm defined on X.

Definition 2.1.8:(Complete Space)
The normed space X is said to be complete if every Cauchy sequence in X converges in$X$.

Definition 2.1.9: (Banach Space)
A Banach space is a complete normed space .

We will give now some examples of Banach space ,in the next section we present other main normed spaces .For all examples we want to show that $\|.\|$ satisfied all properties N1 to N4 of norm .

Example 2.1.2: Euclidian space $\mathbb{R}^n$ is a Banach space with norm defined by

$$\|x\| = \left(\sum_{i=1}^n (x_i)^2 \right)^{\frac{1}{2}} \quad \text{, where } x = (x_1, x_2 \dots, x_n) \in \mathbb{R}^n$$

Example 2.1.3: The Lebesgue Space $\mathrm{L}^p(\Omega)$ where $1 \leq p < \infty$, is normed space with norm defined by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

Solution: Let $f, g \in L^p(\Omega), \alpha \in \mathbb{K}$. So that

N1) Since $|f(x)| \geq 0$ and $1 \leq p < \infty$,then$|f(x)|^p \geq 0$ thus by definition of integral. Since again $1 \leq p < \infty$,then

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \geq 0$$

N2) $\|f\|_p = 0 \ iff \ \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} = 0 \ iff \int_\Omega |f(x)|^p dx = 0$

$iff \ |f(x)|^p = 0 \ \ a.e \ on \ \Omega \ \ iff \ \ |f(x)| = 0 \ \ a.e \ on \ \ \Omega \ \ iff \ f = 0$ since $f \in L^p(\Omega)$.

N3) $\|\alpha f\|_p = \left(\int_\Omega |\alpha f(x)|^p dx\right)^{\frac{1}{p}} = \left(\int_\Omega |\alpha|^p |f(x)|^p dx\right)^{\frac{1}{p}}$

$$= (|\alpha|^p)^{\frac{1}{p}} \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

Since $\alpha \in \mathbb{K}$. So that $\|\alpha f\|_p = |\alpha| \|f\|_p$

N4) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ by Minkowski inequality .

Example 2.1.4: The weighted Lebesgue Space $L^p_\omega(\Omega)$ where $1 \leq p < \infty$, is normed space with norm defined by

$$\|f\|_{p,\omega} = \left(\int_\Omega |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}}$$

Solution: Let $f, g \in L^p_\omega(\Omega), \alpha \in \mathbb{K}$. So that,

N1) Since $|f(x)| \geq 0$, $\omega(x) > 0$, and $1 \leq p < \infty$,then $|f(x)|^p \omega(x) \geq 0$ .Thus $\int_\Omega |f(x)|^p \omega(x) dx \geq 0$ by definition of integral .Since again $1 \leq p < \infty$ then

$$\|f\|_{p,\omega} = \left(\int_\Omega |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}} \geq 0$$

N2) $\|f\|_{p,\omega} = 0 \ iff \ \left(\int_\Omega |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}} = 0 \ iff \int_\Omega |f(x)|^p \omega(x) dx = 0$

$iff \ |f(x)|^p = 0 \ a.e \ on \ \Omega$ because $\omega(x) > 0$. $iff \ \ |f(x)| = 0 \ \ a.e \ on \ \Omega$

$iff \ \ f = 0$ since $f \in L^p_\omega(\Omega)$.

N3) $\|\alpha f\|_{p,\omega} = \left(\int_\Omega |\alpha f(x)|^p \omega(x) dx\right)^{\frac{1}{p}}$

$$= \left(\int_\Omega |\alpha|^p |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}}$$

$$= (|\alpha|^p)^{\frac{1}{p}} \left(\int_\Omega |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}} \text{ since } \alpha \in \mathbb{K}$$

$$= |\alpha| \|f\|_{p,\omega}$$

N4) $\|f+g\|_{p,\omega} \leq \|f\|_{p,\omega} + \|g\|_{p,\omega}$ by Minkowski inequality .

Example 2.1.5: The Lebesgue Space $L^p(\Omega)$ where $p = \infty$ $\left(L^\infty(\Omega)\right)$, is normed space with norm defined by

$$\|f\|_\infty = ess \sup_\Omega |f(x)|$$

Where $ess \sup_\Omega |f(x)| = inf\ \{M : |f(x)| \leq M \quad a.e \text{ on } \Omega\}$

Before we prove that $L^\infty(\Omega)$ is normed space we will give lemma that we use in the prove.

Lemma 2.1.1[2]: If $f \in L^\infty(\Omega)$ then $|f(x)| \leq \|f\|_\infty$ $a.e$ on Ω .

Proof: Let $f \in L^\infty(\Omega)$ then there exist $M > 0$ such that

$$|f(x)| \leq M \quad a.e \text{ on } \Omega$$

But $\|f\|_\infty = inf\ \{M : |f(x)| \leq M \quad a.e \text{ on } \Omega\}$ Therefore

$$|f(x)| \leq \|f\|_\infty \quad a.e \text{ on } \Omega\,.$$

Now we are ready to prove that $L^\infty(\Omega)$ is normed space. Let $f, g \in L^\infty(\Omega)$,

$\alpha \in \mathbb{K}$. So that

N1) Since $\|f\|_\infty = inf\ \{M : |f(x)| \leq M \quad a.e\}$ and $0 < M < \infty$ then

$\|f\|_\infty \geq 0$

N2) $\|f\|_\infty = 0$ iff $\|f\|_\infty = inf\ \{M : |f(x)| \leq M \quad a.e\} = 0$

iff $\mu(\{x \in \Omega : |f(x)| > \|f\|_\infty = 0\}) = 0 \quad iff$

$\mu(\{x \in \Omega : |f(x)| > 0\}) = 0 \quad iff \quad f(x) = 0$ a.e on Ω iff $f = 0$ since $f \in L^\infty(\Omega)$.

N3) The equality is obvious for $\alpha = 0$. Assume $\alpha \neq 0$ then

$|\alpha f(x)| > M$ iff $|f(x)| > S$ where $S = \frac{M}{|\alpha|} > 0$ .Whence

$$\|\alpha f\|_\infty = inf\{M : |\alpha f(x)| \leq M \quad a.e\}$$

$$= inf\{M : \mu(\{x \in \Omega : |\alpha f(x)| > M\}) = 0\}$$

$$= \ inf \ \left\{M: \mu\left(\left\{x \in \Omega: |f(x)| > \frac{M}{|\alpha|}\right\}\right) = 0\right\}$$

$$= inf\{|\alpha|S: \mu(\{x \in \Omega: |f(x)| > S\}) = 0\}$$

$$= |\alpha| \ inf\{S: \mu(\{x \in \Omega: |f(x)| > S\}) = 0\}$$

$$= |\alpha| \ inf \ \{S: |f(x)| \leq S \quad a.e\}$$

$$= \ |\alpha| \|f\|_\infty$$

N4) Let $f, g \ \in L^\infty(\Omega)$ then by lemma 2.1.1 we have $|f(x)| \leq \|f\|_\infty$ $a.e \ \ on \ \Omega$, and $|g(x)| \leq \|g\|_\infty \ \ a.e \ \ on \ \Omega$. Now

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

$a.e$ on Ω . Hence

$$\|f + g\|_\infty = ess \ \sup_{\Omega} |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

Definition 2.1.10: (Converge in Norm)
A sequence $\{x_n\}$ in a normed space X converges in norm(or strongly) to x in X if

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

In this case we write $x_n \to x$

Definition 2.1.11: (Cauchy in Norm)
A sequence $\{x_m\}$ in a normed space X is Cauchy if for every $\varepsilon > 0$ there is an integer N such that for all $m, n \geq N$ we have

$$\|x_m - x_n\| < \varepsilon$$

Example2.1.6. Let X be the set of all continuous real valued function on $[0,1]$ with norm defined

$$\|x\| = \int_0^1 |x(t)| dt$$

And $$x_m(t) = \begin{cases} 0 & , \ 0 \leq t \leq \frac{1}{2} \\ m\left(t - \frac{1}{2}\right) \ , & \frac{1}{2} < t < \frac{1}{2} + \frac{1}{m} \\ 1 \quad , & \frac{1}{2} + \frac{1}{m} \leq t \leq 1 \end{cases}$$

Then $x_m(t)$ is Cauchy sequence .

Solution : Let $\varepsilon > 0$ take $k > \frac{1}{\varepsilon}$, then $\forall$ m, n $\geq$ k we have

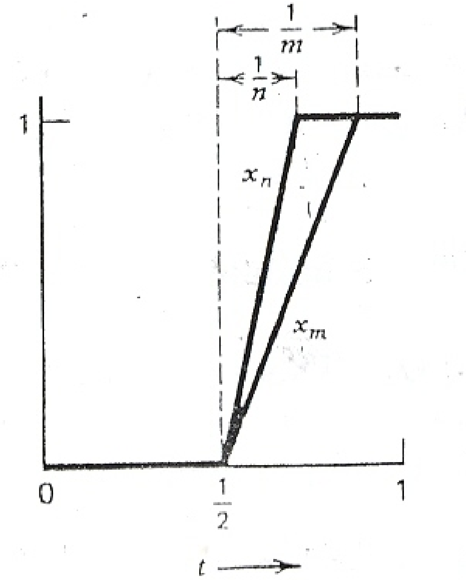
$$\|x_m - x_n\| = \int_0^1 |x_m(t) - x_n(t)| \ dt$$

which is the area of the triangle in fig :1

$$\|x_m - x_n\| = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right) < \frac{1}{2m} < \frac{1}{m} \quad < \frac{1}{k} < \varepsilon$$

Therefore $x_m(t)$ is Cauchy sequence.

Fig :1



Section 2.2 Modular Function

It is better to start with the function called modular which then induces a norm that for investigation of (weighted)(variable) Lebesgue spaces .In some cases the modular has certain advantages compared to the norm, for example in the case of Lebesgue spaces the modular $\int_\Omega |f(x)|^p dx$ compared to the norm

$$\left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

Definition 2.2.1 : (Convex Function)
A function τdefined on open interval $I = (a, b)$ is said to be convex if for each $x, y \in I \text{ and each } \alpha, \beta \geq 0,\ \alpha + \beta = 1$ we have

$$\tau(\alpha x + \beta y) \leq \alpha\tau(x) + \beta\tau(y)$$

Note that If I is any interval, open, closed, or half-open, we say that τ is convex on I if τ is continuity needed on I and convex in the interior.

Lemma 2.2.1[2]: If τ is a function defined on an open interval (a, b) which have a second derivative at each point of (a, b) .Then τ is convex on (a, b) if and only if $\tau''(x) \geq 0$ for each $x \in (a, b)$

Proof : Let $\tau''(x)$ is exist $\forall\, x \in (a,b)$ .If τ is convex on (a,b) , then its left and right -hand derivatives are exist and are monotone increasing on (a, b) so that $\tau'(x)$ is monotone increasing on (a,b). Hence $\tau''(x) \geq 0 \quad \forall\, x \in (a,b)$. Conversely, If $\tau''(x) \geq 0$ $\forall\, x \in (a,b)$ then $\tau'(x)$ is exist and also monotone increasing on (a,b). Since $\tau'(x)$ is exist then $\tau\,(x)$ is continuous on(a,b) .Hence τ is convex on (a,b).

Example 2.2.1 : Show that the function $\varphi(t) = t^p$ is convex on $[0,\infty)$ for $1 \leq p < \infty$.

Solution : Let $\varphi(t) = t^p$ then φ is continuous on $[0,\infty)$ for all p. We want only to show that $\varphi''(t) \geq 0$ on $(0,\infty)$. For $p = 1, \varphi(t) = t$ so that $\varphi''(t) = 0$. And for $1 < p < \infty$ we have $\varphi''(t) = p(p-1)t^{p-2} > 0$ since $t \in (0,\infty)$.Therefore $\varphi''(t) \geq 0$ for each point in $(0,\infty)$ .Thus by lemma 2.2.1 we conclude that $\varphi(t) = t^p$ is convex on $(0,\infty)$ and so φ convex on $[0,\infty)$ for $1 \leq p < \infty$.

Definition 2.2.2 :(Semimodular Function)
Let X be an arbitrary vector space. A function ρ: X $\to [0,\infty]$ is called a semimodular if for arbitrary x, y in X,

S1) $\rho(0) = 0$

S2) $\rho(\lambda x) = \rho(x)$ for every $\lambda \in \mathbb{K}$ with $|\lambda| = 1$

S3) ρ is convex

S4) $\rho\,(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

A semimodular ρ is called

Right continuous, if for every $x \in X$, we have $\lim_{\lambda \to 1^+} \rho(\lambda x) = \rho(x)$.

Left continuous , if for every $x \in X$, we have $\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x)$.

Continuous , if it is both right and left continuous.

Definition 2.2.3:(Modular Function)
Let X be an arbitrary vector space. A function ρ: X $\to [0,\infty]$ is called a modular if for arbitrary x, y in X,

M1) $\rho(x) = 0$ if and only if $x = 0$

M2) $\rho(\lambda x) = \rho(x)$ for every $\lambda \in \mathbb{K}$ with $|\lambda| = 1$

M3) ρ is convex

Remark 2.2.1. Note that our semimodular (modular) are always convex, because we will deal with convex modular in our thesis ,in contrast to some other sources.

Now we give examples of modular functions :

Example 2.2.2 :If $1 \leq p < \infty$. then

$$\rho_p(f) = \int_\Omega |f(x)|^p dx$$

is a modular function on $L^0(\Omega)$.

Solution : Let f be a measurable function on Ω ,So that

M1) $\rho_p(f) = 0 \; iff \; \int_\Omega |f(x)|^p dx = 0 \;\; iff \;\; |f(x)|^p = 0 \;\; a.e \;\; iff \;\; |f(x)| = 0 \;\; a.e$ since $1 \le p < \infty. iff \;\; f(x) = 0 \; a.e \;\; iff \; f = 0$ since $f \in L^0(\Omega)$.

M2) Let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Then

$$\rho_p(\lambda f) = \int_\Omega |\lambda f(x)|^p dx = \int_\Omega |\lambda|^p |f(x)|^p dx$$

Now since $|\lambda| = 1$ and $1 \le p < \infty$ then $|\lambda|^p = 1$ whence

$$\rho_p(\lambda f) = \int_\Omega |f(x)|^p dx = \rho_p(f)$$

M3) Let g be a measurable function on Ω ,$\alpha, \beta \ge 0$, $\alpha + \beta = 1$ .Then

$$\begin{aligned}
\rho_p(\alpha f + \beta g) &= \int_\Omega |(\alpha f + \beta g)(x)|^p dx \\
&= \int_\Omega |(\alpha f)(x) + (\beta g)(x)|^p dx \\
&= \int_\Omega |\alpha f(x) + \beta g(x)|^p dx
\end{aligned}$$

Now since $\varphi(t) = t^p$ is convex on $[0, \infty)$ for $1 \le p < \infty$ then if we let $t = |f(x)|$ we have $\varphi(|f(x)|) = |f(x)|^p$ is convex on $[0, \infty)$ for $1 \le p < \infty$, therefore

$$\varphi(|\alpha f(x) + \beta g(x)|) \le \alpha \, \varphi(|f(x)|) + \beta \, \varphi(|g(x)|)$$

Imply that, $|\alpha f(x) + \beta g(x)|^p \le \alpha |f(x)|^p + \beta |g(x)|^p$ So,

$$\begin{aligned}
\int_\Omega |\alpha f(x) + \beta g(x)|^p dx &\le \int_\Omega (\alpha |f(x)|^p + \beta |g(x)|^p) dx \\
&= \int_\Omega \alpha |f(x)|^p dx + \int_\Omega \beta |g(x)|^p dx \\
&= \alpha \int_\Omega |f(x)|^p dx + \beta \int_\Omega |g(x)|^p dx \\
&= \alpha \rho_p(f) + \beta \rho_p(g)
\end{aligned}$$

Hence, $\rho_p(\alpha f + \beta g) \le \alpha \rho_p(f) + \beta \rho_p(g)$ for $\alpha, \beta \ge 0$, $\alpha + \beta = 1$ it follows that ρ_p is convex and we conclude that $\rho_p(f) = \int_\Omega |f(x)|^p dx$ is a modular function on $L^0(\Omega)$.

