

**OSCILLATION AND NON-OSCILLATION OF ARBITRARY
ORDER DELAY DIFFERENTIAL
EQUATIONS**

By

Fatima M.A. Al-sughayyer

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By

Student Name: Fatima M.A. Al-Sughayyer

Registration No. 20010170

Supervisor: Dr. Taha Abu-Kaff

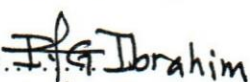
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The names and signature of the examining committee members are as follows:

1- **Dr. Taha Abu-Kaff** Head of committee

Signature.....

2-**Dr.Ibrahim Al-Grouz** Internal Examine

Signature.....

3-**Dr. Ibrahim Al- Massri** External Examine

Signature.....

Al-Quds University

2004

Declaration:

I certify that this thesis submitted for the degree of master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: *Fatima Al-Sughayer*

Fatima M.A.Al-sughayer

Date: 3/5/2004

Dedication

To my husband, my children and my sweet Yusra,

To my parents.

Acknowledgment

I am very grateful to my supervisor Dr. Taha Abu-Kaff for all his help, and excellent guides.

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Abstract

We were interested with differential equations with deviating arguments, especially, delay differential equations of arbitrary order. Our discussion is about the most important issue which is the oscillatory and non-oscillatory behavior of their solutions.

The equation $x^{(n)}(t) + p(t)f(x(g(t))) = h(t)$ was the main equation that discussed mainly throughout this thesis for different values of n .

For $n = 1 \Rightarrow x'(t) + p(t)f(x(g(t))) = h(t)$ discussed in chapter two.

For $n = 2 \Rightarrow (r(t)x'(t))' + p(t)f(x(g(t))) = h(t)$ discussed in chapter three.

For $n = 3 \Rightarrow (b(t)(a(t)x'(t))')' + p(t)f(x(g(t))) = h(t)$ and the n th order

$x^{(n)} + p(t)x(g(t)) = h(t)$ discussed in chapter four.

ملخص بالعربية

لقد تم الاهتمام في هذه الأطروحة بدراسة خاصية التذبذب وعدم التذبذب لحلول بعض المعادلات التفاضلية الاقترانية و التعرف على بعض الشروط الواجب توفرها للحصول على خاصية التذبذب.

و الشكل العام للمعادلات تحت البحث كانت على الصورة التالية

$$x^{(n)}(t) + p(t)f(x(g(t))) = h(t)$$

وذلك لقيم مختلفة من n

$$n = 1 \Rightarrow x'(t) + p(t)f(x(g(t))) = h(t)$$

$$n = 2 \Rightarrow (r(t)x'(t))' + p(t)f(x(g(t))) = h(t)$$

$$n = 3 \Rightarrow (b(t)(a(t)x'(t))')' + p(t)f(x(g(t))) = h(t)$$

$$nth - order \Rightarrow x^{(n)}(t) + p(t)x(g(t)) = h(t).$$

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Introduction

In many applications, the future state of the system is independent of the past states and is determined solely by the present. But in the late thirties and early forties, Minorsky [4], in his study of ship stabilization and automatic steering, pointed out very clearly the importance of the consideration of the delay in the feed back mechanism. The great interest in control theory during these and later years has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state .And also in the late forties and early fifties , a few books appeared which presented the current status of the subject and certainly greatly influenced later developments. Mishkis [4] introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems.

The simplest type of past dependence in a differential equation is that in which the past dependence is through the state variable and not the derivative of the state variable, the so-called retarded functional differential equations or retarded differential equations.

The oscillatory behavior of solutions of differential equations with deviating arguments has been studied by many authors. The problem of oscillations caused by the deviating arguments (delay or advanced) has been the subject of intensive investigations.

Amongst the application areas of differential equations with deviating arguments are in bioscience, economics, material science; medicine, public health; in a number of these there is an underlying problem in control theory.

Among numerous papers dealing with the study of this problem we choose to refer to the papers by M.Kon, Y.G.Sficas, and I.P.Stavroulakis [7], Bingtuan.Li [2], R.Koplatadze and G. Kvinikadze [7], R.S.Dahiya and Olusola Akinyele [9], R.S. Dahiya, Takasi Kusano and Manabu Naito [10], S.R.Grace and B.S.Lalli [11]. In the literature of oscillation theory of functional differential equations, the following are of special interest, either

- (1) all solutions are oscillatory;
- (2) all solutions are non-oscillatory;
- (3) the equation has a non-oscillatory solution;
- (4) the equation has an oscillatory solution;
- (5) the equation has both oscillatory and non-oscillatory solution.

M.Kon, Y.G.Sficas, and I.P.Stavroulakis [7], discussed the oscillatory behavior of first order delay differential equations of the form $x'(t) + p(t)x(\tau(t)) = 0$, $t \geq t_0$

where $p, \tau \in C([t_0, \infty), R^+)$, $R^+ = [0, \infty)$, $\tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$, and

$\lim_{t \rightarrow \infty} \tau(t) = \infty$. They proved that when $L < 1$ and $0 < K \leq \frac{1}{e}$ all solutions $x(t)$ are

oscillatory. In which the condition $L > 2K + \frac{2}{\lambda} - 1$ holds, where

$K = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds$ and $L = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds$ and λ is the smaller root of the

equation $\lambda = e^{K\lambda}$. In [9] R.S. Dahiya and Olusola Akinyele discussed the n -th order functional differential equation of the form $x^{(n)}(t) + p(t)x(g(t)) = h(t)$ where

- (a) $p, g, h : [a, \infty) \rightarrow R$ are continuous.
- (b) $p(t) > 0, g(t)$ is non-decreasing.

(c) $g(t) < t, g(t) \rightarrow \infty$ as $t \rightarrow \infty$.

They provided sufficient conditions for the above functional differential equation to be almost oscillatory in the sense that every solution $x(t)$ is either oscillatory or

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \quad 0 \leq i \leq n-1.$$

One of the basic equations that we seek to discuss for different values of n is

$$x^{(n)}(t) + p(t)f(x(g(t))) = h(t)$$

The first chapter introduces the reader to the basic definition and theorems that are needed later and some applications on delay differential equations. Chapter two is devoted to the discussion about the oscillation, non-oscillation, and the asymptotic behavior of first order delay differential equations of the form $x'(t) + p(t)f(x(g(t))) = h(t)$. In chapter three we investigate the oscillation, and the asymptotic behavior of second order delay differential equations of the form $(r(t)x'(t))' + p(t)f(x(g(t))) = h(t)$. Finally the oscillation of third and n -th order delay differential equations has been studied in chapter four.

The basic observation and the importance of deviating arguments in the sense that their presence causes or destroys the oscillation phenomena and does not merely preserve the oscillatory behavior of equations without deviating arguments.

Remark:

We indicate that we have used to refer to equations by the triple (a,b,c)

where a refers to the chapter number;

and b refers to the section number;

and c refers to the equation number;

For the end of the proof we have used the black dot •

Chapter One

Preliminaries

1.1.Overview

Many important and significant problems in engineering, physical science and social science, when formulated into mathematical term, require the determination of a function satisfying an equation containing one or more derivatives of unknown function, and these derivatives depend on the solution at the present value of the independent variable (t), such equations are called ordinary differential equations of the form

$$x'(t) = f(t, x(t)) \quad (1.1.1)$$

with the initial condition, $x(t_0) = x_0$. Also, some other significant problems in many scientific regions, the function and some of its derivatives depend on its past memory, this kind of differential equations are called differential equations with deviating arguments.

Definition 1.1.1. Differential equations with deviating arguments are differential equations in which the unknown function appears with various values of the arguments. For example

$$x'(t) = f(t, x(t), x(t - \tau(t))) \quad (1.1.2)$$

$$x''(t) = f(t, x(\frac{t}{2}), x'(\frac{t}{2}), x(t), x'(t)) \quad (1.1.3)$$

Definition 1.1.2. A differential equation with retarded (or delay) argument is a differential equation with deviating argument in which the highest – order derivative of the unknown function appears for just one value of the argument, and this

argument is not less than all arguments of the unknown function and its derivatives appearing in the equation .

Equation (1.1.2) is retarded (delay) if $\tau(t) \geq 0$, and (1.1.3) is retarded (delay) if $t \geq 0$.

The study of differential equations with deviating arguments have been intensively investigated for many years, and the area of applications of differential equations with delay argument has greatly expanded in many fields ; In physics and technology, economics and biological sciences, and in medicine , production of real blood cells, etc .This abundance of applications has increased the interest in the theory of differential equations with deviating argument .Consider the differential equation of the form

$$\begin{cases} x'(t) = f(t, x(t), x(t - \tau)), & \tau > 0, \quad t \geq t_0 \\ x(t) = \phi_0(t) \end{cases} \quad (1.1.4)$$

in which the right –hand side depends not only on the instantaneous position $x(t)$, but also on $x(t - \tau)$, the position at τ units back ; the equation has past memory , such an equation is called an ordinary differential equations with a delay or retarded argument. In order to define a solution of (1.1.4) we need to have a known function $\phi_0(t)$ on $[t_0 - \tau, t_0]$, instead of just the initial condition $x(t_0) = x_0$. The basic initial-value problem for (1.1.4) consists of determining a continuous solution $x(t) = \phi_0(t)$ for $t_0 - \tau \leq t \leq t_0$, where $\phi_0(t)$ is a given continuous function, called the initial function. The segment $t_0 - \tau \leq t \leq t_0$ on which the initial function is given is called the initial set and denoted by E_{t_0} . It is usually assumed that $\phi_0(t_0) = x(t_0 + 0)$. In the case of a variable delay $\tau = \tau(t) > 0$ in equation (1.1.4) it is also required to find a solution of this equation for $t > t_0$, such that on the initial set

$E_{t_0} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0\}$, i.e. (the point t_0 and those values of $t - \tau(t)$

less than t_0 for $t \geq t_0$), $x(t)$ coincides with the given initial function $\phi_0(t)$. If it is

required to determine the solution on the interval $[t_0, T]$, then the initial set

$$E_{t_0, T} = \{t - \tau(t) < t_0, t_0 \leq t \leq T\}.$$

Definition 1.1.3. A function x is said to be a solution of (1.1.4) if

$x(t) \in C([t_0 - \tau, T])$ for some t_0 , satisfies equation (1.1.4) for all $t \geq t_0$.

1.2 Method of steps

The most natural method for solving this problem is the so called **method of steps**

(**or method of successive integration**). In this method a continuous solution $x(t)$

is first determined from the differential equation without retardation. To solve (1.1.4)

on $[t_0, t_0 + \tau]$, equation (1.1.4) becomes $x'(t) = f(t, x(t), \phi_0(t - \tau))$ for

$t_0 \leq t \leq t_0 + \tau$, $x(t_0) = \phi_0(t_0)$ since for $t_0 \leq t \leq t_0 + \tau$, the argument $t - \tau$ varies

within the initial set $[t_0 - \tau, t_0]$ and consequently the third argument $x(t - \tau)$ of the

function f equals the initial function $\phi_0(t - \tau)$. Assuming the existence of a solution

$x = \phi_1(t)$ of this initial value problem on the entire segment $[t_0, t_0 + \tau]$ we obtain

$$x'(t) = f(t, x(t), \phi_1(t - \tau)) \quad \text{for } t_0 + \tau \leq t \leq t_0 + 2\tau, \quad x(t_0 + \tau) = \phi_1(t_0 + \tau)$$

$$x'(t) = f(t, x(t), \phi_n(t - \tau)) \quad \text{for } t_0 + n\tau \leq t \leq t_0 + (n+1)\tau,$$

$$x(t_0 + n\tau) = \phi_n(t_0 + n\tau).$$

This method allows us to determine the solution $x(t)$ on some finite segment. And at the same time proves the existence of the solution if the functions f, ϕ are continuous. And also proves its uniqueness if the function f satisfies one of the conditions guaranteeing the uniqueness of the solution of the equation $x'(t) = f(t, x(t), \phi_0(t - \tau))$ without deviating of the argument such as a Lipchitz conditions, as in the following definition.

Definition 1.2.1. f is said to satisfy Lipchitz conditions with Lipchitz constant L if $|f(t, x) - f(t, y)| \leq L|x - y|$ for all $(t, x), (t, y) \in D \subseteq R \times R$.

The process of successive integration determination of the solution on the segment $t_0 + (j-1)\tau \leq t \leq t_0 + j\tau$ cannot be continued in the case when one of the segment $t_0 - \tau \leq t \leq t_0, t_0 + \tau \leq t \leq t_0 + 2\tau, \dots, t_0 + (j-1)\tau \leq t \leq t_0 + j\tau, \dots$ reduce to a single point, this case will be called critical case, such a case arises only when $\tau = 0$ at some point.

Now we will mention the existence theorem.

Theorem 1.2.1. (Existence and uniqueness theorem[6])

If in the equation (1.1.4), $\tau(t)$ is continuous for $t_0 \leq t \leq t_0 + T$ ($T > 0$) and nonnegative, the function f is continuous in the neighborhood of the point $(t_0, \phi_0(t_0))$ and satisfies a Lipchitz condition, and the function $\phi_0(t)$ is continuous on E_{t_0} , then there exists a unique continuous solution $x_{\phi_0}(t)$ i.e. $x(t)$ which depends on the initial function $\phi_0(t)$ of the basic initial-value problem for (1.1.4) for $t_0 \leq t \leq t_0 + h$ (where h is sufficiently small).

One of the problems studied for such functional differential equations, is the oscillatory and non-oscillatory behavior of their solutions.

Definition 1.2.2. A non trivial solution $x(t)$ for a delay differential equations is said to be oscillatory if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros, $\{t_n\}$, of $x(t)$ such that $\lim_{n \rightarrow \infty} t_n = +\infty$. Otherwise, $x(t)$ is said to be non-oscillatory. For non-oscillatory solutions there exists a t_1 such that $x(t) \neq 0$, for all $t \geq t_1$. There is another definition for oscillatory solution.

Definition 1.2.3. A non trivial solution $x(t)$ is said to be oscillatory if it changes sign on (T, ∞) , where T is any number.

Example 1.2.1. Consider the equation of the form

$$x'(t) + x(t - \frac{\pi}{2}) = 0 \quad (1.2.1)$$

It has an oscillatory solutions $x = \sin t$ and $x = \cos t$

Example 1.2.2 consider the equation

$$x'(t) + \frac{1}{2e} x(t - 2) = 0 \quad (1.2.2)$$

It has a non-oscillatory solution $x(t) = \exp(-\frac{1}{2}t)$

Example 1.2.3. Consider the second order equation with delay

$$x''(t) + x(\pi - t) = 0 \quad (1.2.3)$$

it has both an oscillatory solution $x_1 = \sin t$ and a non-oscillatory solution

$$x_2 = e^t - e^{\pi-t}$$

Example 1.2.4. Consider the equation

$$x''(t) + \frac{1}{2}x(t) - \frac{1}{2}x(t - \pi) = 0 \quad t \geq 0 \quad (1.2.4)$$

it has an oscillatory solution $x(t) = 1 - \sin t$, which has an infinite sequence of multiple zeros.

1.3 Fixed point theorem

Fixed point theorems are important in proving the existence of non-oscillatory solutions.

Definition 1.3.1 A subset S of a normed space X is called convex if, for any $x, y \in S$, $ax + (1 - a)y \in S$ for all $a \in [0, 1]$.

Definition 1.3.2 Let X and Y be normed spaces. Suppose T is a linear operator with domain X and range Y . We say that T is compact if, for each bounded sequence $\{x_n\}$ in X , the sequence $\{Tx_n\}$ contains a subsequence converging to some limit in Y .

Theorem 1.3.1: Every continuous mapping of closed bounded convex set in R^n into it self has a fixed point [3].

Theorem 1.3.2. (Lebesgue convergence theorem): Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E

and for almost all x in E we have $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then $\int_E f = \lim \int_E f_n$.

1.4. Mathematical Models

We are concerned with applications in biosciences, but the early use of delay differential equations was to describe technical devices, e.g. control circuits. In that

context the delay is a measurable physical quantity (for example , the time that the signal takes to travels to the controlled object, the reaction time, and the time that the signal takes to return). There are parallels to the reaction of the body to pain.

