

**HOMOGENEOUS SYSTEMS OF LINEAR
DIFFERENTIAL EQUATIONS WITH CONSTANT
OR PERIODIC COEFFICIENTS**

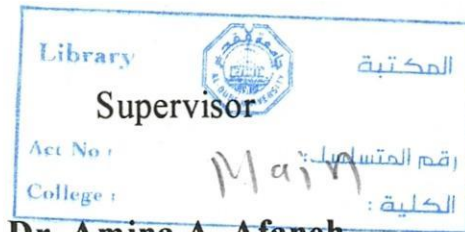
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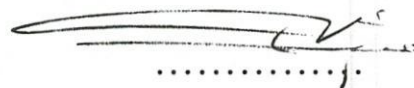
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DEDICATION

**TO
MY
PARENTS**

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ABSTRACT

We study the general properties of linear systems of first-order differential equations, where the coefficients of the unknown variables are continuous functions. Furthermore we study the n -th order homogeneous linear differential equation.

Several analytic methods for determining the general solution of the homogeneous system with constant and periodic coefficients are studied.

We use basic results in linear algebra for determining the general solution.

Furthermore, as an application, Riccati Differential Equation is investigated. Lyapunov transformation and reducible systems are used for solving the periodic system.

Finally, we study the stability of perturbed linear systems; where we remark that the stability of any nonzero solution of the homogeneous linear equation is determined by the stability of the zero solution.

Introduction

Studying simultaneous differential equations plays a major role in solving many problems in Control Theory, Optimization, Fluid Dynamics, and Electrical Engineering.

In chapter one we study the general properties of general linear systems of first-order differential equations, whose coefficients are continuous functions. Furthermore we solve n -th order homogeneous linear differential equation by converting such equation to a system involving n first-order ordinary differential equations.

In chapter two, we turn our attention to the special case of linear systems of ordinary differential equations with constant coefficients. First we study the diagonal system and generalize this method to any linear system with constant coefficients. We use the Jordan Canonical form in the case where eigenvalues are repeated. Furthermore we give different methods for determining the matrix exponentials as solutions of linear homogeneous systems. Finally, these methods are used to solve the matrix Riccati equation.

In chapter three, we use elementary results in linear algebra for solving systems of homogeneous ordinary linear differential equations with constant coefficients.

In chapter four, Lyapunov transformation and reducible systems are used for solving the periodic system.

In the last chapter the stability of linear systems with both constant and periodic coefficients is investigated.

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CHAPTER ONE

SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

In section (1.1) we will give a general discussion of n -dimensional first-order system of ordinary differential equations. Such a system arise naturally in problems involving several dependent variables each of which is a function of a single independent variable. We will denote the independent variable by t , and let $x_1, x_2, x_3, \dots, x_n$ represent dependent variables which are functions of t .

In section (1.2) we will apply the result obtained in section (1.1) to a consideration of the n -th order linear differential equation.

1.1 Linear System Of First-Order Differential Equations

Consider the homogeneous system of linear first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n.\end{aligned}\tag{1.1}$$

If the vector X and the matrix A are defined by

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}$$

where $a_{ij}(t)$ are continuous real-valued functions for t on an closed interval I of real numbers, for $i, j = 1, 2, \dots, n$. Then system (1.1) can be written as

$$\frac{dX}{dt} = A(t)X\tag{1.2}$$

Also if

$$F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

where $f_i(t)$ are continuous functions of $t \in I$.

Then the non-homogeneous system can be written as

$$\frac{dX}{dt} = AX + F(t) \quad (1.3)$$

Definition 1.1.1: A solution vector on an interval I is any column

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

whose entries are differentiable functions satisfying system (1.3) on the interval I . Let t_0 denote a point in an interval I and

$$X(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}, \text{ and } X_0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix},$$

where γ_i are given constants, for $i=1, 2, \dots, n$. Then the problem of finding the solution of (1.3) subject to the initial condition $X(t_0) = X_0$ is called an **initial-value problem**.

Theorem 1.1.2: Let $X_1, X_2, \dots, X_k$ be a set of solution vectors of the homogeneous system (1.2) on the interval I , then the linear combination $X = c_1 X_1 + c_2 X_2 + \dots + c_k X_k$ is also a solution of (1.2) on the interval I , where c_i are arbitrary constants for $i=1, 2, \dots, k$.

Proof : Let $X_1, X_2, \dots, X_k$ be a set of solution vectors of the homogeneous system (1.2) where A is an $n \times n$ matrix. Then (1.2) can be written as

$$\frac{dX}{dt} - A(t)X = 0 \quad (1.4)$$

Let

$X = c_1 X_1 + \dots + c_k X_k$ where $c_1, \dots, c_k$ are arbitrary constants, then

$$\frac{dX}{dt} = c_1 \frac{dX_1}{dt} + c_2 \frac{dX_2}{dt} + \dots + c_k \frac{dX_k}{dt} \quad (1.5)$$

Now system (1.4) becomes as

$$\begin{aligned} & c_1 \frac{dX_1}{dt} + c_2 \frac{dX_2}{dt} + \dots + c_k \frac{dX_k}{dt} - A(t)c_1 X_1 - A(t)c_2 X_2 - \dots - A(t)c_k X_k \\ &= \left(c_1 \frac{dX_1}{dt} - A(t)c_1 X_1 \right) + \left(c_2 \frac{dX_2}{dt} - A(t)c_2 X_2 \right) + \dots + \left(c_k \frac{dX_k}{dt} - A(t)c_k X_k \right) \\ &= c_1 \left(\frac{dX_1}{dt} - A(t)X_1 \right) + c_2 \left(\frac{dX_2}{dt} - A(t)X_2 \right) + \dots + c_k \left(\frac{dX_k}{dt} - A(t)X_k \right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \dots + c_k \cdot 0 = 0. \end{aligned}$$

Hence the linear combination $X = c_1 X_1 + \dots + c_k X_k$ is also a solution on I . ■

Definition 1.1.3: Let $X_1, X_2, \dots, X_k$ be a set of solution vectors of the homogeneous system (1.2) on an interval I . We say that this set is **linearly dependent** on the interval if there exist constants

$c_1, c_2, \dots, c_k$ not all zero, such that $c_1 X_1 + \dots + c_k X_k = 0$ for every $t \in I$. If the set of vectors is not linearly dependent on the interval, it is said to be **linearly independent**.

Theorem 1.1.4 : Let $X_1, X_2, \dots, X_n$ be n solution vectors of the homogeneous system (1.2) on an interval I . A necessary and sufficient condition for the set of solution to be linearly independent is that the Wronskian $W(X_1, X_2, \dots, X_n) \neq 0$ for every $t \in I$.

Proof : Assume that $X_1, X_2, \dots, X_n$ are linearly dependent on I .

Let X be a nontrivial solution of (1.2), so there exist constants $\alpha_1, \dots, \alpha_n$, not all zero, such that

$$X = \sum_{j=1}^n \alpha_j X_j \quad \text{or} \quad X = \Phi C, \quad (1.6)$$

where Φ is a fundamental matrix for (1.2), and $C = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$.

Let $t = \tau \in I$. Then we have

$$X(\tau) = \Phi(\tau)C \quad (1.7)$$

This system of equations has a unique solution for any choice of $X(\tau)$.

Hence, $\det(\Phi(\tau)) \neq 0$. that $\det(\Phi(t)) \neq 0$ for any $t \in I$.

i.e $W(X_1, X_2, \dots, X_n) \neq 0$ for any $t \in I$.

Conversely, let Φ be a solution of the matrix equation (1.2) and assume that $\det(\Phi(t)) \neq 0$ for all $t \in I$. Then the column of Φ , $X_1, \dots, X_n$, are linearly independent for all $t \in I$. Hence, Φ is a fundamental matrix of (1.2). ■

Definition 1.1.5 : (i) Any set $X_1, X_2, \dots, X_n$ of n linearly independent solution vectors of the homogeneous system (1.2) on an interval I is said to be a **fundamental set of solutions** on I .

(ii) For a non-homogeneous system a **particular solution** X_p on an interval I is any vector, free of arbitrary parameters, whose entries are functions satisfying the system (1.3).

Lemma 1.1.6: There exists a unique solution of the initial-value problem

$$\frac{dX}{dt} = A(t)X, \quad X(t_0) = I_n \text{ for } t_0 \in I.$$

Proof: We shall construct the solution by means of

Let

$$X_0(t) = I_n,$$

and

$$X_k(t) = I_n + \int_0^t A(u)X_{k-1}(u)du, \quad k = 1, 2, \dots \quad (1.8)$$

Then

$$\begin{aligned} X_k(t) = I_n + \int_0^t A(u)du + \int_0^t A(t_1)dt_1 \int_0^{t_1} A(t_2)dt_2 \\ + \dots + \int_0^t A(t_1)dt_1 \int_0^{t_1} A(t_2)dt_2 \dots \int_0^{t_{k-1}} A(t_k)dt_k. \end{aligned} \quad (1.9)$$

The convergence of the sequence (1.8) is equivalent to the convergence of the series

$$\begin{aligned} X_0 + (X_1(t) - X_0) + (X_2(t) - X_1(t)) + (X_3(t) - X_2(t)) \\ + \dots + (X_k(t) - X_{k-1}(t)) + \dots \end{aligned} \quad (1.10)$$

By induction, for the terms of this series, we have the estimate

$$\|X_m(t) - X_{m-1}(t)\| \leq \frac{1}{m!} \left(\int_{t_0}^t \|A(u)\| du \right)^m, \quad m = 1, 2, \dots, \quad \text{for } t \in I.$$

For $m = 1$ the estimate is true.

Assume it is true for m , want to show it is true for $m + 1$.

$$\begin{aligned} \|X_{m+1}(t) - X_m(t)\| &= \left\| \int_{t_0}^t A(t_1) dt_1 \int_{t_0}^{t_1} A(t_2) dt_2 \cdots \int_{t_0}^{t_m} A(t_{m+1}) dt_{m+1} \right\| \\ &\leq \left| \int_{t_0}^t A(t_1) dt_1 \frac{1}{m!} \left(\int_{t_0}^t \|A(u)\| du \right)^m \right|. \end{aligned}$$

By subsitute

$$W = \left| \int_{t_0}^{t_1} \|A(u)\| du \right|$$

we get

$$\|X_{m+1}(t) - X_m(t)\| \leq \frac{1}{m+1!} \left(\int_{t_0}^t \|A(u)\| du \right)^{m+1},$$

so the estimate is true for $m + 1$.

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\int_{t_0}^t \|A(u)\| du \right)^k = \exp \left| \int_{t_0}^t \|A(u)\| du \right|,$$

This implies its converges for (1.10).

Denoting the sum of series (1.10) by $X(t)$ we obtain

$$X(t) = I_n + \int_0^t A(u) du + \sum_{k=2}^{\infty} \left(\int_0^t A(t_1) dt_1 \cdots \int_0^{t_{k-1}} A(t_k) dt_k \right) \quad (1.11)$$

Differentiate the series (1.11) term by term, we obtain the same series multiplied on the left by $A(t)$. ■

Theorem 1.1.7: There exists a fundamental set of solutions for the homogeneous system (1.2) on an interval I .

Proof: Let $e_1, e_2, \dots, e_n$ be n -dimensional vectors such that

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ place}.$$

Since

$$\sum_{j=1}^n c_j e_j = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$$

if and only if $c_j = 0$ for $j = 1, 2, \dots, n$.

Hence $e_1, e_2, \dots, e_n$ are linearly independent. For any $t_0 \in I$, Let $X_1(t), X_2(t), \dots, X_n(t)$ be the solutions of (1.2) satisfying $X_j(t_0) = e_j$, $j = 1, 2, \dots, n$. These are all distinct, since each satisfies distinct initial values. Furthermore are linearly independent, For

$$X(t) = \sum_{j=1}^n c_j X_j(t) = \mathbf{0} \text{ for } t \in I.$$

With the c_j not all zero, then

$$X(t_0) = \sum_{j=1}^n c_j X_j(t_0) = \sum_{j=1}^n c_j e_j = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

But this implies $c_1 = \dots = c_n = 0$, which is contradiction.

Therefore $X_1(t), X_2(t), \dots, X_n(t)$ are linearly independent, and form a fundamental set of solutions of (1.2) on I . ■

Theorem 1.1.8: Let $X_1, X_2, \dots, X_n$ be a set of solution vectors of the homogeneous system (1.2) on I and let X_p be any solution vector of the non-homogenous system (1.3) on I then

$X = c_1 X_1 + c_2 X_2 + \dots + c_n X_n + X_p$ is a solution of (1.3) on I .

Proof: Suppose X and X_p are both solutions of

$$\frac{dX}{dt} = A(t)X + F(t) \quad (1.12)$$

if we define the function U by $U(t) = X(t) - X_p(t)$ then

$$\begin{aligned}
\frac{d}{dt}(U(t)) &= \frac{d}{dt}(X(t)) - \frac{d}{dt}(X_p(t)) \\
&= (A(t)X(t) + F(t)) - (A(t)X_p + F(t)) \\
&= A(t)(X(t) - X_p(t)) + (F(t) - F(t)) \\
&= A(t)(X(t) - X_p(t)) \\
&= A(t)U(t).
\end{aligned}$$

Thus $U(t)$ is a solution of the homogenous system.

That is

$$U(t) = c_1 X_1 + c_2 X_2 + \cdots + c_k X_n.$$

Hence

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_k X_n + X_p$$

is a solution of the non-homogeneous system (1.12). ■

Definition 1.1.9: Let X_p be a given solution of the non-homogeneous system (1.8) on I , and let $X = c_1 X_1 + c_2 X_2 + \cdots + c_k X_k + X_p$ denote the **general solution** on the same interval of the corresponding homogeneous system (1.2). The **general solution of the non-homogeneous** system on the interval is defined to be $X = X_c + X_p$. The general solution X_c of the homogeneous System (1.2) is called the **complementary solution** of the non-homogeneous system.

Let $X_1, X_2, \dots, X_n$ be a fundamental set of solutions of the homogeneous system (1.2) on an interval I then its general solution on the interval I is $X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$.

Thus if

$$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{nn} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \dots, X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}.$$

Then

$$X = \begin{pmatrix} c_1 x_{11} + c_2 x_{12} + \cdots + c_n x_{1n} \\ c_1 x_{21} + c_2 x_{22} + \cdots + c_n x_{2n} \\ \vdots \\ c_1 x_{n1} + c_2 x_{n2} + \cdots + c_n x_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Definition 1.1.10 : Let $X_1, X_2, \dots, X_n$ be a fundamental set of n solution vectors of the homogenous system (1.2) on an interval I . The matrix

$$\Phi(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

is said to be a **fundamental matrix** of the system (1.2) on I .

Since the set of solution vectors of the homogeneous system are linearly independent then the inverse of the fundamental matrix $\Phi(t)$ exists for every value of $t \in I$.

Conversely any nonsingular solution of system (1.2) is a fundamental matrix of the system.

Theorem 1.1.11: If $\Phi(t)$ is a fundamental matrix of system (1.2), then any fundamental matrix $\Psi(t)$ of (1.2) has the form $\Psi(t) = \Phi(t) C$, where C is a constant nonsingular matrix.

Proof: Let $\Psi(t)$ be any other fundamental matrix. Consider the product $\Phi^{-1}\Psi$. Since $\det\Phi(t) \neq 0$ for all t , then Φ^{-1} exists for all t .

Differentiate $\Phi \Phi^{-1} = I$ by Leibniz rule, we get :

$$\frac{d\Phi}{dt} \cdot \Phi^{-1} + \Phi \cdot \frac{d}{dt}(\Phi^{-1}) = 0,$$

then

$$\frac{d}{dt}(\Phi^{-1}) = -\Phi^{-1} \cdot \frac{d\Phi}{dt} \cdot \Phi^{-1}.$$

Thus we can compute

$$\begin{aligned} \frac{d}{dt}(\Phi^{-1}\Psi) &= \frac{d}{dt}(\Phi^{-1}) \cdot \Psi + \Phi^{-1} \cdot \frac{d\Psi}{dt} = -\Phi^{-1} \frac{d\Phi}{dt} \Phi^{-1} \cdot \Psi + \Phi^{-1} A \Psi \\ &= -\Phi^{-1} A(t) \Phi \Phi^{-1} \Psi + \Phi^{-1} A(t) \cdot \Psi \\ &= -\Phi^{-1} A(t) \Psi + \Phi^{-1} A(t) \Psi = 0. \end{aligned}$$

Therefore $\Phi^{-1}\Psi$ is a constant matrix say $C := \Phi^{-1}\Psi$ then $\Psi = \Phi C$. ■

Lemma 1.1.12: If $X(t)$ is an $n \times n$ matrix solution of (1.2), then either $\det(X(t)) \neq 0$ or $\det(X(t)) = 0$ for all $t \in I$.

Proof: If there exists a real number t_0 such that the matrix $X(t_0)$ is singular, then there exists a nonzero vector C such that $X(t_0)C = 0$.

For this vector C , linearity of (1.2) implies that $X(t)C$ is a solution of (1.2). Since the function zero is a solution of the equation, uniqueness implies $X(t)C = 0$ for all $t \in I$ and thus $\det(X(t)) = 0$ for all $t \in I$. ■

Corollary 1.1.13: If X_0 is an $n \times n$ nonsingular matrix and if $X(t)$ is the matrix solution of system (1.2) with $X(0) = X_0$, then $X(t)$ is a fundamental matrix solution of (1.2).

Definition 1.1.14: A **principal matrix solution** of system (1.2) at initial condition t_0 is a fundamental matrix $\Phi(t)$ such that $\Phi(t_0) = I_n$, we denote the **principal matrix solution** at t_0 by $\Phi(t, t_0)$.

Theorem 1.1.15: If $\Phi(t)$ is a fundamental matrix of system (1.2), then the general solution of system (1.2) is of the form

$$X(t) = \Phi(t)C \quad \text{where} \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad c_i \in \mathbb{C}.$$

Proof: We have

$$\frac{d(\Phi C)}{dt} = \frac{d\Phi}{dt} \cdot C = (A(t)\Phi)C = A(t)(\Phi C)$$

Hence ΦC is a solution of the system (1.2). ■

Theorem 1.1.16: If system (1.2) is given with initial condition $X(t_0) = X_0$. Then if $\Phi(t, t_0)$ is the principle fundamental matrix of system (1.2) the solution of this initial value problem (denoted by $X(t, t_0)$) is $X(t, t_0) = \Phi(t, t_0)X_0$.

Proof: The general solution of the system (1.2) is of the form

$$X(t) = \Phi(t)C.$$

Then

$$X_0 = X(t_0) = \Phi(t_0, t_0)C = IC = C.$$

Thus

$$X(t, t_0) = \Phi(t, t_0)X_0.$$

1.2 Homogeneous Linear Differential Equation Of n-th Order.

In this section, we consider initial value problems described by n-th order linear ordinary differential equations, which are given by

$$\frac{d^n}{dt^n}(x) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}(x) + \cdots + a_1(t)\frac{d}{dt}(x) + a_0(t)x = 0 \quad (2.1)$$

where $a_k(t)$ are continuous functions on the interval I for $k=0, 1, 2, \dots, n-1$. If we define the linear differential operator L_n by

$$L_n = \frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t) \quad (2.2)$$

Then we can rewrite (2.1) more compactly as

$$L_n x = 0 \quad (2.3)$$

Now we can reduce the study of equation (2.1) to the study of the system of n first order ordinary differential equations as (1.2) as follows:

Let

$$x_1 = x, \quad x_2 = \frac{dx}{dt}, \quad x_3 = \frac{d^2}{dt^2}(x), \quad \dots, \quad x_n = \frac{d^{n-1}}{dt^{n-1}}(x).$$

Hence

$$\frac{dx_1}{dt} = \frac{dx}{dt} = x_2$$

$$\frac{d}{dt}(x_2) = \frac{d^2}{dt^2}(x) = x_3$$

$$\frac{d}{dt}(x_3) = \frac{d^3}{dt^3}(x) = x_4$$

$$\vdots$$

$$\frac{d}{dt}(x_{n-1}) = \frac{d^{n-1}}{dt^{n-1}}(x) = x_n$$

$$\frac{d}{dt}(x_n) = \frac{d^n}{dt^n}(x) = -a_0(t)x_1 - a_1(t)x_2 - \cdots - a_{n-1}(t)x_{n-1}.$$

We can write them in the matrix form as follows :

$$\frac{dX}{dt} = A(t)X, \quad \text{where}$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{pmatrix} \quad (2.4)$$

The matrix given in (2.4) is frequently called a **companion matrix**, (or $A(t)$ is said to be in companion form).