Example 2.2.3 : If $1 \leq p < \infty$, and $\omega(x)$ is weight function .Then

$$\rho_\omega(f) = \int_\Omega |f(x)|^p \omega(x)dx$$

Is a modular function on $L^0(\Omega)$.

Solution : Let f be a measurable function on Ω ,So that

M1) $\rho_\omega(f) = 0 iff \int_\Omega |f(x)|^p \omega(x)dx = 0 iff \ |f(x)|^p \omega(x) = 0 \ a.e$ on Ω
$iff \ |f(x)|^p = 0 \quad a.e \ on \ \Omega$ since $\omega(x) > 0 \quad iff \ |f(x)| = 0 \ a.e$ since$1 \leq p < \infty$.
$iff \ f(x) = 0 \ a.e \ iff \ f = 0$ since $f \in L^0(\Omega)$.

M2) L$et\ \lambda \in \mathbb{K}$ with $|\lambda| = 1$. Then

$$\rho_\omega(\lambda f) = \int_\Omega |\lambda f(x)|^p \omega(x)dx = \int_\Omega |\lambda|^p |f(x)|^p \omega(x)dx$$

now since $|\lambda| = 1$ and $1 \leq p < \infty$ then $|\lambda|^p = 1$ whence

$$\rho_\omega(\lambda f) = \int_\Omega |f(x)|^p \omega(x)dx = \rho_\omega(f)$$

M3) Let g be a measurable function on Ω ,$\alpha, \beta \geq 0, \ \alpha + \beta = 1$ .Then

$$\begin{aligned} \rho_\omega(\alpha f + \beta g) &= \int_\Omega |(\alpha f + \beta g)(x)|^p \omega(x)dx \\ &= \int_\Omega |(\alpha f)(x) + (\beta g)(x)|^p \omega(x)dx \\ &= \int_\Omega |\alpha f(x) + \beta g(x)|^p \omega(x)dx \end{aligned}$$

As we say in example 2.2.1 ,that $\varphi(|f(x)|) = |f(x)|^p$ is convex on $[0,\infty)$ for $1 \leq p < \infty$ therefore

$|\alpha f(x) + \beta g(x)|^p \omega(x) \leq (\alpha |f(x)|^p + \beta |g(x)|^p)\omega(x)$ So,

$$\begin{aligned} \int_\Omega |\alpha f(x) + \beta g(x)|^p \omega(x)dx &\leq \int_\Omega (\alpha |f(x)|^p + \beta |g(x)|^p)\omega(x)dx \\ &= \int_\Omega (\alpha |f(x)|^p \omega(x) + \beta |g(x)|^p \omega(x))dx \\ &= \int_\Omega \alpha |f(x)|^p \omega(x)dx + \int_\Omega \beta |g(x)|^p \omega(x)dx \\ &= \alpha \int_\Omega |f(x)|^p \omega(x)dx + \beta \int_\Omega |g(x)|^p \omega(x)dx \\ &= \alpha \rho_\omega(f) + \beta \rho_\omega(g) \end{aligned}$$

Hence $\rho_\omega(\alpha f+\beta g)\le \alpha\rho_\omega(f)+\beta\rho_\omega(g)$ for $\alpha,\beta\ge 0,\ \alpha+\beta=1$, it follows that ρ_ω is convex .We conclude that $\rho_\omega(f)=\int_\Omega |f(x)|^p\omega(x)dx$ is a modular function on $L^0(\Omega)$.

Example 2.2.4 : Let $X=L^\infty(\Omega)$. X is normed space with norm defined by

$$\|f\|_\infty = ess\ \sup_\Omega |f(x)|$$

Then $\rho_\infty(f)=\|f\|_\infty$ is modular function on $L^0(\Omega)$.

Solution : Let f be a measurable function on Ω ,So that

M1) $\rho_\infty(f)=0\ iff\ \|f\|_\infty=0\ \ iff\ f=0$ as in example 2.1.4

M2) Let $\lambda\in\mathbb{K}$ with $|\lambda|=1$.Then $\rho_\infty(\lambda f)=\|\lambda f\|_\infty=|\lambda|\|f\|_\infty$ .Now since $|\lambda|=1$ thus $\rho_\infty(\lambda f)=\|f\|_\infty=\rho_\infty(f)$.

M3) Let g be a measurable function on Ω ,$\alpha,\beta\ge 0,\ \alpha+\beta=1$ .Then by example 2.1.4 we have,

$\rho_\infty(\alpha f+\beta g)=\|\alpha f+\beta g\|_\infty\le\|\alpha f\|_\infty+\|\beta g\|_\infty$ since $\alpha f,\ \beta g\in L^\infty(\Omega)$.But

$\alpha,\beta\ge 0$ thus $\|\alpha f\|_\infty+\|\beta g\|_\infty=|\alpha|\|f\|_\infty+|\beta|\|g\|_\infty$

$$=\alpha\|f\|_\infty+\beta\|g\|_\infty=\alpha\rho_\infty(f)+\beta\rho_\infty(g)$$

then $\rho_\infty(\alpha f+\beta g)\le\alpha\rho_\infty(f)+\beta\rho_\infty(g)$ and so ρ_∞ is convex .Then we conclude that $\rho_\infty(f)=\|f\|_\infty$ is modular function on $L^0(\Omega)$.

We will give other two modular functions in chapter three .

Let ρ be a semimodular(modular) function on X.Then by convexity , non-negativity of ρ and $\rho(0)=0$ we have $\rho(\lambda x)$ is non-decreasing on $[0,\infty)$ for every $x\in X$. Also,

$$\begin{aligned}&\rho(\lambda x)=\rho(|\lambda|\,x)\le|\lambda|\,\rho(x) && \text{for all } |\lambda|\le 1\\ &\rho(\lambda x)=\rho(|\lambda|\,x)\ge|\lambda|\,\rho(x) && \text{for all } |\lambda|\ge 1\end{aligned}\qquad (2.2.4)$$

Definition 2.2.5 (Modular Space)
If ρ be a semimodular or modular on X , then

$$X_\rho=\{x\in X:\ \rho(\lambda x)<\infty\ for\ some\ \lambda>0\}$$

Is called semimodular space or modular space ,respectively.

Definition 2.2.6: The semimodular(modular) space can be equipped with a norm called the Luxemburg norm, defined by,

$$\|x\|_\rho=inf\left\{\lambda>0:\ \rho\left(\frac{x}{\lambda}\right)\le 1\right\}$$

In the next theorem ,we shall prove that every semimodular (modular) space is normed under Luxemburg norm.

Theorem 2.2.1[4]: Let ρ be a semimodular(modular) on X .Then X_ρ is normed space with the Luxemburg norm ,

$$\|x\|_\rho = inf\left\{\lambda > 0:\ \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

Proof: First we will show that X_ρ is vector space . Let $x, y \in X_\rho$ and $\alpha \in \mathbb{K}\backslash\{0\}$ then we want to show $0 \in X_\rho$, $x + y \in X_\rho$, and $\alpha x \in X_\rho$. $0 \in X_\rho\ since\ \rho(\lambda o) = \rho(o) = 0$ for some $\lambda > 0$. Since $x \in X_\rho$ and by (2.2.4) we have $\rho(\lambda\alpha x)$= $\rho(|\lambda\alpha|x) = \rho(\lambda' x) < \infty$ for some $\lambda' = |\lambda\alpha| > 0$ so that $\alpha x \in X_\rho$.By convexity of ρ we estimate

$$0 \leq \rho\big(\lambda(x + y)\big) = \rho\left(\frac{1}{2}2\lambda(x + y)\right) = \rho\left(\left(\frac{1}{2}2\lambda x\right) + \left(\frac{1}{2}2\lambda y\right)\right)$$

$$\leq \frac{1}{2}\rho(2\lambda x) + \frac{1}{2}\rho(2\lambda y)$$

$$< \infty + \infty = \infty$$

For some $\lambda > 0$ since $x, y \in X_\rho$.Hence $x + y \in X_\rho$ and X_ρ is vector space. Now we say properties of norm .

N1) The set $\left\{\lambda > 0:\ \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}$ is nonempty for all $x \in X_\rho$thus $0 \leq \|x\|_\rho < \infty$ for all $\in X_\rho$.

N2) Let $x = 0$ then $\rho\left(\frac{x}{\lambda}\right) = \rho\left(\frac{0}{\lambda}\right) = 0 \ \leq 1$ for all $\lambda > 0$ .Hence $inf\left\{\lambda > 0:\ \rho\left(\frac{x}{\lambda}\right) \leq 1\right\} = 0$ thus $\|x\|_\rho = 0$. Conversely ,if $\|x\|_\rho = 0$ then $\rho(\alpha x) \leq 1$ for all $\alpha > 0$ .Therefore by (2.2.4) , $\rho(\lambda x) \leq \beta\rho\ \left(\frac{\lambda}{\beta}x\right) \leq \beta(1) = \beta$ for all $\lambda > 0$ and $0 < \beta \leq 1$ .This implies $\rho(\lambda x) = 0$ for all $\lambda > 0$,thus $x = 0$ .We conclude that $\|x\|_\rho = 0\ iff\ x = 0$.

N3) Let $\alpha \in \mathbb{K}$ and $s = \frac{\lambda}{|\alpha|} > 0$ then

$$\|\alpha x\|_\rho = \inf\left\{\lambda > 0:\ \rho\left(\frac{\alpha x}{\lambda}\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0:\ \rho\left(\left|\frac{\alpha}{\lambda}\right| x\right) \leq 1\right\} \qquad \text{by (2.2.4)}$$

$$= \inf\left\{\lambda > 0:\ \rho\left(\frac{|\alpha|}{\lambda}x\right) \leq 1\right\}$$

$$= \inf\left\{s|\alpha| > 0:\ \rho\left(\frac{1}{s}x\right) \leq 1\right\}$$

$$= |\alpha| \inf\left\{s > 0:\ \rho\left(\frac{x}{s}\right) \le 1\right\} = |\alpha| \|x\|_\rho$$

N4) Let $x, y \in X_\rho$ and $\lambda' = \|x\|_\rho < u$ and $\lambda'' = \|y\|_\rho < v$. Therefore $u = \|x\|_\rho + a$ and $v = \|y\|_\rho + b$ where $a > 0$, $b > 0$ .Then

$\rho\left(\frac{x}{u}\right) \le \frac{\lambda'}{u}\, \rho\left(\frac{x}{\lambda'}\right) \le \frac{\lambda'}{u} \le 1$, in the same way $\rho\left(\frac{y}{v}\right) \le 1$. By convexity of ρ we have,

$$\rho\left(\frac{x+y}{u+v}\right) = \rho\left(\frac{x\,u}{(u+v)u} + \frac{yv}{(u+v)v}\right)$$

$$\le \frac{u}{(u+v)}\rho\left(\frac{x}{u}\right) + \frac{v}{(u+v)}\,\rho\left(\frac{y}{v}\right) \le 1$$

Thus $\|x+y\|_\rho = \inf\left\{u+v > 0 : \rho\left(\frac{x+y}{u+v}\right) \le 1\right\} \le u+v$ but $u = \|x\|_\rho + a$ and $v = \|y\|_\rho + b$ then $u + v = \|x\|_\rho + a + \|y\|_\rho + b$. Since $a > 0$, $b > 0$ arbitrary we obtain $\|x+y\|_\rho \le \|x\|_\rho + \|y\|_\rho$ for all $x, y \in X_\rho$.

Example 2.2.5: The Lebesgue Space $L^p(\Omega)$ where $1 \le p < \infty$, is normed space with Luxemburg norm defined by

$$\|f\|_{\rho_p} = inf\left\{\lambda > 0:\ \rho_p\left(\frac{f}{\lambda}\right) \le 1\right\}$$

With modular function $\rho_p(f) = \int_\Omega |f(x)|^p dx$ on $L^0(\Omega)$.

Solution : In example 2.2.2 we prove that $\rho_p(f) = \int_\Omega |f(x)|^p dx$ is modular function on $L^0(\Omega)$ and in theorem 2.2.1 we proved that every modular space is normed space with Luxemburg norm ,then $L^p(\Omega)$ is normed space with $\|f\|_{\rho_p} = inf\left\{\lambda > 0:\ \rho_p\left(\frac{f}{\lambda}\right) \le 1\right\}$.

Example 2.2.6: The weighted Lebesgue Space $L^p_\omega(\Omega)$ where $1 \le p < \infty$, is normed space with Luxemburg norm defined by

$$\|f\|_{\rho_\omega} = inf\left\{\lambda > 0:\ \rho_\omega\left(\frac{f}{\lambda}\right) \le 1\right\}$$

With modular function $\rho_\omega(f) = \int_\Omega |f(x)|^p \omega(x) dx$ on $L^0(\Omega)$.

Solution : In example 2.2.3 we proved that $\rho_\omega(f) = \int_\Omega |f(x)|^p \omega(x) dx$ is modular function on $L^0(\Omega)$ and in theorem 2.2.1 we proved that every modular space is normed space with Luxemburg norm ,then $L^p_\omega(\Omega)$ is normed space with

$$\|f\|_{\rho_\omega} = inf\left\{\lambda > 0:\ \rho_\omega\left(\frac{f}{\lambda}\right) \le 1\right\}$$

Example 2.2.7: $\left(L^{\infty}(\Omega)\right)$, is normed space with norm defined by

$$\|f\|_{\rho_\infty} = inf\left\{\lambda > 0\colon\ \rho_\infty\left(\frac{f}{\lambda}\right) \le 1\right\}$$

With modular function $\rho_\infty(f) = \|f\|_\infty$ on $L^0(\Omega)$.

Solution : In example 2.2.4 we proved that $\rho_\infty(f) = \|f\|_\infty$ is modular function on $L^0(\Omega)$ and in theorem 2.2.1 we proved that every modular space is normed space with Luxemburg norm then $L^{\infty}(\Omega)$ is normed space with $\|f\|_{\rho_\infty} = inf\left\{\lambda > 0\colon\ \rho_\infty\left(\frac{f}{\lambda}\right) \le 1\right\}$

The question! Does $\|f\|_{\rho_p} = \|f\|_p$, $\|f\|_{\rho_\omega} = \|f\|_{p,\omega}$, and $\|f\|_{\rho_\infty} = \|f\|_\infty$ where $\|f\|_p$, $\|f\|_{p,\omega}$, and $\|f\|_\infty$ as we defined them in example 2.2.2, example 2.2.3, and example 2.2.4? The answer is yes .We will prove our claim . Start with

$\|f\|_{\rho_p} = inf\left\{\lambda > 0\colon\ \rho_p\left(\frac{f}{\lambda}\right) \le 1\right\}$,we will try to prove $\|f\|_p = inf\lambda$ for all $\lambda > 0$ such that $\rho_p\left(\frac{f}{\lambda}\right) \le 1$

$$\rho_p\left(\frac{f}{\lambda}\right) \le 1\ iff \int_\Omega \left|\frac{f}{\lambda}(x)\right|^p dx \le 1$$
$$iff \int_\Omega \frac{|f(x)|^p}{\lambda^p}\, dx \le 1\ iff\ \frac{1}{\lambda^p}\int_\Omega |f(x)|^p dx \le 1\ iff$$

$$\int_\Omega |f(x)|^p dx \le \lambda^p\ \ iff \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \le (\lambda^p)^{\frac{1}{p}} = \lambda$$

Hence $\|f\|_p \le \lambda$ for all λ such that $\rho_p\left(\frac{f}{\lambda}\right) \le 1$. It follows that $\|f\|_{\rho_p} = \|f\|_p$. In the same way we can show that $\|f\|_{\rho_\omega} = \|f\|_{p,\omega}$ and $\|f\|_{\rho_\infty} = \|f\|_\infty$.