In most application in the life sciences a delay is introduced when there are some hidden variables and processes which are not well understood but are known to cause a time-lag. In fact , a delay may represent a reaction chain on a transport process. The following are examples of mathematical models.

Example 1.4.1. (mixing of liquids)

Consider a tank containing B gallons of salt water .Fresh water flows in at a rate of q gallons per minute, the brine in the tank is continually stirred and the mixed solution flows out through a hole at the bottom, at the rate of q gallons per minute. Let $x(t)$ be the amount in pounds of salt in the brine in the tank at time t . Now let us agree that the mixing cannot occur instantaneously throughout the tank. Thus the concentration of the brine leaving the tank at time t will equal the average concentration at some earlier instant , say $t - r$, $r > 0$,then the differential equation for x becomes

$$x'(t) = -\frac{q}{B} x(t-r) \quad \text{or setting } c = \frac{q}{B},$$

$$x'(t) = -cx(t-r) \tag{1.4.1}$$

using the method of steps to solve (1.4.1).

One should specify an initial function on some interval of length r say $[t_0 - r, t_0]$, and then try to satisfy equation (1.4.1) for $t \geq t_0$, set $x(t) = \theta(t)$ for $t_0 - r \leq t \leq t_0$ where θ is some given function ,we seek a continuous extension of θ into the future to a function x which satisfy equation(1.4.1) for $t \geq t_0$.Assuming that

the past history of the salt content of the tank is known and represented by θ , take $\theta(t) = \theta_0 > 0$, the tank contained θ_0 pounds of salt mixed in B gallons of brine prior to time t_0 , then at t_0 valves were opened allowing fresh water to flow in at the top, and mixed brine to flow out at bottom each at the rate of q gallons per minute. Now we solve (1.4.1) on $[t_0, t_0 + r]$. Equation (1.4.1) becomes $x'(t) = -c\theta_0$ with initial condition $x(t_0) = \theta_0$. To find $x(t)$ integrate from t_0 to t

$$x(t) - x(t_0) = -c\theta_0(t - t_0) \Rightarrow x(t) = \theta_0 - c\theta_0(t - t_0) \quad t_0 \leq t \leq t_0 + r$$

Now x is known up to $t_0 + r$, we consider the interval $[t_0 + r, t_0 + 2r]$, equation (1.4.1) becomes $x'(t) = -c\theta_0 + c^2\theta_0(t - r - t_0)$ with initial condition

$$x(t_0 + r) = \theta_0 - c\theta_0(t_0 + r - t_0)$$

$$x(t_0 + r) = \theta_0 - cr\theta_0$$

This procedure can be continued as far as desired.

Example 1.4.2. (Hatching and Maturation periods [1])

Hutchinson was one of the first mathematical modelers to introduce a delay in a biological model, he pointed out that the observed oscillation in some kinds of biological phenomena could be explained by a discrete time delay in the crowding or resource term. He studied the simple equation

$$x'(t) = rx(t) \left(1 - \frac{x(t - \tau_*)}{K} \right), \quad (1.4.2)$$

where the non negative parameters r and K are known respectively as the essential growth rate and the environmental carrying capacity. The incorporation of the delay in equation (1.4.2) allows one to model the appearance of sustained oscillations in a single species population, without the predatory interaction of other species. From

equation(1.4.2) a number of questions may be asked (i) How can it be that the present change in population size depends exactly on the population size of time τ units earlier? (ii) Why does the delay enter the removal term $\frac{-x^2}{K}$ and not the production term x ? For the first question, consider the more general equation of Volterra

$$x'(t) = rx(t) \left(1 - \frac{1}{K} \int_{-\tau}^0 x(t+s) d\sigma(s) \right) \quad (1.4.3)$$

The second question can be answered by applying the theory of populations structured by age. The Sharp- Lotka – McKendrick[1] model, and its extension due to Gurtin and MacCamy [1] ,

$$u_t(t, a) + u_a(t, a) + \mu(a, W)u(t, a) = 0$$

$$u(t, 0) = \int_0^{\infty} b(a, W)u(t, a) da \quad (1.4.4)$$

Provide the standard models in the theory of age-structured populations, $u(t, a)$ is the density of the population with respect to age a , the mortality μ and the fertility b depend on age and some functional W , the total population size $W(t) = \int_0^{\infty} u(t, a) da$. The functional $\mu(a)$ and $b(a)$ have the following typical features: The death function $\mu(a)$ may be large (high infant mortality), then level off to some plateau and finally become large for large a . A maximum age can be incorporated by letting $\mu(a) \rightarrow \infty$ for $a \rightarrow a^*$, where a^* is some finite value, or simply discarding individual with ages $a > a^*$. The birth function $b(a)$ is zero below a certain age, then becomes large in an interval of high fecundity, then returns to

zero. This assumption leads to population models in the form of delay equations.

Assume that

(a) There is a maturation age $\tau_* > 0$, separating juveniles from adults;

(b) $\mu(a) = \mu_0 + (\mu_1 - \mu_0)H_{\tau_*}(a)$;

(c) $b(a) = b_1H_{\tau_*}(a) + b_2\delta_{\tau_*}(a)$;

Here $H_\tau(\alpha)$ is the Heaviside function

($H_\tau(\alpha) = 0$ for $\alpha < \tau$, $H_\tau(\alpha) = 1$ for $\alpha \geq \tau$) and the delta function $\delta_{\tau_*}(\cdot)$ is its

generalized derivative. For $t \geq \tau_*$, the population of juveniles $u(t) = \int_0^{\tau_*} u(t, a) da$ and

adults $v(t) = \int_{\tau_*}^{\infty} u(t, a) da$ satisfy a system of delay differential equations

$$u'(t) = b_1v(t) + (b_2 - 1)(b_1 + b_2\mu_1)e^{-\mu_0\tau_*}v(t - \tau_*) + (b_2 - 1)b_2e^{-\mu_0\tau_*}v'(t - \tau_*) - \mu_0u(t) \quad (1.4.5)$$

$$v'(t) = ((b_1 + b_2\mu_1)v(t - \tau_*) + b_2v'(t - \tau_*))e^{-\mu_0\tau_*} - \mu_1v(t) \quad \text{for } t \geq \tau_* \quad (1.4.6)$$

If we have initial data for the partial differential equation $u_0(a)$, and use the corresponding solution $u(t, a)$ to form $u(t), v(t)$, then these two functions do not satisfy the system of equations (1.4.4)-(1.4.5) for $t \in [0, \tau_*]$, but only for $t \geq \tau_*$ (after a time interval of length τ_* which is, in some sense, needed to application of delay equations. Hopfield [1] introduced a continuous "forget" the information contained in the initial data. For $t \in [0, \tau_*]$, the variables $u(t)$ and $v(t)$ satisfy a non-autonomous system of

$$u'(t) = b_1v(t) + (b_2 - 1)u_0(\tau_* - t)e^{-\mu_0t} - \mu_0u(t) \quad (1.4.7)$$

$$v'(t) = u_0(\tau_* - t)e^{-\mu_0 t} - \mu_1 v(t). \quad (1.4.8)$$

Equation (1.4.6) is a neutral delay differential equation for adult's population. The neutral character of the equation is a consequence of the fertility peak at age τ_* and is in no way artificial. If this peak is absent i. e. $b_2 = 0$, then we get the standard delay differential equation

$$v'(t) = b_1 v(t - \tau_*)e^{-\mu_0 \tau_*} - \mu_1 v(t) \quad (1.4.9)$$

This equation allows the following interpretation: adults die with rate μ_1 and produce offspring with rate b_1 ; the offspring enter the equation only at age of maturity, diminished by a factor $e^{-\mu_0 \tau_*}$ which takes into account juvenile mortality.

An understanding of biological phenomena implies the ability to predict and control them. This understanding comes in several related stages: (a) formulation of mathematical models based on biological first principles, (b) qualitative assessment of the models and (c) biological interpretation. The models represent a number of time dependent state variables, each having a recognized biological interpretation, and a corresponding complexity in the parameters.

Chapter Two

First Order Delay Differential Equations

2.1 Introduction:

Because of the fact that the first order differential equations do not have oscillatory solutions, most of the theory focus around the second and higher order ordinary differential equations. But for differential equations with deviating arguments we can have oscillatory solutions, this oscillation caused by the deviating arguments.

In this chapter we will discuss the oscillation, non-oscillation and the asymptotic behavior of the solution of first order delay differential equations of the form

$$x'(t) + p(t)f(x(g(t))) = h(t) \quad (2.1.1)$$

In section two the oscillation of linear homogenous form of (2.1.1) will be discussed, in section three the non-oscillation of linear homogenous form of (2.1.1) will be discussed. In section four we will explain the asymptotic behavior of the equation $x'(t) + p(t)x(t - \tau) = 0$ where $g(t) = t - \tau$. In section five linear non homogenous form of (2.1.1) will be discussed. In section six non-linear homogenous form of (2.1.1) will be discussed. In section seven the asymptotic behavior of $x'(t) + p(t)f(x(t - \tau(t))) = 0$ where $g(t) = t - \tau(t)$ will be studied. Finally in section eight non-linear, non-homogenous form of (2.1.1) will be discussed.

2.2. Oscillation

In this section we shall consider the first order linear, homogenous delay equation of the form

$$x'(t) + p(t)x(g(t)) = 0 \quad (2.2.1)$$

where $p, g \in C([R^+, R^+])$, $g(t) < t$, and $\lim_{t \rightarrow \infty} g(t) = \infty$.

The following theorem states the conditions for which all solutions of the equation (2.2.1) are oscillatory.

Theorem 2.2.1. If

$$\liminf_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > \frac{1}{e} \quad (2.2.2)$$

Then all solutions of (2.2.1) are oscillatory.

We may have oscillatory solution in spite of the previous condition does not satisfies as in the following example

Example 2.2.1. Consider the following delay differential equation

$$x'(t) + 2x\left(t - \frac{1}{10} \sin^2 \sqrt{t} - \frac{1}{2e}\right) = 0 \quad (2.2.3)$$

$$\text{Now } \liminf_{t \rightarrow \infty} \int_{\left(t - \frac{1}{10} \sin^2 \sqrt{t} - \frac{1}{2e}\right)}^t 2 ds = \liminf_{t \rightarrow \infty} \left(\frac{2}{10} \sin^2 \sqrt{t} + \frac{1}{e}\right) = \frac{1}{e}.$$

According to the condition (2.2.2), equation (2.2.3) has non-oscillatory solution. But it has oscillatory solution. And we shall see this later.

Before proving the theorem we need the following two lemmas.

Lemma 2.2.1. If $\liminf_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > \frac{1}{e}$ then the inequality

$$x'(t) + p(t)x(g(t)) \leq 0 \quad (2.2.4)$$

has no eventually positive solutions.

Proof: Assume that $g(t)$ is non-decreasing, otherwise

set $\delta(t) = \max\{g(s) : s \in [0, t]\}$. Then we can see that (2.2.2) is equivalent to

$\lim_{t \rightarrow \infty} \int_{\delta(t)}^t p(s) ds > \frac{1}{e}$. Assume that $x(t)$ is an eventually positive solution of (2.2.4) such

that $x(g(t)) > 0$ for $t \geq t_1$. Because of (2.2.2) there exists a $t_2 \geq t_1$, such that

$$\int_{g(t)}^t p(s) ds \geq d > \frac{1}{e} \quad \text{for } t \geq t_2 \quad (2.2.5)$$

Since $x'(t) < 0$ for $t \geq t_1$, we get from (2.2.4)

$$x'(t) + p(t)x(t) \leq 0 \quad (2.2.6)$$

Dividing (2.2.6) by $x(t)$ and integrating from $g(t)$ to t , we obtain

$$\ln \frac{x(t)}{x(g(t))} + \int_{g(t)}^t p(s) ds \leq 0 \quad t \geq t_2, \text{ and hence}$$

$$\ln \frac{x(g(t))}{x(t)} \geq \int_{g(t)}^t p(s) ds \geq d \quad t \geq t_2$$

Since $e^y \geq ey$, for $y \geq 0$, it follows that

$$\frac{x(g(t))}{x(t)} \geq e^d \geq ed \quad t \geq t_2$$

Repeating the above procedure, there exists a sequence $\{t_n\}$ such that

$$\frac{x(g(t))}{x(t)} \geq (ed)^n \quad t \geq t_n \quad (2.2.7)$$

From (2.2.5), there exists a t^* such that

$$\int_{g(t)}^{t^*} p(s)ds \geq \frac{d}{2} \quad \text{and} \quad \int_i^t p(s)ds \geq \frac{d}{2} \quad \text{for } t \geq t_n$$

Integrating (2.2.4) from $g(t)$ to t^* gives

$$x(t^*) - x(g(t)) + \int_{g(t)}^{t^*} p(s)x(g(s))ds \leq 0, \text{ this implies that}$$

$$x(g(t)) \geq x(g(t^*)) \frac{d}{2}, \tag{2.2.8}$$

also integrating (2.2.4) from t^* to t , we obtain

$$x(t) - x(t^*) + \int_{t^*}^t p(s)x(g(s))ds \leq 0, \text{ and then}$$

$$x(t^*) \geq x(g(t)) \frac{d}{2} \tag{2.2.9}$$

The result obtained by combining (2.2.8) and (2.2.9) is

$$x(t^*) \geq x(g(t^*)) \left(\frac{d}{2}\right)^2 \tag{2.2.10}$$

From (2.2.7) and (2.2.10) we obtain

$$\left(\frac{2}{d}\right)^2 \geq \frac{x(g(t^*))}{x(t^*)} > (ed)^n, \text{ for all } t \geq t_n \tag{2.2.11}$$

Choose n sufficiently large such that $(ed)^n > \left(\frac{2}{d}\right)^2$. Which is possible because

$ed > 1$, therefore (2.2.11) is a contradiction •

Lemma 2.2.2. If (2.2.2) holds then the inequality

$$x'(t) + p(t)x(g(t)) \geq 0 \tag{2.2.12}$$

has no eventually negative solutions.

Proof: The proof is parallel to the proof of lemma 2.2.1. Then if $x(t)$ is a solution of (2.2.12) then $-x(t)$ is a solution of (2.2.4) •

Now the **proof of theorem 2.2.1** is as follows:

By using lemma 2.2.1, $x'(t) + p(t)x(g(t)) \leq 0$, has no eventually positive solutions.

And from lemma 2.2.2, $x'(t) + p(t)x(g(t)) \geq 0$ has no eventually negative solutions.

Then all solutions of (2.2.1) are oscillatory •

Example 2.2.2. Consider the inequality

$$x'(t) + \frac{2}{(e \ln 2)t} x\left(\frac{t}{2}\right) \leq 0 \quad (2.2.13)$$

where $p(t) = \frac{2}{(e \ln 2)t}$, $g(t) = \frac{t}{2}$

$$\lim_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds = \lim_{t \rightarrow \infty} \int_{t/2}^t \frac{2}{(e \ln 2)s} ds$$

$$\int_{t/2}^t \frac{2}{(e \ln 2)s} ds = \frac{2}{e \ln 2} (\ln t - \ln t + \ln 2) = \frac{2}{e} > \frac{1}{e}$$

then (2.2.13) has no eventually positive solution.

Example 2.2.3. Consider the equation

$$x'(t) + \frac{5}{t} x\left(\frac{t}{3}\right) = 0 \quad (2.2.14)$$

$$\lim_{t \rightarrow \infty} \int_{t/3}^t \frac{5}{s} ds = (5 \ln t - 5 \ln t + 5 \ln 3) = 5.49 > 0.3 = 1/e.$$

Then all solutions of (2.2.14) are oscillatory.

Now If $\lim_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds \leq \frac{1}{e}$, the previous theorem does not give any information

about the oscillation of the solution but according to [7], which concerned with the oscillatory behavior of the first – order delay differential equation of the form

$$x'(t) + p(t)x(\tau(t)) = 0 \quad t \geq t_0 \quad (2.2.15)$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$, $\tau(t)$ is non-decreasing, $\tau(t) < t$ for

$$t \geq t_0 \text{ and } \lim_{t \rightarrow \infty} \tau(t) = \infty. \text{ Let } K = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds \text{ and } L = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds$$

It is proved that when $L < 1$ and $0 < K \leq \frac{1}{e}$, all solutions of equation (2.2.15) oscillate

in several cases in which the condition $L > 2K + \frac{2}{\lambda} - 1$ holds, where λ is the

smaller root of the equation $\lambda = e^{K\lambda}$, as in the following theorem.