Since $A(t)$ is continuous on I , then by theorem (1.1.6) there exists a unique solution $\Phi(t)$, for all $t \in I$, to the initial value problem

$$\frac{dX}{dt} = A(t)X, \quad X(\tau) = \xi, \quad \text{where } \tau \in I,$$

and

$$\xi = (\xi_1, \dots, \xi_n)^T \in R^n.$$

The first component of this solution is a solution of $L_n x = 0$ satisfying

$$x(\tau) = \xi_1, \quad \frac{d}{dt}x(\tau) = \xi_2, \quad \dots, \quad \frac{d^{n-1}}{dt^{n-1}}x(\tau) = \xi_n.$$

1.3 Homogeneous n-th Order Linear Differential Equations With Constant Coefficients.

We can solve homogeneous n-th order linear differential equations by converting it to linear system of differential equations as in section (1.2). We shall now find another method [4] of solving the homogeneous n^{th} order linear differential equation by using the operator method, in which

$$D = \frac{d}{dt}.$$

Consider a single homogeneous n^{th} order linear differential equation with constant coefficients :

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = 0 \quad (3.1)$$

using the differential operator notation $D = \frac{d}{dt}$ so the equation (3.1) becomes

$$(D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)x(t) = 0 \quad (3.2)$$

if we can factorize (3.2) as follows

$$(D - \lambda_1)^{n_1} (D - \lambda_2)^{n_2} \dots (D - \lambda_s)^{n_s} x(t) = 0 \quad (3.3)$$

where $n_1, n_2, \dots, n_s$ are the multiplicities corresponding to the distinct roots $\lambda_1, \lambda_2, \dots, \lambda_s$ of the polynomial

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0,$$

and

$$n_1 + n_2 + \dots + n_s = n.$$

In equation (3.1) if we let

$$x_1(t) = (D - \lambda_2)^{n_2} \dots (D - \lambda_s)^{n_s} x(t)$$

we obtain

$$(D - \lambda_1)^{n_1} x_1(t) = 0 \quad (3.4)$$

and if

$$x_2(t) = (D - \lambda_3)^{n_3} \dots (D - \lambda_s)^{n_s} x(t)$$

then

$$(D - \lambda_2)^{n_2} x_2(t) = x_1(t).$$

If we let

$$x_3(t) = (D - \lambda_4)^{n_4} \dots (D - \lambda_s)^{n_s} x(t)$$

we obtain

$$(D - \lambda_s)^{n_3} x_3(t) = x_2(t),$$

we proceed in the same way to get finally

$$(D - \lambda_s)^{n_s} x_s(t) = x_{s-1}(t),$$

where

$$x_s(t) = x(t).$$

So the differential equation (3.3) can be reduced to the system:

$$\begin{aligned} (D - \lambda_1)^{n_1} x_1(t) &= 0 \\ (D - \lambda_2)^{n_2} x_2(t) &= x_1(t) \\ (D - \lambda_3)^{n_3} x_s(t) &= x_2(t) \\ &\vdots \\ (D - \lambda_s)^{n_s} x_s(t) &= x_{s-1}(t) \\ x(t) &= x_s(t). \end{aligned} \tag{3.5}$$

We solve the system (3.5) by using the substitutions

$$\begin{aligned} x_1(t) &= \xi_1(t) e^{\lambda_1 t} \\ x_2(t) &= \xi_2(t) e^{\lambda_2 t} \\ &\vdots \\ x_s(t) &= \xi_s(t) e^{\lambda_s t}. \end{aligned} \tag{3.6}$$

for $i = 1, 2, \dots, s$ where $\xi_i(t)$ are polynomial of t .

Now substitute

$$x_I(t) = \xi_I(t) e^{\lambda_I t}$$

in equation (3.4) so we have

$$0 = (D - \lambda_I)^{n_I} x_I(t) = (D - \lambda_I)^{n_I} \xi_I(t) e^{\lambda_I t}$$

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$$\begin{aligned}
&= (D - \lambda_1)^{n_1-1} (\xi_1(t) \lambda_1 e^{\lambda_1 t} + e^{\lambda_1 t} D(\xi_1(t)) - \lambda_1 e^{\lambda_1 t} \xi_1(t)) \\
&= (D - \lambda_1)^{n_1-1} e^{\lambda_1 t} D(\xi_1(t)) \\
&= (D - \lambda_1)^{n_1-2} (e^{\lambda_1 t} D^2(\xi_1(t)) + D(\xi_1(t)) \lambda_1 e^{\lambda_1 t} - \lambda_1 e^{\lambda_1 t} D(\xi_1(t))) \\
&= (D - \lambda_1)^{n_1-2} e^{\lambda_1 t} D^2(\xi_1(t)) \\
&\quad \vdots \\
&= (D - \lambda_1) e^{\lambda_1 t} D^{n_1-1}(\xi_1(t)) \\
&= e^{\lambda_1 t} D^{n_1}(\xi_1(t)).
\end{aligned}$$

Thus

$$e^{\lambda_1 t} D^{n_1}(\xi_1(t)) = 0$$

but

$$e^{\lambda_1 t} \neq 0$$

for every $t \in R$, then

$$D^{n_1} \xi_1(t) = 0.$$

We substitute the values of $x_1(t)$ and $x_2(t)$ in the equation

$$(D - \lambda_2)^{n_2} x_2(t) = x_1(t),$$

then we get

$$\begin{aligned}
\xi_1(t) e^{\lambda_1 t} &= (D - \lambda_2)^{n_2} \xi_2(t) e^{\lambda_2 t} \\
&= (D - \lambda_2)^{n_2-1} (D - \lambda_2) \xi_2(t) e^{\lambda_2 t} \\
&= (D - \lambda_2)^{n_2-1} e^{\lambda_2 t} D(\xi_2(t)) \\
&= (D - \lambda_2)^{n_2-2} (e^{\lambda_2 t} D^2(\xi_2(t)) + D(\xi_2(t)) \lambda_2 e^{\lambda_2 t} - \lambda_2 e^{\lambda_2 t} D(\xi_2(t))) \\
&= (D - \lambda_2)^{n_2-2} e^{\lambda_2 t} D^2(\xi_2(t)) \\
&\quad \vdots \\
&= (D - \lambda_2) e^{\lambda_2 t} D^{n_2-1}(\xi_2(t)) \\
&= e^{\lambda_2 t} D^{n_2}(\xi_2(t)) + D^{n_2-1}(\xi_2(t)) \lambda_2 e^{\lambda_2 t} - \lambda_2 e^{\lambda_2 t} D^{n_2-1}(\xi_2(t)) \\
&= e^{\lambda_2 t} D^{n_2}(\xi_2(t)).
\end{aligned}$$

Thus

$$D^{n_2}(\xi_2(t)) = \xi_1(t) e^{(\lambda_1 - \lambda_2)t}.$$

We proceed in the same way for the result of the equations in (3.5). Then the system in (3.5) becomes

$$\left. \begin{aligned} D^{n_1} \xi_1(t) &= 0 \\ D^{n_2} \xi_2(t) &= \xi_1(t) e^{(\lambda_1 - \lambda_2)t} \\ &\vdots \\ D^{n_s} \xi_s(t) &= \xi_{s-1}(t) e^{(\lambda_{s-1} - \lambda_s)t} \\ x(t) &= \xi_s(t) e^{\lambda_s t} \end{aligned} \right\} \quad (3.7)$$

To solve the first equation in (3.7) which is $D^{n_1} \xi_1(t) = 0$, we integrate this equation n_1 times, then $\xi_1(t)$ is a polynomial of degree $n_1 - 1$ say that $p_1^1(t)$, i.e. $\xi_1(t) = p_1^1(t)$. Now solve the second equation of (3.7),

$$\begin{aligned} D^{n_2} \xi_2(t) &= \xi_1(t) e^{(\lambda_1 - \lambda_2)t} \\ &= p_1^1(t) e^{(\lambda_1 - \lambda_2)t}. \end{aligned}$$

Denote the solution of the homogeneous equation

$$D^{n_2} \xi_2(t) = 0 \text{ by } p_2^1(t),$$

where $p_2^1(t)$ is a polynomial of degree $n_2 - 1$ and the particular solution of the equation

$$D^{n_2} \xi_2(t) = p_1^1(t) e^{(\lambda_1 - \lambda_2)t} \text{ by } p_2^2(t) e^{(\lambda_1 - \lambda_2)t}$$

then

$$\xi_2(t) = p_2^1(t) + p_2^2(t) e^{(\lambda_1 - \lambda_2)t}.$$

In the same way, we can find that

$$\xi_3(t) = p_3^1(t) + p_3^2(t) e^{(\lambda_2 - \lambda_3)t} + p_3^3(t) e^{(\lambda_1 - \lambda_3)t}.$$

where $p_3^1(t)$ the solution of the homogenous equation $D^{n_3} \xi_3(t) = 0$ and $p_3^2(t) e^{(\lambda_2 - \lambda_3)t} + p_3^3(t) e^{(\lambda_1 - \lambda_3)t}$

is the particular solution of

$$D^{n_3} \xi_3(t) = p_2^1(t) e^{(\lambda_2 - \lambda_3)t} + p_2^2(t) e^{(\lambda_1 - \lambda_2)t}.$$

We proceed in the same way, and the solution of

$$D^{n_s} \xi_s(t) = \xi_{s-1}(t) e^{(\lambda_{s-1} - \lambda_s)t}$$

where

$$\xi_{s-1}(t) = p_{s-1}^1(t) + p_{s-1}^2(t) e^{(\lambda_{s-2} - \lambda_{s-1})t} + \dots + p_{s-1}^{s-1}(t) e^{(\lambda_1 - \lambda_{s-1})t}$$

so the solution of the homogeneous equation

$$D^{n_s} \xi_s(t) = 0 \text{ is } p_s(t)$$

where $p_s(t)$ is a polynomial of degree $n_s - 1$

and the particular solution of the equation

$$D^{n_s} \xi_s(t) = (p_{s-1}^1(t) + p_{s-1}^2(t)e^{(\lambda_{s-2}-\lambda_{s-1})t} + \dots + p_{s-1}^{s-1}(t)e^{(\lambda_1-\lambda_{s-1})t})e^{(\lambda_{s-1}-\lambda_s)t} \\ = p_{s-1}^1(t)e^{(\lambda_{s-1}-\lambda_s)t} + p_{s-1}^2(t)e^{(\lambda_{s-2}-\lambda_s)t} + \dots + p_{s-1}^{s-1}(t)e^{(\lambda_1-\lambda_s)t}$$

is of the form

$$p_1(t)e^{(\lambda_1-\lambda_s)t} + p_2(t)e^{(\lambda_2-\lambda_s)t} + \dots + p_{s-1}(t)e^{(\lambda_{s-1}-\lambda_s)t},$$

thus

$$\xi_s(t) = p_1(t)e^{(\lambda_1-\lambda_s)t} + p_2(t)e^{(\lambda_2-\lambda_s)t} + \dots + p_{s-1}(t)e^{(\lambda_{s-1}-\lambda_s)t} + p_s(t)$$

where each $p_k(t)$ is an arbitrary polynomial of degree $n_k - 1$.

$$x(t) = \xi_s(t)e^{\lambda_s t} \\ = \left[p_1(t)e^{(\lambda_1-\lambda_s)t} + p_2(t)e^{(\lambda_2-\lambda_s)t} + \dots + p_{s-1}(t)e^{(\lambda_{s-1}-\lambda_s)t} + p_s(t) \right] e^{\lambda_s t}$$

Hence the general solution of equation (3.1) is

$$x(t) = p_1(t)e^{\lambda_1 t} + p_2(t)e^{\lambda_2 t} + \dots + p_{s-1}(t)e^{\lambda_{s-1} t} + p_s(t)e^{\lambda_s t}.$$

This general solution is a sum of n linearly independent solutions, because if $x(t) \equiv 0$, then $\xi_s(t) \equiv 0$ then $\xi_{s-1}(t) = 0$ and so on.

Hence $x(t) \equiv 0$ if and only if

$$\xi_s(t) \equiv \xi_{s-1}(t) \equiv \dots \equiv \xi_1(t) \equiv 0$$

that is if and only if

$$p_s(t) \equiv p_{s-1}(t) \equiv \dots \equiv p_1(t) \equiv 0. \blacksquare$$

Example(1): Solve the following n -th order linear homogenous equation

$$\frac{d^6 x}{dt} - 3 \frac{d^5 x}{dt} - 9 \frac{d^4 x}{dt} + 19 \frac{d^3 x}{dt} + 12 \frac{d^2 x}{dt} - 36 \frac{dx}{dt} + 16x = 0$$

Solution: Let $D = \frac{d}{dt}$ then

$$(D^6 - 3D^5 - 9D^4 + 19D^3 + 12D^2 - 36D + 16)x(t) = 0,$$

can be reduced as follows :

$$(D-1)^3 (D+2)^2 (D-4)x(t) = 0.$$

Hence $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 4$ and $n_1 = 3$, $n_2 = 2$, $n_3 = 1$.

So the general solution is

$$X(t) = (a_0 + a_1 x + a_2 x^2) e^t + (b_0 + b_1 x) e^{-2t} + c_0 e^{4t}.$$

CHAPTER TWO

LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In this chapter, we shall discuss how to find a solution of system of the form

$$\frac{dX}{dt} = AX,$$

where A is a matrix with constant entries.

2.1 Diagonalizable system

If the system

$$\frac{dX}{dt} = AX \tag{1.1}$$

is diagonal, that is, A is a diagonal matrix, then it can be written as

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_{11} \\ \frac{dx_2}{dt} &= 0 + a_{22}x_2 \\ &\vdots \\ \frac{dx_n}{dt} &= 0 + \dots + 0 + a_{nn}x_n. \end{aligned} \tag{1.2}$$

This system can be solved in the same way as a single differential equation. We get the solution of (1.2) by separation of variable as

$$\begin{aligned} x_1(t) &= c_1 e^{a_{11}t} \\ x_2(t) &= c_2 e^{a_{22}t} \\ &\vdots \\ x_n(t) &= c_n e^{a_{nn}t} \end{aligned} \tag{1.3}$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants. We can write (1.3) in vector form, that is

$$X(t) = \begin{pmatrix} c_1 e^{a_{11}t} \\ c_2 e^{a_{22}t} \\ \vdots \\ c_n e^{a_{nn}t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{a_{11}t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{a_{22}t} + \dots + c_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} e^{a_{nn}t},$$

this implies that the vector functions

$$X_1(t) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{a_{11}t}, \quad X_2(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{a_{22}t}, \quad \dots, \quad X_n(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} e^{a_{nn}t}$$

form a fundamental set of solutions for the diagonal system (1.2) , thus

$$\Phi(t) = \begin{pmatrix} e^{a_{11}t} & 0 & \dots & 0 \\ 0 & e^{a_{22}t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{a_{nn}t} \end{pmatrix}.$$

Example (1): The diagonal system

$$\frac{dX}{dt} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

can be written as three equations:

$$\frac{dx_1}{dt} = 3x_1$$

$$\frac{dx_2}{dt} = -2x_2$$

$$\frac{dx_3}{dt} = 4x_3.$$

Integrating these after separating variables, we obtain

$$x_1(t) = c_1 e^{3t}, \quad x_2(t) = c_2 e^{-2t}, \quad x_3(t) = c_3 e^{4t},$$

where c_1, c_2, c_3 are arbitrary constants, thus

$$X(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t}$$

is the general solution of the diagonal system and the fundamental matrix

$$\text{is } \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix}.$$

If the system

$$\frac{dX}{dt} = AX \quad (1.4)$$

is not diagonal, suppose A is diagonalizable, that is there exists a non-singular matrix P such that

$$P^{-1}AP = D,$$

where D is diagonal, then multiply (1.4) from the left by P^{-1} we obtain

$$P^{-1} \frac{dX}{dt} = P^{-1}AX,$$

since

$$P^{-1}P = I_n$$

we have

$$P^{-1} \frac{dX}{dt} = P^{-1}APP^{-1}X \quad (1.5)$$

Let

$$U = P^{-1}X$$

so we have

$$\frac{dU}{dt} = P^{-1} \frac{dX}{dt},$$

since P^{-1} is a constant matrix so the system (1.5) can be written as

$$\frac{dU}{dt} = DU \quad (1.6)$$

where

$$D = P^{-1}AP.$$

This is a diagonal system and can be solved where D has the form:

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A and the columns of P are linearly independent eigenvectors of A associated, respectively with $\lambda_1, \dots, \lambda_n$, so the general solution to (1.6) is

$$U(t) = c_1 U_1(t) + c_2 U_2(t) + \cdots + c_n U_n(t),$$

where

$$U_1(t) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_1 t}, \quad U_2(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_2 t}, \quad \dots, \quad U_n(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} e^{\lambda_n t},$$

and $c_1, \dots, c_n$ are arbitrary constants. From

$$U = P^{-1} X$$

we have

$$X = P U,$$

thus if $P_1, P_2, \dots, P_n$ are the columns of P and therefore the corresponding eigenvectors of A associated to $\lambda_1, \dots, \lambda_n$ respectively.

the general solution to (1.6) is

$$X(t) = P U(t) = c_1 P_1 e^{\lambda_1 t} + c_2 P_2 e^{\lambda_2 t} + \cdots + c_n P_n e^{\lambda_n t}$$

So the general solution of the system (1.1) have the basic form

$$X_i = C e^{\lambda_i t}, \quad i = 1, 2, \dots, n,$$

where

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, c_2, \dots, c_n$ are constants.

Theorem 2.1.1: If the $n \times n$ matrix A has n linearly independent eigenvectors $P_1, P_2, \dots, P_n$ associated with the eigenvalue $\lambda_1, \dots, \lambda_n$ respectively, then the general solution to the system of differential equation (1.1) is

$$X(t) = c_1 P_1 e^{\lambda_1 t} + c_2 P_2 e^{\lambda_2 t} + \dots + c_n P_n e^{\lambda_n t}.$$

Proof:

$$\begin{aligned} \frac{dX}{dt} - AX &= \lambda_1 c_1 P_1 e^{\lambda_1 t} + \lambda_2 c_2 P_2 e^{\lambda_2 t} + \dots + \lambda_n c_n P_n e^{\lambda_n t} \\ &\quad - (Ac_1 P_1 e^{\lambda_1 t} + \dots + Ac_n P_n e^{\lambda_n t}) \\ &= (\lambda_1 I - A)c_1 P_1 e^{\lambda_1 t} + \dots + (\lambda_n I - A)c_n P_n e^{\lambda_n t} = 0 \end{aligned}$$

Since $AP_i = \lambda_i P_i$ for all i , that is $(A - \lambda_i I)P_i = 0$ for all $i = 1, 2, \dots, n$. Hence

$$X(t) = c_1 P_1 e^{\lambda_1 t} + \dots + c_n P_n e^{\lambda_n t}$$

is a solution of the homogenous system. ■

$X = Ce^{\lambda t}$ is a solution of (1.1) if and only if $\lambda Ce^{\lambda t} = ACe^{\lambda t}$ that is

$$(A - \lambda I)C = 0.$$

This is equivalent to the simultaneous algebraic equations :

$$\begin{aligned} (a_{11} - \lambda)c_1 + a_{12}c_2 + \dots + a_{1n}c_n &= 0 \\ a_{21}c_1 + (a_{22} - \lambda)c_2 + \dots + a_{2n}c_n &= 0 \\ \vdots & \\ a_{n1}c_1 + a_{n2}c_2 + \dots + (a_{nn} - \lambda)c_n &= 0. \end{aligned} \tag{1.7}$$

Hence C is a non-zero solution vector of (1.7) if and only if λ is a solution of $\det(A - \lambda I) = 0$, expanding the determinant shows that $\det(A - \lambda I) = 0$ is an n -th degree polynomial equation.

Example (2): Solve the system

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + 2x_2 + x_3 \\ \frac{dx_2}{dt} &= 6x_1 - x_2 \\ \frac{dx_3}{dt} &= -x_1 - 2x_2 - x_3 \end{aligned}$$

Solution: First we write this system in matrix form, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix},$$

and

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

That is

$$\frac{dX}{dt} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} X,$$

so

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 12\lambda = 0.$$

That is $\lambda(\lambda + 4)(\lambda - 3) = 0$, then $\lambda_1 = 0$, $\lambda_2 = -4$, $\lambda_3 = 3$.

Thus the system $(A - \lambda I)P = 0$ can be written as :

$$(1 - \lambda)p_1 + 2p_2 + p_3 = 0$$

$$6p_1 - (1 + \lambda)p_2 = 0$$

$$-p_1 - 2p_2 - (1 + \lambda)p_3 = 0$$

we must determine three eigenvectors of the form

$$P_{\lambda_i} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

First, for $\lambda = \lambda_1 = 0$, we get the algebraic system

$$p_1 + 2p_2 + p_3 = 0$$

$$6p_1 - p_2 = 0$$

$$-p_1 - 2p_2 - p_3 = 0.$$

Using Gaussian elimination, we get $p_1 = t \in R$, then $p_2 = 6t$, and $p_3 = -13t$. Choosing $p_1 = 1$, then $p_2 = 6$, and $p_3 = -13$,

hence the first eigenvector is $P_{\lambda_1} = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$.

Second, for $\lambda = \lambda_1 = -4$, we get the algebraic system

$$\begin{aligned} -p_1 - 2p_2 + 3p_3 &= 0 \\ 5p_1 + 2p_2 + p_3 &= 0 \\ 6p_1 + 3p_2 &= 0. \end{aligned}$$

It has infinity many solutions, hence let $p_3 = t \in R$, then $p_2 = 2t$ and $p_1 = -t$. choosing $t = 1$, we get $p_1 = -1$, then $p_2 = 2$ and $p_3 = 2$, hence the second eigenvector is

$$P_{\lambda_2} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Finally, when $\lambda = \lambda_3 = 3$, we get

$$\begin{aligned} -2p_1 + 2p_2 + p_3 &= 0 \\ 6p_1 - 4p_2 &= 0 \\ -p_1 - 2p_2 + 4p_3 &= 0 \end{aligned}$$

the usual choice of $p_1 = -1$ leads to $p_2 = \frac{3}{2}$ and $p_3 = -1$ so that the third

eigenvector is $P_{\lambda_3} = \begin{pmatrix} -1 \\ \frac{3}{2} \\ -1 \end{pmatrix}$. Hence the general solution of the system is

$$X = c_1 \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} -1 \\ \frac{3}{2} \\ -1 \end{pmatrix} e^{3t}.$$

2.2 Repeated Eigenvalues And Jordan Canonical Form.

Consider the system

$$\frac{dX}{dt} = AX \quad (2.1)$$

Where the matrix A has repeated eigenvalues. Suppose A has k distinct eigenvalues $\lambda_1, \dots, \lambda_k$, with multiplicities $n_1, n_2, \dots, n_k$.

We shall find the solution of the homogeneous system by using the Jordan Canonical form.

