OSCILLATION AND NON-OSCILLATION OF ARBITRARY ORDER DELAY DIFFERENTIAL EQUATIONS

By

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Declaration:

I certify that this thesis submitted for the degree of master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Dedication

To my husband, my children and my sweet Yusra,

To my parents.

Acknowledgment

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Abstract

We were interested with differential equations with deviating arguments, especially, delay differential equations of arbitrary order. Our discussion is about the most important issue which is the oscillatory and non-oscillatory behavior of their solutions.

The equation $x^{(n)}(t)+p(t)f(x(g(t)))=h(t)$ was the main equation that discussed mainly throughout this thesis for different values of n.

For $n=1 \Rightarrow x'(t)+p(t)f(x(g(t)))=h(t)$ discussed in chapter two.

For $n=2 \Rightarrow (r(t)x'(t))'+p(t)f(x(g(t)))=h(t)$ discussed in chapter three.

For $n=3 \Rightarrow (b(t)(a(t)x'(t))')'+p(t)f(x(g(t)))=h(t)$ and the nth order $x^{(n)}+p(t)x(g(t))=h(t)$ discussed in chapter four.

ملخص بالعربية

لقد تم الاهتمام في هذه الأطروحة بدراسة خاصية التذبذب وعدم التذبذب لحلول بعض المعادلات التفاضلية الاقترانية و التعرف على بعض الشروط الواجب توفرها للحصول على خاصية التذبذب.

و الشكل العام للمعادلات تحت البحث كانت على الصورة التالية

$$x^{(n)}(t) + p(t)f(x(g(t))) = h(t)$$

وذلك لقيم مختلفة من n

$$n = 1 \Rightarrow x'(t) + p(t)f(x(g(t))) = h(t)$$

$$n = 2 \Rightarrow (r(t)x'(t))' + p(t)f(x(g(t))) = h(t)$$

$$n = 3 \Rightarrow (b(t)(a(t)x'(t))')' + p(t)f(x(g(t))) = h(t)$$
$$nth - order \Rightarrow x^{(n)}(t) + p(t)x(g(t)) = h(t).$$

Contents

Introduction

In many applications, the future state of the system is independent of the past states and is determined solely by the present. But in the late thirties and early forties, Minorsky [4], in his study of ship stabilization and automatic steering, pointed out very clearly the importance of the consideration of the delay in the feed back mechanism. The great interest in control theory during these and later years has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state .And also in the late forties and early fifties , a few books appeared which presented the current status of the subject and certainly greatly influenced later developments. Mishkis [4] introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems.

The simplest type of past dependence in a differential equation is that in which the past dependence is through the state variable and not the derivative of the state variable, the so-called retarded functional differential equations or retarded differential equations.

The oscillatory behavior of solutions of differential equations with deviating arguments has been studied by many authors. The problem of oscillations caused by the deviating arguments (delay or advanced) has been the subject of intensive investigations.

Amongst the application areas of differential equations with deviating arguments are in bioscience, economics, material science; medicine, public health; in a number of these there is an underlying problem in control theory.

Among numerous papers dealing with the study of this problem we choose to refer to the papers by M.Kon,Y.G.Sficas,and I.P.Stavroulakis[7],Bingtuan.Li [2],R.Koplatadze and G. Kvinikadze[7],R.S.Dahiya and Olusola Akinyele [9], R.S. Dahiya, Takasi Kusano and Manabu Naito [10],S.R.Grace and B.S.Lalli [11]. In the literature of oscillation theory of functional differential equations , the following are of special interest, either

(1) all solutions are oscillatory;

(2) all solutions are non-oscillatory;

(3) the equation has a non- oscillatory solution;

(4) the equation has an oscillatory solution;

(5) the equation has both oscillatory and non-oscillatory solution.

M.Kon,Y.G.Sficas,and I.P.Stavroulakis[7], discussed the oscillatory behavior of first order delay differential equations of the form $x'(t)+p(t)x(\tau(t))=0$, $t \geq t_0$ where $p,\tau \in C([t_0,\infty),R^+), R^+=[0,\infty), \tau(t)$ is non-decreasing, $\tau(t)<t$ for $t \geq t_0$, and $\lim_{t\to\infty} \tau(t)=\infty$. They proved that when $L<1$ and $0<K\leq \frac{1}{e}$ all solutions x (t) are oscillatory. In which the condition $L>2K+\frac{2}{\lambda}-1$ holds, where $K=\underset{t\to\infty}{Lim}\inf \int_{\tau(t)}^{t} p(s)ds$ and $L=\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ and λ is the smaller root of the equation $\lambda=e^{K\lambda}$. In [9] R.S. Dahiya and Olusola Akinyele discussed the nth- order functional differential equation of the form $x^{(n)}(t)+p(t)x(g(t))=h(t)$ where

(a) $p,g,h:[a,\infty)\to R$ are continuous.

(b) $p(t)>0, g(t)$ is non-decreasing.

(c) $g(t) < t, g(t) \to \infty \;\; as \; t \to \infty$.

They provided sufficient conditions for the above functional differential equation to be almost oscillatory in the sense that every solution x (t) is either oscillatory or

$\lim_{t\to\infty} x^{(i)}(t) = 0 \qquad 0 \le i \le n-1$.