Theorem 2.2.2. Consider the differential equation (2.2.15) and let

$L < 1, 0 < K \leq \frac{1}{e}$ and there exist $\theta > 0$ such that

$$\int_{\tau(u)}^{\tau(t)} p(s) ds \geq \theta \int_u^t p(s) ds \quad \text{for all } \tau(t) \leq u \leq t \quad (2.2.16)$$

Assume that

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - K - \sqrt{(1 - K)^2 - 4A}}{2} \quad (2.2.17)$$

where λ_1 is the smaller root of the equation $\lambda = e^{K\lambda}$ and A is given by

$(A = \frac{\exp(\lambda_1 \theta K) - \lambda_1 \theta K - 1}{(\lambda_1 \theta)^2})$ then all solutions of (2.2.15) oscillate.

Proof: See [7] •

Remarks : (1) When $\theta = 1$, then $A = \frac{e^{K\lambda_1} - \lambda_1 K - 1}{\lambda_1^2}$, and then (2.2.17) reduces

to

$$L > 2K + \frac{2}{\lambda_1} - 1 \quad (2.2.18)$$

(2) The case that $K = \frac{1}{e}$, then $\lambda_1 = e$ and (2.2.18) becomes $L > (4/e) - 1 \approx 0.4715164$

Example 2.2.4. Consider the delay differential equation

$$x'(t) + px(t - a \sin^2 \sqrt{t} - \frac{1}{pe}) = 0 \quad (2.2.19)$$

where $p > 0$, $a > 0$, and $pa = \frac{1}{2} - \frac{1}{e}$

$$\text{then } L = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p ds = \limsup_{t \rightarrow \infty} p(a \sin^2 \sqrt{t} + \frac{1}{pe}) = \frac{1}{2} - \frac{1}{e} + \frac{1}{e} = \frac{1}{2}$$

$$K = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p ds = \liminf_{t \rightarrow \infty} [(pa \sin^2 \sqrt{t} + \frac{1}{e})] = \frac{1}{e}$$

Thus, according to the remark all solutions of equation (2.2.19) oscillate.

Theorem 2.2.3. If $p, g \in C[\mathbb{R}^+, \mathbb{R}^+]$, $g(t) < t$ and it is non decreasing,

$\lim_{t \rightarrow \infty} g(t) = +\infty$. And

$$\overline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > 1 \quad (2.2.20)$$

Then every solution of (2.2.1) is oscillatory.

Proof: Let $x(t)$ be a non-oscillatory solution, such that $x(g(t)) > 0, t \geq t_1$. Integrating

(2.2.1) from $g(t)$ to t , we have $x(t) - x(g(t)) + \int_{g(t)}^t p(s)x(g(s))ds = 0$ or, its equivalent

to

$$x(t) + x(g(t)) \left[\int_{g(t)}^t p(s)ds - 1 \right] \leq 0 \quad (2.2.21)$$

From (2.2.21), $\int_{g(t)}^t p(s)ds \geq 1$ when t is sufficiently large; which is a contradiction

with (2.2.21). The proof is complete •

Example 2.2.5. Consider the equation

$$x'(t) + \left((\sqrt{2} + \frac{1}{e}) \left(\frac{2}{\pi} \right) + \cos t \right) x(t - \frac{\pi}{2}) = 0 \quad (2.2.22)$$

where $p(t) = \left((\sqrt{2} + \frac{1}{e}) \left(\frac{2}{\pi} \right) + \cos t \right) > 0$ for $t \in R^+$, now

$$\int_{t-\frac{\pi}{2}}^t \left((\sqrt{2} + \frac{1}{e}) \left(\frac{2}{\pi} \right) + \cos s \right) ds = \left(\sqrt{2} + \frac{1}{e} \right) \left(\frac{2}{\pi} \right) t + \sin t - \left((\sqrt{2} + \frac{1}{e}) \left(\frac{2}{\pi} \right) \left(t - \frac{\pi}{2} \right) + \sin \left(t - \frac{\pi}{2} \right) \right)$$

$$= \sin t + \left(\sqrt{2} + \frac{1}{e} \right) - (-\cos t).$$

$$\overline{\lim}_{t \rightarrow \infty} (\sin t + \cos t + \left(\sqrt{2} + \frac{1}{e} \right)) = 2\sqrt{2} + \frac{1}{e} > 1.$$

Then according to (2.2.20) every solution of (2.2.22) is oscillatory.

As a special case of (2.2.1) where $p(t) = p > 0$ and $g(t) = t - \tau, \tau > 0$. We have the following theorem.

Theorem 2.2.4. Let p, τ are positive numbers in (2.2.1). If $p\tau e > 1$, then all solutions of (2.2.1) are oscillatory.

proof: Assume that there exist a non-oscillatory solution of the form

$$x(t) = \exp(\lambda t), \text{ then } F(\lambda) = \lambda + p \exp(-\lambda \tau), \text{ take } \lambda = 0$$

$$F(0) = p > 0.$$

$$\text{Take } \lambda = \frac{-1}{\tau} \Rightarrow F(\lambda) = \frac{-1}{\tau} + pe = \frac{p\tau e - 1}{\tau}.$$

To satisfy the non-oscillatory solution, $p\tau e \leq 1$. But this contradicts the condition $p\tau e > 1$.

The proof is complete •

Example 2.2.6. Consider the equation

$$x'(t) + x(t-9) = 0 \tag{2.2.23}$$

$p\tau e = p\tau e = 24.3 > 1$. Then all solutions of (2.2.23) oscillate.

2.3. Non-Oscillation

In this section we shall discuss the conditions in which the equation (2.2.1) has a non-oscillatory solution i.e. (there exists a t_1 such that $x(t) \neq 0$, for all $t \geq t_1$).

Theorem 2.3.1. If $p, g \in C[\mathbb{R}^+, \mathbb{R}^+]$, $g(t) < t$ and $\lim_{t \rightarrow \infty} g(t) = +\infty$ and

$$\overline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds < \frac{1}{e} \tag{2.3.1}$$

Then equation (2.2.1) has a non oscillatory solution.

Proof : We need to find a solution of (2.2.1) of the form

$$x(t) = \exp \left[\int_{t_0}^t \lambda(s) ds \right] \tag{2.3.2}$$

$$\text{then } \lambda(t) = -p(t) \exp \left[- \int_{g(t)}^t \lambda(s) ds \right] \quad (2.3.3)$$

our aim is to show that there exists a real - valued continuous function $\lambda(t)$ such that $\lambda(t)$, satisfies (2.3.3). So define an operator as follows:

$$(T\lambda)(t) = \begin{cases} -p(t) \exp \left[- \int_{g(t)}^t \lambda(s) ds \right], & t \geq t_0 \\ \phi(t), & t_0 - \tau \leq t \leq t_0, \inf_{t \geq t_0} g(t) = t_0 - \tau \end{cases} \quad (2.3.4)$$

where $\tau > 0$

T is non-decreasing and continuous operator defined on a space of continuous functions $C[t_0 - \tau, +\infty)$ into itself. From (2.3.1), we can find $t_0 \in \mathbb{R}^+$ such that

$$\int_{g(t)}^t p(s) ds < \frac{1}{e} \text{ as } t \geq t_0 \quad (2.3.5)$$

Let $x_0(t) = -ep(t) \leq 0$ and $\phi(t)$ in (2.3.4) satisfy

$$x_0(t) \leq \phi(t) \leq 0 \text{ on } [t_0 - \tau, t_0] \quad (2.3.6)$$

Now, $x_0 \in C[t_0 - \tau, t]$, and from (2.3.4), (2.3.5) (2.3.6), we see that .

$$(Tx_0)(t) = -p(t) \exp \left[- \int_{g(t)}^t x_0(s) ds \right] \geq -p(t)e = x_0(t) \quad t \geq t_0$$

$$\therefore (Tx_0)(t) \geq x_0(t) \quad t \geq t_0$$

Set $z_0(t) = 0$ for $t \in [t_0 - \tau, +\infty)$, then $(Tz_0)(t) \leq z_0(t)$. Note that $x_0 \leq z_0$. Hence

$Tx_0 \leq Tz_0$, and $x_0 \leq Tx_0 \leq Tz_0 \leq z_0$. Let $x_{n+1} = Tx_n$ be an increasing sequence satisfying $x_0 \leq x_n \leq x_{n+1} \leq z_0$. Thus the sequence $\{x_n\}$ increases to a limit λ . By using

Lebesgue convergence theorem, Tx_n converges to $T\lambda$. Therefore $T\lambda = \lambda$. Thus λ is a continuous function on $[t_0 - \tau, +\infty)$, Furthermore,

$$x_0(t) \leq \lambda(t) \leq z_0(t) \quad t \geq t_0 - \tau \quad (2.3.7)$$

This proves that (2.3.3) has a solution $\lambda(t)$ which is continuous on $[t_0 - \tau, +\infty]$ and

$$x(t) = \exp \left[\int_{t_0}^t \lambda(s) ds \right] \text{ is a non-oscillatory solution of (2.2.1)}$$

Example 2.3.1. Consider the equation

$$x'(t) + \frac{1}{(3e)t} x\left(\frac{t}{3}\right) = 0 \quad (2.3.8)$$

$$\overline{\lim}_{t \rightarrow \infty} \int_{t/3}^t \frac{1}{(3e)s} ds = \overline{\lim}_{t \rightarrow \infty} \left(\frac{1}{3e} (\ln t - \ln t + \ln 3) \right) = 0.1 < \frac{1}{e}.$$

Then equation (2.3.8) has a non-oscillatory solution.

Theorem 2.3.2. Assume that p and τ are positive numbers in (2.2.1), and assume that $p\tau e \leq 1$ then (2.2.1) has a non-oscillatory solution.

Proof: Assume that $x(t) = \exp(\lambda t)$. Then

$$F(\lambda) = \lambda + p \exp(-\lambda \tau). \text{ Take } \lambda = 0, \text{ then } F(0) = p > 0.$$

$$\text{Take } \lambda = -\frac{1}{\tau}, F\left(-\frac{1}{\tau}\right) = \frac{-1}{\tau} + pe = \frac{p\tau e - 1}{\tau} \leq 0$$

Hence there exists a negative real root $\lambda \in \left[-\frac{1}{\tau}, 0\right)$ such that $\exp(\lambda t)$ is a non-oscillatory solution of (2.2.1) •

Example 2.3.2. Consider the equation

$$x'(t) + \frac{1}{2} x\left(t - \frac{1}{e}\right) = 0 \quad (2.3.9)$$

$$p\tau e = \left(\frac{1}{2}\right)\left(\frac{1}{e}\right)e = \frac{1}{2} < 1$$

Then (2.3.9) has a non-oscillatory solution.

So, concerning the linear equation

$$x'(t) + p(t)x(g(t)) = 0 \text{ where, } p, g \in ([R^+, R^+]), g(t) < t, \lim_{t \rightarrow \infty} g(t) = \infty, \text{ we can}$$

summarize the following results

$$\text{Let } K = \lim_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds$$

$$L = \overline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds$$

Oscillation	Non-oscillation
<p>All solutions are oscillatory if one of the following is satisfied:</p> <p>(a) $K > \frac{1}{e}$. Theorem 2.2.1</p> <p>(b) $L > 1$. Theorem 2.2.3.</p> <p>(c) $L < 1$ and $0 < K \leq \frac{1}{e}$, in which $L > 2k + \frac{2}{\lambda_1} - 1$. Theorem 2.2.2.</p> <p>(d) $p\tau e > 1$, where $p(t) = p > 0$, $g(t) = t - \tau, \tau > 0$. Theorem 2.2.4.</p>	<p>The equation has a non-oscillatory solution if one of the following is satisfied:</p> <p>(a) $L < \frac{1}{e}$. Theorem 2.3.1</p> <p>(b) $p\tau e \leq 1$ where $p(t) = p > 0$, $g(t) = t - \tau, \tau > 0$. Theorem 2.3.2.</p>

2.4. The asymptotic behavior

We will consider the equation

$$x'(t) + p(t)x(t - \tau) = 0 \tag{2.4.1}$$

where $g(t) = t - \tau, \tau > 0$ is constant, $p \in C[R^+, R^+]$. Let $x(t, t_0, \phi)$ denote the solution of (2.4.1) satisfying the initial condition $x(t) = \phi(t)$ for $t \in E_{t_0}$, ϕ is continuous function on E_{t_0} , where $E_{t_0} = [t_0 - \tau, t_0]$.

We need the following lemmas

Lemma 2.4.1. Let $p(t) \equiv p > 0$ and $0 \leq p\tau < \frac{\pi}{2}$. Then there exist positive

constants M and ν such that

$$|x(t; t_0, \phi)| \leq M \|\phi\| e^{-\nu(t-t_0)} \quad t \geq t_0 \quad (2.4.2)$$

Where $\|\phi\| = \sup_{t_0 - \tau \leq s \leq t_0} |\phi(s)|$, also, if $z(t, t_0, 0)$ denotes the solution of

$$z'(t) + pz(t - \tau) = h(t) \quad t \geq t_0 \quad (2.4.3)$$

with zero initial function at t_0 , then

$$|z(t, t_0, 0)| \leq \frac{M}{\nu} \exp(p + \nu)\tau \max_{t_0 \leq s \leq t} |h(s)| \quad (2.4.4)$$

Lemma 2.4.2. Consider the retarded differential equation

$$x'(t) + x(t - \sigma(t)) = 0 \quad t \geq t_0 \quad (2.4.5)$$

Where $0 \leq \sigma(t) \leq t$ is continuous and $\lim_{t \rightarrow \infty} \sigma(t) = \tau < \infty$. Assume that

$$\tau < \frac{\pi}{2} \quad (2.4.6)$$

Then every solution of (2.4.1) tends to zero as $t \rightarrow \infty$.

Proof: Let $x(t)$ be any solution of (2.4.1). Choose a

$$t_1 \geq t_0 + 2\tau + 2 \quad \text{such that } \sigma(t) \leq \tau + 1 \quad t \geq t_1 \quad \text{and}$$

$$\frac{M}{\nu} \exp(1 + \nu)\tau |\tau - \sigma(t)| \leq \frac{1}{2} \quad t \geq t_1 \quad (2.4.7)$$

Where the constant M and ν are defined in lemma 2.4.1 with $p=1$. Let $y(t)$ be the solution of $y'(t) + y(t - \tau) = 0$ for $t \geq t_1$ with initial function $y_{t_1} = x_{t_1}$ on $E_{t_1} = [t_1 - \tau, t_1]$. Because of condition (2.4.6), by lemma 2.4.1, we have that $y(t)$ tends to zero as $t \rightarrow \infty$. Set $z(t) = x(t) - y(t)$; then $z(t)$ satisfies the following equation: $z'(t) + z(t - \tau) = x(t - \tau) - x(t - \sigma(t))$ $t \geq t_1$ with zero initial function at t_1 . Using (2.4.4), with $p = 1$ and $h(s) = x(s - \tau) - x(s - \sigma(s))$, we find

$$|z(t)| \leq \frac{M}{\nu} \exp(1 + \nu)\tau \max_{t_1 \leq s \leq t} |x(s - \tau) - x(s - \sigma(s))| \quad (2.4.8)$$

By applying mean value theorem to equation (2.4.5), we obtain

$$\begin{aligned} |x(s - \tau) - x(s - \sigma(s))| &= |\sigma(s) - \tau| |x'(\xi)| \\ &= |\sigma(s) - \tau| |x(\xi - \sigma(\xi))| \end{aligned}$$

Where ξ is between $s - \tau$ and $s - \sigma(s)$. Then setting $B_1 = \max_{t_0 \leq s \leq t_1} |x(s)|$ we have

$$\begin{aligned} \max_{t_1 \leq s \leq t} |x(s - \tau) - x(s - \sigma(s))| &\leq \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \max_{t_0 \leq s \leq t_1} |x(s)| \\ &\leq \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| [B_1 + \max_{t_1 \leq s \leq t} |x(s)|] \end{aligned}$$

From (2.4.8), one gets

$$|x(t)| - |y(t)| \leq \frac{M}{\nu} \exp(1 + \nu)\tau \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right] \quad (2.4.9)$$

And, with respect to (2.4.7), it follows that $|x(t)| \leq |y(t)| + \frac{1}{2} \left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right]$

Hence, for every $T \geq t_1$, and $t_1 \leq t \leq T$, we obtain $|x(t)| \leq |y(t)| + \frac{1}{2} \left[B_1 + \max_{t_1 \leq s \leq T} |x(s)| \right]$

Taking the maximum of both sides and rearranging terms, we find

$$\max_{t_1 \leq s \leq T} |x(t)| \leq 2 \max_{t_1 \leq s \leq T} |y(t)| + B_1$$

That is, $x(t)$ is bounded function and so there exists a $B \geq B_1$ such that

$|x(t)| \leq B$ for $t \geq t_0$. Then using (2.4.9), we have

$$|x(t)| \leq |y(t)| + 2B \left(\frac{M}{\nu} \right) \exp(1 + \nu)\tau \max_{t_1 \leq s \leq t} |\sigma(s) - \tau|$$

Because $\lim_{t \rightarrow \infty} \sigma(t) = \tau$, the above inequality implies that $\lim_{t \rightarrow \infty} x(t) = 0$. The proof is

complete •

Now we can prove the following theorem;

Theorem 2.4.1. Assume that $p \in C[R^+, R^+]$, $p > 0$, $\tau > 0$ is constant and

$$\int_{t_0}^{\infty} p(t) dt = +\infty \tag{2.4.10}$$

Suppose further that $\lim_{t \rightarrow \infty} \int_{t-\tau}^t p(s) ds$ exists, and

$$\lim_{t \rightarrow \infty} \int_{t-\tau}^t p(s) ds < \frac{\pi}{2} \tag{2.4.11}$$

Then every solution of (2.4.1) tends to zero as $t \rightarrow \infty$.