Theorem 2.2.2[4]: Let ρ be a semimodular(modular) on X. Then

(a) $\|x\|_\rho \le 1\ and\ \rho(x) \le 1$ are equivalent .

(b) If ρ is continuous , then $\|x\|_\rho < 1$ and $\rho(x) < 1$ are equivalent.

(c) If ρ is continuous , then $\|x\|_\rho = 1\ and\ \rho(x) = 1$ are equivalent .

Proof: (a) If $\rho(x) \le 1$ then $\|x\|_\rho \le 1$ by definition of $\|.\|_\rho$. On the other hand if $\|x\|_\rho \le 1$ Indeed, let $\|x\|_\rho \le u \le 1$ then $\rho\left(\frac{x}{u}\right) \le 1$ thus

$$\rho(x) = \rho\left(u\frac{x}{u}\right) \le u\,\rho\left(\frac{x}{u}\right) \le 1$$

(b) Let ρ be continuous .If $\|x\|_\rho < 1$ then there exist $\lambda < 1$ with $\rho\left(\frac{x}{\lambda}\right) \le 1$. It follows that by(2.2.4) $\rho(x) = \rho\left(\lambda\frac{x}{\lambda}\right) \le \lambda\rho\left(\frac{x}{\lambda}\right) \le \lambda < 1$.On other hand let $\rho(x) < 1$ and since ρ

is continuous then ρ is Right- continuous so that there exist $\gamma > 1$ such that $\rho(\gamma x) = \rho(x) < 1 \leq 1$ hence by (a) we have $\|\gamma x\|_\rho \leq 1$ thus $\|x\|_\rho \leq \frac{1}{\gamma} < 1$.

(c) The equivalence of $\|x\|_\rho = 1$ *and* $\rho(x) = 1$ follows immediately from (a)and(b)

.

Corollary 2.2.1[4]: Let ρ be a semimodular(modular) on X.

(1) If $\|x\|_\rho \leq 1$,then $\rho(x) \leq \|x\|_\rho$
(2) If $\|x\|_\rho > 1$, then $\rho(x) > \|x\|_\rho$

Proof: (1) The claim is obvious for $x = 0$. Assume $0 < \|x\|_\rho \leq 1$ then by (a) in Theorem 2.2.2 and since $\left\|\frac{x}{\|x\|_\rho}\right\|_\rho = 1 \leq 1$ we have $\rho\left(\frac{x}{\|x\|_\rho}\right) \leq 1$,it follows that by (2.2.4)

$\frac{1}{\|x\|_\rho}\rho(x) = \frac{1}{\|x\|_\rho}\rho\left(\|x\|_\rho \frac{x}{\|x\|_\rho}\right) \leq \frac{\|x\|_\rho}{\|x\|_\rho}\rho\left(\frac{x}{\|x\|_\rho}\right) \leq 1$ .Hence

$$\rho(x) \leq \|x\|_\rho$$

(2) If $\|x\|_\rho > 1$,then for $1 < \lambda < \|x\|_\rho$ we get $1 < \rho\left(\frac{x}{\lambda}\right)$. Since

$\lambda < \|x\|_\rho$ then $\|x\|_\rho = \lambda + a$ where $a > 0$ .By (2.2.4),

$1 < \rho\left(\frac{x}{\lambda}\right) \leq \frac{1}{\lambda}\rho(x)$ thus $\lambda \leq \rho(x)$ and since $a > 0$ was arbitrary then $\rho(x) > \|x\|_\rho$.

Lemma2.2.2[4]: Let ρ be a semimodular(modular) on X and $\{x_k\}$ be a sequence in X_ρ .Then $\{x_k\}$ is converges to 0 in norm as $k \to \infty$ if and only if $\rho(\lambda x_k)$ converges to 0 for all $\lambda > 0$ *as* $k \to \infty$.

Proof: Assume $\{x_k\}$ is converges to 0 in norm as $k \to \infty$ then $\|x_k\|_\rho \to 0$ so that for $\lambda > 0$ we have $\|K\lambda x_k\|_\rho < 1$ *for all* $K > 1$ and large k.Thus by (b)in theorem 2.2.2 we have $\rho(K\lambda x_k) < 1$ *for all* $K > 1$ and large k .Hence by (2.2.4) $\rho(\lambda x_k) \leq \frac{1}{K}\rho(K\lambda x_k) \leq \frac{1}{K}$ therefore $\rho(\lambda x_k) \to 0$ *as* $k \to \infty$. Conversely , let $\rho(\lambda x_k) \to 0$ for all $\lambda > 0$ *as* $k \to \infty$, then $\rho(\lambda x_k) \leq 1$ for large k so that by (b)in theorem 2.2.2 we have $\|\lambda x_k\|_\rho \leq 1$. Hence $\|x_k\|_\rho \leq \frac{1}{\lambda}$ for large k .Since $\lambda > 0$ was arbitrary ,we get$\|x_k\|_\rho \to 0$ *as* $k \to \infty$ and then $\{x_k\}$ is converges to 0 in norm as $k \to \infty$.

From our modular space X_ρ,which was induced by the norm, we can define another type of convergence by means of the semimodular as the following definition .

Definition 2.2.7 : (Converge in Modular)
Let ρ be a semimodular(modular) on X and x_k , $x \in X_\rho$.Then we say that x_k is modular

convergent ($\rho -$ convergent) to x if there exist $\lambda > 0$ such that $\rho\big(\lambda(x_k - x)\big) \to 0$ as $k \to \infty$. In this case we write $x_k \overset{\rho}{\to} x$.

We say that from Lemma2.2.2 that modular convergence is weaker than norm convergence. In fact, we have $\lim_{k\to\infty} \rho\big(\lambda(x_k - x)\big) = 0$ for all $\lambda > 0$ in norm convergence ,while .For the modular convergence $\lim_{k\to\infty} \rho\big(\lambda(x_k - x)\big) = 0$ for some $\lambda > 0$. From this point we can ask our self for what conditions that modular convergence and norm convergence are coincide ? This is what the following lemma answers it.

Lemma 2.2.3[4]: Let X_ρ be a semimodular (modular) space and $x_k \in X_\rho$. Then modular convergence and norm convergence are equivalent if and only if $\rho(x_k) \to 0$ implies $(2x_k) \to 0$.

Proof: Let modular convergence and norm convergence are equivalent and let $\rho(x_k) \to 0$ with $x_k \in X_\rho$ then want to show that $\rho(2x_k) \to 0$ .By Lemma 2.2.2 we have $\|x_k\|_\rho \to 0$ also by the second direction of same lemma we have $\rho(\lambda x_k) \to 0$ for all $\lambda > 0$, in particular take $\lambda = 2$ so we have $\rho(2x_k) \to 0$. Conversely ,let $x_k \in X_\rho$with $\rho(x_k) \to 0$.

To show that modular convergence and norm convergence are equivalent we have to show that $\rho(\lambda x_k) \to 0$ for all $\lambda > 0$. For fixed $\lambda > 0$ choose $m \in N$ such that $2^m \geq \lambda$. Since $\rho(x_k) \to 0$ implies $\rho(2x_k) \to 0$,then by repeated application of our assumption we get $\lim_{k\to\infty} \rho(2^m x_k) = 0$. Now by (2.2.4)and $2^m \geq \lambda$,we have $\rho(\lambda x_k) \leq \frac{\lambda}{2^m} \rho(2^m x_k)$ it follows that $0 \leq \lim_{k\to\infty} \rho(\lambda x_k) \leq \frac{\lambda}{2^m} \lim_{k\to\infty} \rho(2^m x_k) = 0$ .Thus $\lim_{k\to\infty} \rho(\lambda x_k) = 0$ and so $\rho(\lambda x_k) \to 0$ for all $\lambda > 0$.We conclude that modular convergence and norm convergence are equivalent.

Definition 2.2.8: (Monotone Complete)
Let X_ρ be a modular space .A $modular\ \rho\ on\ X_\rho$ is said to be monotone complete if

$$0 \leq x_k \uparrow_{k\in\Lambda} \text{ and } \sup_{k\in\Lambda} \rho(x_k) < \infty \text{ imply } \bigcup_{k\in\Lambda} x_k \in X_\rho$$

In the following theorem , we find the necessary conditions that modular functions are equivalent .

Theorem 2.2.3[3]: Let X_ρ be a modular space with two modular functions $\rho_1 and\ \rho_2$ where ρ_1 is monotone complete .Then there exist $\varepsilon, \varepsilon', k, K, \gamma > 0$ such that

(a) $\rho_2(kx) < \gamma$ $\quad for\ all\ x\ with\ \rho_1(x) < \varepsilon'$

(b) $\rho_2(kx) < K\rho_1(x)$ $\quad for\ all\ x\ with\ \varepsilon \leq \rho_1(x) < \varepsilon'$

Proof: (a) by contradiction, assume γ can't be found ,then there exist a sequence $0 \leq x_k \in X_\rho(k = 1,2, \dots)$ such that $\rho_1(x_k) \leq \frac{1}{2^k}$ and

$\rho_2\left(\frac{1}{k}x_k\right) \geq k$. Let $y_n = \bigcup_{k=1}^{n} x_k$ (n=1,2,…) thus $0 \leq y_n \leq y_{n+1}$ therefore $0 \leq y_n \uparrow_{n=1}^{\infty}$ and

$$\rho_1(y_n) = \rho_1\left(\bigcup_{k=1}^{n} x_k\right) \leq \sum_{k=1}^{n} \rho_1(x_k) \leq \sum_{k=1}^{n} \frac{1}{2^k} \leq 1$$

so that $\sup_{k=1,2,\dots} \rho_1(y_n) \leq 1 < \infty$.

Since ρ_1 is monotone complete by assumption, there exist $y_0 \in X_\rho$ such that $y_0 = \bigcup_{n=1}^{\infty} y_n$. On other hand since ρ_2 in non-decreasing then for n$\geq k$ we have $\rho_2\left(\frac{1}{k}y_0\right) \geq \rho_2\left(\frac{1}{k}y_n\right) \geq n \geq k \geq 1$ which contradiction to that ρ_2 is modular function.

(b)For any x replace ε by $\frac{\varepsilon}{2}$ with $\frac{\varepsilon}{2} \leq \rho_1(x) < \varepsilon'$, then by (a) we have $\rho_2(kx) < \gamma$ .Now since $\frac{\varepsilon}{2} \leq \rho_1(x)$, then $1 \leq \frac{2}{\varepsilon}\rho_1(x)$ multiply both side by γ we have $\rho_2(kx) < \gamma \leq \gamma\frac{2}{\varepsilon}\rho_1(x)$ .Choose $K = \gamma\frac{2}{\varepsilon}$ and we conclude that $\rho_2(kx) < K\rho_1(x)$ for all x with $\varepsilon \leq \rho_1(x) < \varepsilon'$.

Section 2.3 Φ- function

In this section we start with function that used in modular function definition also will use in $L^{p(.)}$ or $L_\omega^{p(.)}$ definitions which are modular spaces whether $p(.)$ is constant as in section 2.1 or $p(.)$ is function as will see in sections 3.1, 3.2.This function is real –valued function, if we take integral of this function we will get the modular that exist in modular space definition . The function is called Φ- function.

The modular space which define in this way is called Orlicz spaces which will talk about it later .

Definition 2.3.1:(Φ- function)

A function $\varphi: [0,\infty) \to [0,\infty]$ is called Φ- function if for arbitrary t $\in [0,\infty)$,

F1) φ is convex

F2) $\varphi(0) = 0$

F3) $\lim_{t\to 0^+} \varphi(t) = 0$

F4) $\lim_{t\to\infty} \varphi(t) = \infty$

A Φ- function is called positive if $\varphi(t) > 0$ for all $t > 0$.

In fact, there is a relationship between Φ- functions and semimodular (modular) on $\mathbb{R}$. We will say that in the following lemma .

Lemma 2.3.1[4]: Let $\varphi: [0,\infty) \to [0,\infty]$ and let ρ be a function such that $\rho(t) = \varphi(|t|)$ for all $t \in \mathbb{R}$. Then φ is Φ- function if and only if ρ semimodular on $\mathbb{R}$ with $X_\rho = \mathbb{R}$. Moreover , φ is positive Φ- function if and only if ρ modular on $\mathbb{R}$ with $X_\rho = \mathbb{R}$.

Proof: Suppose φ is Φ- function. Since $\lim_{t\to 0^+} \varphi(t) = 0$,we have $X_\rho = \mathbb{R}$. Now we will check the properties of semimodular :

S1) Since $\varphi(0) = 0$ then $\rho(0) = \varphi(|0|) = 0$
S2) Let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$,then
$\rho(\lambda t) = \varphi(|\lambda t|) = \varphi(|\lambda||t|) = \varphi(|t|) = \rho(t)$ since $|\lambda| = 1$ .Hence $\rho(\lambda t) = \rho(t)$ for

every with $|\lambda| = 1$.

S3) Since $\varphi(t)$ is convex on $[0,\infty)$ so that $\rho(t) = \varphi(|t|)$ is convex on $\mathbb{R}$.

S4) Assume that $\rho(\lambda t_0) = 0$ for all $\lambda > 0$ want to show that $t_0 = 0$, since $\lim_{t\to\infty} \varphi(t) = \infty$ then there exist $t_1 > 0$ with $\varphi(t_1) > 0$. Thus there is no $\lambda > 0$ such that $t_1 = \lambda t_0$ which implies that $t_0 = 0$. We conclude that ρ is semimodular. Let φ is positive and $\rho(s) = 0$ want to show $s = 0$, since $\rho(s) = 0$ then $0 = \rho(s) = \varphi(|s|)$ but φ is positive Φ- function then $\varphi(|s|) = 0 \ iff \ |s| = 0 \ iff \ s = 0$ and so ρ modular on $\mathbb{R}$.

Conversely, let ρ semimodular on $\mathbb{R}$ with $X_\rho = \mathbb{R}$. Since $X_\rho = \mathbb{R}$ then there exist $t_2 > 0$ such that $\varphi(t_2) < \infty$ so for all $0 \le t \le t_2$ and from (2.2.4) we have $0 \le \varphi(t) = \varphi\left(\frac{t_2}{t_2} t\right) \le \frac{t}{t_2}\varphi(t_2) \le \varphi(t_2) < \infty$ thus $\exists\, t > 0$ such that $\varphi(t) = 0$ .Hence $\lim_{t\to 0^+} \varphi(t) = 0$. Since $1 \neq 0$ and $\varphi(0) = 0$ then there exist $\lambda > 0$ such that $\rho(\lambda.1) \neq 0$ .In particular there exist $t_3 > 0$ with $\varphi(t_3) > 0$ and so we get $\varphi(kt_3) \ge k\varphi(t_3) > 0$ for all $k \in \mathbb{N}$.Since k was arbitrary ,we have $\lim_{t\to\infty} \varphi(t) = \infty$.Since $\rho(0) = 0$ then

$\varphi(0) = \varphi(|0|) = \rho(0) = 0$ .And finally since ρ is convex then φ is convex We proved that φ is Φ- function.

Assume that ρ is modular then want to show that φ is positive Φ- function .Since ρ is modular then if $\varphi(t) = \varphi(|t|) = 0$ this imply $t = 0$ .Hence for $t > 0$ we have $\varphi(t) > 0$. So that φ is positive Φ- function .

Examples of Φ- functions :

Example 2.3.1:Let $1 \le p < \infty$. Then $\varphi_p(|f(x)|) = |f(x)|^p$ is positive Φ- function .

Solution: F1) Let $|f(x)| = t$ then $0 \le t < \infty$ so φ is convex as we say in example 2.2.1.