For every $n \times n$ matrix A we can find a nonsingular matrix P such that the matrix $J = P^{-1}AP$ has the Jordan Canonical form for A , which has the form :

$$J = \begin{pmatrix} J_0 & & 0 \\ & J_1 & \\ & & \ddots \\ 0 & & & J_s \end{pmatrix},$$

where J_0 is a diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_k$, not necessarily distinct, that is

$$J_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix},$$

and each J_p , $p = 1, 2, \dots, s$ is an $n_p \times n_p$ matrix of the form

$$J_p = \begin{pmatrix} \lambda_{k+p} & 1 & 0 & \dots & 0 \\ 0 & \lambda_{k+p} & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \lambda_{k+p} \end{pmatrix},$$

where λ_{k+p} need not be different from λ_{k+i} if $p \neq i$ and $k + n_1 + \dots + n_s = n$. The number λ_i , $i = 1, 2, \dots, k + s$ are the eigenvalues of A if λ_1 is a simple eigenvalue of A , it appears in the block J_0 , the blocks $J_0, J_1, \dots, J_s$ are called the Jordan blocks and J is called the **Jordan Canonical form of A** .

Note that a matrix may be similar to a diagonal matrix without having simple eigenvalues. The identity matrix is such an example. To see a procedure for computing the Jordan Canonical Form see Appendix II.

Theorem 2.2.1: Let A be any $n \times n$ constant matrix. Then e^{At} (see appendix I) is a principle fundamental matrix of the system $\frac{dX}{dt} = AX$.

Proof: By definition

$$e^{At} = I_n + \sum_{m=1}^{\infty} \frac{(tA)^m}{m!}, \text{ for any } t \in R.$$

Since

$$\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A \text{ (see Appendix I).}$$

Then if

$$\Phi(t) = e^{At},$$

then

$$\frac{d\Phi}{dt} = Ae^{At} = A\Phi(t).$$

Note that

$$\Phi(0) = e^{At} \big|_{t=0} = I_n,$$

then

$$\det(e^{At}) = e^{\text{tr}(At)} \neq 0 \text{ for any } t \in R.$$

Therefore

$$\Phi(t) = e^{At}$$

is a principal fundamental matrix of the system

$$\frac{dX}{dt} = AX. \blacksquare$$

We now present a method of evaluating e^{At} and of solving the initial-value problem

$$\frac{dX}{dt} = AX, X(0) = X_0.$$

This method involves the transformation of A into a Jordan Canonical form. Let P be a real $n \times n$ nonsingular matrix such that $J = P^{-1}AP$, consider the transformation

$$X = PY \quad \text{or} \quad Y = P^{-1}X,$$

differentiating both sides with respect to t , we obtain

$$\frac{dY}{dt} = P^{-1} \frac{dX}{dt} = P^{-1}APY = JY$$

hence

$$\frac{dY}{dt} = JY, \quad Y(0) = P^{-1}X_0 \quad (2.3)$$

By last theorem the solution of (2.3) is given by

$$Y(t) = e^{Jt}Y(0),$$

since

$$X(0) = PY(0),$$

then

$$Y(0) = P^{-1}X_0,$$

then

$$P^{-1}X(t) = e^{Jt}P^{-1}X_0,$$

so we have

$$X(t) = Pe^{Jt}P^{-1}X_0.$$

Now let us consider the case when A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, if we choose $P = (P_1, P_2, \dots, P_n)$ in such a way that P_i is an eigenvector corresponding to the eigenvalue λ_i for $i = 1, 2, \dots, n$,

then the matrix $J = P^{-1}AP$ assumed the form

$$J = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \text{then } e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}.$$

We therefore have the following expression for the solution of the diagonal system

$$X(t) = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1}X_0 \quad (2.4)$$

In this method all eigenvalues of A assumed to be distinct. One has to determine the eigenvalues of A , compute n eigenvectors corresponding to the eigenvalues, from the matrix P and evaluate (2.4). In the general case

when A has repeated eigenvalues, it is no longer possible to diagonalize A . However, we can generate linearly independent vectors $P_1, P_2, \dots, P_n$ and an $n \times n$ matrix $P = (P_1, P_2, \dots, P_n)$ which transforms A into Jordan Canonical Form. Thus $J = P^{-1}AP$ here J is a block diagonal of the form

$$J = \begin{pmatrix} J_0 & & \mathbf{0} \\ & J_1 & \\ & & \ddots \\ \mathbf{0} & & & J_s \end{pmatrix},$$

where J_0 is a diagonal matrix with diagonal element $\lambda_1, \lambda_2, \dots, \lambda_k$ (not necessarily distinct), and each J_i is an $n_i \times n_i$ matrix of the form

$$J_i = \begin{pmatrix} \lambda_{k+i} & 1 & & \mathbf{0} \\ & \lambda_{k+i} & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda_{k+i} \end{pmatrix}$$

where λ_{k+i} need not be different from λ_{k+j} , $i \neq j$, and

$$k + n_1 + n_2 + \dots + n_s = n.$$

Since for any block diagonal matrix

$$C = \begin{pmatrix} C_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & C_1 \end{pmatrix},$$

we have

$$C^k = \begin{pmatrix} C_1^k & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & C_1^k \end{pmatrix}, \quad k = 0, 1, 2, \dots$$

Then we get for all $t \in (-\infty, \infty)$;

$$e^{Jt} = \begin{pmatrix} e^{J_0 t} & & \mathbf{0} \\ & e^{J_1 t} & \\ & & \ddots \\ \mathbf{0} & & & e^{J_s t} \end{pmatrix},$$

as before, we have

$$e^{J_0 t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix}.$$

Notice that for any J_i , where $i = 1, 2, \dots, s$. We have $J_i = \lambda_{i+k} I_n + N_i$ where I_n is the $n_i \times n_i$ identity matrix and N_i is the $n_i \times n_i$ nilpotent matrix given by

$$N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{pmatrix},$$

hence

$$e^{J_i t} = e^{(\lambda_{k+i} I + N_i)t} = e^{\lambda_{k+i} t I} \cdot e^{N_i t},$$

where N_i^k is zero matrix for all $k \geq n_i$, so

$$e^{N_i t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \dots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \dots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

then

$$e^{J_i t} = e^{\lambda_{k+i} t I} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \dots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \dots & \frac{t^{n_i-3}}{(n_i-3)!} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

so from (2.4) it now follows that the solution of the homogeneous system is given by

$$X(t) = P \begin{pmatrix} e^{J_0 t} & & 0 \\ & e^{J_1 t} & \\ & & \ddots \\ 0 & & & e^{J_s t} \end{pmatrix} P^{-1} X_0.$$

Example (1): Consider the homogenous system

$$\frac{dX}{dt} = AX$$

where

$$A = \begin{pmatrix} -1 & 0 & -1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & -1 & -1 & -6 & 0 \\ -2 & 0 & -1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 2 & 4 & 1 \end{pmatrix}.$$

The characteristic polynomial of A is $P(\lambda) = (1 - \lambda)^7 \lambda$. Following the procedure outlined in appendix II to generate the Jordan canonical form, we obtain $J = P^{-1}AP$ where

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 2 & 0 & -1 & 1 \\ -1 & 0 & 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

We calculate P^{-1} and we obtain

$$P^{-1} = \begin{pmatrix} 0 & 0 & 2 & 1 & 4 & -2 & 2 \\ 0 & 0 & 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & -1 & -3 & 1 & -1 \\ 1 & 0 & 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

Then

$$e^{Jt} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix} e^t,$$

and

$$P^{-1} e^{Jt} P = \begin{pmatrix} 1 & 0 & 0 & 4 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & 0 & t & 0 \\ t & t & 1 & -2t & 0 & -t & t \\ 1 & 0 & 0 & 1 & 0 & -t & 0 \\ \frac{-t^2}{2} & \frac{-t^2}{2} & t & t^2 & 1 & 1 - \frac{3}{2}t & 3t + \frac{-t^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} e^t.$$

2.3 Matrix Exponentials

Let A be $n \times n$ matrix of constants, then the fundamental matrix of solutions of the system of linear differential equations

$$\frac{dX}{dt} = AX \quad (3.1)$$

is given by the matrix e^{At} .

In this section we shall give different methods for computing e^{At} without using Jordan Canonical Form, we use just Caley Hamilton Theorem.

The first method is due to Fulmer [7].

Theorem 2.3.1: Every entry of the matrix e^{At} satisfies the n -th order linear differential equation $\det(A - I_n D)X(t) = 0$.

Proof: $\det(A - I_n D) = D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0$

$$\det(A - I_n D)X(t) = (D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)X(t)$$

If we differentiate the matrix e^{At} k - times we get

$$\begin{aligned} \frac{d}{dt}(e^{At}) &= Ae^{At} \\ \frac{d^2}{dt^2}(e^{At}) &= A^2 e^{At} \\ &\vdots \\ \frac{d^n}{dt^n}(e^{At}) &= A^n e^{At}. \end{aligned} \tag{3.2}$$

Thus for substitute

$$X(t) = e^{At}$$

we have

$$\begin{aligned} \det(A - I_n D)e^{At} &= (D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)e^{At} \\ &= D^n e^{At} + a_{n-1}D^{n-1}e^{At} + \dots + a_1De^{At} + a_0e^{At} \\ &= A^n e^{At} + a_{n-1}A^{n-1}e^{At} + \dots + a_1Ae^{At} + a_0e^{At} \\ &= (A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I)e^{At} \end{aligned}$$

so by Caley-Hamilton Theorem we have

$$A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I = 0,$$

hence

$$\det(A - I_n D)e^{At} = 0.$$

Thus each entry of the matrix e^{At} satisfies the n^{th} order linear differential equation

$$\det(A - I_n D)X(t) = 0. \blacksquare$$

Corollary 2.3.2: e^{At} is the unique solution of the initial-value problem

$$\det(A - I_n D)X(t) = 0, \text{ with}$$

$$\begin{aligned}
X(0) &= I \\
\frac{d}{dt}(X(0)) &= A \\
\frac{d^2}{dt^2}(X(0)) &= A^2 \\
&\vdots \\
\frac{d^{n-1}}{dt^{n-1}}(X(0)) &= A^{n-1}.
\end{aligned} \tag{3.3}$$

Proof: Let

$$X_1 = e^{At_1} \text{ and } X_2 = e^{At_2}$$

be two solutions to the n^{th} order matrix differential equations (3.1) and the initial conditions (3.3), and let $X = X_1 - X_2$, we want to show that

$$X_1 = X_2.$$

Then X satisfies (3.1) with initial conditions

$$X(0) = \frac{d}{dt}(X(0)) = \dots = \frac{d^{n-1}}{dt^{n-1}}(X(0)) = \mathbf{0}.$$

Therefore, each entry of X satisfies the scalar initial value problem

$$\frac{d^n}{dt^n}(x) + c_{n-1} \frac{d^{n-1}}{dt^{n-1}}(x) + \dots + c_1 \frac{d}{dt}(x) + c_0 x = 0$$

with

$$x(0) = \frac{d}{dt}(x(0)) = \dots = \frac{d^{n-1}}{dt^{n-1}}(x(0)) = 0,$$

with solution $x(t) = 0$, and so $X(t) = 0$, for every $t \in \mathbb{R}$. That is

$$X_1 = X_2. \blacksquare$$

Suppose A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, the general solution of the differential equation

$$\det(A - I_n D)X(t) = 0$$

is

$$X(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t}$$

where the C_k are $n \times n$ constants matrices.

The n initial conditions yield a system of n linear equations for the n unknown matrices C_k :

$$\begin{aligned} C_1 + C_2 + \cdots + C_n &= I_n \\ \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_n C_n &= A \\ \lambda_1^2 C_1 + \lambda_2^2 C_2 + \cdots + \lambda_n^2 C_n &= A^2 \\ &\vdots \\ \lambda_1^{n-1} C_1 + \lambda_2^{n-1} C_2 + \cdots + \lambda_n^{n-1} C_n &= A^{n-1}. \end{aligned}$$

We can write this system in the following matrix form

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{n-1} \end{pmatrix}$$

define

$$V := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

which is called the **Vandermonde matrix** of order n , that is determinant is equal to the product of all differences of the form $\lambda_i - \lambda_j (1 \leq i \leq n)$.

So we have

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{pmatrix} = V^{-1} \begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{n-1} \end{pmatrix}.$$

Hence

$$C_i = v_{i1}I + v_{i2}A + v_{i3}A^2 + \cdots + v_{in}A^{n-1} = \sum_{j=1}^n v_{ij}A^{j-1},$$

where v_{ij} are elements in the i -th row of the matrix V^{-1} .

The method is essentially the same when A has repeated eigenvalues. Suppose A has k distinct eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_k$ with multiplicities $n_1, n_2, \dots, n_k$.

Then the general solution of $\det(A - I_n D)X(t) = 0$ is

$$X(t) = (C_{11} + tC_{12} + \dots + t^{n_1-1}C_{1n_1})e^{\lambda_1 t} + \dots + (C_{k1} + tC_{k2} + \dots + t^{n_k-1}C_{kn_k})e^{\lambda_k t} \quad (3.4)$$

The n initial conditions again yield a system of n linear equations for the $n \times n$ matrices C_{ij} , but in this case, the coefficient matrix for the system is called the **Confluent Vandermonde matrix** and written as

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \\ \lambda_1 & 1 & \dots & 0 & \dots & \lambda_k & 1 & \dots & 0 \\ \lambda_1^2 & 2\lambda_1 & \dots & 0 & \dots & \lambda_k^2 & 2\lambda_k & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & (n-1)\lambda_1^{n-2} & \dots & \frac{(n-1)!}{(n-n_1)!}\lambda_1^{n-n_1} & \dots & \lambda_k^{n-1} & (n-1)\lambda_k^{n-2} & \dots & \frac{(n-1)!}{(n-n_k)!}\lambda_k^{n-n_k} \end{pmatrix}$$

The matrices C_{ij} are again obtained as polynomials of at most degree $n-1$ in A , and the coefficients in these polynomials are the entries in the rows of the inverse of the confluent Vandermonde matrix.

Example (1): Compute e^{At} for the system

$$\frac{dX}{dt} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} X.$$

Solution: The eigenvalue of the matrix A are 1, 3, and -4 ; hence the Vandermonde matrix is

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -4 \\ 1 & 9 & 16 \end{pmatrix}$$

Thus,

$$e^{At} = C_1 e^t + C_2 e^{3t} + C_3 e^{-4t},$$

where

$$C_i = \sum_{j=1}^4 v_{ij} A^{j-1}$$

and v_{ij} are entries of

$$V^{-1} = \begin{pmatrix} \frac{18}{5} & -4 & \frac{7}{10} \\ -2 & 3 & \frac{-1}{2} \\ \frac{-3}{5} & 1 & \frac{-1}{5} \end{pmatrix},$$

so

$$C_1 = \sum_{j=1}^3 v_{1j} A^{j-1} = v_{11}I + v_{12}A + v_{13}A^2 = \begin{pmatrix} 16\frac{3}{5} & -14\frac{1}{10} & -2\frac{1}{10} \\ -14\frac{1}{10} & 18\frac{4}{5} & 3\frac{3}{10} \\ -2\frac{1}{10} & 4\frac{7}{10} & -\frac{8}{5} \end{pmatrix}$$

$$C_2 = \sum_{j=1}^3 v_{2j} A^{j-1} = v_{21}I + v_{22}A + v_{23}A^2 = \begin{pmatrix} -\frac{3}{2} & 10\frac{1}{2} & \frac{3}{2} \\ 10\frac{1}{2} & -12 & -2\frac{1}{2} \\ \frac{3}{2} & -3\frac{1}{2} & 3 \end{pmatrix}$$

$$C_3 = \sum_{j=1}^3 v_{3j} A^{j-1} = v_{31}I + v_{32}A + v_{33}A^2 = \begin{pmatrix} -1\frac{3}{5} & 2\frac{2}{5} & \frac{3}{5} \\ 3\frac{3}{5} & -5\frac{2}{5} & -\frac{4}{5} \\ \frac{3}{5} & -1\frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

Example (2): Compute e^{At} for the system

$$\frac{dX}{dt} = \begin{pmatrix} 2 & 2 & -2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} X.$$

Solution: The eigenvalue of the matrix A are 2, 2, 2 and 1; hence the Vandermonde matrix is

$$V_c = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 4 & 4 & 1 & 1 \\ 16 & 12 & 4 & 1 \end{pmatrix}$$

Thus,

$$e^{At} = (C_1 + C_2 t + \frac{t^2}{2!} C_3) e^{2t} + C_4 e^t,$$

where

$$C_i = \sum_{j=1}^4 v_{ij} A^{j-1}$$

and v_{ij} are entries of

$$V_c^{-1} = \begin{pmatrix} \frac{-1}{7} & \frac{4}{7} & \frac{-4}{7} & \frac{-1}{7} \\ \frac{-6}{7} & \frac{3}{7} & \frac{4}{7} & \frac{-1}{7} \\ \frac{20}{7} & \frac{-24}{7} & \frac{3}{7} & \frac{1}{7} \\ \frac{8}{7} & \frac{-4}{7} & \frac{4}{7} & \frac{-1}{7} \end{pmatrix},$$

so we can calculate C_1, C_2, C_3 as in example (1).

This method is not practical for computing e^{At} when A is a large matrix, since it requires the computation of the first $n-1$ power of A .

This alone require $n^4 - 2n^3$ multiplications.

Now we give a method of computing e^{At} due to Leonard [9].

This method depends on corollary (2.3.2) and theorem (2.3.1). And the following theorem gives a method for constructing e^{At} from the solutions of certain scalar initial value problem.

Theorem 2.3.3: Let A be a constant $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0;$$

then

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \dots + x_n(t)A^{n-1}$$

where $x_k(t)$, $k = 1, 2, \dots, n$ are solutions to the n^{th} order scalar differential equation

$$\frac{d^n}{dt^n}(x) + c_{n-1} \frac{d^{n-1}}{dt^{n-1}}(x) + \dots + c_1 \frac{d}{dt}(x) + c_0 x = 0 \quad (3.5)$$

satisfying the following initial conditions:

$$\left. \begin{array}{l} x_1(0) = 1 \\ \frac{d}{dt}(x_1(0)) = 0 \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}}(x_1(0)) = 0 \end{array} \right\} \left. \begin{array}{l} x_2(0) = 0 \\ \frac{d}{dt}(x_2(0)) = 1 \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}}(x_2(0)) = 0 \end{array} \right\} \cdots \left. \begin{array}{l} x_n(0) = 0 \\ \frac{d}{dt}(x_n(0)) = 0 \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}}(x_n(0)) = 1 \end{array} \right\} \quad (3.6)$$

Proof: Let A be a constant $n \times n$ matrix, and let

$$p(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

be its characteristic polynomial. Define

$$\Phi(t) = x_1(t)I + x_2(t)A + x_3(t)A^2 + \cdots + x_n(t)A^{n-1},$$

where $x_k(t), k = 1, 2, \dots, n$, are the unique solutions to the n^{th} order scalar differential equations

$$\frac{d^n}{dt^n}(x) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(x) + \cdots + c_1\frac{d}{dt}(x) + c_0x = 0$$

satisfying the initial condition (3.6).

Then

$$\begin{aligned} & \frac{d^n}{dt^n}(\Phi) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(\Phi) + \cdots + c_1\frac{d}{dt}(\Phi) + c_0\Phi \\ &= \left(\frac{d^n}{dt^n}(x_1) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(x_1) + \cdots + c_1\frac{d}{dt}(x_1) + c_0x_1 \right) I \\ &+ \left(\frac{d^n}{dt^n}(x_2) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(x_2) + \cdots + c_1\frac{d}{dt}(x_2) + c_0x_2 \right) A \\ &+ \left(\frac{d^n}{dt^n}(x_3) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(x_3) + \cdots + c_1\frac{d}{dt}(x_3) + c_0x_3 \right) A^2 \\ &\quad \vdots \\ &+ \left(\frac{d^n}{dt^n}(x_n) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(x_n) + \cdots + c_1\frac{d}{dt}(x_n) + c_0x_n \right) A^{n-1} \\ &= 0.I + 0.A + 0.A^2 + \cdots + 0.A^{n-1} = 0. \end{aligned}$$

Thus

$$\frac{d^n}{dt^n}(\Phi) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}(\Phi) + \cdots + c_1\frac{d}{dt}(\Phi) + c_0\Phi = 0 \text{ for every } t \in R$$

and,

$$\Phi(0) = x_1(0)I + x_2(0)A + x_3(0)A^2 + \cdots + x_n(0)A^{n-1} = I$$

$$\frac{d}{dt}(\Phi(0)) = \frac{d}{dt}(x_1(0))I + \frac{d}{dt}(x_2(0))A + \cdots + \frac{d}{dt}(x_n(0))A^{n-1} = A$$

$$\vdots$$

$$\frac{d^{n-1}}{dt^{n-1}}(\Phi(0)) = \frac{d^{n-1}}{dt^{n-1}}(x_1(0))I + \frac{d^{n-1}}{dt^{n-1}}(x_2(0))A + \cdots + \frac{d^{n-1}}{dt^{n-1}}(x_n(0))A^{n-1} = A^{n-1}$$

Therefore

$$\Phi(t) = x_1(t)I + x_2(t)A + \cdots + x_n(t)A^{n-1}$$

satisfying the initial value problem

$$\frac{d^n}{dt^n}(\Phi) + c_{n-1} \frac{d^{n-1}}{dt^{n-1}}(\Phi) + \cdots + c_1 \frac{d}{dt}(\Phi) + c_0 \Phi = 0$$

$$\Phi(0) = I, \frac{d}{dt}(\Phi(0)) = A, \frac{d^2}{dt^2}(\Phi(0)) = A^2, \dots, \frac{d^{n-1}}{dt^{n-1}}(\Phi(0)) = A^{n-1}.$$

By corollary 2.3.2 the uniqueness of the solution gives that

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \cdots + x_n(t)A^{n-1}, \text{ for every } t \in \mathbb{R}. \blacksquare$$

Example (3): Compute e^{At} for the system of equation

$$\frac{dX}{dt} = AX$$

where the coefficient matrix is

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Solution: The eigenvalue of A is λ of multiplicity 4.