Proof: Set $u = \sigma(t) \equiv \int_{t_0}^t p(s) ds$ $t \geq t_0$, because of (2.4.10), $\sigma^{-1}(t)$ exists, and

$$\lim_{t \rightarrow \infty} u(t) = \infty. \text{ Also } \sigma(t - \tau) = \int_{t_0}^{t-\tau} p(s) ds = \int_{t_0}^t p(s) ds - \int_{t-\tau}^t p(s) ds = u(t) - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s) ds,$$

that is $t - \tau = \sigma^{-1}(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s) ds)$. Then the transformation

$$z(u) = x(\sigma^{-1}(u)) \text{ with } t = \sigma^{-1}(u) \tag{2.4.12}$$

reduces (2.4.1) to $x'(\sigma^{-1}(u)) + p(t)x(\sigma^{-1}(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s) ds)) = 0$

$$z'(u) + z(u) - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s) ds = 0 \quad (2.4.13)$$

According to condition (2.4.11), equation (2.4.13) satisfies the hypotheses of lemma (2.4.2) and therefore $\lim_{t \rightarrow \infty} z(u) = 0$. From (2.4.12), we have $\lim_{t \rightarrow \infty} x(t) = 0$. The proof is complete •

Example 2.3.1. Consider the equation

$$x'(t) + p(2 + \cos t)x(t - 2\pi) = 0 \quad \text{where } 0 < p < \frac{1}{8} \quad (2.4.14)$$

$$p(t) = p(2 + \cos t) \quad \text{Then } \int_0^{\infty} p(2 + \cos t) dt = \infty$$

$$\text{And } \int_{t-2\pi}^t p(2 + \cos s) ds = 4p\pi < \frac{\pi}{2}$$

Therefore every solution of (2.4.14) tends to zero as $t \rightarrow \infty$.

2.5. Non-homogenous equations.

Consider the equation of the form

$$x'(t) + p(t)x(g(t)) = h(t) \quad (2.5.1)$$

Where

- a) $p, g, h : [a, \infty) \rightarrow R$ are continuous.
- b) $p(t) > 0, g(t)$ is non-decreasing, $g(t) < t$.
- c) $\lim_{t \rightarrow \infty} g(t) = \infty$.

Theorem 2.5.1 Assume that

- (i) $h(t), p(t) \geq 0, g(t), h(t)$, are continuous.

(ii) $\rho(t)$ is an oscillatory function such that $\rho'(t) = h(t)$ and there exists two constants q_1, q_2 and sequences $(t'_m), (t''_m)$ such that $\rho(t'_m) = q_1, \rho(t''_m) = q_2, \lim_{t \rightarrow \infty} t'_m = \infty, \lim_{t \rightarrow \infty} t''_m = \infty$, and $q_1 \leq \rho(t) \leq q_2$ for $t \geq 0$.

(iii) $p(t)$ satisfy any one of these conditions

$$\overline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > 1 \quad \text{or}$$

$$\underline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > \frac{1}{e}$$

Then every solution of (2.5.1) oscillates.

Proof: Let $x(t)$ be a non-oscillatory solution such that $x(t) > 0$, then $x(g(t)) > 0$,

for $t \geq t_1$. Let $z(t) = x(t) - \rho(t)$, then $z'(t) = x'(t) - \rho'(t)$

$$= -p(t)x(g(t)) + h(t) - \rho'(t) = -p(t)x(g(t)) \leq 0 \quad \text{for } t \geq t_1.$$

Suppose $z(t) + q_1 \leq 0$ for $t \geq t_2 \geq t_1$

since $z(t) + \rho(t) \equiv x(t) > 0$, especially, $z(t'_m) + \rho(t'_m) = x(t'_m) > 0, t'_m > t_2$. This is

contradiction since $z(t) + \rho(t) = x(t) > 0$, so $z(t) + q_1 > 0$ for all $t \geq t_2$.

Let $y(t) = z(t) + q_1$ then $y'(t) = z'(t) = x'(t) - \rho'(t) = -p(t)x(g(t))$

$$= -p(t) [z(g(t)) + \rho(g(t))]$$

$$\leq -p(t)y(g(t))$$

That is $y'(t) + p(t)y(g(t)) \leq 0$, has an eventual positive solution. But it is

impossible with respect to condition (iii). The proof is complete •

Example 2.5.1. Consider the equation

$$x'(t) + x(t - \frac{\pi}{2}) = \cos t \tag{2.5.2}$$

$$h(t) = \rho'(t) \Rightarrow \rho(t) = \sin t$$

$$t'_m = m\pi \quad t''_m = \frac{\pi}{2} + 2m\pi \quad \text{then}$$

$$\rho(m\pi) = 0 = q_1 \quad \text{and} \quad \rho\left(\frac{\pi}{2} + 2m\pi\right) = 1 = q_2 \Rightarrow 0 \leq \sin t \leq 1$$

$$\lim_{t \rightarrow \infty} \int_{t-\frac{\pi}{2}}^t ds = \frac{\pi}{2} > \frac{1}{e}.$$

Every solution of (2.5.2) oscillates, since all the conditions of theorem 2.5.1 are satisfied.

2.6. Non-linear homogenous equations.

Consider the equation

$$x'(t) + p(t)f(x(g(t))) = 0 \tag{2.6.1}$$

where the following are satisfied .

(a) $g \in C(R^+, R)$, $g(t) < t$ for $t \in R^+$, $g(t)$ is strictly increasing on R^+ and

$$\lim_{t \rightarrow \infty} g(t) = +\infty.$$

(b) $p(t)$ is locally integrable and $p(t) \geq 0$, a, e .

(c) $xf(x) > 0$ for $x \neq 0$, $f \in C(R, R)$, f is non-decreasing and

$$\lim_{x \rightarrow 0} \frac{x}{f(x)} = M < +\infty \tag{2.6.2}$$

Theorem 2.6.1. Assume that conditions a, b, and c hold.

Assume further that

$$\overline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > M \quad (2.6.3)$$

Where M is defined by (2.6.2), then every solution of (2.6.1) is oscillatory.

The following lemma is needed before proving the theorem

Lemma 2.6.1. Assume that (a) holds. Let $\{t_n\}$ be a sequence defined by

$t_n = g^{-1}(t_{n-1})$, t_0 being an arbitrary number, then $t_n \rightarrow \infty$ as $n \rightarrow \infty$.

Proof: If we assume that the claim is false, let $t_n \rightarrow B < +\infty$, since g and g^{-1} are

continuous, we have $B = \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} g^{-1}(t_{n-1}) = \lim_{n \rightarrow \infty} g^{-1}(g(t_n))$

$= g^{-1}(\lim_{n \rightarrow \infty} (g(t_n))) = g^{-1}(B) > B$. Which is a contradiction •

The proof of theorem 2.6.1 : For sufficiently large t^* , we have

$$\int_{g(t)}^t p(s) ds > M + K \quad \text{for } t \geq t^* \quad \text{from (2.6.3)} \quad (2.6.4)$$

Where $K > 0$. Let $x(t)$ be a non oscillatory solution of (2.6.1). Without loss of generality, assume that $x(t) > 0$ for $t > t_0 > g(t^*)$. Then

$$x'(t) = -p(t)f(x(g(t))) \leq 0 \quad \text{for } t > t_1 = g^{-1}(t_0).$$

Thus $x(t)$ is non-increasing and has a finite non-negative limit, say α , as $t \rightarrow \infty$.

We claim that $\alpha = 0$. If not, let $\alpha > 0$ and $f(\alpha) > 0$. Let $s_0 = t_0$, and $s_n = g^{-1}(s_{n-1})$

such that

$$\int_{g(s_n)}^{s_n} p(s) ds > M + K \quad (2.6.5)$$

Integrating (2.6.1) from t_0 to s_n , we get

Example 2.6.1. Consider the equation

$$x'(t) + \frac{2}{(\ln 2)t} x\left(\frac{t}{2}\right) = 0 \quad (2.6.6)$$

then $M=1$ and

$$\overline{\lim}_{t \rightarrow \infty} \int_{t/2}^t \frac{2}{(\ln 2)s} ds = \frac{2}{\ln 2} [\ln t - \ln t + \ln 2] = 2 > 1.$$

Then every solution of (2.6.6) is oscillatory.

Example 2.6.2. Consider the equation

$$x'(t) + \frac{3}{(\ln 3)t} x^{1/3}\left(\frac{t}{3}\right) = 0 \quad (2.6.7)$$

Then $M = \lim_{x \rightarrow 0} \frac{x}{x^{1/3}} = \lim_{x \rightarrow 0} x^{2/3} = 0.$

And $\overline{\lim}_{t \rightarrow \infty} \int_{t/3}^t \frac{3}{(\ln 3)s} ds = \frac{3}{\ln 3} [\ln t - \ln t + \ln 3] = 3 > 0.$

Then every solution of (2.6.7) is oscillatory.

Theorem 2.6.2. Assume that conditions a,b and c hold. Assume further that

$$\underline{\lim}_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > \frac{M}{e}, \quad M \neq 0 \quad (2.6.8)$$

Then every solution of (2.6.1) oscillates.

Proof: Assume that there is a non-oscillatory solution

$x(t) > 0, x(g(t)) > 0$ for $t \geq t_0 \geq 0$. So $x'(t) \leq 0$ and hence

$x(t) \rightarrow \alpha \geq 0$ as $t \rightarrow \infty$. As in theorem 2.6.1, $x(t) \rightarrow 0$ as $t \rightarrow \infty$. There exists

$t^* \in (g(t), t)$ such that

$$\int_{t^*}^t p(s) ds > \frac{M}{2e} \quad \text{and} \quad \int_{g(t)}^{t^*} p(s) ds > \frac{M}{2e} \quad (2.6.9)$$

Now integrating (2.6.1) from t^* to t ,

$$x(t^*) - x(t) = \int_{t^*}^t p(s) f(x(g(s))) ds \geq f(x(g(t))) \int_{t^*}^t p(s) ds > f(x(g(t))) \frac{M}{2e}$$

And from $g(t)$ to t^* , we obtain

$$x(g(t)) - x(t^*) = \int_{g(t)}^{t^*} p(s) f(x(g(s))) ds \geq f(x(g(t^*))) \frac{M}{2e}$$

Which implies that

$$x(t^*) > f(x(g(t))) \frac{M}{2e} \geq \frac{f(x(g(t)))}{x(g(t))} f(x(g(t^*))) \left(\frac{M}{2e}\right)^2. \text{ And hence}$$

$$\frac{x(g(t^*))}{x(t^*)} \leq \frac{x(g(t))}{f(x(g(t)))} \frac{x(g(t^*))}{f(x(g(t^*)))} \left(\frac{2e}{M}\right)^2 < \infty \quad (2.6.10)$$

Because of condition (2.6.2).

Setting $w(t) = \frac{x(g(t))}{x(t)} \geq 1$, $\liminf_{t \rightarrow \infty} w(t) = l \geq 1$, l is finite, because of (2.6.10). From

$$(2.6.1), \text{ we have } \ln w(t) = \int_{g(t)}^t p(s) \frac{f(x(g(s)))}{x(g(s))} w(s) ds = w(\xi) \frac{f(x(g(\xi)))}{x(g(\xi))} \int_{g(t)}^t p(s) ds$$

Where $g(t) < \xi < t$. Taking the limit inferior in the above equation, we obtain

$$\ln l \geq \frac{l}{M} \liminf_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds. \text{ But } \max_{l \geq 1} \frac{\ln l}{l} = \frac{1}{e}, \text{ and therefore } \frac{M}{e} \geq \liminf_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds.$$

This is a contradiction, because of (2.6.8) holds •

Remark 2.6.2. This theorem is not applicable in the case $M = 0$. So we did

assume that $M \neq 0$.

2.7. The asymptotic behavior

Consider the equation

$$x'(t) + p(t)f(x(t - \tau(t))) = 0 \quad (2.7.1)$$

Where $g(t) = t - \tau(t)$ in (2.6.1).

Theorem 2.7.1. Assume that

$$p, \tau \in C[R^+, R^+], p(t) > 0, f \in C[R, R], 0 \leq \tau(t) \leq q, \quad \text{and} \quad xf(x) > 0 \quad \text{for} \quad x \neq 0.$$

If $\int_0^\infty p(t)dt = \infty$, then all non-oscillatory solutions of (2.7.1) tend to zero as $t \rightarrow \infty$.

Proof: Let $x(t) > 0$ be a non-oscillatory solution of (2.7.1) for sufficiently large t .

Then $x'(t) < 0$, and it follows from the hypotheses that $\lim_{t \rightarrow \infty} x(t) = C \geq 0$ exists.

We show that $C = 0$. Otherwise, $C > 0$, and then there exists a $t^* \geq t_0$ such that

$$f(x(t - \tau(t))) \geq d > 0 \quad \text{for} \quad t \geq t^* \quad \text{and} \quad f(C) \geq d > 0. \quad \text{Thus}$$

$$x'(t) \leq -p(t)d \quad \text{for} \quad t \geq t^* \quad (2.7.2)$$

Integrating (2.7.2) from t^* to t , we get $x(t) \leq x(t^*) - d \int_{t^*}^t p(s)ds$. Hence $x(t)$ will

become negative for sufficiently large t . This is a contradiction to the fact that

$$x(t) > 0. \quad \text{Therefore} \quad C = 0 \bullet$$

Example 2.7.1. Consider the equation

$$x'(t) + t^2 x^{1/2}(t - \sqrt{2}) = 0 \quad (2.7.3)$$

Then $M = \lim_{t \rightarrow 0} \frac{x}{x^{1/2}} = 0$, and according to the asymptotic behavior since $\int_0^\infty t^2 dt = \infty$,

then all non-oscillatory solutions of (2.7.3) tend to zero as $t \rightarrow \infty$.

Theorem 2.7.2. Assume that

$$p, \tau \in C[R^+, R^+], p(t) > 0, f \in C[R, R], 0 \leq \tau(t) \leq q \text{ and } xf(x) > 0 \text{ for } x \neq 0.$$

If either $p(t) \rightarrow 0$ or $b(t)\tau(t) \rightarrow 0$ as $t \rightarrow \infty$, where $b(t) = \sup_{s \in [0, t]} p(s)$.

Then all bounded oscillatory solutions of (2.7.1) tend to zero as $t \rightarrow \infty$.