F2) Let $|f(x)| = 0$ then $\varphi(0) = \varphi_p(|f(x)|) = |f(x)|^p = 0^{\mathrm{p}} = 0$

F3) $\lim_{|f(x)|\to 0^+} \varphi_p(|f(x)|) = \lim_{|f(x)|\to 0^+} |f(x)|^p = 0$

F4) $\lim_{|f(x)|\to\infty} \varphi_p(|f(x)|) = \lim_{|f(x)|\to\infty} |f(x)|^p = \infty^p = \infty$

Moreover for $|f(x)| > 0$ we get $\varphi_p(|f(x)|) = |f(x)|^p > 0$ since $1 \le p < \infty$ so that $\varphi_p(|f(x)|) = |f(x)|^p$ is positive Φ- function.

Example 2.3.2: Let $1 \le p < \infty$ *and* $\omega(x)$ is weight function. Then $\varphi_\omega(|f(x)|) = |f(x)|^p \omega(x)$ is positive Φ- function .

Solution: F1) Since $\varphi(|f(x)|) = |f(x)|^p$ is convex as in Example 2.3.1 then

$$\varphi_\omega\big(\alpha f(x) + \beta g(x)\big) = |\alpha f(x) + \beta g(x)|^p \omega(x)$$

$$\le (\alpha|f(x)|^p + \beta|g(x)|^p)\omega(x)$$

$$= \alpha|f(x)|^p\omega(x) + \beta|g(x)|^p\omega(x)$$

$$\alpha\varphi_\omega(|f(x)|) + \beta\varphi_\omega(|g(x)|)$$

So $\varphi_\omega$is convex .

F2) Let $|f(x)| = 0$ then $\varphi_\omega(0) = \varphi_\omega(|f(x)|) = |f(x)|^p\omega(x) = 0^{\mathrm{p}}\omega(x) = 0$

F3) $\lim_{|f(x)| \to 0^+} \varphi_\omega(|f(x)|) = \lim_{|f(x)| \to 0^+} |f(x)|^p \omega(x) = 0$

F4) $\lim_{|f(x)| \to \infty} \varphi_\omega(|f(x)|) = \lim_{|f(x)| \to \infty} |f(x)|^p \omega(x) = \infty^p \omega(x) = \infty$ since $(x) > 0$.

Moreover for $|f(x)| > 0$ we get $\varphi_\omega(|f(x)|) = |f(x)|^p\omega(x) > 0$ since $1 \le p < \infty$ and $\omega(x) > 0$ so that $\varphi_\omega(|f(x)|) = |f(x)|^p\omega(x)$ is positive Φ- function .

Remark 2.3.1: Let φ be a Φ- function .Then by convexity of φ and $\varphi(0) = 0, \varphi$ is non-decreasing .Moreover the following properties are useful:

$$\varphi(rt) \le r\varphi(t), \tag{2.3.2}$$

$$\varphi(st) \ge s\varphi(t),$$

for any $r \in [0,1]$, $s \in [1, \infty)$ *and* $t \ge 0$. Furthermore inequality (2.3.2) implies

$$\varphi(a) + \varphi(b) \le \frac{a}{a+b}\varphi(a+b) + \frac{b}{a+b}\varphi(a+b)$$

$$= \varphi(a+b) \le \tfrac{1}{2}\varphi(2a) + \tfrac{1}{2}\varphi(2b)$$

for all $a, b \ge 0$ *and* $a + b > 0$

Although Φ- function using in many function spaces ,these are not general for what our need .In the case of variable exponent Lebesgue space (see CH.3) we need to generalize Φ- function that may depend on the space variable.

Definition 2.3.2: (Generalized Φ- Function)
Let (Ω, F, μ) be σ- finite ,complete measure space and μ is a Lebesgue measure. A real function $\varphi: \Omega \times [0,\infty) \to [0,\infty]$ is said to be generalized Φ- function on (Ω, F, μ) if:

(a) $\varphi(x,t)$ is Φ- function of the variable $t \geq 0$ for every $x \in \Omega$.
(b) $\varphi(x,t)$ is measurable function of x for all $t \geq 0$.

If φ is a generalized Φ- function on (Ω, F, μ) ,we write $\varphi \in \Phi(\Omega)$ and in this case we say that φ is a generalized Φ- function on Ω .

Note that every Φ- function is generalized Φ- function if we set

$$\varphi(x,t) := \varphi(t) \text{ for } x \in \Omega \text{ and } t \in [0,\infty) .$$

In next theorem, we show that every generalized Φ- function (positive) generates a semimodular (modular) respectively, on $L^0(\Omega)$

Theorem 2.3.1[4]: If $\varphi \in \Phi(\Omega)$ and $f \in L^0(\Omega)$,then $\varphi(x,|f(x)|)$ is a measurable function of x and

$$\rho_\varphi(f) := \int_\Omega \varphi(x,|f(x)|)\ dx$$

is a semimodular on $L^0(\Omega)$ .If φ is positive ,then ρ_φ is a modular .We call ρ_φ the semimodular induced by φ.

Proof: By splitting the function into its positive and negative part it suffices to consider the case $f \geq 0$. Let $f_k \to f$ where $f_k \geq 0$ are simple functions. so $f_k(x) = \sum_{j=1}^{n} \alpha_j^k \chi_{E_j^k}(x)$ where $E_j = \{x : f_k(x) = \alpha_j^k\}$ Thus

$$\varphi(x,|f_k(x)|) = \textstyle\sum_{j=1}^{n} \varphi(x,\alpha_j^k).\chi_{E_j^k}(x)$$

is measurable function of x by definition of simple function then $\varphi(x,|f_k(x)|) \to \varphi(x,|f(x)|)$ therefore $\varphi(x,|f(x)|)$ is a measurable function of x .

Now will show that $\rho_\varphi(f)$ is a semimodular on $L^0(\Omega)$.

S1) Since $\varphi(x,|f(x)|)$ is Φ-function of $|f(x)|$ then $\varphi(x,0) = 0$ and so

$$\rho_\varphi(0) = \int_\Omega \varphi(x,|o(x)|)dx$$

$$= \int_\Omega \varphi(x,|0|)\ dx$$

$$= \int_\Omega \varphi(x,0)\ dx = \int_\Omega 0\ dx = 0$$

S2) Let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ then

$$\rho_\varphi(\lambda f) = \int_\Omega \varphi(x, |\lambda f(x)|)\, dx = \int_\Omega \varphi(x, |\lambda||f(x)|)\, dx$$

But $|\lambda| = 1$ then

$$\rho_\varphi(\lambda f) = \int_\Omega \varphi(x, |f(x)|)\, dx = \rho_\varphi(f)$$

$S3$) Let $\alpha, \beta \geq 0$ and $\alpha + \beta = 1.$ Since $\varphi(x, |f(x)|)$ is convex of $|f(x)|$ we have

$$\begin{aligned} \rho_\varphi(\alpha f + \beta g) &= \int_\Omega \varphi(x, |\alpha f + \beta g(x)|)\ dx \\ &\leq \int_\Omega [\alpha\, \varphi(x, |f(x)|) + \beta\, \varphi(x, |g(x)|)]\ dx \\ &= \alpha \int_\Omega \varphi(x, |f(x)|) dx + \beta \int_\Omega \varphi(x, |g(x)|)\, dx \\ &= \alpha\, \rho_\varphi(f) + \beta \rho_\varphi(g) \quad \textit{so that } \rho_\varphi \textit{ is convex} \end{aligned}$$

$S4$) Let $\rho\,(\lambda f) = 0$ for all $\lambda > 0$ then

$$\rho_\varphi(\lambda f) = \int_\Omega \varphi(x, |\lambda f(x)|)\, dx = 0$$

thus $\varphi(x, |\lambda f(x)|) = 0\ a.e$ on Ω by theorem1.4.1(5) .But $\lambda > 0$ then $\varphi(x, \lambda|f(x)|)\ 0 = a.e$ on Ω and since $\lim_{t\to\infty} \varphi(x,t) = \infty$ for all $x \in \Omega$ then $|f(x)| = 0\ a.e$ on Ω thus $f(x) = 0\ a.e\ on\ \Omega$ implies $f = 0$. We conclude that ρ_φ is a semimodular on $L^0(\Omega)$.

Assume now that φ is positive and $\rho_\varphi(f) = 0$ then

$$\int_\Omega \varphi(x, |f(x)|)\ dx = 0$$

Thus $\varphi(x, |f(x)|) = 0\ \ a.e$ on Ω.Since φ is positive then

$\varphi(x, |f(x)|) > 0$ for all $|f(x)| > 0$ implies $|f(x)| = 0\ \ a.e$ on Ω and we have $f = 0$.This proves that ρ_φ is a modular on $L^0(\Omega)$.

Section 2.4 Orlicz Spaces

The aim of this section is provide to basic results about Orlicz spaces. The Orlicz spaces is extending the usual L^p space with $p \geq 1$ where t^p function which enter in definition of L^p is replaced by a more general function, Φ- function. Also this section will present the modular space corresponding to $\varphi \in \Phi(\Omega)$ with φ positive .This modular space is called Musielak- Orlicz spaces .

Definition 2.4.1 (Orlicz Space)
Let (Ω, F, μ) be σ- finite ,complete measure space .Let φ be positive Φ- function and ρ_φ be given by

$$\rho_\varphi(f) \coloneqq \int_\Omega \varphi(|f(x)|)\ dx$$

is modular on $L^0(\Omega)$. Then the modular space

$$L^\varphi(\Omega, \mu) = \{f \in L^0(\Omega): \rho_\varphi(\lambda f) < \infty \ for\ some\ \lambda > 0\}$$

will be called Orlicz space and denoted by $L^\varphi(\Omega)$ *or* L^φ, for short .

Example 2.4.1: For $1 \leq p < \infty$,we saw in Example 2.2.2 that $\rho_p(f) = \int_\Omega |f(x)|^p dx$ is a modular function on $L^0(\Omega)$ and we saw in Example 2.3.1 that $\varphi_p(|f(x)|) = |f(x)|^p$ is positive Φ- function then

$$L^{\varphi_p}(\Omega) = \{f \in L^0(\Omega): \rho_p(\lambda f) < \infty \ for\ some\ \lambda > 0\}$$

is Orlicz space .

Example 2.4.2: For $1 \leq p < \infty$ and weight function $\omega(x)$, we saw in example 2.2.3 that $\rho_\omega(f) = \int_\Omega |f(x)|^p \omega(x) dx$ is a modular function on $L^0(\Omega)$ and we saw in Example 2.3.2 that $\varphi_\omega(|f(x)|) = |f(x)|^p \omega(x)$ is positive Φ- function then

$$L^{\varphi_\omega}(\Omega) = \{f \in L^0(\Omega): \rho_\omega(\lambda f) < \infty \ for\ some\ \lambda > 0\}$$

is Orlicz space .

By theorem 2.2.1, $L^\varphi(\Omega)$ can be equipped Luxemburg norm ,

$$\|f\|_\varphi = inf\left\{\lambda > 0:\ \rho_\varphi\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

Definition 2.4.2: (Musielak -Orlicz Space)
Let (Ω, F, μ) be σ- finite ,complete measure space .Let $\varphi_* \in \Phi(\Omega)$, φ_* is positive and ρ_{φ_*} be given by

$$\rho_{\varphi_*}(f) \coloneqq \int_\Omega \varphi_*(x, |f(x)|)\ dx$$

is modular on $L^0(\Omega)$. Then the modular space

$$L^{\varphi_*}(\Omega, \mu) = \{f \in L^0(\Omega): \rho_{\varphi_*}(\lambda f) < \infty \ for\ some\ \lambda > 0\}$$

will be called Musielak- Orlicz space and denoted by $L^{\varphi_*}(\Omega)$ *or* L^{φ_*}, for short .The Musielak- Orlicz spaces is also called generalized Orlicz spaces .

Example 2.4.3: $L^{\varphi_p}(\Omega)$ and $L^{\varphi_\omega}(\Omega)$ in examples 2.4.1 and examples 2.4.2 are also Musielak- Orlicz spaces when define $\varphi_*(x,|f(x)|) \coloneqq \varphi_p(|f(x)|)$ in definition $L^{\varphi_p}(\Omega)$ and $\varphi_*(x,|f(x)|) \coloneqq \varphi_\omega(|f(x)|)$ in definition $L^{\varphi_\omega}(\Omega)$.

In general , every Orlicz space $L^{\varphi}(\Omega)$ is Musielak- Orlicz spaces when define $\varphi_*(x,|f(x)|) \coloneqq \varphi(|f(x)|)$.

Also by theorem 2.2.1, $L^{\varphi_*}(\Omega)$ can be equipped Luxemburg norm ,

$$\|f\|_{\varphi_*} = inf\left\{\lambda > 0\colon \rho_{\varphi_*}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

Theorem 2.2.1 proved that every semimodular (modular) space is normed space with Luxemburg norm .This is achieved for Musielak- Orlicz space L^{φ_*}.In fact L^{φ_*}=$\left(L^{\varphi_*}, \|.\|_{\varphi_*}\right)$ is complete. We need to prove two lemmas then we can show that L^{φ_*} is a Banach space.

Lemma 2.4.1[4]: Let $\varphi_* \in \Phi(\Omega)$ and $\mu(\Omega) < \infty$ .Then every $\|.\|_{\varphi_*}$- Cauchy sequence is also a Cauchy sequence with respect to convergence in measure .

Proof: Fixed $\varepsilon > 0$ and let $V_t = \{x \in \Omega :\ \varphi_*(x,t) = 0\}$ for $t > 0$.Since $\varphi_*(x,t)$ is measurable function of x then V_t is measurable. For all $x \in \Omega$, $\varphi_*(x,t)$ is Φ- function of the variable t then $\varphi_*(x,t)$ is non-decreasing with respect to t and $\lim_{t\to\infty} \varphi_*(x,t) = \infty$, so $V_t \downarrow \emptyset$ as $t \to \infty$ .Therefore $\mathrm{Lim}_{t\to\infty}\, \mu(V_t) = \mu(\emptyset) = 0$ .Since $\mu(\Omega) < \infty$ and $V_t \subseteq \Omega$ then there exist $K \in \mathbb{N}$ such that $\mu(V_K) < \varepsilon$. Let $E \subset \Omega$ be a μ -measurable set and define the measure

$$\nu_K(E) \coloneqq \rho_{\varphi_*}(K\chi_E) = \int_E \varphi_*(x,K)d\mu$$

If E is μ -measurable set with $\nu_K(E) = 0$ then $\varphi_*(x,K) = 0\ \ a.e\ [\mu]$
$x \in$ E. Now since $V_K = \{x \in \Omega :\ \varphi_*(x,K) = 0\}$ and $x \in E$ we have $\mu(E\backslash V_K) = 0$ .Hence $\mu|_{\Omega\backslash V_K} \ll \nu_K$. Because $(\Omega\backslash V_K) \subseteq \Omega$ we get $\mu(\Omega\backslash V_K) \le \mu(\Omega) < \infty$ .Then there exist $\delta \in (0,1)$ such that $\nu_K(E) \le \delta$ implies $\mu(E\backslash V_K) \le \varepsilon$ (ref[6], Theorem 30.B) . Suppose $\{f_n\}$ is a Cauchy sequence in $\|.\|_{\varphi_*}$ then there exist $k_0 \in \mathbb{N}$ such that far all $m,n \ge k_0$ we have $\left\|K\,\varepsilon^{-1}\delta^{-1}(f_m - f_n)\right\|_{\varphi_*} \le 1$. For $m,n \ge k_0$,by Theorem2.2.2 and by (2.2.4)we have

$$\rho_{\varphi_*}\left(K\,\varepsilon^{-1}(f_m - f_n)\right) \le \delta\rho_{\varphi_*}\left(K\,\varepsilon^{-1}\delta^{-1}(f_m - f_n)\right) \le \delta$$

Let us set $E_{m,n,\varepsilon} \coloneqq \{x \in \Omega\colon |f_m(x) - f_n(x)| \ge \varepsilon\}$ want to show that $\mu\left(E_{m,n,\varepsilon}\right) < \varepsilon'$ for $all\ \varepsilon' > 0$.Now

$$\nu_K\left(E_{m,n,\varepsilon}\right) = \rho_{\varphi_*}\left(K\chi_{E_{m,n,\varepsilon}}\right)$$

$$= \int_{E_{m,n,\varepsilon}} \varphi_*(x,K)d\mu \ \le \rho_{\varphi_*}\left(K\,\varepsilon^{-1}(f_m - f_n)\right) \le \delta$$

As above since $\mu|_{\Omega\backslash V_K} \ll \nu_K$ and $\nu_K\left(E_{m,n,\varepsilon}\right) \le \delta$ then

$\mu(E_{m,n,\varepsilon} \backslash V_K) \leq \varepsilon$ .With $\mu(V_K) < \varepsilon$ we have $\mu(E_{m,n,\varepsilon}) < 2\varepsilon$,choose $\varepsilon' = 2\varepsilon$ and since $\varepsilon > 0$ was arbitrary .This proves that $\{f_n\}$ is a Cauchy sequence in measure.