The general solution to

$$\frac{d^4}{dt^4}(x) + c_3 \frac{d^3}{dt^3}(x) + c_2 \frac{d^2}{dt^2}(x) + c_1 \frac{d}{dt}(x) + c_0 x = 0$$

is given by

$$x(t) = a_1 e^{\lambda t} + a_2 t e^{\lambda t} + a_3 t^2 e^{\lambda t} + a_4 t^3 e^{\lambda t}$$

and therefore

(i) The solution satisfying the initial conditions

$$x(0) = 1, \frac{d}{dt}(x(0)) = 0, \frac{d^2}{dt^2}(x(0)) = 0, \frac{d^3}{dt^3}(x(0)) = 0 \text{ is}$$

$$x_1(t) = (1 - \lambda t - \frac{\lambda^2}{3!} t^3) e^{\lambda t}.$$

(ii) While the second solution satisfying the initial condition

$$x(0) = 0, \frac{d}{dt}(x(0)) = 1, \frac{d^2}{dt^2}(x(0)) = 0, \frac{d^3}{dt^3}(x(0)) = 0$$

is

$$x_2(t) = (t - \frac{\lambda}{2} t^2 - \frac{\lambda^2}{3!} t^3) e^{\lambda t}.$$

(iii) And the third solution satisfies the initial conditions

$$x(0) = 0, \frac{d}{dt}(x(0)) = 0, \frac{d^2}{dt^2}(x(0)) = 1, \frac{d^3}{dt^3}(x(0)) = 0$$

is

$$x_3(t) = \frac{-(\lambda^2 + \lambda)}{3!} t^3 e^{\lambda t}.$$

(iv) Finally, the forth solution satisfies the initial conditions

$$x(0) = 0, \frac{d}{dt}(x(0)) = 0, \frac{d^2}{dt^2}(x(0)) = 0, \frac{d^3}{dt^3}(x(0)) = 1$$

is

$$x_4(t) = \frac{(1 - \lambda^2)}{3!} t^3 e^{\lambda t}.$$

Also

$$A^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & 0 \\ 0 & \lambda^2 & 2\lambda & 1 \\ 0 & 0 & \lambda^2 & 2\lambda \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}.$$

therefore

$$e^{At} = (1 - \lambda t - \frac{\lambda^2}{2!}t^2) e^{\lambda t} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (t - \frac{\lambda}{2}t^2 - \frac{\lambda^2}{3!}t^3) e^{\lambda t} \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$= (\frac{(\lambda^2 + \lambda)}{3!}t^3) e^{\lambda t} \begin{pmatrix} \lambda^2 & 2\lambda & 1 & 0 \\ 0 & \lambda^2 & 2\lambda & 1 \\ 0 & 0 & \lambda^2 & 2\lambda \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix} + (\frac{(1-\lambda^2)}{3!}t^3) e^{\lambda t} \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}$$

Then

$$e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} e^{\lambda t}.$$

In the previous method it requires to solve, for $n \times n$ matrix n initial value problems for an n^{th} order linear differential equation with constant coefficients.

Now we present a result due to Liz [10], which is deduced from the previous theorem in which the practical method to calculate the exponential of a matrix is simpler.

Consider the homogeneous differential equation

$$\frac{d^n}{dt^n}(x) + c_{n-1} \frac{d^{n-1}}{dt^{n-1}}(x) + \cdots + c_1 \frac{d}{dt}(x) + c_0 x = 0 \quad (3.7)$$

Let

$$\{\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)\}$$

be a fundamental set of solutions of it, now we define

$$B_t = \begin{pmatrix} \varphi_1(t) & \frac{d}{dt}(\varphi_1(t)) & \dots & \frac{d^{n-1}}{dt^{n-1}}(\varphi_1(t)) \\ \varphi_2(t) & \frac{d}{dt}(\varphi_2(t)) & \dots & \frac{d^{n-1}}{dt^{n-1}}(\varphi_2(t)) \\ \vdots & \vdots & & \vdots \\ \varphi_n(t) & \frac{d}{dt}(\varphi_n(t)) & \dots & \frac{d^{n-1}}{dt^{n-1}}(\varphi_n(t)) \end{pmatrix}$$

so B_t is invertible for every $t \in \mathbb{R}$.

Theorem 2.3.4: Let A be a constant $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0;$$

then

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \dots + x_n(t)A^{n-1},$$

where

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = B_0^{-1} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix} \quad (3.8)$$

Proof: Consider the n^{th} order differential equation (3.7).

By theorem (2.3.3)

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \dots + x_n(t)A^{n-1}$$

where $x_k(t)$ is the solution of (3.7) for $k = 1, 2, \dots, n$, note that

$\{x_1(t), x_2(t), \dots, x_n(t)\}$ is also a fundamental set of solutions of (3.7).

since $W(x_1(t), \dots, x_n(t)) = 1$ at $t = 0$. Moreover, by using the uniqueness of solution for the initial value problem (3.7) and (3.8), we obtain that for each $k = 1, \dots, n$:

$$\varphi_k(t) = \varphi_k(0)x_1(t) + \frac{d}{dt}(\varphi_k(0))x_2(t) + \dots + \frac{d^{n-1}}{dt^{n-1}}(\varphi_k(0))x_n(t)$$

that is,

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix} = B_0 \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

since B_0 is invertible, we obtain the equality (3.8). ■

Example(4): Find e^{At} where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is

$P(\lambda) = (\lambda^2 + 1)(\lambda - 1)$, and hence a fundamental system for the differential equation whose characteristic equation is $P(\lambda)$ is given by

$$S = \{\cos t, \sin t, e^t\}.$$

The matrices B_0 and its inverse are

$$B_0 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_0^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = B_0^{-1} \begin{pmatrix} \cos t \\ \sin t \\ e^t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos t - \sin t + e^t \\ 2 \sin t \\ -\cos t - \sin t + e^t \end{pmatrix}.$$

$$\begin{aligned} e^{At} &= x_1(t)I + x_2(t)A + x_3(t)A^2 \\ &= \begin{pmatrix} x_1(t) - x_3(t) & x_2(t) & 0 \\ -x_2(t) & x_1(t) - x_3(t) & 0 \\ 0 & 0 & x_1(t) + x_2(t) + x_3(t) \end{pmatrix} \\ e^{At} &= \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}. \end{aligned}$$

2.4 Application: (Matrix Riccati Equations)

There are many applications of the system of linear differential equations one of this is the matrix Riccati differential equation, the paper [6] contains an extended version of this note.

Definition 2.4.1: For any $m \times n$ matrix W we can define the matrix differential equation by :

$$\frac{dW}{dt} = B_{21} + B_{22}W - WB_{11} - WB_{12}W \quad (4.1)$$

which is called the **matrix Riccati differential equation** denoted by (RDE) , where B_{11} , B_{12} , B_{21} , B_{22} are constant matrices of dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively.

The question is: how can we find a solution for (RDE) ? In the following we show that, finding a solution of such equations is equivalent of finding a solution of a homogeneous system of differential equations.

Let

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} Q \\ P \end{pmatrix}$$

with $n \times n$ and $m \times n$ matrix function Q and P respectively.

It will be shown that the solutions of (4.1) can be obtained from the solutions of the linear system of differential equations

$$\frac{dY}{dt} = BY \quad (4.2)$$

If

$$Y = \begin{pmatrix} Q \\ P \end{pmatrix}$$

is a solution of (4.2), then PQ^{-1} is a solution of (4.1),

Now,

$$\frac{d}{dt}(PQ^{-1}) = \frac{d}{dt}(P)Q^{-1} + P \frac{d}{dt}(Q^{-1}),$$

since

$$\frac{dY}{dt} = \frac{d}{dt} \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}(Q) \\ \frac{d}{dt}(P) \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} B_{11}Q + B_{12}P \\ B_{21}Q + B_{22}P \end{pmatrix}$$

we have

$$\frac{d}{dt}(P) = B_{21}Q + B_{22}P,$$

$$\frac{d}{dt}(Q) = B_{11}Q + B_{12}P.$$

Want to compute $\frac{d}{dt}(Q^{-1})$ from the formula $QQ^{-1} = I$.

$$0 = \frac{d}{dt}(I) = \frac{d}{dt}(QQ^{-1}) = \frac{d}{dt}(Q) \cdot Q^{-1} + Q \cdot \frac{d}{dt}(Q^{-1}),$$

so we get

$$\frac{d}{dt}(Q^{-1}) = -Q^{-1} \frac{d}{dt}(Q) Q^{-1}.$$

Hence

$$\begin{aligned} \frac{dW}{dt} &= \frac{d}{dt}(PQ^{-1}) = (B_{21}Q + B_{22}P)Q^{-1} + P(-Q^{-1}B_{11} - Q^{-1}B_{12}PQ^{-1}) \\ &= B_{21} + B_{22}PQ^{-1} - PQ^{-1}B_{11} - PQ^{-1}B_{12}PQ^{-1} \\ &= B_{21} + B_{22}W - WB_{11} - WB_{12}W. \end{aligned}$$

Therefore $W = PQ^{-1}$ is a solution of (4.1). ■

Example(1): Solve the matrix Riccati differential equation

$$\begin{aligned} \frac{dW}{dt} &= \begin{pmatrix} -2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 4 & 1 \end{pmatrix} W - W \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix} \\ &\quad - W \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -6 & 0 \end{pmatrix} W \end{aligned} \quad (4.3)$$

Solution: The solution can be obtained from the solution of the linear system of differential equations

$$\frac{dY}{dt} = BY,$$

where

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\text{and } B_{11} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -6 & 0 \end{pmatrix},$$

$$B_{21} = \begin{pmatrix} -2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 4 & 1 \end{pmatrix}.$$

From Example (1) page (29) we see that

$$Y(t) = \begin{pmatrix} 1 & 0 & 0 & 4 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & 0 & t & 0 \\ t & t & 1 & -2t & 0 & -t & t \\ 1 & 0 & 0 & 1 & 0 & -t & 0 \\ \frac{-t^2}{2} & \frac{-t^2}{2} & t & t^2 & 1 & 1 - \frac{3}{2}t & 3t + \frac{-t^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} e^t.$$

Thus

$$Y(t) = \begin{pmatrix} Q \\ P \end{pmatrix},$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & t & 1 \end{pmatrix} e^t,$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{t^2}{2} & \frac{t^2}{2} & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$W = PQ^{-1} = \begin{pmatrix} e^{2t} & 0 & 0 \\ \frac{t^2}{2}e^{2t} - t^2 & -\frac{t^2}{2} & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a solution of the matrix Riccati equation (4.3). ■

CHAPTER THREE

Elementary Treatment Of Systems Of Homogeneous Ordinary Linear Differential Equations With Constant Coefficients

Introduction

In this chapter it will be shown that, the general solution of homogeneous linear differential equation with constant coefficients can be solved by simple method, it will be shown that one need only elementary results from linear algebra to give a full treatment. It will also be shown that Fulmer's method for calculating the fundamental solution, can be proved without assuming the Caley-Hamilton Theorem. This method is due to Besjes [4].

3.1 Diagonalization By Means Of Multiplication.

In this section we shall find the general solution of a system of n homogeneous first order linear differential equations with constant coefficients

$$\frac{dX}{dt} = AX \quad (1.1)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Suppose that A is diagonalizable and B is non-singular matrix such that $B^{-1}AB = D$ where D is diagonal, then (1.1) can be converted as follows

Multiply both sides of (1.1) by B^{-1} , we get

$$B^{-1} \frac{dX}{dt} = B^{-1} A B B^{-1} X,$$

let

$$Y = B^{-1} X$$

then

$$\frac{dY}{dt} = (B^{-1} A B) Y \quad (1.2)$$

Hence (1.2) is a diagonal system which can be solved.

But there is not always a matrix B such that $B^{-1}AB$ is a diagonal matrix (and it is not simple to find such a matrix B , if it exists). However, one could try to treat the system the same way as single differential equations.

If A is not diagonalizable (or if it is difficult to find B such that $B^{-1}AB$ is diagonal matrix), then we use diagonalization by mean of multiplication: We write the system (1.1) by using the differential operator notation

$$D = \frac{d}{dt} \quad \text{as follows } DX = AX \text{ i.e. } AX - DX = 0. \text{ Then}$$

$$(A - I_n D)X(t) = 0 \quad (1.3)$$

where I_n is the $n \times n$ identity matrix.

Consider the matrix $Q(D) := (q_{kl})$ such that $(-1)^{k+l} q_{kl}$ is the determinant of the matrix obtained by deleting the l^{th} row and k^{th} column of the matrix $(A - I_n D)$.

The matrix $Q(D)$ is well defined, for its elements are polynomials in D .

Theorem 3.1.1: The following identities holds :

$$(A - I_n D)Q(D) = I_n \det(A - I_n D) = Q(D)(A - I_n D).$$

Proof: First we proof the first equality

$$(A - I_n D)Q(D) = I_n \det(A - I_n D)$$

Denote the determinants of the matrix $(A - I_n D)$ after deleting i^{th} row and j^{th} column by $|A - I_n D|_{ij}$. Let $C = (A - I_n D)Q(D)$ then

$$C_{ii} = \sum_{k=1}^n (a_{ik} - D\delta_{ik})q_{ki} = \sum_{k=1}^n (a_{ik} - D\delta_{ik})(-1)^{k+i} |A - I_n D|_{ik}$$

$$= \det(A - I_n D), \text{ where } \delta_{ij} = \begin{cases} 1 & , \text{ if } i = j \\ 0 & , \text{ otherwise.} \end{cases}$$

The last expression is the expansion of $\det(A - I_n D)$ through the i^{th} row.

Hence $C_{ii} = \det(A - I_n D)$. Now if $i \neq j$

$$C_{ij} = \sum_{k=1}^n (a_{ik} - D\delta_{ik})q_{kj} = \sum_{k=1}^n (a_{ik} - D\delta_{ik})(-1)^{k+j} |A - I_n D|_{jk}.$$

We need to show that $C_{ij} = 0$.

Consider the matrix B obtained from $(A - I_n D)$ by replacing the j^{th} row of $(A - I_n D)$ by its i^{th} row, hence

$$B = \begin{pmatrix} a_{11} - D & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - D & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} - D & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} - D & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ a & a & \cdots & a_{nn} - D \end{pmatrix}.$$

Thus B is a matrix having two identical rows, the i^{th} and j^{th} row. Then $\det(B) = 0$. Now expand $\det(B)$ on the j^{th} row, the elements of the j^{th} row of B are $a_{j1}, a_{j2}, \dots, a_{jj} - D, \dots, a_{jn}$. Thus

$$0 = \det(B) = \sum_{k=1}^n (a_{jk} - D\delta_{jk})(-1)^{k+j} |A - I_n D|_{jk}.$$

Thus $C_{ij} = 0$ for $j \neq i$. Hence

$$(A - I_n D)Q(D) = \begin{pmatrix} \det(A - I_n D) & & & 0 \\ & \det(A - I_n D) & & \\ 0 & & \ddots & \\ & & & \det(A - I_n D) \end{pmatrix}$$

$$= I_n \det(A - I_n D).$$

In the same way, we can show the second equality:

$$Q(D)(A - I_n D) = I_n \det(A - I_n D). \blacksquare$$

Corollary 3.1.2: From the first identity the system $(A - I_n D)X(t) = 0$ is equivalent to the diagonal system

$$\det(A - I_n D)V(t) = 0, \quad X(t) = Q(D)V(t).$$

Proof: If $V(t)$ is a solution of $\det(A - I_n D)V(t) = 0$.

Then $Q(D)V(t)$ is a solution of $(A - I_n D)X(t) = 0$.

Conversely, given any solution $X(t)$ of $(A - I_n D)X(t) = 0$, so there exist at least one function $V(t)$, such that $X(t) = Q(D)V(t)$ then

$$\begin{aligned} (A - I_n D)X(t) &= (A - I_n D)Q(D)V(t) \\ &= I_n \det(A - I_n D)V(t) = 0 \end{aligned}$$

thus $V(t)$ is a solution of $\det(A - I_n D)V(t) = 0$, and $Q(D)V(t)$ is a solution of $(A - I_n D)X(t) = 0$. ■

Corollary 3.1.3: From the second identity

$$Q(D)(A - I_n D) = I_n \det(A - I_n D),$$

$(A - I_n D)X(t) = 0$, can be reduced to

$$Q(D)(A - I_n D)X(t) = I_n \det(A - I_n D)X(t) = 0.$$

This is a diagonal system, so each component $X(t)$ of the vector $X(t)$ has to be of the form $p_1(t)e^{\lambda_1 t} + p_2(t)e^{\lambda_2 t} + \dots + p_s(t)e^{\lambda_s t}$ where λ_k $k=1, \dots, s$ are the distinct eigenvalues, and $p_k(t)$ are polynomials of degree $n_k - 1$.

Hence the form of $X(t)$ is

$$X(t) = \sum_{k=1}^s p_k(t) e^{\lambda_k t}.$$

Then

$$X(t) = \sum_{k=1}^s \left(c_0^{(k)} + c_1^{(k)} t + \frac{1}{2!} c_2^{(k)} t^2 + \dots + \frac{1}{(n_k-1)!} c_{n_k-1}^{(k)} t^{n_k-1} \right) e^{\lambda_k t}.$$

Hence,

$$0 = (A - I_n D)X(t)$$

$$\begin{aligned} &= \sum_{k=1}^s (A - I_n D) \left(c_0^{(k)} + c_1^{(k)} t + \frac{1}{2!} c_2^{(k)} t^2 + \dots + \frac{1}{(n_k-1)!} c_{n_k-1}^{(k)} t^{n_k-1} \right) e^{\lambda_k t} \\ &= \sum_{k=1}^s \left((Ac_0^{(k)} - \lambda_k I_n c_0^{(k)}) + (Ac_1^{(k)} t - I_n c_1^{(k)} t \lambda_k - I_n c_1^{(k)}) + \left(\frac{1}{2!} Ac_2^{(k)} t^2 \right. \right. \\ &\quad \left. \left. - I_n \frac{c_2^{(k)}}{2!} t^2 \lambda_k - I_n c_2^{(k)} t \right) + \dots + \left(A \frac{1}{(n_k-1)!} c_{n_k-1}^{(k)} t^{n_k-1} - I_n \frac{c_{n_k-1}^{(k)}}{(n_k-1)!} t^{n_k-1} \lambda_k \right. \right. \\ &\quad \left. \left. - I_n \frac{c_{n_k-1}^{(k)}}{(n_k-2)!} t^{n_k-2} \right) \right) e^{\lambda_k t} \\ &= \sum_{k=1}^s \left((Ac_0^{(k)} - \lambda_k I_n c_0^{(k)} - I_n c_1^{(k)}) + (Ac_1^{(k)} t - \lambda_k I_n c_1^{(k)} t - I_n c_2^{(k)} t) \right. \\ &\quad \left. + \left(\frac{1}{2!} Ac_2^{(k)} t^2 - \lambda_k I_n \frac{c_2^{(k)}}{2!} t^2 - \frac{1}{2!} I_n c_3^{(k)} t^2 \right) + \dots + \right. \\ &\quad \left. \left(A \frac{1}{(n_k-1)!} c_{n_k-1}^{(k)} t^{n_k-1} - \lambda_k I_n \frac{c_{n_k-1}^{(k)}}{(n_k-1)!} t^{n_k-1} \right) \right) e^{\lambda_k t} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^s \left((A - \lambda_k I_n) c_0^{(k)} - c_1^{(k)} \right) + \left((A - \lambda_k I_n) c_1^{(k)} - c_2^{(k)} \right) t \\
&\quad + \frac{1}{2!} \left((A - \lambda_k I_n) c_2^{(k)} - c_3^{(k)} \right) t^2 + \cdots + \\
&\quad \frac{1}{(n_k-1)!} \left((A - \lambda_k I_n) c_{n_k-1}^{(k)} t^{n_k-1} \right) \Big) e^{\lambda_k t}
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= \sum_{k=1}^s \left(\left((A - \lambda_k I_n) c_0^{(k)} - c_1^{(k)} \right) + \left((A - \lambda_k I_n) c_1^{(k)} - c_2^{(k)} \right) t \right. \\
&\quad \left. + \frac{1}{2!} \left((A - \lambda_k I_n) c_2^{(k)} - c_3^{(k)} \right) t^2 + \cdots + \right. \\
&\quad \left. \frac{1}{(n_k-2)!} \left((A - \lambda_k I_n) c_{n_k-2}^{(k)} - c_{n_k-1}^{(k)} \right) t^{n_k-2} \right. \\
&\quad \left. + \frac{1}{(n_k-1)!} \left((A - \lambda_k I_n) c_{n_k-1}^{(k)} \right) t^{n_k-1} \right) e^{\lambda_k t}
\end{aligned}$$

so all the coefficients have to be zeros, so we get the following theorem:

Theorem 3.1.4: The system of differential equations $(A - I_n D)X(t) = 0$

(i.e. $\frac{dX}{dt} = AX$) is equivalent to the following system of linear algebraic equations, where $k = 1, 2, \dots, s$,

$$\begin{aligned}
(A - \lambda_k I_n) c_{n_k-1}^{(k)} &= 0 \\
(A - \lambda_k I_n) c_{n_k-2}^{(k)} &= c_{n_k-1}^{(k)} \\
&\vdots \\
(A - \lambda_k I_n) c_0^{(k)} &= c_1^{(k)}
\end{aligned} \tag{1.4}$$

where

$$X(t) = \sum_{k=1}^s \left(c_0^{(k)} + c_1^{(k)} t + \cdots + \frac{1}{(n_k-1)!} c_{n_k-1}^{(k)} t^{n_k-1} \right) e^{\lambda_k t}$$

is the solution of $\det(A - I_n D)x(t) = 0$. ■

3.2 System Of Linear Algebraic Equations.

Let λ be an eigenvalue of $n \times n$ matrix A and m its multiplicity. Consider the following system of m linear algebraic equations:

$$\begin{aligned}(A - \lambda I_n)d_{m-1} &= 0 \\(A - \lambda I_n)d_{m-2} &= d_{m-1} \\(A - \lambda I_n)d_{m-3} &= d_{m-2} \\&\vdots \\(A - \lambda I_n)d_0 &= d_1\end{aligned}\tag{2.1}$$

Theorem 3.2.1: The system (2.1) has m linearly independent solution chains $[d_{m-1}, d_{m-2}, \dots, d_0]$.