Proof: Let $x(t)$ be a bounded oscillatory solution. Assume that the conclusion of the theorem is false. This implies that there exists an $\varepsilon > 0$ and sequences

$\{t_n\}, \{t_n^*\} \rightarrow \infty$ such that for each n , either

$$x(t_n) = 0, x(t_n^*) = \varepsilon, x'(t_n^*) \geq 0 \quad \text{and} \quad 0 < x(t) < \varepsilon \quad \text{whenever} \quad t_n < t < t_n^* < t_{n+1}$$

or

$$x(t_n) = 0, x(t_n^*) = -\varepsilon, x'(t_n^*) \leq 0 \quad \text{and} \quad -\varepsilon < x(t) < 0 \quad \text{whenever} \quad t_n < t < t_n^* < t_{n+1}$$

The following argument holds for both cases. We will assume the first case. Integrate

$$(2.7.1) \text{ from } t_n \text{ to } t_n^*, \text{ we obtain } \varepsilon = x(t_n^*) - x(t_n) \leq \int_{t_n}^{t_n^*} p(s) |f(x(s - \tau(s)))| ds$$

$$\leq M \int_{t_n}^{t_n^*} p(s) ds \quad (2.7.4)$$

Where M is the bounded of $|f(x(\cdot))|$. Suppose $b(t)\tau(t) \rightarrow 0$ as $t \rightarrow \infty$. Since

$b(t)$ is continuous and monotone increasing we have

$$\varepsilon \leq M \int_{t_n}^{t_n^*} b(s) ds \leq Mb(t_n^*)(t_n^* - t_n) \quad \text{or} \quad t_n^* - t_n \geq \frac{\varepsilon}{Mb(t_n^*)}$$

Let n be chosen sufficiently large that $b(t_n^*)\tau(t_n^*) < \frac{\varepsilon}{M}$.

Then $\tau(t_n^*) < \frac{\varepsilon}{Mb(t_n^*)} \leq t_n^* - t_n$, which implies $t_n < t_n^* - \tau(t_n^*) \leq t_n^*$.

Thus $x(t_n^* - \tau(t_n^*)) > 0$ and we reach $x'(t_n^*) = -p(t_n^*)f(x(t_n^* - \tau(t_n^*))) < 0$

Which is a contradiction with $x'(t_n^*) \geq 0$. Now suppose $p(t) \rightarrow 0$ as $t \rightarrow \infty$. from (2.7.4) we get

$$\varepsilon \leq M \int_{t_n}^{t_n^*} p(s) ds = Mp(\xi)(t_n^* - t_n) \quad \text{or}$$

$$t_n^* - t_n \geq \frac{\varepsilon}{Mp(\xi)}$$

Where $t_n < \xi < t_n^*$. Since $p(t) \rightarrow 0$, it follows that $p(\xi) \rightarrow 0$ as $n \rightarrow \infty$.

Then n can be chosen sufficiently large that $t_n^* - t_n > q$. It follows that

$t_n < t_n^* - \tau(t_n^*) \leq t_n^*$. Thus $x(t_n^* - \tau(t_n^*)) > 0$, and we reach to

$x'(t_n^*) = -p(t_n^*)f(x(t_n^* - \tau(t_n^*))) < 0$. This is a contradiction with $x'(t_n^*) \geq 0$.

The proof is complete •

Example 2.7.2. Consider the equation

$$x'(t) + e^{-t}x(t - \frac{t}{2}) = 0 \tag{2.7.5}$$

Now since $\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} e^{-t} = e^{-\infty} = 0$. Then all bounded oscillatory solution of

(2.7.5) tends to zero as $t \rightarrow \infty$.

2.8. Non linear equations with forcing term

In this section, we present oscillation and non-oscillation for non-homogenous differential equations with delay argument.

Consider the equation

$$x'(t) + p(t)f(x(g(t))) = h(t) \tag{2.8.1}$$

Theorem 2.8.1. Assume that

- 1) $p, h \in C[R^+, R], p(t) \geq 0$;
- 2) $g \in C^1[R^+, R], g(t) < t, \lim_{t \rightarrow \infty} g(t) = \infty, g'(t) \geq 0$;
- 3) $f \in C[R, R], xf(x) > 0$ for $x \neq 0, f(x)$ is non-decreasing;
- 4) $\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds > M, \text{ where } M = \lim_{x \rightarrow 0} \frac{x}{f(x)}$;
- 5) There exists a function $Q \in C^1[R^+, R]$ such that $Q'(t) = h(t)$ for $t \geq 0$;

Then

(a) $\lim_{t \rightarrow \infty} Q(t) = 0$. implies that every solution $x(t)$ of (2.8.1) is either oscillatory or

$$\lim_{t \rightarrow \infty} x(t) = 0 \tag{2.8.2}$$

(b) If there exist constants q_1, q_2 and sequences $\{t'_m\}, \{t''_m\}$, such that

$$\lim_{m \rightarrow \infty} t'_m = \lim_{m \rightarrow \infty} t''_m = \infty \text{ and } Q(t'_m) = q_1, Q(t''_m) = q_2, q_1 \leq Q(t) \leq q_2 \text{ for } t \geq 0, \text{ then}$$

every solution of (2.8.1) is oscillatory or such that

$$\lim_{t \rightarrow \infty} [x(t) - Q(t)] = -q_1 \text{ or } -q_2 \tag{2.8.3}$$

Proof: Assume the existence of a non-oscillatory solution $x(t)$ of (2.8.1), let $x(t) > 0$

for $t \geq t_0$ and choose a $t_1 \geq t_0$ such that $x(g(t)) > 0$ for $t \geq t_1$. If we put

$y(t) \equiv x(t) - Q(t)$, then $y(t)$ is a solution of

$$y'(t) + p(t)f(y(g(t)) + Q(g(t))) = 0 \tag{2.8.4}$$

Such that $y(g(t)) + Q(g(t)) > 0$ for $t \geq t_1$. From (2.8.4) we obtain

$y'(t) \leq 0$ for $t \geq t_1$. So we have $\lim_{t \rightarrow \infty} y(t) = c$, where c is constant. Suppose that the

case (a) holds. If $c < 0$, then we get a contradiction that $x(t) < 0$ for sufficiently large t .

If $c > 0$, then we obtain $x(g(t)) = y(g(t)) + Q(g(t))$ and $\lim_{t \rightarrow \infty} x(g(t)) = \lim_{t \rightarrow \infty} y(g(t)) = c$,

which implies $x(g(t)) \geq \frac{c}{2}$, $y(g(t)) \geq \frac{c}{2}$. From (2.8.4) it follows that

$$y'(t) + p(t)f\left(\frac{c}{2}\right) \leq 0 \quad (2.8.5)$$

Integrate (2.8.5) from $g(t)$ to t , we have

$$y(t) - y(g(t)) + \left(\int_{g(t)}^t p(s) ds \right) f\left(\frac{c}{2}\right) \leq 0 \quad (2.8.6)$$

By taking the lim sup of (2.8.6) as $t \rightarrow \infty$, we get a contradiction to (4). Hence we

conclude that $c = 0$. From this, we see that $\lim_{t \rightarrow \infty} x(t) = 0$. Suppose that the case (b)

holds; put

$s(t) = y(t) + q_1$. Then we have

$$\lim_{t \rightarrow \infty} s(t) = \lim_{t \rightarrow \infty} (y(t) + q_1) = c + q_1 = d. \quad (-\infty < d < \infty) \quad (2.8.7)$$

If $d < 0$ in (2.8.7), then $y(t) + q_1 < 0$, for sufficiently large t , say $t \geq t_1$. This leads to

a contradiction to the fact that $y(t'_s) + q_1 = y(t'_s) + Q(t'_s) = x(t'_s) > 0$ for $t'_s > t$. If

$d > 0$, then

$$x(t) = y(t) + Q(t) \geq y(t) + q_1 = s(t) > \frac{d}{2} \quad (2.8.8)$$

for sufficiently large t , say $t \geq t_2$. By using (2.8.4) and (2.8.8), we obtain, setting

$s(t) = y(t) + q_1$, that

$$s'(t) + p(t)f(s(g(t))) \leq 0 \quad (2.8.9)$$

has a positive solution. By the application of theorem (2.6.1), this is a contradiction.

The proof is complete •

Example 2.8.1. Consider the following delay equation

$$x'(t) + e^t x^{\frac{1}{3}}\left(\frac{t}{3}\right) = e^{-t} \quad (2.8.10)$$

$$p(t) = e^t, f(x) = x^{\frac{1}{3}}, g(t) = \frac{t}{3}, h(t) = e^{-t}.$$

$$\overline{\lim}_{t \rightarrow \infty} \int_{\frac{t}{3}}^t e^s ds = e^t - e^{\frac{t}{3}} > \lim_{x \rightarrow 0} \frac{x}{x^{\frac{1}{3}}} = 0.$$

$Q'(t) = e^{-t} \Rightarrow Q(t) = -e^{-t}$. And $\lim_{t \rightarrow \infty} Q(t) = \lim_{t \rightarrow \infty} -e^{-t} = 0$. Which implies that $x(t)$ is

either oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Chapter Three

Second Order Delay Differential Equations

3.1 Introduction

In this chapter we will consider the oscillatory and asymptotic behavior of solutions of second order delay equation of the form

$$(r(t)x'(t))' + p(t)f(x(g(t))) = h(t) \quad (3.1.1)$$

where $r, h \in C([t_0, \infty), R)$, $r(t) > 0$, $f: R \rightarrow R$ is continuous. $g(t) \rightarrow \infty$ as $t \rightarrow \infty$. Assume $xf(x) > 0$, $x \neq 0$. $p(t) \geq 0$ is not identically zero in any half line of

the form (τ, ∞) for some $\tau \geq 0$ and $g(t) < t$, $g'(t) > 0$ and $\int_{\tau}^{\infty} \frac{dt}{r(t)} = \infty$. In section

two we shall consider the linear homogenous case of (3.1.1), and in section three the non-linear homogenous case of (3.1.1) will be discussed. In section four we shall consider the non-linear, non-homogenous case of (3.1.1). Finally, in section five the asymptotic behavior of (3.1.1) will be discussed.

3.2 Linear homogenous second order delay differential equations

Consider the linear, homogenous second order delay equation of the form

$$(r(t)x'(t))' + p(t)x(g(t)) = 0 \quad (3.2.1)$$

Theorem 3.2.1: Suppose there exists a differentiable function $\rho(t) \in C([t_0, \infty), R)$

$\rho(t) > 0$ such that $\int_{t_0}^{\infty} [\rho(t)p(t) - \frac{[\rho'(t)]^2 r(g(t))}{4g'(t)\rho(t)}] dt = \infty$ then equation (3.2.1) is

oscillatory.

Proof: Suppose there exists a non-oscillatory solution $x(t)$ of (3.2.1), since $g(t) \rightarrow \infty$ as $t \rightarrow \infty$, there exists a $t_1 \geq t_0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$. Now from (3.2.1), we have $(r(t)x'(t))' = -p(t)x(g(t))$ since $p(t)$ is non-negative and $x(g(t)) > 0$ for $t \geq t_1$, the right hand side becomes non-positive, therefore, we have $(r(t)x'(t))' \leq 0$ for $t \geq t_1$.

Thus $x(t)$, $x'(t)$ are monotonic and eventually of one-signed, now we want to show that there is a $t_2 \geq t_1$ such that for $t \geq t_2$, $x'(t) > 0$. Suppose on the contrary that $x'(t) \leq 0$. Since $p(t)$ is not identically zero, then there is a $t_3 \geq t_2$ such that $r(t_3)x'(t_3) < 0$, then for $t > t_3$,

$r(t)x'(t) \leq r(t_3)x'(t_3) < 0$. Dividing by $r(t)$ and integrating between t_3 and t we

obtain $\int_{t_3}^t x'(s) ds \leq r(t_3)x'(t_3) \int_{t_3}^t \frac{ds}{r(s)}$. Letting $t \rightarrow \infty$, $x(t) \rightarrow -\infty$. But this is a

contradiction since $x(t) > 0$, $x'(t)$ having eventually positive sign ($x'(t) > 0$). Now

define the function $z(t)$ by

$$z(t) = \frac{r(t)x'(t)}{x(g(t))} \rho(t) \Rightarrow z(t) > 0 \text{ since } r(t) > 0, x'(t) > 0, x(g(t)) > 0 \text{ for } t \geq t_2$$

$$z'(t) = \frac{r(t)x'(t)}{x(g(t))} \rho'(t) + \rho(t) \left[\frac{x(g(t))(r(t)x'(t))' - r(t)x'(t)x'(g(t))g'(t)}{[x(g(t))]^2} \right]$$

$$= \frac{\rho'(t)}{\rho(t)} z(t) + -p(t)\rho(t) - \frac{x'(g(t))g'(t)}{x(g(t))} z(t)$$

$$(e^{-t}x')' + 6e^t x(t - 2\pi) = 0 \quad (3.2.2)$$

Now $r(t) = e^{-t}$, $p(t) = 6e^t$, $g(t) = t - 2\pi$, $\rho(t) = 1 > 0$ then

$$\int_{t_0}^{\infty} \left[\rho(t)p(t) - \frac{[\rho'(t)]^2 r(g(t))}{4g'(t)\rho(t)} \right] dt = \int_{t_0}^{\infty} 6e^t dt = 6e^{\infty} = \infty.$$

The hypothesis of the theorem is satisfied, and then equation (3.2.2) is oscillatory.

3.3 Non linear homogenous second order delay differential equations

Consider the equation of the form

$$(r(t)x'(t))' + p(t)f(x(g(t))) = 0 \quad (3.3.1)$$

Theorem 3.3.1. Let $f'(x) \geq \lambda$ for some $\lambda > 0$, suppose there exists a

differentiable function $\rho \in C([t_0, \infty), \mathbb{R})$ $\rho(t) > 0$ such that

$$\int_{t_0}^{\infty} \left[p(t)\rho(t) - \frac{[\rho'(t)]^2 r(g(t))}{4\lambda g'(t)\rho(t)} \right] dt = \infty$$

Then equation (3.3.1) is oscillatory

Proof: The beginning of the proof is similar to the proof of theorem (3.2.1); until

we reach to the definition of the function $z(t)$, so define $z(t)$ as follows

$$z(t) = \frac{r(t)x'(t)}{f(x(g(t)))} \rho(t) \Rightarrow z(t) > 0 \text{ since } r(t) > 0, x'(t) > 0, f(x(g(t))) > 0. \text{ for } t \geq t_2$$

then

$$z'(t) = -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{f'(x(g(t)))x'(g(t))g'(t)}{f(x(g(t)))} z(t).$$

From $(r(t)x'(t))' \leq 0$ for $t \geq t_1$, $r(t)x'(t)$ is decreasing we get

$r(g(t))x'(g(t)) \geq (r(t)x'(t))$ substituting this equation in $z'(t)$ and using

$f'(x) \geq \lambda > 0$ we obtain

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)}z(t) - \frac{\lambda g'(t)}{r(g(t))\rho(t)}z^2(t)$$

Completing the square as before will leads to

$$z'(t) \leq -p(t)\rho(t) + \frac{(\rho')^2(t)r(g(t))}{4\lambda g'(t)\rho(t)}$$

$$\text{Integrating between } t_2 \text{ and } t, \Rightarrow z(t) - z(t_2) \leq \int_{t_2}^t -p(s)\rho(s) + \frac{(\rho')^2(s)r(g(s))}{4\lambda g'(s)\rho(s)} ds$$

Letting $t \rightarrow \infty$, we see that $\lim z(t) = -\infty$. Which is a contradiction with $z(t) > 0$,

$x'(t) > 0$. The proof is complete •

Example 3.3.1: Consider the following delay differential equation

$$(t^{-2}x'(t))' + f(x(g(t))) = 0 \tag{3.3.2}$$

$f(x) = xe^{|x|}$; $x \in R$, $g(t) = t - c$ for $c > 0$; $\rho(t) = 1$; then

$$\int_1^{\infty} 1 dt = \infty$$

Then equation (3.3.2) is oscillatory .