If $\|f_n\|_{\varphi_*} \to 0$, then as above there exist $K \in \mathbb{N}$ such that

$$\mu(\{|f_n| \geq \varepsilon\}) \leq 2\varepsilon \text{ for all } n \geq K \text{ thus } f_n \underset{\mu}{\to} 0 .$$

Lemma 2.4.2[4]: Let $\varphi_* \in \Phi(\Omega)$ .Then every $\|.\|_{\varphi_*}$ - Cauchy sequence $\{f_n\} \subset L^{\varphi_*}$ has a subsequence which converge almost everywhere to a measurable function f.

Proof : Recall that μ is σ- finte . Let $\Omega = \bigcup_{i=1}^{\infty} A_i$with A_i pairwise disjoint and $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$. Then by Lemma 2.4.1 , $\{f_n\}$ is a Cauchy sequence with respect to convergence in measure on A_1 therefore there exist a measurable function $f: A_1 \to \mathbb{K}$ such that $f_n \underset{\mu}{\to} f$ and by Theorem 1.5.1 there exist a subsequence $\{f_{n_1}\}$ of $\{f_n\}$ which converge $a.e$ to f.Rrepeating this argument for every A_i and passing to the diagonal sequence we get a subsequence $\{f_{n_j}\}$ and a measurable function $f: \Omega \to \mathbb{K}$ such that $f_{n_j} \to f$ $a.e$ on Ω.

Theorem 2.4.1[4]: Let $\varphi_* \in \Phi(\Omega)$. Then $L^{\varphi_*}(\Omega)$ is a Banach space .

Proof: Let $\{f_n\}$ be a Cauchy sequence in L^{φ_*} then want to show $\{f_n\}$ is convergent in L^{φ_*}. By Lemma 2.4.2 there exist a subsequence $\{f_{n_j}\}$ and a measurable function $f: \Omega \to \mathbb{K}$ such that $f_{n_j} \to f$ $a.e$ on Ω so that $\left|f_{n_j}(x) - f(x)\right| \to 0$ $a.e$ on Ω .This implies $\varphi_*\left(x, \left|f_{n_j}(x) - f(x)\right|\right) \to 0$ $a.e\ on\ \Omega$ .Let > 0 and $0 < \varepsilon < 1$, since $\{f_n\}$ is a Cauchy sequence thus there exist $K = K(\lambda, \varepsilon) \in \mathbb{N}$ such that $\|\lambda\,(f_m - f_n)\|_{\varphi_*} < \varepsilon < 1$ for all $m, n \geq K$ which implies $\rho_{\varphi_*}(\lambda\,(f_m - f_n)) \leq \varepsilon$ by Theorem 2.2.2. Therefore by Fatou's Lemma (Theorem1.5.2)

$$\begin{aligned}
\rho_{\varphi_*} \quad (\lambda\,(f_m - f)) &= \int_\Omega \varphi_*(x, |\lambda\,(f_m - f)(x)|)\ dx \\
&= \int_\Omega \varphi_*\left(x, \left|\lambda\left(f_m - \lim_{j\to\infty} f_{n_j}\right)(x)\right|\right) dx \\
&= \int_\Omega \lim_{j\to\infty} \varphi_*\left(x, \lambda\left|\left(f_m - f_{n_j}\right)(x)\right|\right) dx \\
&\leq \underline{\lim} \int_\Omega \varphi_*\left(x, \left|\lambda\left(f_m - f_{n_j}\right)(x)\right|\right) dx \\
&= \underline{\lim}\, \rho_{\varphi_*}(\lambda\,(f_m - f_{n_j})) \leq \varepsilon
\end{aligned}$$

Hence $\rho_{\varphi_*}(\lambda\,(f_m - f)) \to 0\ as\ m \to \infty$ for all $\lambda > 0$ and by Lemma 2.2.2 we have $\|f_n - f\|_{\varphi_*} \to 0$ thus $\{f_n\}$ is converges in L^{φ_*} and since $\{f_n\}$ is arbitrary then L^{φ_*} is complete .we conclude that $L^{\varphi_*}(\Omega)$ is a Banach space .

The next lemma has the analogues of the classical Lebesgue integral convergence results .

Lemma 2.4.3[4]: Let $\varphi_* \in \Phi(\Omega)$ and $f_k, f, g \in L^0(\Omega)$ .Then

(a) If $f_k \to f \quad a.e$ on Ω, $then\ \rho_{\varphi_*}(f) \le \underline{\lim}\ \rho_{\varphi_*}(f_k)$.

(b) If $|f_k| \uparrow |f| \quad a.e$ on Ω $,then\ \rho_{\varphi_*}(f) = \lim_{k\to\infty} \rho_{\varphi_*}(f_k)$.

(c) If $f_k \to f \quad a.e$ on Ω , $|f_k| \le |g| \quad a.e$ on Ω , and $\rho_{\varphi_*}(\lambda g) < \infty$

for every $\lambda > 0$, then $f_k \to f$ in L^{φ_*} .

These properties are called Fatou's lemma (for the modular) , Monotone Convergence Theorem(for the modular),Lebesgue Convergence Theorem(for the modular).

Proof: (a) By Fatou's lemma (Theorem1.5.2)

$$\rho_{\varphi_*}(f) = \int_\Omega \varphi_*(x, |f(x)|)\ dx$$

$$= \int_\Omega \varphi_*\left(x, \left|\lim_{k\to\infty} f_k(x)\right|\right)\ dx$$

$$= \int_\Omega \varphi_*\left(x, \lim_{k\to\infty} |f_k(x)|\right)\ dx$$

$$= \int_\Omega \lim_{k\to\infty} \varphi_*(x, |f_k(x)|)\ dx$$

$$\le \underline{\lim} \int_\Omega \varphi_*(x, |f_k(x)|)\ dx$$

$$= \underline{\lim}\ \rho_{\varphi_*}(f_k)$$

(b) Let $|f_k| \uparrow |f| \quad a.e$ .Since $\varphi_*(x,t)$ is non-decreasing of t we have

$$0 \le \varphi_*(x, |f_k(x)|) \uparrow \varphi_*(x, |f(x)|) \quad a.e \text{ on } \Omega$$

So by Monotone Convergence Theorem we get ,

$$\rho_{\varphi_*}(f) = \int_\Omega \varphi_*(x, |f(x)|)\ dx$$

$$= \int_\Omega \varphi_*\left(x, \lim_{k\to\infty} |f_k(x)|\right)\ dx$$

$$= \int_\Omega \lim_{k\to\infty} \varphi_*(x, |f_k(x)|)\, dx$$

$$= \lim_{k\to\infty} \int_\Omega \varphi_*(x, |f_k(x)|)\ dx = \lim_{k\to\infty}\ \rho_{\varphi_*}(f_k)$$

(c) Assume $f_k \to f \quad a.e$,$|f_k| \le |g| \quad a.e$, and $\rho_{\varphi_*}(\lambda g) < \infty$ for every $\lambda > 0$ then $|f_k - f \ | \to 0$ a.e $on\ \Omega$, $|f| = \lim_{k\to\infty}|f_k(x)| \le \lim_{k\to\infty}|g| = |g|$ and $|f_k - f \ | \le 2|g|$.Since $\rho_{\varphi_*}(2\lambda g) < \infty$ then by Lebesgue Convergence Theorem we have

$$\lim_{k\to\infty}\ \rho_{\varphi_*}(\lambda|f_k - f|) = \lim_{k\to\infty} \int_\Omega \varphi_*(x, \lambda|(f_k - f)(x)|)\ dx$$

$$= \lim_{k\to\infty} \int_\Omega \varphi_*(x, \lambda|f_k(x) - f(x)|)\ dx$$

$$= \int_\Omega \lim_{k\to\infty} \varphi_*(x, \lambda|f_k(x) - f(x)|)\ dx$$

$$= \int_\Omega \varphi_*\left(x, \lambda \lim_{k\to\infty}|f_k(x) - f(x)|\right)$$

$$= \int_\Omega \varphi_*(x, 0) = 0$$

Since $\lambda > 0$ was arbitrary then by Lemma 2.2.2 we have $\| f_k - f\|_{\varphi_*} \to 0$ which implies that $f_k \to f$ in L^{φ_*} .

Note that if $\varphi_* \in \Phi(\Omega)$, φ_* is positive then in addition to modular condition of modular

$$\rho_{\varphi_*}(f) \coloneqq \int_\Omega \varphi_*(x, |f(x)|)\ dx$$

is also monotone complete modular function.

Theorem 2.4.2[3]: Let φ ,$\psi \in \Phi(\Omega)$,ψ, φ are positive .Then $L^\varphi(\Omega) \subseteq L^\psi(\Omega)$ if and only if there exist $k, K > 0$ and $c(x) \in L^1(\Omega)$ such that

$$\psi(x, kt) \le K\varphi(x, t) + c(x) \qquad (*)$$

For all $t \ge 0$ and $a.e$ on Ω .

Proof: Assume $(*)$ is satisfied and let $f \in L^\varphi(\Omega)$ then want to show that $f \in L^\psi(\Omega)$. Since $f \in L^\varphi(\Omega)$ then $f: \Omega \to \mathbb{R}$ is measurable function and

$$\int_\Omega \varphi(x, |\lambda f(x)|)\ dx < \infty\ \ for\ some\ \lambda > 0$$

Then

$$\int_\Omega \varphi(x, \lambda|f(x)|)\ dx < \infty\ for\ some\ \lambda > 0$$

But $(*)$ is satisfied for all $t \geq 0$ and $a.e$ on Ω thus if we take $k = \lambda$ and $t = |f(x)|$ then we have

$$\psi(x, \lambda|f(x)|) \leq K\varphi(x, |f(x)|) + c(x) \qquad a.e \text{ on } \Omega$$

And since $\varphi(x,t)$ is non- decreasing of t then $\varphi(x, |f(x)|) \leq \varphi(x, \lambda|f(x)|)$ for some $\lambda > 0$ so that

$$\psi(x, \lambda|f(x)|) \leq K\varphi(x, \lambda|f(x)|) + c(x) \qquad a.e \text{ on } \Omega$$

Hence ,

$$\int_\Omega \psi(x, |\lambda f(x)|)\ dx \leq\ K \int_\Omega \varphi(x, \lambda|f(x)|)\ dx + \int_\Omega c(x)\ dx$$

Now $c(x) \in \mathrm{L}^1(\Omega)$ thus $\int_\Omega |c(x)|\ dx < \infty$, moreover $\int_\Omega \varphi(x, |\lambda f(x)|)\ dx < \infty$

$for\ some\ \ \lambda > 0$, therefore

$$\int_\Omega \psi(x, |\lambda f(x)|)\ dx < \infty\ \ for\ some\ \lambda > 0$$

Which implies that $f \in \mathrm{L}^\psi(\Omega)$. This proves that $\mathrm{L}^\varphi(\Omega) \subseteq \mathrm{L}^\psi(\Omega)$.

Assume $\mathrm{L}^\varphi(\Omega) \subseteq \mathrm{L}^\psi(\Omega)$,want to show that $\psi(x, kt) \leq K\varphi(x,t) + c(x)$ $a.e$ on Ω .Let $0 \leq \alpha_i (i = 1,2, \dots)$ be the sequence of all positive rational numbers .For any measurable set E with $\mu(E) < \infty$ and for $\varepsilon, k, K > 0$ in Theorem 2.2.3 (b),we put

$$E_i = \{x: \psi(x, \alpha_i t) > K\varphi(x, \alpha_i)\} \cap E \qquad (1)$$

And

$$f_i(x) = \alpha_i \chi_{E_i}(x) \qquad (2)$$

respectively , consider $\mu(E_i) \neq 0$ .Since $\varphi(x, \alpha_i) < \infty\ \ on\ E_i$ we can define

$E_{i,n} = \{x: \varphi(x, \alpha_i) < n\} \cap E_i$ .So that $E_{i,n} \uparrow_{n=1}^{\infty} E_i$ for all $n \geq n_0$ where n_0 is sufficiently large such that $\mu(E_{i,n}) \neq 0$,then $\rho_\varphi(\ \alpha_i \chi_{E_{i,n}}) < \varepsilon\ \ and\ \ \alpha_i \chi_{E_{i,n}} \in \mathrm{L}^\varphi$.Since if otherwise we have $\rho_\varphi\big(\ \alpha_i \chi_{E_{i,n}}\big) \geq \varepsilon$ and so from eq. (1)

$$\begin{aligned} \rho_\psi\big(\ k\alpha_i \chi_{E_{i,n}}\big) &= \int_\Omega \psi\big(x,\ k\alpha_i \chi_{E_{i,n}}\big) dx \\ &= \int_{E_{i,n}} \psi(x,\ k\alpha_i) dx \\ &>\ K \int_{E_{i,n}} \varphi(x, \alpha_i) dx \end{aligned}$$

$$= K\int_{\Omega}\varphi(x,\ \alpha_i\chi_{E_{i,n}})dx$$
$$= K\rho_{\varphi}(\ \alpha_i\chi_{E_{i,n}})$$

Which contradiction to Theorem2.2.3(b). Therefore

$$\alpha_i\chi_{E_{i,n}} \uparrow_{n=1}^{\infty}\ \ and\ \rho_{\varphi}(\ \alpha_i\chi_{E_{i,n}}) < \varepsilon$$

We have

$$f_i = \bigcup_{n=1}^{\infty}\alpha_i\chi_{E_{i,n}} \in \mathrm{L}^{\varphi}\ and\ \ \rho_{\varphi}(f_i) < \varepsilon$$

by eq. (1) and Theorem2.2.3(b) as above . Here putting $y_n = \bigcup_{i=1}^{\infty} f_i$, hence we have a sequence of step functions $0 \le\ y_n \uparrow_{n=1}^{\infty}$ and $y_n \in \mathrm{L}^{\varphi}$.Now since y_n is step function and every step function is simple function then we have $y_n = \sum_{r=1}^{n}\alpha_{i_r}\chi_{E^{(r)}}$ for $\alpha_{i_1} < \alpha_{i_2} \dots < \alpha_{i_n}$ with $(i_r = r = 1,2, \dots n)$ and for the sequence of disjoint sets $E^{(r)} = \left[E_{i_r} - \left(\bigcup_{p=1}^{r-1} E_{i_r}\right)\right] \cap E_{i_r}$ which are subset of E_i . Now by eq. (1) and Theorem 2.2.3(b) we have for $n \ge 1$,

$$\rho_{\psi}(ky_n) = \int_{\Omega}\psi(x,\ ky_n(x))dx$$
$$= \int_{\Omega}\psi\left(x, k\sum_{r=1}^{n}\alpha_{i_r}\chi_{E^{(r)}}(x)\right)dx$$
$$= \sum_{r=1}^{n}\int_{\Omega}\psi(x, k\alpha_{i_r}\chi_{E^{(r)}}(x))dx \qquad \text{(by Theorem 1.5.4)}$$
$$= \sum_{r=1}^{n}\int_{E^{(r)}}\psi(x, k\alpha_{i_r})dx$$
$$> \sum_{r=1}^{n}\int_{E^{(r)}}K\varphi\ (x, \alpha_{i_r})dx$$
$$= K\int_{E^{(r)}}\varphi\left(x, \sum_{r=1}^{n}\alpha_{i_r}\right)dx$$
$$= K\int_{\Omega}\varphi\left(x, \sum_{r=1}^{n}\alpha_{i_r}\chi_{E^{(r)}}(x)\right)dx$$
$$= K\int_{\Omega}\varphi(x,\ y_n(x))dx =\ K\rho_{\varphi}(y_n)$$