Proof: By induction, for $m=1$ we have $(A - \lambda I_n)d_0 = 0$, so the system $(A - \lambda I_n)d_0 = 0$ has a solution, then so it is true for $m=1$ and for any n .

Now assume it is true for $m-1$ and any n . Let $\mathbf{c}$ be an eigenvector of A having λ as an eigenvalue. By definition $\mathbf{c} \neq 0$, assume that $c_1 = 1$, (where c_1 is the first component of the vector $\mathbf{c}$).

We can write

$$\begin{aligned}d_{m-1} &= \mathbf{c}v_{m-1} \\d_{m-2} &= \mathbf{c}v_{m-2} + (0, w_{m-2}) \\d_{m-3} &= \mathbf{c}v_{m-3} + (0, w_{m-3}) \\d_{m-4} &= \mathbf{c}v_{m-4} + (0, w_{m-4}) \\&\vdots \\d_1 &= \mathbf{c}v_1 + (0, w_1) \\d_0 &= (0, w_0)\end{aligned}\tag{2.2}$$

where $(0, w_0), (0, w_1), (0, w_2), \dots, (0, w_{m-2})$ are column vectors whose first coordinate is equal to 0, and $v_1, v_2, \dots, v_{m-1}$ are constants. substitute each equation of (2.2) in (2.1) respectively so we obtain

$$0 = (A - \lambda I_n)\mathbf{c}v_{m-1}$$

$$\begin{aligned}
\mathbf{C}v_{m-1} &= (A - \lambda I_n)(\mathbf{C}v_{m-2} + (0, w_{m-2})) \\
&= (A - \lambda I_n)\mathbf{C}v_{m-2} + (A - \lambda I_n)(0, w_{m-2}) \\
&= 0 + (A - \lambda I_n)(0, w_{m-2}) \\
&= (A - \lambda I_n)(0, w_{m-2}).
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\mathbf{C}v_{m-2} + (0, w_{m-2}) &= (A - \lambda I_n)(\mathbf{C}v_{m-3} + (0, w_{m-3})) \\
&= (A - \lambda I_n)\mathbf{C}v_{m-3} + (A - \lambda I_n)(0, w_{m-3}) \\
&= 0 + (A - \lambda I_n)(0, w_{m-3}) \\
&= (A - \lambda I_n)(0, w_{m-3}).
\end{aligned}$$

$\vdots$

$$\mathbf{C}v_1 + (0, w_1) = (A - \lambda I_n)(0, w_0).$$

So we can write the system (2.3) as the following

$$\begin{aligned}
(A - \lambda I_n)(0, w_{m-2}) &= \mathbf{C}v_{m-1} \\
(A - \lambda I_n)(0, w_{m-3}) &= \mathbf{C}v_{m-2} + (0, w_{m-2}) \\
(A - \lambda I_n)(0, w_{m-4}) &= \mathbf{C}v_{m-3} + (0, w_{m-3}) \\
&\vdots \\
(A - \lambda I_n)(0, w_0) &= \mathbf{C}v_1 + (0, w_1)
\end{aligned} \tag{2.4}$$

If we write

$$\mathbf{C} = \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix}, \quad \text{and} \quad (0, w_s) = \begin{pmatrix} 0 \\ w_s^{(2)} \\ \vdots \\ w_s^{(j)} \\ \vdots \\ w_s^{(n)} \end{pmatrix}.$$

We compute $(A - \lambda I_n)(0, w_{m-2}) = \mathbf{C}v_{m-1}$;

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} - \lambda & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} 0 \\ w_{m-2}^{(2)} \\ w_{m-2}^{(3)} \\ \vdots \\ w_{m-2}^{(j)} \\ \vdots \\ w_{m-2}^{(n)} \end{pmatrix} = \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} v_{m-1}.$$

We obtain:

$$(a_{11} - \lambda) \cdot 0 + a_{12} w_{m-2}^{(2)} + a_{13} w_{m-2}^{(3)} + \dots + a_{1j} w_{m-2}^{(j)} + \dots + a_{1n} w_{m-2}^{(n)} = v_{m-1},$$

that is $\sum_{j=2}^n a_{1j} w_{m-2}^{(j)} = v_{m-1}$. Now for $i > 1$ we obtain

$$\sum_{j=2}^n (a_{ij} - \lambda \delta_{ij}) w_{m-2}^{(j)} = c_i v_{m-1},$$

The second equation of (2.4) becomes :

$$\sum_{j=2}^n a_{1j} w_{m-3}^{(j)} = v_{m-2} \quad \text{and for } i > 1 \quad \sum_{j=2}^n (a_{ij} - \lambda \delta_{ij}) w_{m-3}^{(j)} = c_i v_{m-2} + w_{m-2}^{(i)},$$

so if we continue this procedure we get

$$\sum_{j=2}^n a_{1j} w_0^{(j)} = v_1 \quad \text{and for } i > 1 \quad \sum_{j=2}^n (a_{ij} - \lambda \delta_{ij}) w_0^{(j)} = c_i v_1 + w_1^{(i)}$$

Then for $i > 1$, we have

$$\sum_{j=2}^n (a_{ij} - \lambda \delta_{ij}) w_{m-2}^{(j)} = c_i \sum_{j=2}^n a_{1j} w_{m-2}^{(j)} = \sum_{j=2}^n c_i a_{1j} w_{m-2}^{(j)},$$

$$\text{hence } \sum_{j=2}^n (a_{ij} - \lambda \delta_{ij} - c_i a_{1j}) w_{m-2}^{(j)} = 0.$$

Also for $i > 1$, we get

$$\begin{aligned} \sum_{j=2}^n (a_{ij} - \lambda \delta_{ij}) w_{m-3}^{(j)} &= c_i v_{m-2} + w_{m-2}^{(i)} \\ &= c_i \sum_{j=2}^n a_{1j} w_{m-3}^{(j)} + w_{m-2}^{(i)}, \end{aligned}$$

$$\text{then } \sum_{j=2}^n (a_{ij} - \lambda \delta_{ij} - c_i a_{1j}) w_{m-3}^{(j)} = w_{m-2}^{(i)}.$$

Finally we have

$$\begin{aligned}\sum_{j=2}^n (a_{ij} - \lambda \delta_{ij}) w_0^{(j)} &= c_i v_1 + w_1^{(i)} \\ &= c_i \sum_{j=2}^n a_{1j} w_0^{(j)} + w_1^{(i)}\end{aligned}$$

$$\text{then } \sum_{j=2}^n (a_{ij} - \lambda \delta_{ij} - a_{1j}) w_0^{(j)} = w_1^{(i)}.$$

Hence we get the system

$$\begin{aligned}\sum_{j=2}^n a_{1j} w_{m-2}^{(j)} &= v_{m-1} \quad ; \quad \sum_{j=2}^n (a_{ij} - c_i a_{1j} - \lambda \delta_{ij}) w_{m-2}^{(j)} = 0 \\ \sum_{j=2}^n a_{1j} w_{m-3}^{(j)} &= v_{m-2} \quad ; \quad \sum_{j=2}^n (a_{ij} - c_i a_{1j} - \lambda \delta_{ij}) w_{m-3}^{(j)} = w_{m-2}^{(i)} \\ &\vdots \\ \sum_{j=2}^n a_{1j} w_0^{(j)} &= v_1 \quad ; \quad \sum_{j=2}^n (a_{ij} - c_i a_{1j} - \lambda \delta_{ij}) w_0^{(j)} = w_1^{(i)}\end{aligned} \tag{2.5}$$

Now consider the $(n-1) \times (n-1)$ matrix $B = (a_{ij} - c_i a_{1j})$, for $i, j = 2, 3, \dots, n$. If the unknowns $v_{m-1}, \dots, v_1$ are defined by

$$\begin{aligned}v_{m-1} &= \sum_{j=2}^n a_{1j} w_{m-1}^{(j)} \\ &\vdots \\ v_1 &= \sum_{j=2}^n a_{1j} w_0^{(j)}.\end{aligned}$$

Then the algebraic problem in (2.4) reduced to

$$\begin{aligned}(B - \lambda I_{n-1}) w_{m-2} &= 0 \\ (B - \lambda I_{n-1}) w_{m-3} &= w_{m-2} \\ &\vdots \\ (B - \lambda I_{n-1}) w_0 &= w_1\end{aligned} \tag{2.6}$$

with

$$B = (a_{ij} - c_i a_{1j}), \quad i, j = 2, \dots, n$$

an $(n-1) \times (n-1)$ matrix.

In the following lemma, we see that λ is an eigenvalue of B of multiplicity $(m-1)$.

Lemma 3.2.2: We have $\det(A - \mu I_n) = (\lambda - \mu) \det(B - \mu I_{n-1})$.

Proof:

$$\det(A - \mu I_n) = \begin{vmatrix} a_{11} - \mu & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} - \mu & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jj} - \mu & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} - \mu \end{vmatrix}$$

For $j > 1$ now multiply the j -th column in $\det(A - \mu I_n)$ by c_j and add them successively to the first column so we get

$$\begin{pmatrix} a_{11} - \mu + c_j a_{1j} & a_{12} & a_{13} & a_{14} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} + c_j a_{2j} & a_{22} - \mu & a_{23} & a_{24} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{j1} + c_j(a_{jj} - \mu) & a_{j2} & a_{j3} & a_{j4} & \cdots & a_{jj} - \mu & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} + c_j a_{nj} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nj} & \cdots & a_{nn} - \mu \end{pmatrix}.$$

Do this procedure for $j = 2, 3, \dots, n$, and since $A\mathbf{C} = \lambda\mathbf{C}$, we get :
11-component is

$$\begin{aligned} & (a_{11} - \mu) + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1j}c_j + \cdots + a_{1n}c_n \\ &= \sum_{j=1}^n (a_{1j} - \mu\delta_{1j})c_j \\ &= \lambda c_1 - \mu c_1 \\ &= \lambda - \mu. \end{aligned}$$

21-component is

$$\begin{aligned} & a_{21} + (a_{22} - \mu)c_2 + a_{23}c_3 + \cdots + a_{2j}c_j + \cdots + a_{2n}c_n \\ &= \sum_{j=1}^n (a_{2j} - \mu\delta_{2j})c_j \\ &= \lambda c_2 - \mu c_2 \\ &= (\lambda - \mu)c_2. \end{aligned}$$

il-component is

$$\begin{aligned}
a_{i1} + a_{i2}c_2 + a_{i3}c_3 + \cdots + (a_{ii} - \mu)c_i + \cdots + a_{in}c_n &= \sum_{j=1}^n (a_{ij} - \mu\delta_{ij})c_j \\
&= \sum_{j=1}^n (a_{ij} - \mu\delta_{ij})c_j = \lambda c_i - \mu c_i.
\end{aligned}$$

Hence

$$\det(A - \mu I_n) = \begin{vmatrix} \lambda - \mu & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ \lambda c_2 - \mu c_2 & a_{22} - \mu & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda c_i - \mu c_i & a_{i2} & a_{i3} & \cdots & a_{ij} - \mu & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda c_n - \mu c_n & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} - \mu \end{vmatrix}$$

Next for $i > 1$ multiply the first row by c_i ; subtract it from the i -th row and observe, doing simultaneously for $i = 2, 3, \dots, n$ that $\det(A - \mu I_n)$ is

$$\begin{vmatrix} \lambda - \mu & a_{12} & a_{13} & \cdots & a_{1i} & \cdots & a_{1n} \\ 0 & a_{22} - \mu - c_2 a_{12} & a_{23} - c_2 a_{13} & \cdots & a_{2i} - c_2 a_{1i} & \cdots & a_{2n} - c_2 a_{1n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & a_{i2} - c_i a_{12} & a_{i3} - c_i a_{13} & \cdots & a_{ii} - \mu - c_i a_{1i} & \cdots & a_{in} - c_i a_{1n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & a_{n2} - c_n a_{12} & a_{n3} - c_n a_{13} & \cdots & a_{ni} - c_n a_{1i} & \cdots & a_{nn} - \mu - c_n a_{1n} \end{vmatrix}$$

Therefore

$$\det(A - \mu I_n) = (\lambda - \mu) \det(B - \mu I_{n-1}),$$

where

$$B = \begin{pmatrix} a_{22} - \mu - c_2 a_{12} & a_{23} - c_2 a_{13} & \cdots & a_{2i} - c_2 a_{1i} & \cdots & a_{2n} - c_2 a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i2} - c_i a_{12} & a_{i3} - c_i a_{13} & \cdots & a_{ii} - \mu - c_i a_{1i} & \cdots & a_{in} - c_i a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n2} - c_n a_{12} & a_{n3} - c_n a_{13} & \cdots & a_{ni} - c_n a_{1i} & \cdots & a_{nn} - \mu - c_n a_{1n} \end{pmatrix}.$$

So by induction there exists $(m-1)$ linearly independent chains

$$\left[w_{m-2}, w_{m-3}, \dots, w_0 \right], \text{ which satisfy system (2.6).}$$

Now, we want to show that $d_{m-1}, d_{m-2}, \dots, d_0$ are linearly independents.

If there exists $\alpha_0, \alpha_1, \dots, \alpha_{m-1}$ constants such that

$$\alpha_0 d_0 + \alpha_1 d_1 + \dots + \alpha_{m-2} d_{m-2} + \alpha_{m-1} d_{m-1} = 0$$

Then

$$\begin{aligned} & \alpha_0(0, w_0) + \alpha_1(cv_1 + (0, w_1)) + \alpha_2(cv_2 + (0, w_2)) \\ & + \dots + \alpha_{m-2}(cv_{m-2} + (0, w_{m-2})) + \alpha_{m-1}cv_{m-1} = 0. \end{aligned}$$

That is

$$\begin{aligned} & \alpha_0 \begin{pmatrix} 0 \\ \omega_0^{(2)} \\ \omega_0^{(3)} \\ \vdots \\ \omega_0^{(n)} \end{pmatrix} + \alpha_1 v_1 \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ \omega_1^{(2)} \\ \omega_1^{(3)} \\ \vdots \\ \omega_1^{(n)} \end{pmatrix} + \alpha_2 v_2 \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ \omega_2^{(2)} \\ \omega_2^{(3)} \\ \vdots \\ \omega_2^{(n)} \end{pmatrix} \\ & + \dots + \alpha_{m-2} v_{m-2} \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} + \alpha_{m-2} \begin{pmatrix} 0 \\ \omega_{m-2}^{(2)} \\ \omega_{m-2}^{(3)} \\ \vdots \\ \omega_{m-2}^{(n)} \end{pmatrix} + \alpha_{m-1} v_{m-1} \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = 0. \end{aligned}$$

Then

$$\begin{aligned} & \alpha_0 0 + \alpha_1 v_1 + \alpha_1 \cdot 0 + \alpha_2 v_2 + \alpha_2 \cdot 0 + \dots + \alpha_{m-2} v_{m-2} + \\ & \alpha_{m-2} \cdot 0 + \alpha_{m-1} v_{m-1} = 0 \end{aligned}$$

Denote $\begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}$ by e then

$$\begin{aligned} & \alpha_0 w_0 + \alpha_1 v_1 e + \alpha_1 w_1 + \alpha_2 v_2 e + \alpha_2 w_2 \\ & + \dots + \alpha_{m-2} v_{m-2} e + \alpha_{m-2} w_{m-2} + \alpha_{m-1} v_{m-1} e = 0. \end{aligned}$$

Thus

$$\begin{aligned} & \alpha_0 w_0 + \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_{m-2} w_{m-2} + \\ & (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{m-1} v_{m-1}) e = 0 \end{aligned}$$

Since

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{m-1} v_{m-1} = 0,$$

then we obtain

$$\alpha_0 w_0 + \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_{m-2} w_{m-2} = 0.$$

Since $w_0, w_1, w_2, \dots, w_{m-2}$ are linearly independent then

$$\alpha_0 = \alpha_1 = \alpha_2 = \dots = \alpha_{m-2} = 0.$$

Hence from $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{m-1} v_{m-1} = 0$ and α_{m-1} is also zero. ■

To find the general solution of the system

$$\frac{dX}{dt} = AX \quad (2.7)$$

We could find first the fundamental solution $\Phi(t)$, which is the matrix solution of the initial-value problem

$$\frac{dX}{dt} = AX, \quad \Phi(t_0) = I_n \quad (2.8)$$

where t_0 is arbitrary constant but fixed. Then the general solution of (2.7) is given by $X(t) = \Phi(t)C$, where C is a constants vector.

The system (2.7) can be written in the differential operator form

$(A - I_n D)X(t) = 0$, and previous theorem reduce it to the system $0 = Q(D)(A - I_n D)X(t) = I_n \det(A - I_n D)X(t)$.

And therefore the matrix $X(t)$ is of the form

$$X(t) = C_1 X_1(t) + C_2 X_2(t) + \dots + C_n X_n(t) \quad (2.9)$$

where $C_1, C_2, \dots, C_n$ are $n \times n$ matrices with t-independent element (see (3.4) page 27).

But from

$$\frac{d}{dt}(X(t)) = AX(t)$$

it follows that

$$\frac{d^k}{dt^k}(X(t)) = A^k X(t), \quad k = 1, 2, \dots, n-1.$$

Hence

$$\frac{d^k}{dt^k}(X(t))|_{t=0} = A^k, \quad k = 1, 2, \dots, n-1.$$

And the initial-value problem (2.8) is equivalent to the algebraic problem

$$C_1 X_1(t_0) + C_2 X_2(t_0) + \dots + C_n X_n(t_0) = I_n$$

$$C_1 \frac{d}{dt}(X_1(t_0)) + C_2 \frac{d}{dt}(X_2(t_0)) + \dots + C_n \frac{d}{dt}(X_n(t_0)) = A$$

$$\vdots$$

$$C_1 \frac{d^{n-1}}{dt^{n-1}}(X_1(t_0)) + C_2 \frac{d^{n-1}}{dt^{n-1}}(X_2(t_0)) + \dots + C_n \frac{d^{n-1}}{dt^{n-1}}(X_n(t_0)) = A^{n-1}.$$

Thus, we have proved theorem (2.3.1) without using Caley-Hamilton Theorem. ■

CHAPTER FOUR

LINEAR SYSTEMS WITH PERIODIC COEFFICIENTS

In this chapter we study the general solution of linear systems with periodic coefficients. Consider the linear homogeneous system

$$\frac{dX}{dt} = A(t)X, \quad -\infty < t < \infty \quad (1.1)$$

where the elements of the matrix A are continuous functions on R and where $A(t) = A(t + \omega)$ for some $\omega > 0$. System (1.1) is called a **periodic system** and ω is called a **period** of A .

We give also the concept of the logarithm of any matrix, The Floquet matrix for the systems with periodic coefficients.

4.1 Logarithm Of Matrices.

First we state some properties and results about fundamental matrices of periodic systems.

Definition 4.1.1: Let A and B be $n \times n$ matrices, we say that A is the **Logarithm** of B (denoted by $\ln(B)$) if $e^A = B$. In order to define A , we shall use the expansion formula:

$$\ln(1+x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p, \quad |x| < 1$$

as follows :

$$\ln(B) = \ln(I_n + (B - I_n)) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} (B - I_n)^p.$$

Remark 4.1.2: If Λ is a diagonal matrix with diagonal elements

$$d_{11}, d_{22}, \dots, d_{nn},$$

then

$$\ln(\Lambda) = \begin{pmatrix} \ln(d_{11}) & & & \\ & \ln(d_{22}) & & \\ & & \ddots & \\ & & & \ln(d_{nn}) \end{pmatrix},$$

where $d_{11}, d_{22}, \dots, d_{nn} > 0$.

Since

$$\begin{aligned}
 \ln(\Lambda) &= \ln(I_n + (\Lambda - I_n)) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} (\Lambda - I_n)^p \\
 &= (\Lambda - I_n) - \frac{(\Lambda - I_n)^2}{2} + \frac{(\Lambda - I_n)^3}{3} - \frac{(\Lambda - I_n)^4}{4} + \dots \\
 &= \begin{pmatrix} d_{11} - 1 & & \\ & d_{22} - 1 & \\ & & \ddots \\ 0 & & & d_{nn} - 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} (d_{11} - 1)^2 & & \\ & (d_{22} - 1)^2 & \\ & & \ddots \\ 0 & & & (d_{nn} - 1)^2 \end{pmatrix} \\
 &\quad + \frac{1}{3} \begin{pmatrix} (d_{11} - 1)^3 & & \\ & (d_{22} - 1)^3 & \\ & & \ddots \\ 0 & & & (d_{nn} - 1)^3 \end{pmatrix} - \dots \\
 &= \begin{pmatrix} S_{11} & & \\ & S_{22} & \\ & & \ddots \\ 0 & & & S_{nn} \end{pmatrix},
 \end{aligned}$$

where

$$\begin{aligned}
 S_{11} &= (d_{11} - 1) - \frac{1}{2}(d_{11} - 1)^2 + \frac{1}{3}(d_{11} - 1)^3 - \dots = \ln(d_{11}) \\
 S_{22} &= (d_{22} - 1) - \frac{1}{2}(d_{22} - 1)^2 + \frac{1}{3}(d_{22} - 1)^3 - \dots = \ln(d_{22}) \\
 &\vdots \\
 S_{nn} &= (d_{nn} - 1) - \frac{1}{2}(d_{nn} - 1)^2 + \frac{1}{3}(d_{nn} - 1)^3 - \dots = \ln(d_{nn})
 \end{aligned}$$

Therefore

$$\ln(\Lambda) = \begin{pmatrix} \ln(d_{11}) & & & \\ & \ln(d_{22}) & & 0 \\ & & \ddots & \\ 0 & & & \ln(d_{nn}) \end{pmatrix} \quad \blacksquare$$

Remark 4.1.3: If A is a nonsingular matrix, then all eigenvalues of A are distinct from zero. Since if there exist j such that $\lambda_j = 0$ where λ_j is an eigenvalue of A then $0 = \det(A - \lambda_j I) = \det(A)$, which contradicts that $\det(A) \neq 0$.