3.4. Non-homogenous, non linear second order delay differential equations.

Consider the equation of the form

$$(r(t)x'(t))' + p(t)f(x(g(t))) = h(t) \tag{3.4.1}$$

Definition 3.4.1. Equation (3.4.1) is called almost oscillatory if every solution

$x(t)$ of (3.4.1) is either oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Theorem 3.4.1: Let $f'(x) \geq k > 0$ for $x \neq 0$, (3.4.2)

and assume that there exist an oscillatory function $\psi : [t_0, \infty) \rightarrow R$ and differentiable

function $\rho(t) : [t_0, \infty) \rightarrow (0, \infty)$ such that

$$(r(t)\psi'(t))' = h(t), \quad \psi(t) \rightarrow 0 \text{ and } \psi'(t) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (3.4.3)$$

$$g'(t) > 0 \text{ and } g(t) \rightarrow \infty \text{ as } t \rightarrow \infty \quad (3.4.4)$$

If $\rho'(t) \geq 0$ and for each $d, 0 < d < 1$

$$\int_{t_0}^{\infty} \left[\rho(s)p(s) - \frac{r(g(s))\rho'^2(s)}{4kd\rho(s)g'(s)} \right] ds = \infty \quad (3.4.5)$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(t)\rho(t)} \int_{t_0}^t \rho(s)p(s) ds dt = \infty \quad (3.4.6)$$

Then every solution $x(t)$ of (3.4.1) is either oscillate or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: Let $x(t)$ be a non-oscillatory solution, we may assume without loss of generality that $x(t) > 0$ for $t \geq t_0$, then there exists $t_1 \geq t_0$ so that $x(g(t)) > 0$ for every $t \geq t_1$. Consider

$$y(t) = x(t) - \psi(t) \quad t \geq t_1 \quad (3.4.7)$$

Then from (3.4.1) we have

$$(r(t)y'(t))' = -p(t)f(x(g(t))), \quad t \geq t_1 \quad (3.4.8)$$

so that $(r(t)y'(t))'$ is eventually negative for $t \geq t_1$, hence $y(t), y'(t)$ are monotone and of one signed for large t , say $t \geq t_1$ in fact, $y(t)$ is eventually positive, since if $y(t) < 0$, then $\psi(t) > x(t)$. And since $\psi(t) \rightarrow 0, x(t) < 0$, which is a contradiction.

Now consider the following two cases:

Case (1): Suppose $y'(t)$ is eventually positive, then $y(t)$ is increasing and eventually positive, on the other hand, since $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$ and $y(t) = x(t) - \psi(t)$, there exists a $t_2 \geq t_1$ such that $x(g(t)) \geq dy(g(t))$ for $t \geq t_2$ and $d \in (0,1)$. Since f is increasing function, we obtain

$$f(x(g(t))) \geq f(dy(g(t))) \text{ for } t \geq t_2. \quad (3.4.9)$$

Define $z(t)$ by

$$z(t) = \frac{r(t)y'(t)}{f(dy(g(t)))} \rho(t), \Rightarrow z(t) > 0 \text{ for } t \geq t_2. \text{ And}$$

$$z'(t) = -\rho(t)p(t) \frac{f(x(g(t)))}{f(dy(g(t)))} + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{df'(dy(g(t)))y'(g(t))g'(t)}{f(dy(g(t)))} z(t) \quad (3.4.10)$$

from (3.4.4) and (3.4.8) we have

$$r(t)y'(t) \leq r(g(t))y'(g(t)) \quad (3.4.11)$$

now using (3.4.2), (3.4.9) and (3.4.11) in (3.4.10) we obtain

$$z'(t) \leq -\rho(t)p(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{dkg'(t)}{r(g(t))\rho(t)} z^2(t), t \geq t_2$$

Completing the square as follows

$$\begin{aligned} & \frac{-dkg'(t)}{r(g(t))\rho(t)} \left[z^2 - \frac{\rho'(t)r(g(t))}{dkg'(t)} z \right] - \rho(t)p(t) \\ &= \frac{-dkg'(t)}{r(g(t))\rho(t)} \left[z^2 - \frac{\rho'(t)r(g(t))}{dkg'(t)} z + \frac{(\rho'(t))^2 r^2(g(t))}{4d^2 k^2 (g'(t))^2} \right] + \frac{(\rho'(t))^2 r(g(t))}{4dk\rho(t)g'(t)} \end{aligned}$$

$$z'(t) \leq -\rho(t)p(t) + \frac{dkg'(t)}{r(g(t))\rho(t)} \left[z - \frac{\rho'(t)r(g(t))}{2dkg'(t)} \right]^2 + \frac{(\rho'(t))^2 r(g(t))}{4dk\rho(t)g'(t)}$$

$$z'(t) \leq - \left[\rho(t)p(t) - \frac{r(g(t))\rho'^2(t)}{4kd\rho(t)g'(t)} \right] \quad (3.4.12)$$

integrating (3.4.12) from t_2 to t we get

$$\int_{t_2}^t \left[\rho(s)p(s) - \frac{r(g(s))\rho'^2(s)}{4kd\rho(s)g'(s)} \right] ds \leq z(t_2) - z(t)$$

Letting $t \rightarrow \infty$ we have $\lim z(t) = -\infty$. Which is contradiction with $z(t) > 0$.

Case (2): Suppose $y'(t)$ is eventually negative, since $y(t)$ is eventually positive and decreasing, $\lim_{t \rightarrow \infty} y(t) = c$. We prove that $c = 0$. Let $c > 0$, from (3.4.3) and (3.4.7)

we see that $\lim_{t \rightarrow \infty} x(g(t)) = \lim_{t \rightarrow \infty} y(g(t)) = c$

Hence there exists a $t_3 \geq t_2$ such that

$$x(g(t)) \geq \frac{c}{2} \quad \text{for every } t \geq t_3 \quad (3.4.13)$$

Put $w(t) = r(t)y'(t)\rho(t), t \geq t_3$

$$\begin{aligned} \text{Thus } w'(t) &= r(t)(y'(t)\rho'(t) + \rho(t)y''(t)) + y'(t)\rho(t)r'(t) \\ &= r(t)y'(t)\rho'(t) + \rho(t)(r(t)y''(t) + r'(t)y'(t)) \\ &= r(t)y'(t)\rho'(t) + \rho(t)[-p(t)f(x(g(t)))] \\ &= -\rho(t)p(t)f(x(g(t))) + r(t)y'(t)\rho'(t) \end{aligned}$$

Using (3.4.13) and the fact that $\rho'(t) > 0$, we obtain

$$w'(t) \leq -f\left(\frac{c}{2}\right)\rho(t)p(t), t \geq t_3$$

Integrating the above inequality from t_3 to t we get

$$\int_{t_3}^t w'(s) ds \leq \int_{t_3}^t -f\left(\frac{c}{2}\right) \rho(s) p(s) ds$$

$$w(t) - w(t_3) \leq -f\left(\frac{c}{2}\right) \int_{t_3}^t \rho(s) p(s) ds$$

$$r(t)y'(t)\rho(t) - r(t_3)y'(t_3)\rho(t_3) \leq -f\left(\frac{c}{2}\right) \int_{t_3}^t \rho(s) p(s) ds \quad \text{it follows that for } t > t_3, \text{ one}$$

has

$$y'(t) \leq -f\left(\frac{c}{2}\right) \frac{1}{r(t)\rho(t)} \int_{t_3}^t \rho(s) p(s) ds \quad (3.4.14)$$

Integrating the above inequality from t_3 to t

$$\int_{t_3}^t y'(t) \leq -f\left(\frac{c}{2}\right) \int_{t_3}^t \frac{1}{r(s)\rho(s)} \int_{t_3}^s \rho(s) p(s) ds$$

$$y(t) - y(t_3) \leq -f\left(\frac{c}{2}\right) \int_{t_3}^t \frac{1}{r(s)\rho(s)} \int_{t_3}^s \rho(s) p(s) ds$$

Letting $t \rightarrow \infty$ and making use of (3.4.6) we get a contradiction with $y(t) > 0$ for

$t \geq t_1$. It follows that $\lim_{t \rightarrow \infty} x(t) = 0$. But $x(t)$ is monotone, thus $c = 0$, and

$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0, i = 0, 1$. Which means that $x(t)$ is almost oscillatory. The proof is

complete •

Example 3.4.1: Consider the following differential equation

$$(t^{-2}x'(t))' + f(x(g(t))) = \frac{1}{t} \cos(\log t) \quad \text{for } t \geq t_0 = 1 \quad (3.4.15)$$

$$f(x) = x \log^2(\mu + |x|), \quad x \in \mathbb{R} \quad \text{for } \mu > 0;$$

$$g(t) = c_1 t; \quad 0 < c_1 \leq 1; \quad \rho(t) = 1$$

Now

$$(t^2 \psi'(t))' = h(t) \Rightarrow (t^2 \psi'(t))' = \frac{1}{t} \cos(\log t)$$

then $t^2 \psi'(t) = \int \frac{1}{t} \cos(\log t) dt$

let $u = \log t \Rightarrow du = \frac{1}{t} dt$. Then $\int \cos u du = \sin u = \sin(\log t) + c$.

Now $t^2 \psi'(t) = \sin(\log t) + c$

$$\psi'(t) = \frac{1}{t^2} \sin(\log t) dt \Rightarrow \psi(t) = \int \frac{1}{t^2} \sin(\log t) dt$$

Integrating by parts. Let $u = \sin(\log t) \Rightarrow du = \frac{1}{t} \cos(\log t)$

$$dv = \frac{1}{t^2} dt \Rightarrow v = -\frac{1}{t}. \text{ Then } \int \frac{1}{t^2} \sin(\log t) dt = -\frac{1}{t} \sin(\log t) + \int \frac{1}{t^2} \cos(\log t) dt$$

one more time, we get

$$u = \cos(\log t) \Rightarrow du = -\frac{1}{t} \sin(\log t)$$

$$dv = \frac{1}{t^2} dt \Rightarrow v = -\frac{1}{t}$$

$$\int \frac{1}{t^2} \sin(\log t) dt = -\frac{1}{t} \sin(\log t) - \frac{1}{t} \cos(\log t) - \int \frac{1}{t^2} \sin(\log t) dt$$

$$2 \int \frac{1}{t^2} \sin(\log t) dt = -\frac{1}{t} \sin(\log t) - \frac{1}{t} \cos(\log t)$$

$$\int \frac{1}{t^2} \sin(\log t) dt = -\frac{(\sin(\log t) + \cos(\log t))}{2t} = \psi(t)$$

Now $\int_1^{\infty} 1 \cdot dt = \infty$. And also

$$\int_{t_0}^{\infty} t^2 \int_{t_0}^t ds dt = \int_{t_0}^{\infty} t^2 (t - t_0) dt = \infty$$

All conditions of theorem (3.4.1) are satisfied. Then every solution of (3.4.15) is either oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

3.5. Asymptotic behavior

Consider the equation

$$(r(t)x'(t))' + p(t)f(x(g(t))) = h(t)$$

Where

(a) $\lim_{t \rightarrow \infty} g(t) = \infty$;

(b) $r(t) > 0$ and $\lim_{t \rightarrow \infty} R(t) = \infty$, where $R(t) = \int_T^t \frac{ds}{r(s)}$

(c) $xf(x) > 0$ for $x \neq 0$, $\liminf_{x \rightarrow \infty} f(x) > 0$ and $\limsup_{x \rightarrow -\infty} f(x) < 0$.

Theorem 3.5.1. Assume that $p(t) \geq 0$ and

$$\int_{\infty}^{\infty} p(t) dt = \infty \tag{3.5.1}$$

$$\int_{\infty}^{\infty} R(t)|h(t)| dt < \infty \tag{3.5.2}$$

Then all non-oscillatory solutions of (3.4.1) tend to zero as $t \rightarrow \infty$.

We need the following lemma before proving the theorem

Lemma 3.5.1. Consider the differential equation

$$u' - \frac{\rho'(t)}{\rho(t)}u + \frac{\rho'(t)}{\rho(t)}\phi(t) = 0, \tag{3.5.3}$$

where $\phi(t)$ is continuous on $[T, \infty)$, $\rho(t)$ is continuously differentiable

on $[T, \infty)$, and $\rho(t) > 0, \rho'(t) > 0, \lim_{t \rightarrow \infty} \rho(t) = \infty$. Let $u(t)$ be the solution of (3.5.3)

on $[T, \infty)$ satisfying $u(T)=0$. If $\lim_{t \rightarrow \infty} |\phi(t)| = \phi^*$ exists in the extended real line $R^\#$.

Then $\lim_{t \rightarrow \infty} |u(t)| = u^*$ exists in $R^\#$. In particular, $\phi^* = \infty$ implies $u^* = \infty$

proof: See [8] •

Now the proof of theorem 3.5.1:

Let $x(t)$ be a non-oscillatory solution of (3.4.1) such that $x(g(t)) > 0$ for $t \geq t_1$.

Integrate (3.4.1) from t_1 to t

$$r(t)x'(t) - r(t_1)x'(t_1) + \int_{t_1}^t p(s)f(x(g(s)))ds = \int_{t_1}^t h(s)ds \quad (3.5.4)$$

Multiplying both sides of equation (3.4.1) by $R(t)$ and integrating from

t_1 to t ,

$$\int_{t_1}^t R(s)(r(s)x'(s))' ds + \int_{t_1}^t R(s)p(s)f(x(g(s)))ds = \int_{t_1}^t R(s)h(s)ds \quad (3.5.5)$$

Suppose that

$$\int_{t_1}^{\infty} R(t)p(t)f(x(g(t)))dt = \infty \quad (3.5.6)$$

Letting $t \rightarrow \infty$ in (3.5.5) and using (3.5.2).

We see that the first integral in (3.5.5) tends to $-\infty$ as $t \rightarrow \infty$.

So that $\lim_{t \rightarrow \infty} \phi(t) = \infty$, where is

$$\phi(t) = - \int_{t_1}^t R(s)(r(s)x'(s))' ds - R(t_1)r(t_1)x'(t_1)$$

from the lemma (3.5.1), it follows that

$x(t) = \int_{t_1}^t x'(s) ds$ tends to ∞ as $t \rightarrow \infty$. Consequently, we have

$$\lim_{t \rightarrow \infty} x(t) = \infty \quad (3.5.7)$$

Combining (3.5.7) with (c) and (3.5.1), we find

$$\int_{t_1}^{\infty} p(t) f(x(g(t))) dt = \infty \quad (3.5.8)$$

since (3.5.2) implies that the right-hand side of (3.5.4) is bounded, we have from

(3.5.4) and (3.5.8), $\lim_{t \rightarrow \infty} r(t)x'(t) = -\infty$. By using (b), we conclude that

$\lim_{t \rightarrow \infty} x(t) = -\infty$, but this contradicts the fact that $x(t)$ is positive, hence (3.5.6) must be

$$\int_{t_1}^{\infty} R(t)p(t)f(x(g(t)))dt < \infty \quad (3.5.9)$$

Letting $t \rightarrow \infty$ in (3.5.5), we see that $\lim_{t \rightarrow \infty} \phi(t)$ exists and finite. According to the

lemma (3.5.1) it follows $\lim_{t \rightarrow \infty} u(t)$ exists in $R^{\#}$. This limit must be finite, since

otherwise we reach to a contradiction with the positivity of $x(t)$. Therefore, there

exists a finite limit, $\lim_{t \rightarrow \infty} x(t)$. On the other hand, (3.5.9) implies

$$\int_{t_1}^{\infty} p(t) f(x(g(t))) dt < \infty \quad (3.5.10)$$

from (c), (3.5.1), and (3.5.10), it is verified that $\liminf_{t \rightarrow \infty} x(g(t)) = \liminf_{t \rightarrow \infty} x(t) = 0$.