Thus we have $\rho_{\varphi}(y_n) < \varepsilon$ which implies that $\sup_n \rho_{\varphi}(y_n) < \varepsilon < \infty$. Since $0 \le\ y_n \uparrow_{n=1}^{\infty}$ and $\sup_n \rho_{\varphi}(y_n) < \infty$ then $y_E = \bigcup_{n=1}^{\infty} y_n \in \mathrm{L}^{\varphi}$ because ρ_{φ} is monotone complete ,and furthermore $\rho_{\varphi}(y_E) < \varepsilon$ then by Theorem 2.2.3 (a) we have $\rho_{\psi}(ky_E) < \gamma$. By our hypothesis $\mathrm{L}^{\varphi} \subseteq \mathrm{L}^{\psi}$ and since $y_E \in \mathrm{L}^{\varphi}$ then $y_E \in \mathrm{L}^{\psi}$.If we let $S_n = \{x: \psi(x, \alpha_n t) > K\varphi(x, \alpha_n)\}$ then for $n \ge 1$we get

$$\psi(x,k\alpha_n) \le \begin{cases} \psi\big(x,ky_E(x)\big) & for\ all\ x \in E_n \\ \\ K\varphi(x,\alpha_n) & for\ all\ x \in S_n^c \cap E \end{cases}$$

The sequence $\{y_E\}$ in which every y_E is depending on E with measure $\mu(E) < \infty$ therefore we can construct a sequence $0 \le\ y_E \uparrow_{\mu(E)<\infty}$. Since $\rho_\varphi(y_E) < \varepsilon$ then $\sup_E \rho_\varphi(y_E) < \infty$ then exist $y_0 = \bigcup_{\mu(E)<\infty} y_E \in \mathrm{L}^\varphi$ so that $y_0 \in \mathrm{L}^\psi$and $\rho_\psi(ky_0) < \gamma$ by the same reason state above . Thus for all positive $t \ge 0$ we have

$$\psi(x,kt) \le K\varphi(x,t) + \psi\big(x,ky_0(x)\big) \qquad a.e \text{ on } \Omega .$$

Since $\rho_\psi(ky_0) < \gamma$ then $\psi\big(x,ky_0(x)\big) \in \mathrm{L}^1(\Omega)$. The prove is done $if\ \ we\ put\ \ c(x) =\ \psi\big(x,ky_0(x)\big)$.

Chapter Three
(Weighted)Variable Exponent Lebesgue Spaces

In this chapter we define Lebesgue spaces with variable exponents, $L^{p(.)}$.They differ from classical L^p spaces in that the exponent p is not constant but a function from Ω to $[1,\infty]$.Also we define weighted variable exponent Lebesgue Spaces $L_{\omega}^{P(.)}(\Omega)$ and noneffective weights in Variable Lebesgue Spaces , $L^{p(.)}$. The spaces $L^{p(.)}$ fit into the framework of Musielak–Orlicz spaces L^{φ_*} and are therefore also semimodular spaces.

In Section 3.1 we study $L^{p(.)}$ properties. We have collection of properties that satisfied immediately from properties of L^{φ_*} .In section 3.2 we study noneffective weights in Variable Lebesgue Spaces .We have more results which are linking noneffective weights by constant weights almost everywhere on any subset Ω of $\mathbb{R}^n$. We will denote the set of all measurable functions from Ω to $\mathbb{R}$ by $L^0(\Omega)$.

Section3.1 variable exponent Lebesgue spaces

For the definition of the variable exponent Lebesgue spaces it is necessary to introduce the kind of variable exponents $p(.)$ that we are interested in. Let us also mention that main results on the basic properties on $L^{p(.)}$ in this section is satisfied immediately from section 2.2 ,section 2.4 .

Definition 3.1.1: Let $\Omega \subset \mathbb{R}^n$ and let (Ω, F, μ) be a σ-finite, complete measure space. We define $P(\Omega, \mu) := P(\Omega)$ to be the set of all measurable functions $p: \Omega \to [1,\infty]$. These functions $p \in P(\Omega)$ are called variable exponents on Ω. We define also ,for $E \subset \Omega$,that $p_+(E) = ess\ \sup_E p(E)$and $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$ .For brevity ,we denote $p_+ = p_+(\Omega)$ .We define also $P1(\Omega)$ to be the set of all measurable functions $p: \Omega \to [1,\infty)$.

In section 2.3 we mentioned the generalized Φ- function .Now we will give an example of it and an other example in section 3.2.

Example 3.1.1: Let $p \in P1(\Omega)$ and $f \in L^0(\Omega)$.Then $\varphi_{p(.)}(x, |f(x)|) = |f(x)|^{p(x)}$ is positive generalized Φ- function .

Solution : Since in example 2.3.1 $\varphi_{p(.)}(x, |f(x)|)$ is positive Φ- function of $|f(x)|$ for every $x \in \Omega$ and since $\varphi_{p(.)}(x, |f(x)|)$ is measurable function of x for all $|f(x)| \geq 0$ by Theorem 2.3.1 . We conclude that $\varphi_{p(.)}(x, |f(x)|) = |f(x)|^{p(x)}$ is positive generalized Φ- function .

We will give other example of modular function which defined it in section 2.2 .

Example 3.1.2:If $p \in P1(\Omega)$ then

$$\rho_{p(.)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx$$

is a modular function on $L^0(\Omega)$.

Solution: Since $\varphi_{p(.)}(x,|f(x)|) = |f(x)|^{p(x)} \in \Phi(\Omega), \varphi_{p(.)}$ is positive by Example 3.1.1 and $f \in \mathrm{L}^0(\Omega)$ then by Theorem 2.3.1

$$\rho_{p(.)}(f) = \int_{\Omega} \varphi_{p(.)}(x,|f(x)|)dx = \int_{\Omega} |f(x)|^{p(x)}dx$$

is a modular function on $\mathrm{L}^0(\Omega)$.

Definition 3.1.2:(Variable Exponent Lebesgue Space $\mathbf{L}^{p(.)}(\Omega)$)
Let Ω, F , μ) be a σ-finite, complete measure space and $p(.) \in \mathrm{P}(\Omega)$ Then we defined variable exponent Lebesgue Space $\mathrm{L}^{p(.)}(\Omega,\mu)$ by

$$\mathrm{L}^{p(.)}(\Omega,\mu) = \{f \in \mathrm{L}^0(\Omega) \ |\rho_{p(.)}(\lambda f) < \infty \, for\ some\ \lambda > 0\}$$

With modular function

$$\rho_{p(.)}(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{p(x)}dx + \|f\|_{\mathrm{L}^\infty(\Omega_\infty)}$$

To simplify , we write $\mathrm{L}^{p(.)}(\Omega)$, or $\mathrm{L}^{p(.)}$ when the measure space has been specified .

Note that if $\mu(\Omega_\infty) = 0$ then $\|f\|_{\mathrm{L}^\infty(\Omega_\infty)}$ =0 and so

$$\rho_{p(.)}(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{p(x)}dx$$

And if $\mu(\Omega/\Omega_\infty) = 0$ then by Theorem 1.4.1(4) we have

$$\int_{\Omega/\Omega_\infty} |f(x)|^{p(x)}dx = 0$$

So that

$$\rho_{p(.)}(f) = \|f\|_{\mathrm{L}^\infty(\Omega_\infty)}$$

Since $\mathrm{L}^{p(.)}(\Omega)$ is $\mathrm{L}^{\varphi_*}(\Omega)$ then $\mathrm{L}^{p(.)}(\Omega)$ can be equipped Luxemburg norm ,

$$\|f\|_{p(.)} = inf\left\{\lambda > 0\colon \ \rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

The next result follows from Theorem 2.2.2 and Corollary 2.2.1 . We will deal with variable exponent instead constant exponent in definition of modular function $\rho_{p(.)}(f)$.

Theorem 3.1.1[4]: If $p \in \mathrm{P}(\Omega)$ and $f \in \mathrm{L}^{p(.)}(\Omega)$ then

(a) $\|f\|_{p(.)} \leq 1\ and\ \rho_{p(.)}(f) \leq 1$ are equivalent .

(b) If $\rho_{p(.)}$ is continuous ,then $\|f\|_{p(.)} < 1$ and $\rho_{p(.)}(f) < 1$ are equivalent.

(c) If $\rho_{p(.)}$ is continuous, then $\|f\|_{p(.)} = 1$ and $\rho_{p(.)}(f) = 1$ are equivalent .

Proof: (a) If $\rho_{p(.)}(f) \leq 1$ then $\|f\|_{p(.)} \leq 1$ by definition of $\|.\|_{p(.)}$ .On

other hand if $\|f\|_{p(.)} \leq 1$ Indeed, let $\|f\|_{p(.)} \leq u \leq 1$ then

$\rho_{p(.)}\left(\frac{f}{u}\right) \leq 1$ thus

$$\rho_{p(.)}(f) = \rho_{p(.)}\left(u\frac{f}{u}\right) \leq u\, \rho_{p(.)}\left(\frac{f}{u}\right) \leq 1$$

Let $\rho_{p(.)}$ is continuous .If $\|f\|_{p(.)} < 1$ then there exist $\lambda < 1$ with

$\rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1$. It follows that by (2.2.4) $\rho_{p(.)}(f) = \rho_{p(.)}\left(\lambda\frac{f}{\lambda}\right) \leq \lambda\rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq \lambda < 1$ .On other hand let $\rho_{p(.)}(f) < 1$ and since $\rho_{p(.)}$ is continuous then $\rho_{p(.)}$ is right- continuous so that there exist $\gamma > 1$ such that

$$\rho_{p(.)}(\gamma f) = \rho_{p(.)}(f) < 1 \leq 1$$

Hence by (a) we have $\|\gamma f\|_{p(.)} \leq 1$ thus $\|f\|_{p(.)} \leq \frac{1}{\gamma} < 1.$

(c) The equivalence of $\|f\|_{p(.)} = 1$ and $\rho_{p(.)}(f) = 1$ follows immediately from (a)and(b) .

Corollary 3.1.1[4]: If $p \in \mathrm{P}(\Omega)$ and $f \in \mathrm{L}^{p(.)}(\Omega)$ then

(1) If $\|f\|_{p(.)} \leq 1$, then $\rho_{p(.)}(f) \leq \|f\|_{p(.)}$
(2) If $\|f\|_{p(.)} > 1$, then $\rho_{p(.)}(f) > \|f\|_{p(.)}$

Proof: (1) The claim is obvious for $f = 0$. Assume $0 < \|f\|_{p(.)} \leq 1$ then by (a) in Theorem 3.1.1 and since $\left\|\frac{f}{\|f\|_{p(.)}}\right\|_{p(.)} = 1 \leq 1$ we have

$\rho_{p(.)}\left(\frac{f}{\|f\|_{p(.)}}\right) \leq 1$,it follows that by (2.2.4)

$$\frac{1}{\|f\|_{p(.)}}\rho_{p(.)}(f) = \frac{1}{\|f\|_{p(.)}}\rho_{p(.)}\left(\|f\|_{p(.)}\frac{f}{\|f\|_{p(.)}}\right)$$

$$\leq \frac{\|f\|_{p(.)}}{\|f\|_{p(.)}}\rho_{p(.)}\left(\frac{f}{\|f\|_{p(.)}}\right) \leq 1$$

Hence

$$\rho_{p(.)}(f) \leq \|f\|_{p(.)}$$

(2) If $\|f\|_{p(.)} > 1$, then for $1 < \lambda < \|f\|_{p(.)}$ we get $1 < \rho_{p(.)}\left(\frac{f}{\lambda}\right)$. Since $\lambda < \|f\|_{p(.)}$ then $\|f\|_{p(.)} = \lambda + a$ where $a > 0$ .By (2,2,4) ,

$$1 < \rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq \frac{1}{\lambda}\,\rho_{p(.)}(f)$$

Thus $\lambda \leq \rho_{p(.)}(x)$ and since $a > 0$ was arbitrary then $\rho_{p(.)}(f) > \|f\|_{p(.)}$.

Let us give those properties of $L^{p(.)}(\Omega)$ derived directly by applying the results of Chap. 2 .From Theorem 2.4.1we derive

Theorem 3.1.2[4]: If $p \in P(\Omega)$ then $L^{p(.)}(\Omega)$ is a Banach space.

Proof: Let $p \in P(\Omega)$. Since $L^{p(.)}(\Omega)$ *is* $L^{\varphi_*}(\Omega)$ where $\varphi_*(f) = |f(x)|^{p(.)}$ or $\varphi_*(f) = \|f\|_{L^\infty(\Omega_\infty)}$,then by Theorem 2.4.1we have $L^{p(.)}(\Omega)$ is a Banach space .

As Lemma 2.4.3, In analogy with the properties for the integral, the next lemma will be called Fatou's lemma (for the modular), monotone convergence and dominated convergence, respectively.

Lemma 3.1.1[4]: Let $p \in P(\Omega)$ and $f_k, f, g \in L^0(\Omega)$ .Then

(a) If $f_k \to f$ $a.e\ \Omega$, then $\rho_{p(.)}(f) \leq \underline{\lim}\, \rho_{p(.)}(f_k)$.

(b) If $|f_k| \uparrow |f|$ $a.e\ \Omega$, then $\rho_{p(.)}(f) = \lim_{k\to\infty} \rho_{p(.)}(f_k)$.

(c) If $f_k \to f$ $a.e$ on Ω , $|f_k| \leq |g|$ $a.e\ \Omega$, and $\rho_{p(.)}(\lambda g) < \infty$ for every $\lambda > 0$ then $f_k \to f$ in $L^{p(.)}$.

Proof: By the same reason in Theorem 3.1.2 and by Lemma 2.4.3 then the proof will be done immediately .

Section 3.2 Noneffective Weights in Variable Lebesgue Spaces

We define in section 2.1 , $L^p_\omega(\Omega)$ where $1 \leq p < \infty$ is constant and $\omega(x)$ is weighted function . Also we define in section 3.1, $L^{p(.)}(\Omega)$ where $p \in P(\Omega)$. Below we will define $L^{p(.)}_\omega(\Omega)$ space where $p \in P(\Omega)$ we also see that this space is Banach space .

Our aim in this section is to study noneffective weights in the framework of variable exponent Lebesgue spaces, This means what are the necessary and sufficient conditions that needed to get the variable exponent Lebesgue spaces is equal to weighted variable exponent Lebesgue space (i. e $L^{p(.)} = L^{p(.)}_\omega(\Omega)$) up to the equivalence of norms.

As is section 3.1 we give an other example of generalized Φ- function as we defined in section 2.3.

Example 3.2.1: Let $p \in P1(\Omega)$, $\omega(x)$is weight function and $f \in L^0(\Omega)$. Then $\varphi_{\omega,p(.)}(x, |f(x)|) = |f(x)|^{p(x)}\omega(x)$ is positive generalized Φ- function.

Solution : Since in example 2.3.2 $\varphi_{\omega,p(.)}(x, |f(x)|)$ is positive Φ- function of $|f(x)|$ for every $x \in \Omega$ and since $\varphi_{\omega,p(.)}(x, |f(x)|)$ is measurable function of x for all $|f(x)| \geq 0$

by Theorem 2.3.1 . We conclude that $\varphi_{\omega,p(.)}(x,|f(x)|) = |f(x)|^{p(x)}\omega(x)$ is positive generalized Φ- function .

We will give an other example of modular function which we defined in section 2.2 .