Theorem 4.1.4: Let B be a nonsingular $n \times n$ matrix. Then there exists an $n \times n$ matrix A , such that $A = \ln B$ i.e $e^A = B$.

Proof: Let B^1 be similar to B . Then there is a nonsingular matrix P such that

$$P^{-1}BP = B^1.$$

If

$$e^{A^1} = B^1,$$

then

$$B = PB^1P^{-1} = Pe^{A^1}P^{-1} = e^{PA^1P^{-1}},$$

let $A = PA^1P^{-1}$ then $B = e^A$.

hence PA^1P^{-1} is a logarithm of B . Now we consider the case, when B is in a Jordan canonical form. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of B with multiplicities $n_1, n_2, \dots, n_k$ respectively. Then B has the form

$$B = \begin{pmatrix} B_0 & & & \\ & B_1 & & 0 \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix},$$

where B_0 is a diagonal matrix with diagonal element λ_i , $i = 1, 2, \dots, k$, and $B_j = \lambda_j I_{n_j} + N_j$, where N_j is the following $n_j \times n_j$ nilpotent matrix,

$$N_j = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}.$$

Then $\ln(B_0)$ is a diagonal matrix with diagonal elements $\ln(\lambda_i)$.

Since B is nonsingular we have $\lambda_j \neq 0$ for all j , then

$$B_j = \lambda_j (I_{n_j} + \frac{1}{\lambda_j} N_j).$$

Let

$$A_j = \ln(B_j) = \ln(\lambda_j (I_{n_j} + \frac{1}{\lambda_j} N_j)) = I_{n_j} \ln \lambda_j + \ln(I_{n_j} + \frac{1}{\lambda_j} N_j)$$

$$= I_{n_j} \ln \lambda_j + \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \left(\frac{N_j}{\lambda_j} \right)^p,$$

since $N_j^{n_j} = 0$ we have

$$A_j = I_{n_j} \ln(\lambda_j) + \sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{N_j}{\lambda_j} \right)^p, \quad j=1, 2, \dots, k \quad (1.2)$$

Note that $\ln(\lambda_j)$ is defined since $\lambda_j \neq 0$.

Now

$$\begin{aligned} e^{A_j} &= \exp(I_{n_j} \ln \lambda_j) \exp \left(\sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{N_j}{\lambda_j} \right)^p \right) \\ &= \lambda_j (I_{n_j} + \frac{N_j}{\lambda_j}) = B_j, \quad j=1, 2, \dots, k. \end{aligned}$$

Now let

$$A = \begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$$

where A_j is defined in (1.2). So we have

$$e^A = \begin{pmatrix} e^{A_0} & 0 \\ \vdots & \\ 0 & e^{A_k} \end{pmatrix} = \begin{pmatrix} B_0 & 0 \\ B_1 & \\ & \ddots \\ 0 & B_k \end{pmatrix} = B.$$

Note that A is not unique since for example

$$e^{A+2\pi k i I} = e^A \cdot e^{2\pi k i I} = e^A I = e^A = B.$$

4.2 Fundamental Matrices Of Periodic Systems.

Consider the periodic system (1.1), we shall now give some properties for a fundamental matrix of periodic systems

Theorem 4.2.1: i) If Φ is a fundamental matrix of (1.1) and if C is any nonsingular constant $n \times n$ matrix, then ΦC is also a fundamental matrix of (1.1).

ii) If Ψ is any other fundamental matrix of (1.1) then there exists a constant $n \times n$ nonsingular matrix P such that $\Psi = \Phi P$.

Proof: i) We have

$$\frac{d(\Phi C)}{dt} = \frac{d\Phi}{dt} \cdot C = (A(t)\Phi)C = A(t)(\Phi C)$$

Hence ΦC is a solution of the system (1.1). And so

$$\det(\Phi C) = \det\Phi \cdot \det C \neq 0,$$

hence ΦC is a fundamental matrix.

ii) Let Ψ be any other fundamental matrix. Consider the product

$\Phi^{-1}\Psi$. Since $\det\Phi(t) \neq 0$ for all t , then Φ^{-1} exists for all t .

Differentiate $\Phi\Phi^{-1} = I$ by Leibniz rule, we get :

$$\frac{d\Phi}{dt}\Phi^{-1} + \Phi \frac{d}{dt}(\Phi^{-1}) = 0,$$

then

$$\frac{d}{dt}(\Phi^{-1}) = -\Phi^{-1} \frac{d\Phi}{dt} \Phi^{-1}.$$

Thus we can compute

$$\begin{aligned}
\frac{d}{dt}(\Phi^{-1}\Psi) &= \frac{d}{dt}(\Phi^{-1})\Psi + \Phi^{-1}\frac{d\Psi}{dt} \\
&= -\Phi^{-1}\frac{d\Phi}{dt}\Phi^{-1}\Psi + \Phi^{-1}A\Psi \\
&= -\Phi^{-1}A(t)\Phi\Phi^{-1}\Psi - \Phi^{-1}A(t)\Psi \\
&= -\Phi^{-1}A(t)\Psi - \Phi^{-1}A(t)\Psi = 0.
\end{aligned}$$

Therefore $\Phi^{-1}\Psi$ is a constant matrix say

$$P := \Phi^{-1}\Psi \text{ then } \Psi = \Phi P. \blacksquare$$

Theorem 4.2.2: (Floquet) (i) Consider system (1.1), if $\Phi(t)$ is a fundamental matrix for (1.1), then so is $\Phi(t + \omega)$ for all $t \in (-\infty, \infty)$.

ii) Corresponding to every fundamental matrix Φ , there exists a nonsingular matrix P which is also periodic with period ω and a constant matrix R , such that $\Phi(t) = P(t)e^{tR}$.

Proof: i) Since

$$\frac{d\Phi}{dt} = A(t)\Phi(t), \text{ for } t \in (-\infty, \infty),$$

it follows that

$$\frac{d\Phi(t + \omega)}{dt} = A(t + \omega)\Phi(t + \omega) = A(t)\Phi(t + \omega), \text{ for } t \in (-\infty, \infty).$$

Hence, $\Phi(t + \omega)$ is a solution of (1.1), and it is a fundamental matrix, since $\Phi(t + \omega)$ is nonsingular for all $t \in (-\infty, \infty)$.

(ii) Since $\Phi(t + \omega)$ is a fundamental matrix, then by the last theorem there exists a constant nonsingular matrix C such that $\Phi(t + \omega) = \Phi(t)C$. By theorem (4.1.4) there exists a constant matrix R such that $e^{\omega R} = C$.

Therefore

$$\Phi(t + \omega) = \Phi(t)e^{\omega R}.$$

Now define a matrix P by

$$\begin{aligned}
P(t) &:= \Phi(t)e^{-tR} \text{ so,} \\
P(t + \omega) &= \Phi(t + \omega)e^{-(t + \omega)R} \\
&= \Phi(t)e^{\omega R}e^{-(t + \omega)R} \\
&= \Phi(t)e^{\omega R}e^{-tR}e^{-\omega R} = P(t).
\end{aligned}$$

Then $P(t)$ is periodic. Also $P(t)$ is nonsingular for all $t \in (-\infty, \infty)$. $\blacksquare$

Note 4.2.3: Now if $\Phi(t)$ is known only over the interval $[t_0, t_0 + \omega]$. Since

$$\Phi(t + \omega) = \Phi(t)C, \quad C = e^{\omega R}$$

we have by setting $t = t_0$,

$$C = \Phi^{-1}(t_0)\Phi(t_0 + \omega)$$

and R is given by $\omega^{-1} \ln(C)$. Thus $P(t) = \Phi(t)e^{-tR}$ is now determined over $(0, \omega)$. However $P(t)$ is periodic over $(-\infty, \infty)$. Therefore $\Phi(t)$ is given over $(-\infty, \infty)$ by $\Phi(t) = P(t)e^{tR}$. Thus we conclude from the last theorem that the determination of a fundamental matrix Φ of (1.1) over any interval of length ω , leads at once to the determination of Φ over $(-\infty, \infty)$.

Note 4.2.4: (i) Let Φ be a fundamental matrix for (1.1) and if Φ_I is any other fundamental matrix for (1.1) with $A(t + \omega) = A(t)$. Then

$\Phi = \Phi_I S$ for some constant nonsingular matrix S .

Since

$$\Phi(t + \omega) = \Phi(t)e^{\omega R},$$

we have

$$\Phi_I(t + \omega)S = \Phi(t + \omega) = \Phi(t)e^{\omega R} = \Phi(t)S^{-1}Se^{\omega R} = \Phi_I(t)Se^{\omega R}.$$

Hence

$$\Phi_I(t + \omega)S = \Phi_I(t)Se^{\omega R}$$

or

$$\Phi_I(t + \omega) = \Phi_I(t)(Se^{\omega R}S^{-1}) = \Phi_I(t)e^{S\omega RS^{-1}} = \Phi_I(t)e^{\omega(SRS^{-1})}.$$

Hence

$$\Phi_I(t + \omega) = \Phi_I(t)e^{\omega(SRS^{-1})} \quad (2.1)$$

Thus by (2.1) every fundamental matrix Φ , determine a matrix $Se^{\omega R}S^{-1}$, which is similar to $e^{\omega R}$.

(ii) Conversely, let S be any constant nonsingular matrix. Then there exists fundamental matrix of (1.1) such that (2.1) holds.

Thus from (i) and (ii) although Φ does not determine R uniquely, the set of all fundamental matrices of (1.1) and hence of A , determine uniquely all quantities associated with $e^{\omega R}$ which are invariant under a similarity transformation. Specifically, the set of all fundamental matrices of A determine a unique set of eigenvalues of the matrix $e^{\omega R}$ say $\lambda_1, \lambda_2, \dots, \lambda_k$,

which are called the **multipliers** associated with A (or Floquet multipliers associated with A).

None of these vanish since

$$\det(e^{\omega R} - \lambda I) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n) \text{ so}$$

$$\det(e^{\omega R}) = \det(e^{\omega R} - 0I) = \lambda_1 \dots \lambda_n = \prod \lambda_i$$

Hence

$$\det(e^{\omega R}) = \prod \lambda_i \neq 0.$$

The eigenvalues of R are called the **characteristic exponents**.

Note 4.2.5: By Floquet Theorem, the fundamental solution of the system (1.1) is of the form $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is periodic with period ω and R is a constant matrix. Then the dynamics of solutions is determined by the matrix e^{tR} , which generates the fundamental solution of the system (1.1). Let $\rho_1, \rho_2, \dots, \rho_n$ be the eigenvalues of R , then they are connected with the multipliers $\lambda_1, \lambda_2, \dots, \lambda_n$ of the system (1.1) in the following way:

$$\rho_i = \frac{1}{\omega} \ln(\lambda_i) \text{ or } \lambda_i = e^{\omega \rho_i}.$$

In the following we will find the explicit form that a set of n linearly independent solution vectors of (1.1) assumes.

Let Q be a constant nonsingular matrix such that $J = Q^{-1}RQ$, where J is the Jordan canonical form of R . i.e.

$$J = \begin{pmatrix} J_0 & & & \\ & J_1 & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & J_s \end{pmatrix}.$$

Let $\Phi_I = \Phi Q$ and $P_I = PQ$.

By theorem (4.2.2) and from $\Phi = P(t)e^{tR}$ we get :

$$\Phi_I(t) = \Phi Q = P(t)e^{tR} Q = P(t)Q Q^{-1} e^{tR} Q = P_I(t) e^{tJ} \quad (2.2)$$

and

$$P_I(t + \omega) = P_I(t).$$

Let $\rho_1, \rho_2, \dots, \rho_n$ be the eigenvalues of R . Then

$$e^{tJ} = \begin{pmatrix} e^{tJ_0} & & \\ & e^{tJ_1} & 0 \\ & \ddots & \\ 0 & & e^{tJ_s} \end{pmatrix},$$

where

$$e^{tJ_0} = \begin{pmatrix} e^{t\rho_1} & & \\ & e^{t\rho_2} & 0 \\ & \ddots & \\ 0 & & e^{t\rho_q} \end{pmatrix}$$

and

$$e^{tJ_i} = e^{t\rho_{q+i}} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{r_i-1}}{(r_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{r_i-2}}{(r_i-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$i = 1, 2, \dots, s, q + \sum_{i=1}^s r_i = n.$$

Thus even though the ρ_i are not uniquely determined, their real parts are. By (2.2) we have

$$\Phi_I(t) = P_I(t) e^{tJ}.$$

We shall find the column vectors of matrix Φ_I which are linearly independent denoted by $\phi_1, \dots, \phi_n$.

Let $p_1, \dots, p_n$ denote the periodic column vectors of P_I . Then

$$\Phi_1 = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix},$$

$$P_1 = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1q} & p_{1q+1} & p_{1q+2} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2q} & p_{2q+1} & p_{2q+2} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nq} & p_{nq+1} & p_{nq+2} & \cdots & p_{nn} \end{pmatrix}$$

$$e^{tJ} = \begin{pmatrix} e^{t\rho_1} & & & & & & & \\ & \ddots & & & & & & \\ & & e^{t\rho_q} & & & & & \\ & & & 1 & t & \cdots & \frac{t^{r_1-1}}{(r_1-1)!} & \\ & & & 0 & 1 & \cdots & \frac{t^{r_1-2}}{(r_1-2)!} & (e^{t\rho_{q+1}}) \\ & & & \vdots & \ddots & \vdots & \vdots & \\ & & & 0 & \cdots & 0 & 1 & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & t & \cdots & \frac{t^{r_s-1}}{(r_s-1)!} \\ & & & & & & 0 & 1 & \cdots & \frac{t^{r_s-2}}{(r_s-2)!} & (e^{t\rho_{q+s}}) \\ & & & & & & \vdots & \ddots & \vdots & \vdots & \\ & & & & & & 0 & \cdots & 0 & 1 & \end{pmatrix}.$$

Now if we multiply the matrix P_1 by the matrix e^{tJ} we get

$$\begin{aligned}
\phi_1(t) &= e^{t\rho_1} p_1(t), \\
\phi_2(t) &= e^{t\rho_2} p_2(t), \\
&\vdots \\
\phi_q(t) &= e^{t\rho_q} p_q(t), \\
\phi_{q+1}(t) &= e^{t\rho_{q+1}} p_{q+1}(t), \\
\phi_{q+2}(t) &= e^{t\rho_{q+2}} (tp_{q+1}(t) + p_{q+2}(t)), \quad \dots \quad (2.3) \\
&\vdots \\
\phi_{q+r_1}(t) &= \left(\frac{t^{r_1-1}}{(r_1-1)!} p_{q+1} + \frac{t^{r_1-2}}{(r_1-2)!} p_{q+2} + \dots + tp_{q+r_1-1}(t) + p_{q+r_1}(t) \right) e^{t\rho_{q+1}}, \\
&\vdots \\
\phi_{n-r_s+1}(t) &= e^{t\rho_{q+s}} p_{n-r_s+1}(t), \\
&\vdots \\
\phi_n(t) &= \left(\frac{t^{r_s-1}}{(r_s-1)!} p_{n-r_s+1} + \dots + tp_{n-1}(t) + p_n(t) \right) e^{t\rho_{q+s}}.
\end{aligned}$$

Where $p_i(t)$ is the i^{th} column in p_1 , and ϕ_i is the i^{th} column in Φ_1 .
From (2.3) when $Re \rho_i \equiv \alpha_i < 0$, then

$$\|\phi_i(t)\| = \|p_i(t)\| \cdot |e^{\rho_i t}|, \text{ for } i=1, 2, \dots, q.$$

Note that

$$\begin{aligned}
|e^{\rho_i t}| &= |e^{t\alpha_i + it\beta_i}| \\
&= |e^{t\alpha_i}| \cdot |e^{it\beta_i}| \\
&= |e^{t\alpha_i}| \cdot |\cos t\beta_i + i \sin t\beta_i| \\
&= |e^{t\alpha_i}| = e^{t\alpha_i}.
\end{aligned}$$

Thus

$$\|\phi_i(t)\| = \|p_i(t)\| \cdot e^{\alpha_i t}.$$

Now we want to calculate $\|p_i(t)\|$.

All the elements of the column vector $p_i(t)$ are continuous functions on $[t_0, t_0 + \omega]$ and periodic over $(-\infty, \infty)$.

Hence

$$\|p_i(t)\| = \sum_{j=1}^n \|p_{ji}(t)\| \leq k.$$

So

$$\|\phi_i(t)\| \leq k e^{\alpha_i t} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \text{ since } \alpha_i < 0.$$

In other words, if the eigenvalues ρ_i , $i = 1, 2, \dots, n$, of R have negative real parts, then the norm of any solution (1.1) tends to zero as $t \rightarrow \infty$ at an exponential rate.

From $\Phi(t) = P(t)e^{tR}$ we have

$$\frac{d\Phi(t)}{dt} = \frac{d}{dt}(P(t))e^{tR} + P(t)Re^{tR}$$

then

$$\begin{aligned} A\Phi(t) &= \frac{d}{dt}(P(t))e^{tR} + P(t)Re^{tR} \\ AP(t)e^{tR} &= \frac{d}{dt}(P(t))e^{tR} + P(t)Re^{tR}, \end{aligned}$$

hence

$$AP(t) = \frac{d}{dt}(P(t)) + P(t)R, \text{ or } AP(t) - \frac{dP(t)}{dt} = P(t)R.$$

Let X be a fundamental solution of (1.1). Thus, for the transformation

$$X = P(t)Y \tag{2.4}$$

We compute

$$\begin{aligned} \frac{dX}{dt} &= A(t)X(t) \\ &= A(t)P(t)Y(t), \\ \frac{dX}{dt} &= \frac{d}{dt}(P(t)Y(t)) \\ &= \frac{d}{dt}(P(t))Y(t) + P(t)\frac{dY(t)}{dt}. \end{aligned}$$

Also,

$$\begin{aligned} A(t)P(t)Y(t) &= \frac{dP(t)}{dt}Y(t) + P(t)\frac{dY(t)}{dt} \\ A(t)P(t)Y(t) - \frac{dP(t)}{dt}Y(t) &= P(t)\frac{dY(t)}{dt} \\ (A(t)P(t) - \frac{dP(t)}{dt})Y(t) &= P(t)\frac{dY(t)}{dt}. \end{aligned}$$

hence

$$\begin{aligned}\frac{dY(t)}{dt} &= P^{-1}(t) \left(A(t)P(t) - \frac{dP(t)}{dt} \right) Y(t) \\ &= P^{-1}(t)P(t)RY(t) \\ &= RY(t)\end{aligned}\tag{2.5}.$$

Thus we conclude the following theorem.

Theorem 4.2.5: The transformation (2.4) reduces the linear, homogenous, periodic system (1.1) to linear homogenous systems with constant coefficients

$$\frac{dY(t)}{dt} = R Y(t).$$

Note 4.2.6: If $A(t)$ is real (so that $C = e^{tR}$ is real), the matrices $P(t)$ and R may be complex.

4.3 Lyapunov Transformations And Reducible Systems.

The most simple and the most studied systems of linear differential equation of first order are system with constant coefficients. For this reason systems which can be reduced to systems with constant coefficients by means of a Lyapunov transformations are of interest, Consider the system

$$\frac{dX}{dt} = A(t)X\tag{3.1}$$

Where the elements of the matrix $A(t)$ are periodic continuous functions on the interval $[t_0, \infty)$.

Consider the transformation

$$X = L(t)Y,\tag{3.2}$$

with nonsingular matrix $L(t)$, which is continuous and differentiable for $t \geq t_0$. If we differentiate (3.2) we get

$$\frac{dX}{dt} = \frac{d}{dt}(L(t))Y + L(t)\frac{dY}{dt}\tag{3.3}$$

Now substitute (3.3) in (3.1) we get

$$\frac{d}{dt}(L(t))Y + L(t)\frac{dY}{dt} = A(t)X = A(t)L(t)Y.$$

Hence

$$\frac{dY}{dt} = L^{-1}(t)A(t)L(t)Y - L^{-1}(t)\frac{d}{dt}(L(t))Y,$$

or

$$\frac{dY}{dt} = \left(L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}(L(t)) \right) Y.$$

Let

$$B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}L(t). \quad (3.4)$$

Then

$$\frac{dY}{dt} = B(t)Y.$$

Definition 4.3.1: The transformation (3.2) is called the **Lyapunov transformation** if i) $L(t)$ is nonsingular, continuous and differentiable for $t \geq t_0$.

ii) $L(t)$, $L^{-1}(t)$ and $\frac{d}{dt}(L(t))$ are bounded for $t \geq t_0$.

Theorem 4.3.2: Lyapunov transformations form a group.

Proof: First: we want to show that if $L(t)$ is a Lyapunov matrix, then $L^{-1}(t)$ is also a Lyapunov matrix.

Since $L(t)$ is a nonsingular matrix, then $L^{-1}(t)$ exists and

$$L^{-1}(t) = \frac{1}{\det(L(t))} \text{adj}(L(t)) \text{ with } \det(L(t)) \neq 0.$$

Hence $L^{-1}(t)$ is continuous and differentiable for all $t \geq t_0$.

$L^{-1}(t)$ and $(L^{-1}(t))^{-1} = L(t)$ are bounded for $t \geq t_0$.