Thus it follows that $\lim_{t \rightarrow \infty} x(t) = 0$, and the proof is complete •

Example 3.5.1. Consider the equation

$$\left(\frac{1}{t} x'(t)\right)' + t^{-1} x^3\left(\frac{t}{2}\right) = 11t^{-4} \quad \text{for } t \geq 1 \quad (3.5.11)$$

$$p(t) = t^{-1} \geq 0; \int_T^{\infty} t^{-1} dt$$

$$\int_T^{\infty} \frac{1}{t} dt = \ln t = \ln \infty = \infty$$

$$R(t) = \int_T^t \frac{ds}{r(s)} = \int_T^t s ds = \frac{1}{2} (t^2 - T^2)$$

$$\int_T^{\infty} |R(t)| |1/t^{-4}| dt = \frac{11}{2} \int_T^{\infty} (t^{-2} - T^2 t^{-4}) dt < \infty$$

equation (3.5.11) has a non oscillatory solution t^{-1} , this solution tends to zero as $t \rightarrow \infty$. Since all conditions of theorem (3.5.1) are satisfied then all non-oscillatory solutions of (3.5.11) tend to zero as $t \rightarrow \infty$.

Remark:

For ordinary differential equations of the second order the following criterion is well known: If an equation has oscillatory solutions then all its solutions are oscillatory. It is known that for delay differential equations this statement is not true.

Chapter Four

Third and Higher Order Delay Differential Equations

4.1 Introduction.

In this chapter we shall discuss the oscillation of n-th order delay differential equations of the form

$$x^{(n)}(t) + p(t)f(x(g(t))) = h(t) \quad (4.1.1)$$

where

- (1) $p, g, h, : [t_0, \infty) \rightarrow R$ are continuous,
- (2) $p(t) > 0, h(t)$ is non-decreasing,
- (3) $g(t) < t$, and $g(t) \rightarrow \infty$ as $t \rightarrow \infty$.

As a special case of nth- order we will consider the third order delay differential equation of the form

$$(b(t)(a(t)x'(t))')' + p(t)f(x(g(t))) = h(t) \quad (4.1.2)$$

where $a, b, h \in C([t_0, \infty), R)$

- (a) $a(t), b(t) > 0, f : R \rightarrow R$ continuous.
- (b) $g(t) \rightarrow \infty$ as $t \rightarrow \infty, g'(t) > 0$.
- (c) $p(t) \geq 0$ is not identically zero in any half line of the form (τ, ∞) for some $\tau \geq 0$ and

$$(d) b'(t) \geq 0 \text{ and } \int_{t_0}^{\infty} \frac{dt}{b(t)} = \infty, \int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty. \quad (4.1.3)$$

In section two, linear homogenous form of (4.1.2) will be discussed. In section three, non homogenous form of (4.1.2) has been discussed .In section four , non-linear non-

homogenous form of (4.1.2) will be discussed. In the last section , we will give important results about the general nth-order equations in the two cases , homogenous and non-homogenous linear cases.

4.2 Linear homogenous third order delay differential equations

In this section we shall discuss the oscillation of the linear, homogenous delay equation of the form

$$(b(t)(a(t)x'(t)))' + p(t)x(g(t)) = 0 \quad (4.2.1)$$

Theorem 4.2.1. Suppose that there exists a differentiable function

$\rho \in C([t_0, \infty), R)$, $\rho(t) > 0$ such that

$$\int_{t_0}^{\infty} \left[p(t)\rho(t) - \frac{b(t)[\rho'(t)]^2 a(g(t))}{4(g(t)-T)g'(t)p(t)} \right] dt = \infty \quad (4.2.2)$$

And that
$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) dr > 1 \quad (4.2.3)$$

Then equation (4.2.1) is oscillatory.

proof: Let $x(t)$ be a non-oscillatory solution of (4.2.1). Assume $x(t)$ is eventually positive. Since $g(t) \rightarrow \infty$ as $t \rightarrow \infty$, there exists a $t_1 \geq t_0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$. Now from (4.2.1) we have

$$(b(t)(a(t)x'(t)))' = -p(t)x(g(t)) \quad (4.2.4)$$

Since $p(t)$ is not negative and $x(g(t)) > 0$ is positive for $t \geq t_1$, the right hand side becomes non-positive. Therefore, $(b(t)(a(t)x'(t)))' \leq 0$ for $t \geq t_1$. Thus, $x(t)$, $x'(t)$, $(a(t)x'(t))'$ are monotonic and eventually one-signed.

Now we want to show that there is a $t_2 \geq t_1$ such that for $t \geq t_2$

$$(a(t)x'(t))' > 0 \quad (4.2.5)$$

Suppose this is not true, then $(a(t)x'(t))' \leq 0$. Since $p(t)$ is not identically zero and

$b(t) > 0$, then there is a $t_3 \geq t_2$ such that $b(t_3) (a(t_3)x'(t_3))' < 0$. Then for $t > t_3$

we have

$$b(t)(a(t)x'(t))' \leq b(t_3) (a(t_3)x'(t_3))' < 0 \quad (4.2.6)$$

Dividing (4.2.6) by $b(t)$, we have $(a(t)x'(t))' \leq \frac{b(t_3)(a(t_3)x'(t_3))'}{b(t)}$, and then

integrating between t_3 and t , we obtain

$$a(t)x'(t) - a(t_3)x'(t_3) < b(t_3)(a(t_3)x'(t_3))' \int_{t_3}^t \frac{1}{b(s)} ds. \quad (4.2.7)$$

Letting $t \rightarrow \infty$ in (4.2.7) and because of (4.1.2) we have $a(t)x'(t) \rightarrow -\infty$ as

$t \rightarrow \infty$. Thus there is a $t_4 \geq t_3$ such that $a(t_4)x'(t_4) < 0$. Using $(a(t)x'(t))' \leq 0$, we

have for $t \geq t_4$

$$a(t)x'(t) \leq a(t_4)x'(t_4) \quad (4.2.8)$$

Dividing (4.2.8) by $a(t)$, we have $x'(t) \leq \frac{a(t_4)x'(t_4)}{a(t)}$. Integrating from t_4 to t ,

$$x(t) - x(t_4) < a(t_4)x'(t_4) \int_{t_4}^t \frac{1}{a(s)} ds, \text{ letting } t \rightarrow \infty. \text{ Then } x(t) \rightarrow -\infty.$$

But this is a contradiction with $x(t) > 0$, then (4.2.5) holds.

Now we consider two cases :

Case 1. $x'(t) > 0$, for $t \geq t_2$. Define the function $z(t)$ by

$$z(t) = \frac{b(t)(a(t)x'(t))'}{x(g(t))} \rho(t) \Rightarrow z(t) > 0 \text{ for } t \geq t_2$$

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{x'(g(t))g'(t)}{x(g(t))} z(t) \quad (4.2.9)$$

Using $(b(t)(a(t)x'(t))')' \leq 0$, $b'(t) \geq 0$ and (4.2.5), we can find

$$(a(t)x'(t))'' \leq 0 \quad (4.2.10)$$

Using (4.2.10) and the equality

$$a(t)x'(t) = a(T)x'(T) + \int_T^t (a(s)x'(s))' ds \quad (4.2.11)$$

we have

$$a(t)x'(t) \geq (t-T)(a(t)x'(t))' \quad \text{for } T \geq t_2 \quad (4.2.12)$$

Since $(a(t)x'(t))'$ is non-increasing we obtain

$$a(g(t))x'(g(t)) \geq (g(t)-T)(a(t)x'(t))' \quad (4.2.13)$$

Multiplying both sides of (4.2.13) by $\frac{g'(t)}{a(g(t))}$, and then we have

$$x'(g(t))g'(t) \geq \frac{(g(t)-T)}{a(g(t))} g'(t) (a(t)x'(t))' \quad (4.2.14)$$

Using (4.2.14) in (4.2.9) it follows that

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{(g(t)-T)}{a(g(t))} \frac{g'(t)(a(t)x'(t))'}{x(g(t))} z(t)$$

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{(g(t)-T)g'(t)}{a(g(t))} \frac{z^2(t)}{b(t)\rho(t)}$$

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{(g(t)-T)g'(t)}{b(t)\rho(t)} z^2(t)$$

Completing the square will leads to

$$z'(t) \leq -p(t)\rho(t) + \frac{b(t)(\rho'(t))^2 a(g(t))}{(g(t)-T)g'(t)4\rho(t)} \quad (4.2.15)$$

Integrating (4.2.15) between T and t

$$z(t) - z(T) \leq \int_T^t \left[-p(t)\rho(t) + \frac{b(t)(\rho'(t))^2 a(g(t))}{(g(t)-T)g'(t)4\rho(t)} \right] dt$$

Letting $t \rightarrow \infty$, we see that $\lim_{t \rightarrow \infty} z(t) = -\infty$. This is contradicts $z(t)$ being eventually positive.

Case 2. If $x'(t)$ is eventually negative. Integrate (4.2.1) from t to ∞ and since

$b(t) (a(t)x'(t))' > 0$. We have

$$-b(t)(a(t)x'(t))' + \int_t^\infty p(r)x(g(r))dr \leq 0 \quad (4.2.16)$$

Dividing (4.2.16) by $b(t)$ we have $-(a(t) x'(t))' + \frac{1}{b(t)} \int_t^\infty p(r)x(g(r))dr \leq 0$.

Then integrating from t to ∞ we have

$$a(t) x'(t) + \int_t^\infty \left(\int_t^r \frac{1}{b(u)} du \right) p(r)x(g(r))dr \leq 0 \quad (4.2.17)$$

Dividing (4.2.17) by $a(t) \Rightarrow x'(t) + \frac{1}{a(t)} \int_t^\infty \left(\int_t^r \frac{1}{b(u)} du \right) p(r)x(g(r))dr \leq 0$

Then integrating from t to ∞

$$\int_t^\infty \left[\int_t^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r)x(g(r))dr \leq x(t) \quad (4.2.18)$$

Replacing t by $g(t)$ in (4.2.18) we will give

$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r)x(g(r))dr \leq x(g(t)) \quad (4.2.19)$$

Using the fact that $g(t) < t$ and $x(t)$ is decreasing in (4.2.19) we obtain.

$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r)dr \leq 1$$

This is a contradiction to (4.2.3), this complete the proof, and (4.2.1) is oscillatory •

Example 4.2.1: Consider the following functional differential equation

$$x'''(t) + e^t x(t - \pi) = 0 \quad (4.2.20)$$

Now

$$a(t) = b(t) = 1, \quad p(t) = e^t, \quad g(t) = t - \pi$$

$\rho(t) = c$, c is any positive constant

then

$$\int_{-\infty}^{\infty} [e^t \cdot c - \text{zero}] dt = \int_{-\infty}^{\infty} ce^t dt = e^{\infty} c = \infty, c > 0.$$

$$\text{and } \int_{t-\pi}^t \left[\int_{t-\pi}^r 1 \cdot du \left(\int_u^r 1 \cdot dv \right) \right] e^r dr$$

$$= \int_{t-\pi}^t \left[\int_{t-\pi}^r 1 \cdot du (r - u) \right] e^r dr = \int_{t-\pi}^t \left[r^2 - \frac{r^2}{2} - r(t - \pi) + \frac{(t - \pi)^2}{2} \right] e^r dr$$

$$= \int_{t-\pi}^t \left[\frac{r^2}{2} - r(t - \pi) + \frac{(t - \pi)^2}{2} \right] e^r dr = \int_{t-\pi}^t \left[\frac{r^2}{2} - rc + \frac{c^2}{2} \right] e^r dr$$

$$= \int_{t-\pi}^t \left(\frac{r^2}{2} e^r - re^r c + e^r \frac{c^2}{2} \right) dr. \text{ By taking } t \text{ sufficiently large, (4.2.3) is satisfied.}$$

Then equation (4.2.20) is oscillatory.

4.3. Nonlinear homogenous third order delay differential equations.

In this section we shall discuss the oscillation of nonlinear, homogenous third order delay equation of the form

$$(b(t)(a(t)x'(t))')' + p(t)f(x(g(t))) = 0 \quad (4.3.1)$$

Theorem 4.3.1. Let $f'(x) \geq \lambda$ for some $\lambda > 0$. Suppose that there exists a differentiable function $\rho \in C([t_0, \infty), R)$, $p(t) > 0$, such that

$$\int^{\infty} \left[p(t)\rho(t) - \frac{b(t)(\rho'(t))^2 a(g(t))}{4(g(t)-T)g'(t)\lambda\rho(t)} \right] dt = \infty \quad (4.3.2)$$

And

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) dr = \infty \quad (4.3.3)$$

Then equation (4.3.1) is oscillatory.

Proof : The first part is similar to the proof of theorem (4.2.1) until we reach the two possible cases .

Case 1. Suppose $x'(t)$ is eventually positive. Then we can define.

$$z(t) = \frac{b(t)(a(t)x'(t))'}{f(x(g(t)))} \rho(t) > 0. \text{ Then } z(t) > 0 \text{ for } t \geq t_2 \text{ and}$$

$$z'(t) = -\frac{p(t)f(x(g(t)))}{f(x(g(t)))} \rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{f'(x(g(t)))x'(g(t))g'(t)}{f(x(g(t)))} z(t)$$

Then

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{f'(x(g(t)))x'(g(t))g'(t)}{f(x(g(t)))} z(t) \quad (4.3.4)$$

Now since $(b(t)(a(t)x'(t))')' \leq 0$, then (4.2.5) holds and $b'(t) \geq 0$, we can obtain

$(a(t)x'(t))'' \leq 0$. Using this inequality and the equality

$$\int_T^t (a(t)x'(s))' ds = a(t)x'(t) - a(T)x'(T).$$

This equation will lead to $a(t)x'(t) \geq (t-T)(a(t)x'(t))'$.

Now using the non-increasing nature of $(a(t)x'(t))'$ we obtain

$$a(g(t))x'(g(t)) \geq (g(t)-T)(a(t)x'(t))' \quad (4.3.5)$$

Multiplying both sides of (4.3.5) by $\frac{g'(t)}{a(g(t))}$

$$g'(t)x'(g(t)) \geq \frac{(g(t)-T)}{a(g(t))} g'(t)(a(t)x'(t))' \quad (4.3.6)$$

Substituting (4.3.6) in (4.3.4)

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \lambda \frac{(g(t)-T)g'(t)}{b(t)\rho(t)} z^2(t)$$

Completing the square leads to

$$z'(t) \leq -p(t)\rho(t) + \frac{b(t)(\rho'(t))^2}{(g(t)-T)g'(t)4\rho(t)}$$

Integrating between T to t and letting $t \rightarrow \infty$ we see that $\lim_{t \rightarrow \infty} z(t) = -\infty$.

This is a contradiction with $z(t)$ being eventually positive.

Case 2. If $x'(t) < 0$. And also as in the proof of theorem 4.2.1 we will end up with

$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r)f(x(g(r)))dr \leq x(g(t)) \quad (4.3.7)$$

Using the fact that $g(t) < t$, $f(x)$ is increasing and $x(t)$ is decreasing in the (4.3.7)

we obtain.