One of the basic equations that we seek to discuss for different values of n is

$x^{(n)}(t) + p(t)f(x(g(t))) = h(t)$

The first chapter introduces the reader to the basic definition and theorems that needed later and some applications on delay differential equations. Chapter two is devoted to the discussion about the oscillation, non-oscillation, and the asymptotic behavior of first order delay differential equations of the form $x'(t) + p(t)f(x(g(t))) = h(t)$. In chapter three we investigate the oscillation, and the asymptotic behavior of second order delay differential equations of the form $(r(t)x'(t))' + p(t)f(x(g(t))) = h(t)$ .Finally the oscillation of third and n-th order delay differential equations has been studied in chapter four.

The basic observation and the importance of deviating arguments in the sense that their presence causes or destroys the oscillation phenomena and does not merely preserves the oscillatory behavior of equations without deviating arguments.

Remark:

We indicate that we have used to refer to equations by the triple (a,b,c)

where a refers to the chapter number;

and b refers to the section number;

and c refers to the equation number;

For the end of the proof we have used the black dot ●

Chapter One

Preliminaries

1.1.Overview

Many important and significant problems in engineering, physical science and social science, when formulated into mathematical term, require the determination of a function satisfying an equation containing one or more derivatives of unknown function, and these derivatives depend on the solution at the present value of the independent variable (t), such equations are called ordinary differential equations of the form

$$x'(t) = f(t, x(t)) \qquad (1.1.1)$$

with the initial condition, $x(t_0) = x_0$ .Also, some other significant problems in many scientific regions , the function and some of its derivatives depend on its past memory , this kind of differential equations are called differential equations with deviating arguments.

Definition 1.1.1. Differential equations with deviating arguments are differential equations in which the unknown function appears with various values of the arguments. For example

$$x'(t) = f(t, x(t), x(t - \tau(t))) \qquad (1.1.2)$$

$$x''(t) = f(t, x(\frac{t}{2}), x'(\frac{t}{2}), x(t), x'(t)) \qquad (1.1.3)$$

Definition1.1.2. A differential equation with retarded (or delay) argument is a differential equation with deviating argument in which the highest – order derivative of the unknown function appears for just one value of the argument, and this argument is not less than all arguments of the unknown function and it's derivatives appearing in the equation .

Equation (1.1.2) is retarded (delay) if $\tau(t) \geq 0$, and (1.1.3) is retarded (delay) if $t \geq 0$.

The study of differential equations with deviating arguments have been intensively investigated for many years, and the area of applications of differential equations with delay argument has greatly expanded in many fields ; In physics and technology, economics and biological sciences, and in medicine , production of real blood cells, etc .This abundance of applications has increased the interest in the theory of differential equations with deviating argument .Consider the differential equation of the form

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau)), \quad \tau > 0, \ t \geq t_0 \\ x(t) = \phi_0(t) \end{cases} \tag{1.1.4}$$

in which the right –hand side depends not only on the instantaneous position x(t) , but also on $x(t-\tau)$, the position at τ units back ; the equation has past memory , such an equation is called an ordinary differential equations with a delay or retarded argument. In order to define a solution of (1.1.4) we need to have a known function $\phi_0(t) \ on \ [t_0 - \tau, t_0]$, instead of just the initial condition $x(t_0) = x_0$. The basic initial-value problem for (1.1.4) consists of determining a continuous solution $x(t) = \phi_0(t)$ for $t_0 - \tau \leq t \leq t_0$, where $\phi_0(t)$ is a given continuous function, called the initial function. The segment $t_0 - \tau \leq t \leq t_0$ on which the initial function is given is called the initial set and denoted by E_{t_0} . It is usually assumed that $\phi_0(t_0) = x(t_0 + 0)$. In the case of a variable delay $\tau = \tau(t) > 0$ in equation (1.1.4) it is also required to find a solution of this equation for $t > t_0$, such that on the initial set

$E_{t_0} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0,\ \ t \ge t_0\}$, i,e (the point t_0 and those values of $t - \tau(t)$ less than t_0 for $t \ge t_0$), x(t) coincides with the given initial function $\phi_0(t)$.If it is required to determine the solution on the interval $[t_0, T]$, then the initial set $E_{t_0 T} = \{t - \tau(t) < t_0,\ \ t_0 \le t \le T\}$.

Definition 1.1.3. A function x is said to be a solution of (1.1.4) if

$x(t) \in C([t_0 - \tau, T])$ for some t_0, satisfies equation (1.1.4) for all $t \ge t_0$.

1.2 Method of steps

The most natural method for solving this problem is the so called **method of steps (or method of successive integration).** In this method a continuous solution x(t) is first determined from the differential equation without retardation .To solve (1.1.4) on $[t_0, t_0 + \tau]$,equation (1.1.4) becomes $x'(t) = f(t, x(t), \phi_0(t - \tau))$ for $t_0 \le t \le t_0 + \tau$, $x(t_0) = \phi_0(t_0)$ since for $t_0 \le t \le t_0 + \tau$, the argument $t - \tau$ varies within the initial set $[t_0 - \tau, t_0]$ and consequently the third argument $x(t - \tau)$ of the function f equals the initial function $\phi_0(t - \tau)$. Assuming the existence of a solution $x = \phi_1(t)$ of this initial value problem on the entire segment $[t_0, t_0 + \tau]$ we obtain

$$x'(t) = f(t, x(t), \phi_1(t - \tau)) \quad for \quad t_0 + \tau \le t \le t_0 + 2\tau \ , \quad x(t_0 + \tau) = \phi_1(t_0 + \tau)$$

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$$x'(t) = f(t, x(t), \phi_n(t - \tau)) \quad for \quad t_0 + n\tau \le t \le t_0 + (n+1)\tau,$$

$$x(t_0 + n\tau) = \phi_n(t_0 + n\tau).$$