Now we want to show that $\frac{d}{dt}(L^{-1}(t))$ is bounded for $t \geq t_0$.

Since $I = L(t)L^{-1}(t)$, then

$$0 = \frac{d}{dt}(I) = \frac{d}{dt}(L(t))L^{-1}(t) + L(t)\frac{d}{dt}(L^{-1}(t)),$$

hence

$$\frac{d}{dt}(L^{-1}(t)) = -L^{-1}(t)\frac{d}{dt}(L(t))L^{-1}(t).$$

So $\frac{d}{dt}(L^{-1}(t))$ is also bounded.

Second: we want to show that if $L(t) = I$, then $L(t)$ is a Lyapunov matrix. $\det(L(t)) = \det(I) = 1 \neq 0$, therefore $L(t)$ is a nonsingular matrix and continuous, differentiable for all $t \geq t_0$.

$L^{-1}(t) = I$, therefore $L(t) = I$, $L^{-1}(t) = I$, and $\frac{d}{dt}(L(t)) = 0$ are bounded for $t \geq t_0$

Third: we want to show that if $L_1(t)$ and $L_2(t)$ are Lyapunov matrices, then $L(t) = L_1(t)L_2(t)$ is also a Lyapunov matrix.

Now

$\det(L(t)) = \det(L_1(t)L_2(t)) = \det(L_1(t)) \det(L_2(t)) \neq 0$, therefore $L(t)$ is nonsingular. $L(t)$ is also continuous and differentiable for $t \geq t_0$. The matrix $L(t) = L_1(t)L_2(t)$ is bounded, since $L_1(t)$ and $L_2(t)$ are a Lyapunov matrices. And

$$L^{-1}(t) = (L_1(t)L_2(t))^{-1} = L_2^{-1}(t)L_1^{-1}(t),$$

Thus $L^{-1}(t)$ is also bounded. Also $\frac{d}{dt}(L(t))$ is bounded for all $t \geq t_0$,

since

$$\frac{d}{dt}(L(t)) = \frac{d}{dt}(L_1(t)L_2(t)) = \frac{d}{dt}(L_1(t)) \cdot L_2(t) + L_1(t) \frac{d}{dt}(L_2(t)).$$

Definition 4.3.3: A system (3.1) is said to be **reducible** to a system

$$\frac{dY}{dt} = B(t)Y \quad (3.5)$$

if there exists a Lyapunov transformation $X = L(t)Y$ such that

$$B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}(L(t)).$$

Theorem 4.3.4: Reducibility is an equivalence relation.

Proof: Transitivity: If a system (3.1) is reducible to (3.5) and the latter is reducible to a system

$$\frac{dZ}{dt} = C(t)Z.$$

Then there exist Lyapunov matrices $L_1(t)$ and $L_2(t)$ such that

$$X = L_1(t)Y$$

where

$$B(t) = L_1^{-1}(t)A(t)L_1(t) - L_1^{-1}(t)\frac{d}{dt}(L_1(t))$$

and

$$Y(t) = L_2(t)Z$$

where

$$C(t) = L_2^{-1}(t)B(t)L_2(t) - L_2^{-1}(t)\frac{d}{dt}(L_2(t))$$

Now we have

$$\begin{aligned} C(t) &= L_2^{-1}(t)B(t)L_2(t) - L_2^{-1}(t)\frac{d}{dt}(L_2(t)) \\ &= L_2^{-1}(t)\left(L_1^{-1}(t)A(t)L_1(t) - L_1^{-1}(t)\frac{d}{dt}(L_1(t))\right)L_2(t) - L_2^{-1}(t)\frac{d}{dt}(L_2(t)) \\ &= L_2^{-1}(t)L_1^{-1}(t)A(t)L_1(t)L_2(t) - L_2^{-1}(t)L_1^{-1}(t)\frac{d}{dt}(L_1(t))L_2(t) \\ &\quad - L_2^{-1}(t)\frac{d}{dt}(L_2(t)) \\ &= (L_1(t)L_2(t))^{-1}A(t)(L_1(t)L_2(t)) - \\ &\quad \left(L_2^{-1}(t)L_1^{-1}(t)\frac{d}{dt}(L_1(t))L_2(t) + L_2^{-1}(t)\frac{d}{dt}(L_2(t))\right) \end{aligned}$$

Now we want to show that

$$((L_1(t)L_2(t))^{-1}\frac{d}{dt}(L_1(t)L_2(t))) = L_2^{-1}(t)L_1^{-1}(t)\frac{d}{dt}(L_1(t))L_2(t) + L_2^{-1}(t)\frac{d}{dt}(L_2(t)).$$

we have

$$\begin{aligned} ((L_1(t)L_2(t))^{-1}\frac{d}{dt}(L_1(t)L_2(t))) &= \\ &= (L_2^{-1}(t)L_1^{-1}(t))\left(\frac{d}{dt}(L_1(t))L_2(t) + L_1(t)\frac{d}{dt}(L_2(t))\right) \\ &= L_2^{-1}(t)L_1^{-1}(t)\frac{d}{dt}(L_1(t))L_2(t) + L_2^{-1}(t)L_1^{-1}(t)L_1(t)\frac{d}{dt}(L_2(t)) \\ &= L_2^{-1}(t)L_1^{-1}(t)\frac{d}{dt}(L_1(t))L_2(t) + L_2^{-1}(t)\frac{d}{dt}(L_2(t)) \end{aligned}$$

$$\text{Hence } C(t) = (L_1(t)L_2(t))^{-1}A(t)(L_1(t)L_2(t)) - (L_1(t)L_2(t))^{-1}\frac{d}{dt}(L_1(t)L_2(t)),$$

therefore the system (3.1) is reducible to the system

$$\frac{dZ}{dt} = C(t)Z$$

by the transformation

$$X = (L_1(t)L_2(t))Z.$$

2) Symmetry: If a system (3.1) is reducible to the system (3.5), then there exists a nonsingular matrix $L(t)$ such that

$$B = L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}(L(t))$$

$$L(t)B = A(t)L(t) - \frac{d}{dt}(L(t))$$

$$L(t)BL^{-1}(t) = A(t) - \frac{d}{dt}(L(t))L^{-1}(t)$$

Hence

$$A(t) = L(t)BL^{-1}(t) + \frac{d}{dt}(L(t))L^{-1}(t),$$

Since

$$\frac{d}{dt}(L^{-1}(t)) = -L^{-1}(t)\frac{d}{dt}(L(t))L^{-1}(t)$$

We have

$$\frac{d}{dt}(L(t))L^{-1}(t) = -L(t)\frac{d}{dt}(L^{-1}(t))$$

Hence

$$\begin{aligned} A(t) &= L(t)BL^{-1}(t) - L(t)\frac{d}{dt}(L^{-1}(t)) \\ &= (L^{-1}(t))^{-1}B(t)L^{-1}(t) - (L^{-1}(t))^{-1}\frac{d}{dt}(L^{-1}(t)) \end{aligned}$$

so the system (3.5) is reducible to the system (3.1). 

Definition 4.3.5: If for a system (3.1) there exists a Lyapunov transformation $X = L(t)Y$ such that the matrix

$$B = L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}(L(t))$$

is constant, then the system is said to be **reducible to a system with constant coefficients**.

Theorem 4.3.6: (Erugin). A system (3.1) is reducible to a system with constant coefficients if and only if it has a fundamental matrix $X(t)$ of the form

$$X(t) = L(t)e^{Bt} \quad (3.6)$$

where $L(t)$ is a Lyapunov and B is a constant matrix [1].

Proof: ($\Rightarrow$) Let the transformation $X = L(t)Y$ reduce the system (3.1) to the system

$$\frac{dY}{dt} = BY$$

with a constant matrix B .

Then by theorem 2.2.1 e^{Bt} is fundamental matrix of the system

$$\frac{dY}{dt} = BY.$$

Hence

$$X = L(t)e^{Bt}.$$

($\Leftarrow$) If

$$X(t) = L(t)e^{Bt}$$

then

$$L(t) = X(t)e^{-Bt}$$

Thus the desired transformation is $X = L(t)Y$, where $L(t) = X(t)e^{-Bt}$. Since

$$\begin{aligned} & (X(t)e^{-Bt})^{-1} A(t) (X(t)e^{-Bt}) - (X(t)e^{-Bt})^{-1} \frac{d}{dt} (X(t)e^{-Bt}) \\ &= e^{Bt} X^{-1}(t) A(t) X(t) e^{-Bt} - e^{Bt} X^{-1}(t) \left(\frac{dX(t)}{dt} e^{-Bt} - X(t) e^{-Bt} B \right) \\ &= B. \blacksquare \end{aligned}$$

Remark 4.3.7: If one of the fundamental matrices $\Phi(t)$ of the system

$$\frac{dX}{dt} = A(t)X$$

can be represented in the form (3.6), then any other $\Phi_1(t)$ has analogous form. For the fundamental matrix $\Phi_1(t)$ there exists a nonsingular matrix C such that $\Phi_1(t) = \Phi(t)C$. Therefore

$$\Phi_1(t) = L(t)e^{Bt} C = L(t)CC^{-1}e^{Bt}C = L_1(t)e^{Bt}.$$

where

$$L_1(t) = L(t)C, \quad B_1 = C^{-1}BC.$$

Using Erugin's theorem, we conclude:

Theorem 4.3.8: (Lyapunov Theorem): Any linear system with periodic coefficients is reducible to a system with constant coefficients.

Proof: Any fundamental matrix $\Phi(t)$ of a system with periodic coefficients can be represented in the form

$$\Phi(t) = P(t)e^{tR}, \quad P(t + \omega) = P(t), \quad R = \text{constant matrix}.$$

We need to show that $P(t)$ is a Lyapunov matrix.

$P(t)$ is continuous and differentiable on $\mathbb{R}$, $P(t)$ and $\frac{d}{dt}(P(t))$ are bounded as periodic matrices. Since

$$|\det P(t)| \geq \min_{0 \leq t \leq \omega} |\det P(t)| = \alpha \neq 0.$$

Thus $P^{-1}(t)$ is bounded. Therefore $P(t)$ is a Lyapunov matrix.

Hence by **Erugin** theorem implies reducibility to a system with constant coefficients. ■

CHAPTER FIVE

STABILITY OF LINEAR SYSTEMS

5.1 On Stability Of Linear Systems.

Consider the system

$$\frac{dX}{dt} = A(t)X \quad \text{for } t \geq t_0 \quad (1.1)$$

where X is unknown vector function and $A(t)$ is a continuous real matrix for $t \geq t_0$.

Denote the solution on the initial condition $X(t_0) = X_0$ by $X(t, t_0, X_0)$, we call this solution **unperturbed**, the solution on the initial condition $X(t_0) = X_1$ **perturbed**, and the difference $X_0 - X_1$ the **perturbation**.

Definition 5.1.1: The unperturbed solution $X(t, t_0, X_0)$ is said to be **Lyapunov stable** if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the condition

$$\|X_0 - X_1\| < \delta \quad \text{implies} \quad \|X(t, t_0, X_0) - X(t, t_0, X_1)\| < \varepsilon \quad \text{for all } t \geq t_0.$$

Geometrically: Lyapunov stability means that the solutions, which were once near together, remain the same for all functions of time.

Definition 5.1.2: The solution $X(t, t_0, X_0)$ is said to be **asymptotically stable** if: 1) it is **Lyapunov stable**.

2) if there exists a $\delta > 0$ such that the condition $\|X_0 - X_1\| < \delta$ implies that $\lim_{t \rightarrow \infty} \|X(t, t_0, X_0) - X(t, t_0, X_1)\| = 0$.

Definition 5.1.3: A solution $X(t, t_0, X_0)$ is said to be **uniformly stable** if for any $\varepsilon > 0$ there exists a $\delta > 0$, independent choice of t_0 , such that the condition $\|X_0 - X_1\| < \delta$ implies the inequality

$$\|X(t, t_0, X_0) - X(t, t_0, X_1)\| < \varepsilon, \quad t \geq t_0 \geq 0.$$

In Fig. (5.1) we depict the behavior of the trajectories in the vicinity of a stable for the case $x \in R^2$. By choosing the initial points a sufficiently small spherical neighborhood, we can force the graph of the solution for $t \geq t_0$ to lie entirely inside a given cylinder.

In definition (5.1.3), δ depends on ε and t_0 (i.e. $\delta = \delta(\varepsilon, t_0)$).

If δ is independent of t_0 , i.e. $\delta = \delta(\varepsilon)$, then the system (1.1) is said to be uniformly stable.

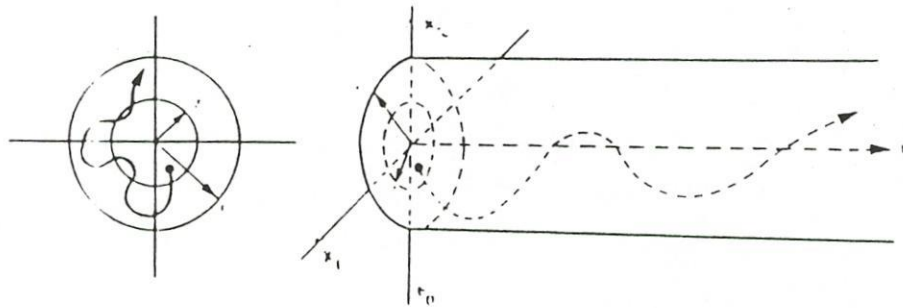


Figure 5.1

Lemma 5.1.4: The stability of any solution of system (1.1) is equivalent to the stability of the trivial solution of the homogenous system.

Proof: Let $\Phi(t, t_0)$ be the fundamental matrix of system (1.1) with $\Phi(t_0, t_0) = I$, then the general solution is

$$X(t, t_0, X_0) = \Phi(t, t_0)X_0 \quad (1.2)$$

Consider the difference of the perturbed and unperturbed solutions

$$z(t) = X(t, t_0, X_0) - X(t, t_0, X_1) = \Phi(t, t_0)(X_0 - X_1).$$

The difference is the solution of the homogenous system with an initial condition $z(t_0) = X_0 - X_1$. Therefore, the problems of stability of the unperturbed solution $X(t, t_0, X_0)$ are related to the following :

for any $\varepsilon > 0$ there exists a $\delta > 0$ such the condition

$\|z(t_0)\| < \delta$ implies the condition $\|z(t)\| < \varepsilon$ for all $t \geq t_0$. This is the problem of stability of the trivial solution of the homogenous system. ■

Note 5.1.5: If some solution of system (1.1) is stable, then the trivial solution of (1.1) is stable, the stability of the trivial solution implies the stability of any solution of system (1.1).

Theorem 5.1.6(a): All solution of system (1.1) are simultaneously stable or unstable.

(b): A linear homogeneous system is stable if and only if each of its solution $X(t)$ is bounded for $t \geq t_0$.

Proof: ($\Rightarrow$) we show that the stability of the system implies the boundedness of any its solution.

Suppose that there exist a solution $z(t)$ of system (1.1) unbounded for $t \geq t_0$, so $\|z(t_0)\| \neq 0$.

Given $\varepsilon > 0$ so

$$X(t) = \frac{z(t)}{\|z(t_0)\|} \cdot \frac{\delta}{2}$$

is a solution for $\delta > 0$, hence

$$\|X(t_0)\| = \frac{\|z(t_0)\|}{\|z(t_0)\|} \cdot \frac{\delta}{2} = \frac{\delta}{2} < \delta$$

so by the stability we have $\|X(t)\| < \varepsilon$ for $t \geq t_0$.

Therefore

$$\|z(t)\| = \|X(t)\| \cdot \|z(t_0)\| \cdot \frac{2}{\delta} < \frac{2\varepsilon}{\delta} \|z(t_0)\| = \varepsilon$$

So we have $z(t)$ is bounded, this contradicts our assumption that $z(t)$ is unbounded.

($\Leftarrow$) Suppose $\Phi(t, t_0)$ is the fundamental matrix of (1.1), its columns are solutions of the system, and they are bounded and n in number; therefore, there exists a constant K such that $\|\Phi(t, t_0)\| < K$ for $t \geq t_0$.

Any solution $X(t)$ is written in the form $X(t) = \Phi(t, t_0)X(t_0)$.

Hence

$$\|X(t)\| = \|\Phi(t, t_0)\| \|X(t_0)\| \leq K \|X(t_0)\|$$

Therefore

$$\|X(t)\| \leq K \|X(t_0)\|.$$

Take $\varepsilon > 0$, so we have for $\|X(t_0)\| < \delta$ we have

$$\|X(t)\| \leq K\delta \text{ for } t \geq t_0, \text{ hence } \delta = \frac{\varepsilon}{K}.$$

Thus the solution $X(t)$ is stable and system (1.1) is also stable. -

Corollary 5.1.7: All the solutions of a stable linear homogeneous system are simultaneously either bounded or unbounded for $t \geq t_0$.

Theorem 5.1.8: A linear homogeneous system is asymptotically stable if and only if all its solutions $X(t)$ tends to zero as $t \rightarrow \infty$, i. e

$$\lim_{t \rightarrow \infty} \|X(t)\| = 0.$$

Proof: ($\Rightarrow$) Let the trivial solution be asymptotically stable i.e there exists a $\delta > 0$ such that $\|X(t_0)\| < \delta$ implies $\lim_{t \rightarrow \infty} \|X(t)\| = 0$.

Take an arbitrary solution $X(t)$ and write it in the following way

$$\begin{aligned} X(t) &= \frac{X(t)}{\|X(t_0)\|} \cdot \frac{\|X(t_0)\|}{\delta/2} \cdot \frac{\delta}{2} \\ &= z(t) \cdot \frac{2\|X(t_0)\|}{\delta}, \end{aligned}$$

where

$$z(t) = \frac{X(t)}{\|X(t_0)\|} \cdot \frac{\delta}{2}.$$

So we have $z(t_0) = \frac{\delta}{2} < \delta$, therefore $\|z(t)\| \rightarrow 0$ as $t \rightarrow \infty$,

and thus $\|X(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

($\Leftarrow$) Let $\|X(t)\| \rightarrow 0$ as $t \rightarrow \infty$ for any solution $X(t)$. Then for each solution there exists $T > t_0$ such that $\|X(t)\| < 1$ for $t \geq T$.

From theorem (5.1.6) $X(t)$ is stable. But for any $\|X(t_0)\| < \infty$ we have $\|X(t)\| \rightarrow 0$ as $t \rightarrow \infty$. i.e. take $\delta = \infty$ and the asymptotically stable is proved. ■

Theorem 5.1.9: System (1.1) is uniformly stable if and only if there exists a constant $K > 0$ such that for all $0 \leq s \leq t \leq \infty$ the inequality

$$\|\Phi(t, s)\| = \|\Phi(t)\Phi^{-1}(s)\| \leq K \quad (1.3)$$

holds, where $\Phi(t, s)$ is a fundamental matrix of (1.1).

Proof: ($\Leftarrow$) Any solution $X(t, t_0)$ can be written as

$$X(t, t_0) = \Phi(t, t_0)X(t_0)$$

therefore by (1.3), we obtain

$$\begin{aligned} \|X(t, t_0)\| &= \|\Phi(t, t_0)X(t_0)\| = \|\Phi(t, t_0)\| \|X(t_0)\| \\ &\leq K \|X(t_0)\|. \end{aligned}$$

hence

$$\|X(t, t_0)\| \leq K \|X(t_0)\|.$$

For any $\varepsilon > 0$ we choose $\delta = \frac{\varepsilon}{K}$, and, independently of the specific value of $t_0 \geq 0$, we have

$$\|X(t, t_0)\| \leq K\delta = K\left(\frac{\varepsilon}{K}\right) = \varepsilon, \quad t \geq t_0.$$

hence

$$\|X(t, t_0)\| \leq \varepsilon, \quad t \geq t_0.$$

Thus the trivial solution is uniformly stable, hence system (1.1) is uniformly stable.

($\Rightarrow$) Take $\varepsilon > 0$, choose $\delta > 0$ independently of t_0 according to it, and consider the solution $X(t, t_0)$ for any $t_0 \geq 0$ such that

$$X(t_0) = \begin{pmatrix} \frac{\delta}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e. the first component $x_1(t_0) = \frac{\delta}{2}$, and the i -th component $x_i(t_0) = 0$, $i \geq 2$.

At the same time, $X(t, t_0) = \Phi(t, t_0)X(t_0)$. So by the uniform stability we have

$$\|\Phi(t, t_0)X(t_0)\| < \varepsilon, \quad t \geq t_0 \geq 0,$$

that is the first column of the matrix $\Phi(t, t_0)$ multiplied by $\frac{\delta}{2}$. This implies

that the norm of this column not exceed the value $\frac{2\varepsilon}{\delta}$. If the initial vector $X(t, t_0)$ is chosen so that

$$x_i(t, t_0) = \frac{\delta}{2}, \quad x_j(t, t_0) = 0, \quad i \neq j.$$

Then we obtain a similar estimate for the i -th column of the matrix $\Phi(t, t_0)$, this shows that

$$\|\Phi(t, t_0)\| \leq \frac{2\varepsilon}{\delta}.$$

Hence

$$\|\Phi(t, t_0)\| = \|X(t, t_0)X^{-1}(t_0)\| \leq \frac{2\varepsilon}{\delta}. \quad \blacksquare$$

Corollary 5.1.10: For a linear system with constant coefficients the notations of stability and uniform stability are equivalent.

Proof: The stability of linear system with constant coefficients implies that the fundamental matrix e^{At} is bounded for $t \geq 0$. -

5.2 On Stability Of Linear Homogeneous System With Constant And Periodic Coefficients.

Recall that, for a function $g : \mathbb{R} \rightarrow \mathbb{R}$, and for $\beta \geq 0$, the notation

$$g(x) = O(|x|^\beta) \text{ as } |x| \rightarrow \infty$$

means that

$$\limsup_{|x| \rightarrow \infty} \frac{|g(x)|}{|x|^\beta} < \infty.$$

If in the above, the continuous variable x is replaced by an integer valued variable $m \geq 0$. Then

$$g(m) = O(m^\beta) \text{ as } m \rightarrow \infty.$$

Theorem 5.2.1: A linear homogeneous system with constant coefficients is stable if and only if all the eigenvalues of the coefficient matrix have non positive real parts, and simple elementary divisor correspond to the eigenvalue with zero real parts.