Example 3.2.2: If $p \in P1(\Omega)$ and $\omega(x)$is weight function then

$$\rho_{\omega,p(.)}(f) = \int_{\Omega} |f(x)|^{p(x)}\omega(x)dx$$

Is a modular function on $L^0(\Omega)$.

Solution: Since $\varphi_{\omega,p(.)}(x,|f(x)|) = |f(x)|^{p(x)}\omega(x) \in \Phi(\Omega), \varphi_{p(.)}$ is postive by Example 3.2.1 and $f \in L^0(\Omega)$ then by Theorem 2.3.1

$$\rho_{\omega,p(.)}(f) = \int_{\Omega} \varphi_{\omega,p(.)}(x,|f(x)|)dx = \int_{\Omega} |f(x)|^{p(x)}\omega(x)dx$$

is a modular function on $L^0(\Omega)$.

Definition 3.2.1 : (weighted variable Lebesgue space ; $L_{\omega}^{p(.)}(\Omega)$)

Let (Ω, F, μ) be a σ-finite, complete measure space , $p(.) \in P(\Omega)$ and $\omega(x)$ is weight function .Then we define weighted variable exponent Lebesgue Space $L_{\omega}^{p(.)}(\Omega,\mu)$ by

$$L_{\omega}^{p(.)}(\Omega,\mu) = \left\{f \in L^0(\Omega) \ \middle|\rho_{\omega,p(.)}(\lambda f) < \infty \ for\ some\ \lambda > 0\right\}$$

With modular function

$$\rho_{\omega,p(.)}(f) = \int_{\Omega/\Omega_{\infty}} |f(x)|^{p(x)}\omega(x)dx + \|f\omega\|_{L^{\infty}(\Omega_{\infty})}$$

To simplify , we write $L_{\omega}^{p(.)}(\Omega)$, or $L_{\omega}^{p(.)}$ when the measure space has been specified .
Note that if $\mu(\Omega_{\infty}) = 0$ then $\|f\omega\|_{L^{\infty}(\Omega_{\infty})} = 0$ and so

$$\rho_{\omega,p(.)}(f) = \int_{\Omega/\Omega_{\infty}} |f(x)|^{p(x)}\omega(x)dx$$

And if $\mu(\Omega/\Omega_{\infty}) = 0$ then by Theorem 1.4.1(4) we have

$$\int_{\Omega/\Omega_{\infty}} |f(x)|^{p(x)}\omega(x)dx = 0$$

so that

$$\rho_{\omega,p(.)}(f) = \|f\omega\|_{L^{\infty}(\Omega_{\infty})}$$

Since $L_{\omega}^{p(.)}(\Omega)$ is $L^{\varphi}(\Omega)$ where $\varphi \in \Phi(\Omega)$ then $L_{\omega}^{p(.)}(\Omega)$ can be equipped Luxemburg norm ,

$$\|f\|_{\omega,p(.)} = inf\left\{\lambda > 0:\ \rho_{\omega,p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

We can see that if $\omega(x) = 1$ then $\mathrm{L}_{\omega}^{p(.)}(\Omega) = \mathrm{L}^{p(.)}(\Omega)$ and so in this case we say that $\omega(x)$ is noneffective weight variable exponent Lebesgue space . We will give general case of $\omega(x)$ to be noneffective at the end of this section .

We see that in Example 3.2.1 that if $p \in \mathrm{P1}(\Omega)$, $\omega(x)$ is weight function and $f \in \mathrm{L}^0(\Omega)$ then $\varphi_{\omega,p(.)}(x, |f(x)|) = |f(x)|^{p(.)}\omega(x)$ is positive generalized Φ- function.We begin with following theorem, which is a consequence of results in this example and in Theorem 2.4.2 .

Theorem 3.2.1[1]: Let $\Omega \subset \mathbb{R}^n$, $p_1(.)$, $p_2(.) \in \mathrm{P}(\Omega)$ where $(p_1)_+ < \infty$, $(p_2)_+ < \infty$ and $\omega_1(.), \omega_2(.)$ are weights functions . Then

$$\mathrm{L}_{\omega_1(.)}^{p_1(.)}(\Omega) \subset \mathrm{L}_{\omega_2(.)}^{p_2(.)}(\Omega)$$

if and only if there exist positive constants K_1, K_2 and $c(x) \in \mathrm{L}^1(\Omega)$ such that

$$t^{p_2(x)}\omega_2(x) \leq K_1(K_2 t)^{p_1(x)}\omega_1(x) + c(x)$$

For all $t > 0$ and $a.e$ on Ω .

Proof: If $p_1(.), p_2(.) \in \mathrm{P}(\Omega)$ where $(p_1)_+ < \infty$, $(p_2)_+ < \infty$ then

$p_1(.) < \infty$, $p_2(.) < \infty$ so that $p_1(.), p_2(.) \in \mathrm{P1}(\Omega)$ thus

$$\varphi_{\omega_1,p_1(.)}(x, |f(x)|) = |f(x)|^{p_1(x)}\omega_1(x)$$

And

$$\varphi_{\omega_2,p_2(.)}(x, |f(x)|) = |f(x)|^{p_2(x)}\omega_2(x)$$

are positive generalized Φ- function . Since $\mathrm{L}_{\omega}^{p(.)}(\Omega)$ is $\mathrm{L}^{\varphi}(\Omega)$ where $\varphi \in \Phi(\Omega)$ and by Theorem 2.4.2 we have $\mathrm{L}_{\omega_1(.)}^{p_1(.)}(\Omega) \subset \mathrm{L}_{\omega_2(.)}^{p_2(.)}(\Omega)$ if and only if there exist $k, K_1 > 0$ and $c(x) \in \mathrm{L}^1(\Omega)$ such that

$$\varphi_{\omega_2,p_2(.)}(x, kt) \leq K_1\varphi_{\omega_1,p_1(.)}(x, t) + c(x) \qquad (1)$$

For all $t = |f(x)| > 0$ and $a.e$ on Ω . Since $t \leq K_2 t$ and $t \leq kt$ for $K_2, k > 0$ and φ is non decreasing we have

$$\varphi_{\omega_2,p_2(.)}(x, t) \leq \varphi_{\omega_2,p_2(.)}(x, kt) \qquad (2)$$

And

$$\varphi_{\omega_1,p_1(.)}(x, t) \leq \varphi_{\omega_1,p_1(.)}(x, K_2 t) \qquad (3)$$

Thus by (1),(2),(3) we get

$$\varphi_{\omega_2,p_2(.)}(x, t) \leq K_1\varphi_{\omega_1,p_1(.)}(x, K_2 t) + c(x)$$

But

$$\varphi_{\omega_2,p_2(.)}(x,t) = t^{p_2(x)}\omega_2(x)$$

And

$$\varphi_{\omega_1,p_1(.)}(x,K_2t) = (K_2t)^{p_1(x)}\omega_1(x)$$

Which imply,

$L^{p_1(.)}_{\omega_1(.)}(\Omega) \subset L^{p_2(.)}_{\omega_2(.)}(\Omega)$ if and only if there exist positive constants K_1, K_2 and $c(x) \in L^1(\Omega)$ such that

$$t^{p_2(x)}\omega_2(x) \le K_1(K_2t)^{p_1(x)}\omega_1(x) + c(x)$$

For all $t > 0$ and $a.e$ on Ω .

In the following theorem we consider the case $p_+ < \infty$ that the noneffective weights must hold.

Theorem 3.2.2[1]: Let $\Omega \subset \mathbb{R}^n$, $p(.) \in P(\Omega)$,where$(p_+) < \infty$ and $\omega(.)$ is weight function .Then $L^{p(.)}_{\omega}(\Omega) = L^{p(.)}(\Omega)$ if and only if ω is constant $a.e$ on Ω .

Proof : Let ω is constant $a.e$ on Ω .Want to show that $L^{p(.)}_{\omega}(\Omega) = L^{p(.)}(\Omega)$. Since ω is constant $a.e$ on Ω then $\exists$ a constant $c > 0$ such that $\omega(x) = c$ $a.e$ on Ω. Since $(p_+) < \infty$ then $p < \infty$ so $x \in \Omega/\Omega_\infty$. If $f \in L^{P(.)}_{\omega}(\Omega)$ and $\alpha = \frac{1}{c}$ then

$$\rho_{\omega,p(.)}(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{p(x)}\omega(x)dx$$

Therefore

$$\|f\|_{\omega,p(.)} = inf\left\{\lambda > 0\colon \rho_{\omega,p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

$$= inf\left\{\lambda > 0\colon \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)\right|^{p(x)} \omega(x)\, dx \le 1\right\}$$

$$= inf\left\{\lambda > 0\colon \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)\right|^{p(x)} c\ dx \le 1\right\}$$

$$= \frac{1}{c} inf\left\{\lambda > 0\colon \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)\right|^{p(x)} dx \le 1\right\}$$

$$= \alpha\ inf\left\{\lambda > 0\colon \rho_{p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\} = \alpha\|f\|_{p(.)}$$

And so that $L^{P(.)}_{\omega}(\Omega) = L^{p(.)}(\Omega)$. Conversely , assume $L^{p(.)}_{\omega}(\Omega) = L^{p(.)}(\Omega)$ then $L^{p(.)}_{\omega}(\Omega) \subset L^{p(.)}(\Omega)$ and $L^{p(.)}(\Omega) \subset L^{p(.)}_{\omega}(\Omega)$,want to show that $\omega(x) = c$ $a.e$ on Ω .If

$L^{p(.)}(\Omega) \subset L_{\omega}^{p(.)}(\Omega)$ then by Theorem 3.2.1 let $p_1(.) = p_2(.) = p(.) \in P(\Omega)$ where $p(.) < \infty$, $\omega_1(.) = 1$ and $\omega_2(.) = \omega(.)$. Thus $L^{p(.)}(\Omega) \subset L_{\omega}^{p(.)}(\Omega)$ if and only if there exist positive constants $c_2, K_2 = 1$ and $h_1(x) \in L^1(\Omega)$ such that

$$t^{p(x)}\omega(x) \leq c_2 t^{p(x)} + h_1(x) \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega$$

So that ,

$$\frac{t^{p(x)}}{t^{p(x)}}\omega(x) \leq c_2 \frac{t^{p(x)}}{t^{p(x)}} + \frac{h_1(x)}{t^{p(x)}} \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega$$

Hence,

$$\omega(x) \leq c_2 + \frac{h_1(x)}{t^{p(x)}} \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega$$

From which, letting $t \to \infty$ we have $\frac{h_1(x)}{t^{p(x)}} \to 0$, therefore

$$\omega(x) \leq c_2 \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega \qquad (1)$$

On other hand , if $L_{\omega}^{p(.)}(\Omega) \subset L^{p(.)}(\Omega)$ then also by Theorem 3.2.1 let $p_1(.) = p_2(.) = p(.) \in P(\Omega)$ where $p(.) < \infty$, $\omega_2(.) = 1$ and $\omega_1(.) = \omega(.)$.Thus $L_{\omega}^{p(.)}(\Omega) \subset L^{p(.)}(\Omega)$ if and only if there exist positive constants $K_1, K_2 = 1$ and $h_2(x) \in L^1(\Omega)$ such that

$$t^{p(x)} \leq K_1 t^{p(x)}\omega(x) + h_2(x) \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega$$

So that ,

$$\frac{t^{p(x)}}{t^{p(x)}} \leq K_1 \frac{t^{p(x)}}{t^{p(x)}}\,\omega(x) + \frac{h_2(x)}{t^{p(x)}} \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega$$

Hence ,

$$1 \leq K_1\omega(x) + \frac{h_2(x)}{t^{p(x)}} \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega$$

From which, letting $t \to \infty$ we have $\frac{h_2(x)}{t^{p(x)}} \to 0$ .Choose $c_1 = \frac{1}{K_1}$,therefore

$$c_1 \leq \omega(x) \qquad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega \qquad (2)$$

Thus by (1) and (2) we have $\omega(x)$ is constant $a.e$ on Ω .

Example 3.2.3: Let $\Omega = (1,2) \subset \mathbb{R}$, $p(x) = x^2$, $\omega(x) = 2$ a.e on

Ω .Then show that $\omega(x)$ is noneffective that means :

$$L_{\omega}^{p(.)}(1,2) = L^{p(.)}(1,2)$$

Solution: Since $p(.) = x^2$ is bounded on $(1,2)$ then $p_+ < \infty$ and $\omega(x)$ is constant a.e on Ω .Thus by Theorem 3.2.2 we conclude that

$$L_{\omega}^{p(.)}(1,2) = L^{p(.)}(1,2)$$

Theorem 3.2.2 is a special case of the following more general result when we consider the case $p_+(\Omega/\Omega_\infty) < \infty$ the noneffective weights must hold.

Theorem 3.2.3[1]: Let $\Omega \subset \mathbb{R}^n$, $p(.) \in \mathrm{P}(\Omega)$ where $p_+(\Omega/\Omega_\infty) < \infty$ and $\omega(.)$ is weight function .Then $L_{\omega}^{p(.)}(\Omega) = L^{p(.)}(\Omega)$ if and only if ω is constant $a.e$ on Ω.

Proof: If $p(.) \in \mathrm{P}(\Omega)$, where $p_+(\Omega/\Omega_\infty) < \infty$ then there exist $M > 0$ such that

$$|p(x)| \leq M < \infty \quad a.e \text{ on } \Omega/\Omega_\infty$$

but $\Omega/\Omega_\infty = \{x \in \Omega: p(x) < \infty\}$

Then

$$|p(x)| \leq M < \infty \quad a.e \text{ on } \Omega$$

So that $p_+(\Omega) < \infty$ thus by Theorem 3.2.2 we have $L_{\omega}^{p(.)}(\Omega) = L^{p(.)}(\Omega)$ if and only if ω is constant $a.e$ on Ω .

We consider now the case if $p = \infty$,as in the previous case that the noneffective weights must hold .

First we claim that if ω is not constant $a.e$ on Ω then $\omega \notin L^{\infty}(\Omega)$ or $\frac{1}{\omega} \notin L^{\infty}(\omega)$.

For If $\omega \neq c$ $a.e$ on Ω where c is constant . Then either $\mu\{x: \omega(x) > c\} > 0$ or $\mu\left\{x: \frac{1}{\omega(x)} > c\right\} > 0$ .Assume that $\mu\{x: \omega(x) > c\} > 0$ and $\omega \in L^{\infty}(\Omega)$. If $\omega \in L^{\infty}(\Omega)$ then ω is essentially bounded ,so there exist $0 < M < \infty$ such that

$$|\omega(x)| \leq M \quad a.e \text{ on } \Omega$$

And

$$\|\omega\|_{L^{\infty}(\Omega)} = ess\ \sup_{\Omega}|\omega(x)| = inf\ \{M: |\omega(x)| \leq M \quad a.e\}$$

Therefore $\omega(x) \leq \|\omega\|_{L^{\infty}(\Omega)}$ $a.e$ on Ω which is a contradiction to $\mu\{x: \omega(x) > c\} > 0$. Hence if $\mu\{x: \omega(x) > c\} > 0$ then $\omega \notin L^{\infty}(\Omega)$.Also if $\mu\left\{x: \frac{1}{\omega(x)} > c\right\} > 0$ and $\frac{1}{\omega} \in L^{\infty}(\Omega)$ thus there exist $0 < M1 < \infty$ such that

$$\left|\frac{1}{\omega(x)}\right| \leq M1 \quad a.e\ on\ \Omega$$

And

$$\left\|\frac{1}{\omega}\right\|_{L^{\infty}(\Omega)} = ess\ \sup_{\Omega}\left|\frac{1}{\omega(x)}\right| = inf\ \left\{M: \left|\frac{1}{\omega(x)}\right| \leq M \quad a.e\right\}$$

Therefore $\frac{1}{\omega}(x) \leq \left\|\frac{1}{\omega}\right\|_{\mathrm{L}^{\infty}(\Omega)}$ $a.e$ on Ω which is a contradiction to $\mu\left\{x: \frac{1}{\omega(x)} > c\right\} > 0$ .Thus if $\mu\left\{x: \frac{1}{\omega(x)} > c\right\} > 0$ then $\frac{1}{\omega} \notin \mathrm{L}^{\infty}(\Omega)$.

Theorem 3.2.4 [1]: Let $\Omega \subset \mathbb{R}^n$ and $\omega(.)$ is weight function .Then $\mathrm{L}^{\infty}_{\omega}(\Omega) = \mathrm{L}^{\infty}(\Omega)$ if and only if ω is constant $a.e$ on Ω .

Proof: Let $\omega = c$ $a.e$ on Ω , $\alpha = \frac{M}{c}$ and $f \in \mathrm{L}^{\infty}_{\omega}(\Omega)$ then

$$\rho_{\infty,p(.)}(f) = \|f\omega\|_{\mathrm{L}^{\infty}(\Omega_{\infty})}$$

Therefore

$$\begin{aligned}
\|f\|_{\omega,\infty} &= inf\left\{\lambda > 0: \left\|\frac{f}{\lambda}\omega\right\|_{\mathrm{L}^{\infty}(\Omega_{\infty})} \leq 1\right\} \\
&= inf\left\{\lambda > 0: \frac{1}{\lambda}\|f\omega\|_{\mathrm{L}^{\infty}(\Omega_{\infty})} \leq 1\right\} \\
&= inf\{\lambda > 0: \|f\omega\|_{\mathrm{L}^{\infty}(\Omega_{\infty})} \leq \lambda\} \\
&= \|f\omega\|_{\infty} \\
&= ess\ \sup_{\Omega_{\infty}}|f\omega(x)| \\
&= \inf\{M: |f\omega(x)| \leq M \quad a.e\} \\
&= inf\ \{M: |fc| \leq M \quad a.e\} \\
&= inf\ \left\{M: |f| \leq \frac{M}{c} \quad a.e\right\} \\
&= inf\ \{c\alpha: |f| \leq \alpha \quad a.e\} \\
&= c\ inf\ \{\alpha: |f| \leq \alpha \quad a.e\} = c\,\|f\|_{\infty}
\end{aligned}$$

Hence $\mathrm{L}^{\infty}_{\omega}(\Omega) = \mathrm{L}^{\infty}(\Omega)$ .On other hand let $\mathrm{L}^{\infty}_{\omega}(\Omega) = \mathrm{L}^{\infty}(\Omega)$ then want to show that ω is constant $a.e$ on Ω . By contradiction , assume $\mathrm{L}^{\infty}_{\omega}(\Omega) = \mathrm{L}^{\infty}(\Omega)$ and ω is not constant $a.e$ on Ω then either $\omega \notin \mathrm{L}^{\infty}(\Omega)$ or $\frac{1}{\omega} \notin \mathrm{L}^{\infty}(\Omega)$. If $\omega \notin \mathrm{L}^{\infty}(\Omega)$ then set

$$E_n = \{x \in \Omega: \omega(x) > n\} \ , n \in \mathbb{N}$$

So that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ .Now since $\mathrm{L}^{\infty}_{\omega}(\Omega) = \mathrm{L}^{\infty}(\Omega)$ then there exist $c_1, c_2 > 0$ such that

$$c_1\|f\|_{\infty} \leq \|f\omega\|_{\infty} \leq c_2\|f\|_{\infty}$$

Taking the supremum in the case $\|f\omega\|_{\infty} \leq c_2\|f\|_{\infty}$ with $f = \chi_{E_n}$ we have

$$\sup_{\|f\|_{\infty}=1}\|f\omega\|_{\infty} \leq \sup_{\|f\|_{\infty}=1} c_2\|f\|_{\infty}$$

Thus

$$\|\omega\|_{L^{\infty}(E_n)} \leq c_2$$

But $f = \chi_{E_n}$ so that

$$n < \|\omega\|_{L^{\infty}(E_n)} \leq c_2$$

for all $n \in \mathbb{N}$ which is a contradiction to $\omega \notin L^{\infty}(\Omega)$ .In the second case If $\frac{1}{\omega} \notin L^{\infty}(\Omega)$ then set

$$E_n = \{x \in \Omega: \omega(x) < n\} \ , n \in \mathbb{N}$$

So that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ .Now since $L^{\infty}_{\omega}(\Omega) = L^{\infty}(\Omega)$ then there exist $c_1, c_2 > 0$ such that

$$c_1\|f\|_{\infty} \leq \|f\omega\|_{\infty} \leq c_2\|f\|_{\infty}$$

Taking the supremum in the case $c_1\|f\|_{\infty} \leq \|f\omega\|_{\infty}$ with $f = \chi_{E_n}$ we have

$$\sup_{\|f\|_{\infty}=1} c_1\|f\|_{\infty} \leq \sup_{\|f\|_{\infty}=1} \|f\omega\|_{\infty}$$

Thus

$$c_1 \leq \|\omega\|_{L^{\infty}(E_n)}$$

But $f = \chi_{E_n}$ so that

$$c_1 \leq \|\omega\|_{L^{\infty}(E_n)} < n$$

for all $n \in \mathbb{N}$ which is a contradiction to $\frac{1}{\omega} \notin L^{\infty}(\Omega)$ .So that ω is constant $a.e$ on Ω .

Example 3.2.4: Let $\Omega = (1,2) \subset \mathbb{R}$, $p(x) = \infty$, $\omega(x) = 2$ a.e on Ω. Then show that $\omega(x)$ is noneffective weight that means :

$$L^{\infty}_{\omega}(1,2) = L^{\infty}(1,2)$$

Solution: Since $\omega(x)$ is constant a.e on Ω . Thus by Theorem 3.2.4 we conclude that

$$L^{\infty}_{\omega}(1,2) = L^{\infty}(1,2)$$

Also this can be checked directly ,

$$\rho_{\infty,p(.)}(f) = \|f\omega\|_{L^{\infty}(\Omega_{\infty})}$$

Let $\alpha = \frac{M}{c}$,we have

$$\begin{aligned} \|f\|_{\omega,\infty} &= inf\left\{\lambda > 0: \ \left\|\frac{f}{\lambda}2\right\|_{L^{\infty}(\Omega_{\infty})} \leq 1\right\} \\ &= \|2f\|_{\infty} \end{aligned}$$

$$= ess \sup_{\Omega_\infty} |2f|$$

$$= inf\ \{M: |2f| \le M \quad a.e\}$$

$$= inf\ \left\{M: |f| \le \frac{M}{2} \quad a.e\right\}$$

$$= inf\ \{2\alpha: |f| \le \alpha \quad a.e\}$$

$$= 2\ inf\ \{\alpha: |f| \le \alpha \quad a.e\} = 2\ \|f\|_\infty$$

Thus $L^\infty_\omega(1,2) = L^\infty(1,2)$ and so $\omega(x)$ is noneffective weight .

The following result is our main theorem that give us the conditions where noneffective weights have been introduced. Theorem 3.2.2 and Theorem 3.2.4 are consequences of our general theorem ,where in Ω/Ω_∞, which possibly empty, an unbounded exponent is allowed .

Theorem 3.2.5[1]: Let $\Omega \subset \mathbb{R}^n$, $p(.) \in P(\Omega)$,and $\omega(.)$ is weight function. Then $L^{P(.)}_\omega(\Omega) = L^{p(.)}(\Omega)$ if and only if

(1) $\omega(x)^{\frac{1}{p(x)}}$ is constant $a.e$ on Ω/Ω_∞ .

(2) $\omega(x)$ is constant $a.e$ on Ω_∞ .

Proof : We have four cases .The two cases $\mu(\Omega_\infty) = 0$ and $\mu(\Omega/\Omega_\infty) > 0$ the same result and the two cases $\mu(\Omega_\infty) > 0$ and $\mu(\Omega/\Omega_\infty) = 0$ the same result .

If $\mu(\Omega/\Omega_\infty) = 0$ then $x \in \Omega_\infty$ so that $p(.) = \infty$ and since $\omega(x)$ is weight function thus by Theorem 3.2.4 we have $L^\infty_\omega(\Omega) = L^\infty(\Omega)$ if and only if $\omega(x)$ is constant $a.e$ on Ω_∞. So want to show only if $p(.) < \infty$ then $L^{p(.)}_\omega(\Omega) = L^{p(.)}(\Omega)$ if and only if $\omega(x)^{\frac{1}{p(x)}}$ is constant $a.e$ on Ω/Ω_∞ .

If $\mu(\Omega_\infty) = 0$ and $L^{p(.)}_\omega(\Omega) = L^{p(.)}(\Omega)$ then $x \in \Omega/\Omega_\infty$ and $p(.) < \infty$ .Since

$L^{P(.)}_\omega(\Omega) \subset L^{p(.)}(\Omega)$ so that $L^{P(.)}_\omega(\Omega/\Omega_\infty) \subset L^{P(.)}(\Omega/\Omega_\infty)$ and therefore by applying Theorem 3.2.1 ,there exist positive constants K_1 , K_2, and

$h_1 \in L^1(\Omega/\Omega_\infty)$,such that

$$t^{p(x)} \le K_1 (K_2 t)^{p(x)} \omega(x) + h_1(x)$$

For all $t > 0$ and $a.e$ on Ω/Ω_∞ . So that

$$1 \le \frac{t^{p(x)} {K_2}^{p(x)} \omega(x)}{t^{p(x)}} K_1 + \frac{h_1(x)}{t^{p(x)}} \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

Thus ,

$$1 \leq K_1 K_2{}^{p(x)} \omega(x) + \frac{h_1(x)}{t^{p(x)}} \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

From which, letting $t \to \infty$ we have $\frac{h_1(x)}{t^{P(x)}} \to 0$ .Hence $1 \leq K_1 K_2{}^{p(x)} \omega(x)$ So

$$K_1{}^{-1} K_2{}^{-p(x)} \leq \omega(x) \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

We get the existence of constant $K_0 = K_1 = K_2$ .Then

$$K_0{}^{-(1+p(x))} \leq \omega(x) \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty \quad (1)$$

Starting from the opposite way, that is if $L^{p(.)}(\Omega) \subset L_\omega^{P(.)}(\Omega)$ thus also by Theorem 3.2.1, there exist positive constants K_3 , K_4, and $h_2 \in L^1(\Omega/\Omega_\infty)$ such that

$$t^{p(x)} \omega(x) \leq K_3 (K_4 t)^{p(x)} + h_2(x) \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

For all $t > 0$ and $a.e$ on Ω/Ω_∞ . So that

$$\omega(x) \leq \frac{t^{p(x)} K_4{}^{p(x)}}{t^{p(x)}} K_3 + \frac{h_2(x)}{t^{p(x)}} \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

Thus ,

$$\omega(x) \leq K_3 K_4{}^{p(x)} + \frac{h_2(x)}{t^{p(x)}} \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

From which, letting $t \to \infty$ we have $\frac{h_2(x)}{t^{P(x)}} \to 0$ .Hence

$$\omega(x) \leq K_3 K_4{}^{p(x)} \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty$$

We get the existence of constant $K_5 = K_3 = K_4$ .Then

$$\omega(x) \leq K_5{}^{1+p(x)} \quad \text{For all } t > 0 \text{ and } a.e \text{ on } \Omega/\Omega_\infty \quad (2)$$

By (1) and (2) we have ,

$$\omega(x)^{\frac{1}{p(x)+1}} \text{is constant } a.e \text{ on } \Omega/\Omega_\infty$$

But this equivalent to

$$\omega(x)^{\frac{1}{p(x)}} \text{ is constant } a.e \text{ on } \Omega/\Omega_\infty$$

That because,

$$\frac{1+p(x)}{p(x)} \text{ is constant} \qquad \text{For } a.e \text{ on } \Omega/\Omega_\infty$$

Conversely , assume that $\omega(x)^{\frac{1}{p(x)}}$ is constant $a.e$ on Ω/Ω_∞ want to show that if $p(x) < \infty$ then $L_\omega^{p(.)}(\Omega) = L^{p(.)}(\Omega)$. Since $p(x) < \infty$ then $x \in \Omega/\Omega_\infty$. Let $s = \frac{\lambda}{c}$. If $f \in L_\omega^{p(.)}(\Omega)$ then

$$\rho_{\omega,p(.)}(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{p(x)}\omega(x)dx$$

Therefore

$$\|f\|_{\omega,p(.)} = inf\left\{\lambda > 0:\ \rho_{\omega,p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

$$= inf\left\{\lambda > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)\right|^{p(x)} \omega(x)\, dx \le 1\right\}$$

$$= inf\left\{\lambda > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)[\omega(x)]^{\frac{1}{p(x)}}\right|^{p(x)} dx \le 1\right\}$$

$$= inf\left\{\lambda > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)c\right|^{p(x)} dx \le 1\right\}$$

$$= inf\left\{sc > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{s}(x)\right|^{p(x)} dx \le 1\right\}$$

$$= c\ \ inf\left\{s > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{s}(x)\right|^{p(x)} dx \le 1\right\}$$

$$= c\ \ inf\left\{s > 0:\ \rho_{p(.)}\left(\frac{f}{s}\right) \le 1\right\} = c\|f\|_{p(.)}$$

And so that $L_\omega^{p(.)}(\Omega) = L^{p(.)}(\Omega)$.

Example 3.2.5: Let $\Omega = (0,1) \subset \mathbb{R}$, $p(x) = \frac{1}{x}$, $\omega(x) = 2^{\frac{1}{x}}$ a.e on Ω .Then show that $\omega(x)$ is noneffective weight that means :

$$L_\omega^{p(.)}(0,1) = L^{p(.)}(0,1)$$

Solution: Since $p(x) = \frac{1}{x}$ then $p(x) < \infty\ \forall x \in (0,1)$,so that $x \in \Omega/\Omega_\infty$ .And since $\omega(x)^{\frac{1}{p(.)}} = 2$ then $\omega(x)^{\frac{1}{p(.)}}$ is constant a.e on Ω/Ω_∞ . Then by Theorem 3.2.5 we have

$$L_\omega^{p(.)}(0,1) = L^{p(.)}(0,1)$$

Also this can be checked directly,

Let $s = \frac{\lambda}{2}$. If $f \in L_\omega^{p(.)}(0,1)$ then $\rho_{\omega,p(.)}(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{\frac{1}{x}} 2^{\frac{1}{x}} dx$

Therefore

$$\|f\|_{\omega,p(.)} = inf\left\{\lambda > 0:\ \rho_{\omega,p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

$$= inf\left\{\lambda > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)\right|^{\frac{1}{x}} 2^{\frac{1}{x}}\, dx \le 1\right\}$$

$$= inf\left\{\lambda > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{\lambda}(x)2\right|^{\frac{1}{x}} dx \le 1\right\}$$

$$= inf\left\{2s > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{s}(x)\right|^{\frac{1}{x}} dx \le 1\right\}$$

$$= 2\ inf\left\{s > 0:\ \int_{\Omega/\Omega_\infty} \left|\frac{f}{s}(x)\right|^{\frac{1}{x}} dx \le 1\right\}$$

$$= 2\ inf\left\{s > 0:\ \rho_{p(.)}\left(\frac{f}{s}\right) \le 1\right\} = 2\|f\|_{p(.)}$$

Thus $L_\omega^{p(.)}(0,1) = L^{p(.)}(0,1)$.

Conclusion

We have studied in this thesis the background of the concept of noneffective Weights in Variable Lebesgue Space, $L_{\omega}^{p(.)}$, by proving two theorems using more general theorem.

In the beginning we have introduced the concept of measure space and Lebesgue integral and given the most important theorems on convergence of integrals of sequence of real valued functions. After that we described the modular functions that are used in some spaces like modular spaces and other spaces definition called Orlicz spaces, L^{φ}. The second concept L^{φ} depends on finite modular function $\rho_{\varphi}(\lambda f)$ for some real positive λ. We use this definition in definitions of $L^{p(.)}$ and $L_{\omega}^{p(.)}$.

Finally, we presented the main theorem which states how the weight function can be noneffective on Variable Lebesgue Space.

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