Proof: We use the Jordan canonical form to compute e^{At} .

Let $J = P^{-1}AP$ and define $X = PY$. Then

$$\frac{dY}{dt} = P^{-1} \frac{dX}{dt} = P^{-1}AX = (P^{-1}AP)Y = JY$$

hence

$$\frac{dY}{dt} = JY.$$

We can assume without loss of generality that the “system matrix A” is in Jordan canonical form, i.e. A is in block diagonal form

$$A = \begin{pmatrix} J_0 & 0 & \cdots & 0 \\ 0 & J_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_s \end{pmatrix},$$

where

$$J_0 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix}$$

and $J_k = \lambda_{k+i}I + N_i$ for the Jordan blocks $J_1, \dots, J_s$.

As in the previous chapter we see that

$$e^{tJ_0} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_k t} \end{pmatrix}$$

and

$$e^{J_0 t} = e^{\lambda_{k+i} t} \begin{pmatrix} 1 & t & t^2 & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{aligned} \|e^{J_0 t}\| &= \sum_{i=1}^n \|e^{\lambda_i t}\| = \|e^{\lambda_1 t}\| + \|e^{\lambda_2 t}\| + \cdots + \|e^{\lambda_k t}\| \\ &= \|e^{(\operatorname{Re} \lambda_1 + i \operatorname{Im} \lambda_1)t}\| + \|e^{(\operatorname{Re} \lambda_2 + i \operatorname{Im} \lambda_2)t}\| + \cdots + \|e^{(\operatorname{Re} \lambda_k + i \operatorname{Im} \lambda_k)t}\| \\ &= e^{\operatorname{Re} \lambda_1 t} \|e^{i \operatorname{Im} \lambda_1 t}\| + e^{\operatorname{Re} \lambda_2 t} \|e^{i \operatorname{Im} \lambda_2 t}\| + \cdots + e^{\operatorname{Re} \lambda_k t} \|e^{i \operatorname{Im} \lambda_k t}\|. \end{aligned}$$

Hence

$$\|e^{J_0 t}\| \leq \sum_{i=1}^n e^{t\beta},$$

if $\operatorname{Re} \lambda_i \leq \beta$ for all $i = 1, 2, \dots, k$ then

$$\limsup_{t \rightarrow \infty} \frac{\|e^{J_0 t}\|}{e^{\beta t}} \leq \infty. \text{ Hence } \|e^{J_0 t}\| = O(e^{\beta t}).$$

Also for $t > 0$ and $\operatorname{Re} \lambda_i \leq \beta$ we have

$$\begin{aligned} \|e^{J_0 t}\| &\leq \|e^{\lambda_{k+i} t}\| \cdot (c_1 + c_2 t + \cdots + c_{n_i} t^{n_i-1}) \\ &\leq e^{\beta t} \cdot (c_1 + c_2 t + \cdots + c_{n_i} t^{n_i-1}) \\ &= O(e^{\beta t} t^{n_i-1}) \leq O(e^{(\beta+\varepsilon)t}), \text{ for } \varepsilon > 0. \end{aligned}$$

Therefore $\|e^{A t}\| \leq K$ for some constant $K > 0$. Thus the linear homogenous system is stable. ■

Theorem 5.2.2: A linear homogenous system with periodic coefficients is stable if and only if all its multipliers belong to a closed unit disk, and simple elementary divisor correspond to the multipliers lying on the unit circle.

Proof: We see any periodic system with period ω can be reduced to a system of constant coefficients. If $\Phi(t)$ is a fundamental matrix for the periodic system

$$\frac{dX}{dt} = A(t)X \quad (2.1)$$

Then $\Phi(t) = P(t)e^{tB}$, where $P(t)$ is a nonsingular matrix which is also periodic, and B is a constant matrix.

Thus

$$P(t) = \Phi(t) e^{-tB} \text{ and } AP(t) - \frac{d}{dt}(P(t)) = P(t)B.$$

If we use the Lyapunov transformation $X = L(t)Y$, then system (2.1) can be reduced to the linear homogenous system with constant coefficient

$$\frac{dY}{dt} = BY \quad (2.2)$$

From theorem (5.2.1) the system (2.2) is stable if and only if all eigenvalues of B have non positive real parts and every eigenvalues of B with a zero real part is a simple zero of characteristic polynomial of B .

Let λ be an eigenvalue of B then $\rho = e^{\lambda\omega}$ is a characteristic multiplier of the Mondromy matrix $e^{B\omega}$.

If $\operatorname{Re} \lambda = 0$, then $\|\rho\| = \|e^{\lambda\omega}\| = \|e^{\operatorname{Im} \lambda \omega i}\| = 1$, that is the characteristic multiplier lies on a unit circle. i.e. $\rho = e^{\lambda\omega}$ is the root of $\det(e^{B\omega} - e^{\lambda\omega} I) = 0$.

Since $\operatorname{Re} \lambda \leq 0$, we have

$$\|e^{\lambda\omega}\| = \|e^{\operatorname{Re} \lambda \omega} e^{\operatorname{Im} \lambda \omega i}\| = e^{\operatorname{Re} \lambda \omega} \|e^{\operatorname{Im} \lambda \omega i}\| = e^{\operatorname{Re} \lambda \omega} \leq 1.$$

Let λ is an eigenvalue of B then $\operatorname{Re} \lambda \leq 0$ if and only if $|\rho| = |e^{\lambda T}| \leq 1$, so the characteristic multipliers lies in a unit disk. ■

Example(1): Consider the system

$$\frac{dX}{dt} = A(t)X \quad (3.7)$$

where

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2} \cos^2 t & 1 - \frac{3}{2} \cos t \sin t \\ -1 - \frac{3}{2} \sin t \cos t & -1 + \frac{3}{2} \sin^2 t \end{pmatrix},$$

then the eigenvalues of A are $\lambda_1(t) = \frac{-1 + i\sqrt{7}}{4}$ and $\lambda_2(t) = \frac{-1 - i\sqrt{7}}{4}$.

And, in particular, the real parts of the eigenvalues have negative real parts.

$$X(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} e^{\frac{t}{2}}$$

is a solution of (3.7). $X(t)$ is unbounded as $t \rightarrow \infty$.

The characteristic multipliers are e^{Π} , $e^{-2\Pi}$.

Theorem 5.2.3: If system (1.1) is stable and reducible to a system with constant coefficients, then it is uniformly stable.

Proof: Consider the Lyapunov transformation $X = L(t)Y$ which reduces system (1.1) to the linear system

$$\frac{dY}{dt} = (L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}(L(t)))Y = BY \quad (2.3)$$

where

$$B = (L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}(L(t))).$$

The stability of the initial system implies that its fundamental matrix $\Phi(t)$ is bounded; therefore,

the fundamental matrix $Y(t) = L^{-1}(t)X(t)$ of system (2.3) is also bounded, hence, this system is stable. By corollary (5.1.10) system (2.3) is uniformly stable, i.e., there exists a constant $D > 0$ such that we have

$$\|Y(t, s)\| \leq D \text{ for all } t \geq s \geq 0.$$

By theorem (5.1.9) we have

$$\|\Phi(t, s)\| = \|\Phi(t)\Phi^{-1}(s)\| = \|L(t)Y(t)Y^{-1}(s)L^{-1}(s)\| \leq C^2 D,$$

where

$$\|L(t)\| \leq C, \quad \|L^{-1}(t)\| \leq C \quad \text{for } t \in R^+.$$

Therefore (1.1) is uniformly stable. ■

Corollary 5.2.4: If a system with periodic coefficients is stable then it is uniformly stable.

Proof: By Lyapunov theorem and last theorem. ■

5.3 Two Dimensional Linear Systems.

We study the stability properties of second order linear homogeneous systems given by

$$\frac{dX}{dt} = A(t)X$$

where $A(t)$ is 2×2 continuous real matrix and $\det(A(t)) \neq 0$.

If λ_1, λ_2 designate the eigenvalues of A , there are many cases to consider,

Case 1a: Negative roots, $\lambda_2 < \lambda_1 < 0$ (Stable node).

All solutions approach zero as $t \rightarrow \infty$. The straight lines L_1 and L_2

are the lines that contain the eigenvector P_1 , P_2 , respectively.

The origin is stable and called stable node (see Fig 5.2).

Case 1b: Positive roots, $\lambda_2 < \lambda_1 < 0$ (**Unstable node**), (see Fig 5.3).

Case 1c: One positive and one negative root, $\lambda_2 < 0 < \lambda_1$ (**Saddle point**)
(See Fig 5.4).

Case 2a: Pure imaginary roots (Center).

Case 2b: Complex root with negative real parts (Stable focus)
(See Fig 5.5).

Case 2b: Complex root with positive real parts (Unstable focus)
(See Fig. 5.6).

Case 3: Both eigenvalues equal (Improper node)
(See Fig 5.7)

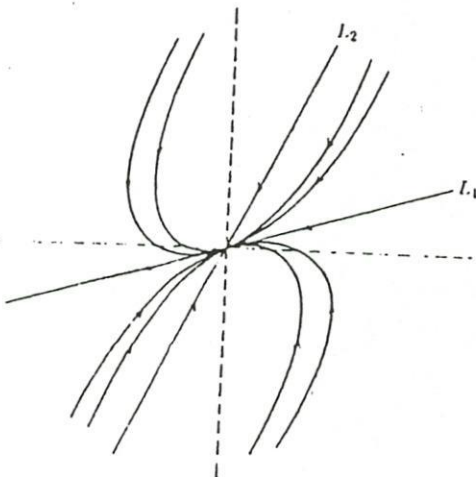


Figure 5.2 Stable node

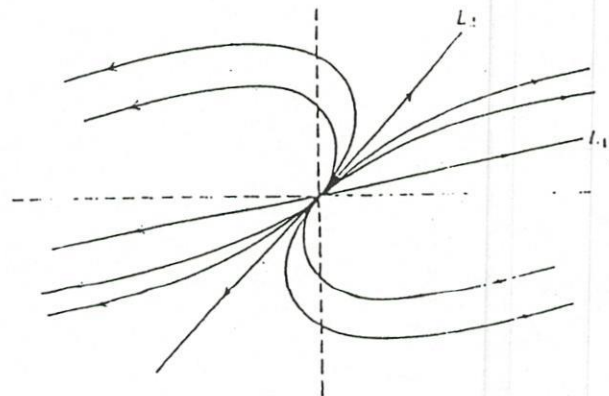


Figure 5.3 Unstable node

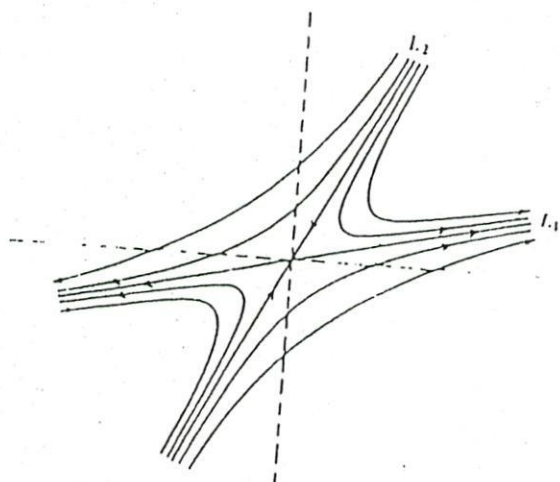


Figure 5.4 Saddle point

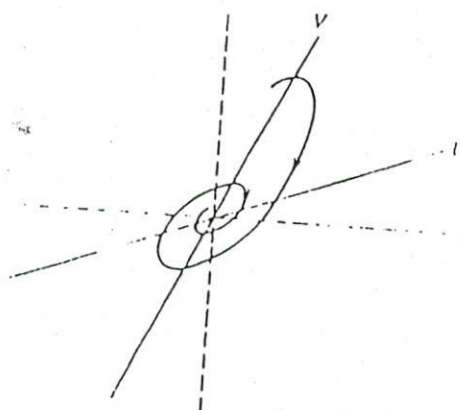


Figure 5.5 Stable focus

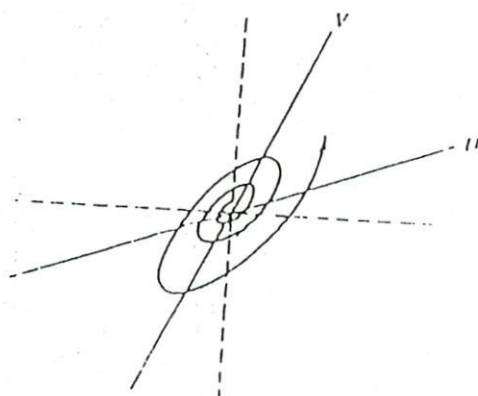


Figure 5.6 Unstable focus

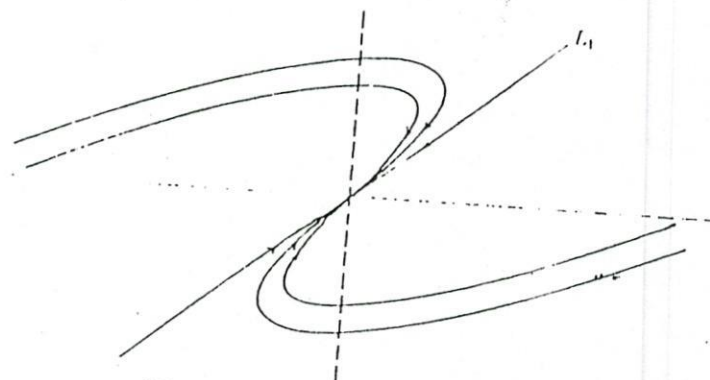


Figure 5.7 Stable Improper node

Appendix I:

If X is a vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $\|X\| = \sum_{i=1}^n \|x_i\|$.

Let A be $n \times n$ matrix of complex numbers. Define the norm of A (denoted by $\|A\|$) by $\|A\| := \sum_{i,j=1}^n |a_{ij}|$.

If $\det(A) = 0$, then A is said to be singular. If A is nonsingular matrix possesses an inverse say A^{-1} . Which satisfies $AA^{-1} = A^{-1}A = I_n$, where I_n is the identity $n \times n$ matrix.

The polynomial $P(\lambda) = \det(A - \lambda I_n)$ is called the characteristic polynomial of A , and the roots of $P(\lambda)$ are called the eigenvalues of A .

If λ_i are roots of $P(\lambda) = 0$, then $P(\lambda) = \det(A - \lambda I_n) = \prod_{i=1}^n (\lambda - \lambda_i)$.

We define the exponential of a matrix A by

$$e^A := I_n + \sum_{m=1}^{\infty} \frac{A^m}{m!} \quad \text{where } A^m = \underbrace{AA \cdots A}_{m\text{-times}}.$$

The series defining e^{At} is convergent.

$$e^{At} := I_n + \sum_{m=1}^{\infty} \frac{(At)^m}{m!}, \quad \text{for } t \in \mathbb{R}. \quad \text{In particular } e^{At}|_{t=0} = I_n.$$

$$(1) \quad e^{tA+sB} = e^{tA} \cdot e^{sB}. \quad (2) \quad (e^{At})^{-1} = e^{-At} \quad \text{for every } t \in \mathbb{R}.$$

$$(3) \quad Ae^{At} = e^{At}A. \quad \text{Since the series of } e^{At} \text{ is convergent, then we can differentiate it term by term.}$$

$$(4) \frac{d}{dt}(e^{At}) = Ae^{At} \quad (5) \det(e^{At}) = e^{tr(At)}. \quad (\text{By abel's formula when } \tau = 0)$$

By abel's formula

$$“\det(\Phi(t)) = \det(\Phi(\tau)) \cdot \exp\left(\int_{\tau}^t \text{tr}(A(s)) ds\right)”.$$

Hence for $\tau = 0$ we have

$$\begin{aligned} \det(e^{At}) &= \det(e^{A\tau}) \Big|_{\tau=0} \cdot \exp\left(\int_0^t \text{tr}(A(s)) ds\right) \\ &= I_n \cdot \exp(\text{tr}(A)) S \Big|_0^t = I_n \cdot \exp(\text{tr}(A)t) = e^{tr(A)t} \end{aligned}$$

$$(6) \text{ If } B = PAP^{-1} \text{ then } e^B = Pe^A P^{-1} = e^{PAP^{-1}}.$$

Appendix II:

Derivation Of Jordan Canonical Form.

Defenition(1): Two $n \times n$ matrices A and B are said to be **similar** if there is a nonsingular matrix P such that $A = P^{-1}BP$.

If the matrix A has repeated eigenvalues, then it is not always possible to diagonalize it, so we introduce the concept of generalized eigenvector.

Definition(2): A vector V is called a **generalized eigenvector of rank k** of a matrix A associated with an eigenvalue if and only if

$$(A - \lambda I)^k V = 0 \text{ and } (A - \lambda I)^{k-1} V \neq 0$$

Now let V be a generalized eigenvector of rank k associated with the eigenvalue λ , define:

$$V_k = V$$

$$V_{k-1} = (A - \lambda I_n)V = (A - \lambda I_n)V_k$$

$$V_{k-2} = (A - \lambda I_n)^2 V = (A - \lambda I_n)V_{k-1}$$

$$\vdots$$

$$V_1 = (A - \lambda I_n)^{k-1} V = (A - \lambda I_n)V_2.$$

Then for each $1 \leq i \leq k$, V_i is a generalized eigenvector of rank i , we call the set of vectors $\{V_1, V_2, \dots, V_k\}$ a chain of generalized eigenvectors.

We now give a procedure for computing a set of basis vectors which yields the Jordan canonical form J of an $n \times n$ matrix A and the required nonsingular transformation P which relates A to J :

Step1: We compute the eigenvalues of A , let $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues of A with multiplicity $n_1, \dots, n_m$ respectively.

Step2: We compute n_1 linearly independent generalized eigenvectors of A associated with λ_1 as follows:

we compute $(A - \lambda_1 I)^i$ for $i = 1, 2, \dots$, until

$$\text{rank}(A - \lambda_1 I)^k = \text{rank}(A - \lambda_1 I)^{k+1}.$$

Then find a generalized eigenvector of rank k , say u .

Define

$$u_i = (A - \lambda_1 I)^{k-i} u, \quad i = 1, 2, \dots, k, \quad \text{if } k = n_1,$$

proceed to step 3.

If $k < n_1$, find another linearly independent generalized eigenvector with the largest possible rank, i.e. try to find another generalized eigenvector with rank k , if is not possible try $k-1$, and so forth, until n_1 linearly independent generalized eigenvectors are determined, note that if $\text{rank}(A - \lambda_1 I) = r$, then there are totally $(n-r)$ chains of generalized eigenvectors associated with λ_1 .

Step 3 : Repeat step 2 for $\lambda_2, \dots, \lambda_m$.

Step 4 : Let $U_1, U_2, \dots, U_k, \dots$ be the new basis, obtained from the generalized eigenvectors. Which yields a representation of A with respect to the new basis as follows:

$$J = \left(\begin{array}{cccc} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_1 \end{array} \right) \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} k$$

$\underbrace{\hspace{10em}}_k$

$\ddots$

Note that each chain of generalized eigenvectors generate a Jordan block whose order equals the length of the chain.

Step 5 : The similarity transformation which yields $J = P^{-1}AP$ is given by $P = (U_1 \ U_2 \ \cdots \ U_k \ \cdots)$

Step 6: Rearrange the Jordan blocks in the desired order to yield the canonical form and the corresponding similarity transformation P.

Example : The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is given by $\det(A - \lambda I) = (\lambda - 2)^5 \lambda = 0$. Thus A has eigenvalues λ_1 with multiplicity 1, λ_2 with multiplicity 5.

Now compute $(A - \lambda_2 I)^i$, $i = 1, 2, \dots$

$$(A - 2I) = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad (A - 2I)^2 = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{pmatrix}$$

$$(A - 2I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 & 4 & -4 \end{pmatrix}, \quad (A - 2I)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 0 & -8 & 8 \end{pmatrix}$$

Thus $\text{rank}(A - 2I) = 4$, $\text{rank}(A - 2I)^2 = 2$, $\text{rank}(A - 2I)^3 = 1$, and $\text{rank}(A - 2I)^4 = 1$, since $\text{rank}(A - 2I)^3 = \text{rank}(A - 2I)^4 = 1$, we stop at $(A - 2I)^3$. So choose

$$U = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ as the solution of } (A - 2I)^3 U = 0, \text{ and } (A - 2I)^2 U = 0.$$

Hence U is a generalized eigenvector of rank 3. So we define

$$U_1 = (A - 2I)^2 U = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$U_2 = (A - 2I)U = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and}$$

$$U_3 = U = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since we have only three generalized eigenvectors for $\lambda_2 = 2$ and since the multiplicity of $\lambda_2 = 2$ is five, we have to find two more linearly

independent eigenvectors for $\lambda_2 = 2$. So let us try to find a generalized eigenvectors of rank 2.

Choose V to be the solution of $(A - 2I)^2 V = 0$ and $(A - 2I)V \neq 0$.

Take $V = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$. Hence V is another linearly independent generalized

eigenvector of rank 2. Define

$$V_2 = (A - 2I)V = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

Next we compute an eigenvector associated with $\lambda_1 = 0$ since

$$W = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

is a solution of $(A - \lambda_1 I)W = 0$, the vector W will do.

Finally with respect to the basis $W, U_1, U_2, U_3, V_1, V_2$ the Jordan canonical form of A is given by

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\text{and } P = (W, U_1, U_2, U_3, V_1, V_2) = \begin{pmatrix} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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