$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) dr \leq \frac{x(g(t))}{f(x(g(t)))} \quad (4.3.8)$$

Since $x(t)$ is decreasing and positive it is approaching a finite non-negative number as $t \rightarrow \infty$. According to equation (4.3.3) and (4.3.8) it is not possible that

$\lim_{t \rightarrow \infty} x(t) > 0$. Suppose $\lim_{t \rightarrow \infty} x(t) = 0$ then

$$\lim_{t \rightarrow \infty} \frac{x(g(t))}{f(x(g(t)))} = \lim_{t \rightarrow \infty} \frac{1}{f'(x(g(t)))} = \frac{1}{f'(0)} \leq \frac{1}{\lambda}$$

This is a contradiction to (4.3.3) therefore, the proof is complete •

Example 4.3.1. Consider the following delay equation

$$(e^{-t} x')'' + 3e^{-t} x^2(t - 3\pi) = 0 \quad (4.3.9)$$

$a(t) = e^{-t}$, $b(t) = 1$, $p(t) = 3e^{-t}$, $g(t) = t - 3\pi$, $\rho(t) = e^t$, $f(x) = x^2$ $x \in (0, \infty)$.

$$\int_{t-3\pi}^t \left[3e^{-t} e^t - \frac{e^{2t} e^{-(t-3\pi)}}{(t-3\pi-T)4e^t} \right] dt$$

$$\int_{t-3\pi}^t \left[3 - \frac{e^{2t} \cdot e^{-t} \cdot e^{3\pi}}{4(t-3\pi-T)e^t} \right] dt = \int_{t-3\pi}^t \left[3 - \frac{e^{3\pi}}{4(t-3\pi-T)} \right] dt$$

$$= 3t - \frac{e^{3\pi}}{4} \ln(t-3\pi-T) = \infty \quad \text{as } t \rightarrow \infty$$

$$\limsup_{t \rightarrow \infty} \int_{t-3\pi}^t \left[\int_{t-3\pi}^r e^u du \left(\int_u^r 1 dv \right) \right] 3e^{-r} dr = \limsup_{t \rightarrow \infty} \int_{t-3\pi}^t \left[\int_{t-3\pi}^r e^u du (r-u) \right] 3e^{-r} dr$$

$$\int_{t-3\pi}^t \left[\int_{t-3\pi}^r e^u u - ue^u \right] du \Big] 3e^{-r} dr = \int_{t-3\pi}^t \left(\left[re^u - ue^u + e^u \right]_{t-3\pi}^r \right) 3e^{-r} dr$$

$$\begin{aligned}
& \int_{t-3\pi}^t ([re^r - re^r + e^r - re^{t-3\pi} + (t-3\pi)e^{t-3\pi} - e^{t-3\pi}]) 3e^{-r} dr \\
& \int_{t-3\pi}^t ([e^r + e^{t-3\pi} (-r+t-3\pi-1)]) 3e^{-r} dr \\
& \int_{t-3\pi}^t 3 + e^{t-3\pi} e^{-r} (-r+t-3\pi-1) dr = \int_{t-3\pi}^t 3 + 3e^{t-3\pi-r} (-r+t-3\pi-1) dr \\
& \int_{t-3\pi}^t 3 + 3e^{-r+c} (-r+c-1) dr = \int_{t-3\pi}^t 3 - 3re^{-r+c} + 3e^{-r+c} (c-1) dr \\
& = 3r - 3re^{-r+c} - e^{-r+c} - 3e^{-r+c} \Big|_{t-3\pi}^t = \\
& (3t - 3te^{-t+c} - 3e^{-t+c} - e^{-t+c}) - (3(t-3\pi) - 3(t-3\pi)e^{-(t-3\pi)+(t-3\pi)} - 4e^{-(t-3\pi)+(t-3\pi)}) \\
& = (3t - 3te^{-3\pi} - 4e^{-3\pi}) - (3(t-3\pi) - 3(t-3\pi) - 4).
\end{aligned}$$

As $t \rightarrow \infty$, (4.3.3) is satisfied. Then (4.3.9) is oscillatory.

4.4. Non linear, non homogenous third order delay differential equations.

In this section we shall discuss the oscillatory behavior of non-linear, non-homogenous third order delay equation of the form

$$(b(t)(a(t)x'(t)))' + p(t)f(x(g(t))) = h(t) \quad (4.4.1)$$

Theorem 4.4.1: Suppose that $f'(x) \geq \lambda$ for some $\lambda > 0$, and

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) dr = \infty \quad (4.4.2)$$

and also, suppose that there exist a continuously differentiable function

$\rho \in C([t_0, \infty), \mathbb{R})$, $\rho(t) > 0$ and an oscillatory function $\varphi(t)$ such that.

$$\int^{\infty} \left[p(t)\rho(t) - \frac{b(t)(\rho'(t))^2 a(g(t))}{4(g(t)-T)g'(t)\lambda d\rho(t)} \right] dt = \infty.$$

for some $d \in (0,1)$ and for every $T \geq 0$ and

$$(b(t)(a(t)\varphi'(t)))' = h(t), \quad \lim_{t \rightarrow \infty} \varphi^{(i)} = 0, \quad i = 0,1,2.$$

Then equation (4.4.1) is almost oscillatory.

Proof: Let $x(t)$ be a non-oscillatory solution of (4.4.1), assume that $x(t)$ is eventually positive. Consider $y(t) = x(t) - \varphi(t)$, $y(t)$ is eventually positive, if not, $x(t) < \varphi(t)$ and it is a contradiction with oscillatory behavior of $\varphi(t)$. Now $((b(t)(a(t)y'(t)))' \leq 0$. And also as in the proof of theorem 4.2.1, there exists a $t_1 \geq 0$, such that for $t \geq t_1$, $(a(t)y'(t))' > 0$ and $(a(t)y'(t))'' \leq 0$. Consider again two cases:

Case1. Suppose $y'(t) > 0$, then $y(t)$ is increasing and eventually positive, since

$\varphi(t) \rightarrow 0$ and $y(t) = x(t) - \varphi(t)$, there exists a $t_2 \geq t_1$ such that $x(g(t)) \geq d y(g(t))$ for $t \geq t_2$ and $d \in (0,1)$. Since f is an increasing function $f(x(g(t))) \geq f(dy(g(t)))$ for $t \geq t_2$. Define $z(t)$ by

$$z(t) = \frac{b(t)(a(t)y'(t))'}{f(dy(g(t)))} \rho(t) \Rightarrow z(t) > 0 \text{ for } t \geq t_2 \text{ and}$$

$$z'(t) = -\frac{p(t)f(x(g(t)))}{f(dy(g(t)))} \rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{df'(dy(g(t)))y'(g(t))g'(t)}{f(dy(g(t)))} z(t).$$

Using $f'(x) \geq \lambda > 0$, we obtain

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{d\lambda(y'(g(t)).g'(t))}{f(dy(g(t)))} z(t) \tag{4.4.3}$$

and also as in the proof of theorem 4.2.1 we show that

$$g'(t)y'(g(t)) \geq \frac{(g(t)-T)}{a(g(t))} g'(t) \cdot (a(t)y'(t))' \quad (4.4.4)$$

Substitute (4.4.4) in (4.4.3) , we have

$$z'(t) \leq -p(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{d\lambda(g(t)-T)g'(t)a(g(t))}{b(t)\rho(t)} z^2(t)$$

Completing the square, we get

$$z'(t) \leq -p(t)\rho(t) + \frac{b(t)(\rho'^2)(t)}{\frac{(g(t)-T)}{a(g(t))} g'(t)4\lambda d\rho(t)} \quad (4.4.5)$$

Integrating (4.4.5) from T to t and letting $t \rightarrow \infty$, see that $z(t) \rightarrow -\infty$ as $t \rightarrow \infty$.

This contradicts $z(t)$ being eventually positive.

Case2. Suppose $y'(t)$ is eventually negative. Since y is eventually positive, and decreasing, $\lim_{t \rightarrow \infty} y(t) = c$, where c is a non-negative number. Therefore,

$\lim_{t \rightarrow \infty} x(t) = c$. Now integrating (4.4.1), three times, as we did in the proof of theorem

4.2.1. We will end up with.

$$\int_{g(t)}^{\infty} \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) f(x(g(r))) dr \leq y(t).$$

Thus we have

$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) f(x(g(r))) dr \leq y(t).$$

$$\int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] p(r) dr \leq \frac{x(t)}{f(x(g(t)))}.$$

Hence we conclude that $\liminf_{t \rightarrow \infty} x(t) = 0$. But $x(t)$ is monotone, so $\lim_{t \rightarrow \infty} x(t) = 0$.

Thus $c = 0$, and by using $\left(b(t)(a(t)\varphi'(t))' \right)' = h(t)$, $\lim_{t \rightarrow \infty} \varphi^{(i)}(t) = 0$ and

$y(t) = x(t) - \varphi(t)$. $\lim_{t \rightarrow \infty} x^{(i)}(t) = 0$, $i = 0, 1, 2$. Which means that $x(t)$ is almost

oscillatory. The proof is complete •

Example 4.4.1. Consider the following delay equation.

$$x'''(t) + (e^t)2x^4(t-1) = e^{-t} \sin t \quad t \geq t_0 \geq 0 \quad (4.4.6)$$

$$\begin{aligned} a(t) &= b(t) = 1 \\ f(x) &= 2x^4, x \in (0, \infty) \quad f'(x) = 8x^3 > 0 \\ g(t) &= t - 1 \\ p(t) &= e^t \\ h(t) &= e^{-t} \sin t \\ \rho(t) &= 1 \end{aligned}$$

And also $\varphi'''(t) = e^{-t} \sin t \Rightarrow \varphi''(t) = \int e^{-t} \sin t$. Integrate by parts

$$\begin{aligned} \text{Let} \quad u &= e^{-t} & dv &= \sin t \\ du &= -e^{-t} dt & v &= -\cos t \end{aligned}$$

Then

$$\int e^{-t} \sin t = -e^{-t} \cdot \cos t - \int \cos t e^{-t} dt.$$

Integrate another time by parts.

$$\begin{aligned} \text{Let} \quad u &= e^{-t} & dv &= \sin t \\ du &= e^{-t} dt & v &= +\cos t \end{aligned}$$

$$\int \cos t e^{-t} dt = e^{-t} \sin t + \int \sin t e^{-t} dt$$

$$\int e^{-t} \sin t dt = -e^{-t} \cos t - e^{-t} \sin t - \int \sin t e^{-t} dt$$

$$\int e^{-t} \sin t dt = \frac{-e^{-t} \cos t - e^{-t} \sin t}{2} = \frac{e^{-t}}{2} (-\cos t - \sin t)$$

$$\varphi''(t) = \int e^{-t} \sin t \, dt = \frac{-e^{-t}}{2} (\cos t + \sin t)$$

$$\varphi'(t) = \frac{-1}{2} \int e^{-t} \cos t + e^{-t} \sin t \, dt$$

Integrate the first term by parts.

$$\varphi'(t) = \frac{-1}{2} \left(\int e^{-t} \cos t \, dt + \int e^{-t} \sin t \, dt \right)$$

$$\int e^{-t} \cos t \, dt = e^{-t} \sin t + \int \sin t e^{-t} \, dt = e^{-t} \sin t - e^{-t} \cos t - \int \cos t e^{-t} \, dt$$

$$\int e^{-t} \cos t = \frac{e^{-t} \sin t - e^{-t} \cos t}{2}$$

$$\therefore \varphi'(t) = \frac{-1}{2} \left[\frac{e^{-t}}{2} (\sin t - \cos t) - \frac{e^{-t}}{2} (\cos t + \sin t) \right] = \frac{-1}{2} (-e^{-t} \cos t) =$$

$$\frac{e^{-t}}{2} \cos t = \varphi'(t)$$

$$\varphi(t) = \frac{1}{2} \int e^{-t} \cos t \, dt = \frac{1}{2} \frac{e^{-t}}{2} (\sin t - \cos t) = \frac{e^{-t}}{4} (\sin t - \cos t)$$

$$\limsup_{t \rightarrow \infty} \int_{t-1}^t \left[\int_{t-1}^r du \left(\int_u^r 1 \, dv \right) \right] e^r \, dr$$

$$\int_{t-1}^t \left[\int_{t-1}^r du (r-u) \right] (e^r) \, dr$$

$$= \int_{t-1}^t \left(r^2 - \frac{r^2}{2} - r(t-1) + \frac{(t-1)^2}{2} \right) (e^r) \, dr = \int_{t-1}^t \left(\frac{r^2}{2} - r(t-1) + \frac{(t-1)^2}{2} \right) (e^r) \, dr$$

As $t \rightarrow \infty$

$$\therefore \limsup_{t \rightarrow \infty} \int_{t-1}^t \frac{r^2}{2} e^r - r e^r (t-1) + \frac{(t-1)^2}{2} e^r \, dr = \infty$$

In addition.

$$\int e^t dt = e^t = e^\infty = \infty.$$

All conditions of theorem 4.4.1 are satisfied, then equation (4.4.4) is almost oscillatory.

4.5: Nth- order Delay Differential Equations

This section is concerned with the oscillation of linear homogenous , non-homogenous simplest form of n-th order delay differential equations. Since *nth* – order delay differential equations is so difficult for studying ,and having many complicated conditions.

Consider the following linear homogenous form of nth-order delay differential equation

$$x^{(n)}(t) + p(t)x(g(t)) = 0 \tag{4.5.1}$$

where $p(t), g(t)$ are continuous function on $([0, \infty), R)$, $p(t) > 0$, $g(t) < t$, $g(t)$ is non-decreasing and $\lim_{t \rightarrow \infty} g(t) = \infty$.

As we mentioned before a solution $x(t)$ of (4.5.1) is called oscillatory if it has arbitrarily large zeros, and it is called non-oscillatory otherwise.

The following lemma is needed which is essentially *Kiguarde'z* lemma.

Lemma 4.5.1[14]: Let $x(t)$ be a solution of (4.5.1), $x(t) > 0$ for $t \in [0, \infty)$. And

let $x^{(n)}(t) \leq 0$ for $t \in [0, \infty)$.Then there exist a $t_1 \in [0, \infty)$ and an integer

$l \in \{0, 1, \dots, n-1\}$ such that $l+n$ is odd (when n is even l is odd and when n is odd

l is even).

$$x^{(i)}(t) > 0 \text{ for } t \in [t_1, \infty) (i = 0, \dots, l-1), (-1)^{i+l} x^{(i)}(t) > 0 \quad (4.5.2)$$

for $t \in [t_1, \infty) (i = l, \dots, n-1)$

$$(t-t_1) |x^{(l-i)}(t)| \leq (1+i) |x^{(l-i-1)}(t)| \text{ for } t \in [t_1, \infty) (i = 0, \dots, l-1), 1 \leq l \leq n-1. \quad (4.5.3)$$

Analogous statement can be made if $x(t) < 0$ and $x^n(t) \geq 0$ for $t \in [0, \infty)$.

Theorem 4.5.1. Suppose that for every $l \in \{0, 1, \dots, n-1\}$ such that $n+l$ is odd and for some $d_l \in \{0, 1, \dots, n-l-1\}$ it holds

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t [s-g(t)]^{n-l-d_l-1} [g(t)-g(s)]^{d_l} [g(s)]^l p(s) ds > l!(n-l-d_l-1)!d_l! \quad (4.5.4)$$

Then every solution of equation (4.5.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (4.5.1) such that $x(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geq 0$. Then by using lemma 4.5.1, there exist $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \dots, n-1\}$ such that $n+l$ is odd and (4.5.2) and (4.5.3) hold. For sufficiently large $t_2 \in [t_1, \infty)$ in view of (4.5.3) we have.

$$|x^{(l-i-1)}(t)| \geq \frac{t-t_1}{1+i} |x^{(l-i)}(t)|, \quad i = 0, 1, \dots, l-1, \quad 1 \leq l \leq n-1.$$

$$\Rightarrow x(t) \geq \frac{(t-t_1)^l}{l!} x^{(l)}(t), \quad t \in [t_2, \infty)$$

$$x(g(t)) \geq \frac{(g(t)-t_1)^l}{l!} x^{(l)}(g(t)), \quad t \geq t_2, \quad 1 \leq l \leq n-1 \quad (4.5.5)$$

Then by applying Taylor's formula with remainder applied to $z^{(j)}(t)$, we get.

$$z^{(j)}(t) = \sum_{i=j}^{k-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} z^{(i)}(s) + \frac{(-1)^{k-j}}{(k-j-1)!} \int_t^s (u-t)^{k-j-1} z^{(k)}(u) du, \quad s \geq t \geq t_2, \quad \text{for}$$

$k = n-l$ we get

$$z^{(j)}(t) = \sum_{i=j}^{n-l-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} z^{(i)}(s) + \frac{(-1)^{n-l-j}}{(n-l-j-1)!} \int_t^s (u-t)^{n-l-j-1} z^{(n-l)}(u) du. \quad (4.5.6)$$

Choose $z(t) = x^{(l)}(t)$. Now for $j = d_l, d_l \in \{0, 1, \dots, n-l-1\}$, and from (4.5.6) with regard to (4.5.2) we have

$$|z^{(d_l)}(g(t))| \geq \frac{1}{(n-l-d_l-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_l-1} |z^{(n-l)}(u)| du \quad (4.5.7)$$

From (4.5.6) for $u \in [g(t), t]$, $d_l \in \{0, 1, \dots, n-l-1\}$, $j = 0$, we get

$$|z(g(u))| \geq \frac{[g(t)-g(u)]^{d_l}}{d_l!} |z^{(d_l)}(g(t))|. \quad (4.5.8)$$

Also, from (4.5.7) in view of equation (4.5.1) we obtain.

$$|z^{(d_l)}(g(t))| \geq \frac{1}{(n-l-d_l-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_l-1} |x(g(u))| p(u) du$$

From the last inequality using (4.5.5) and (4.5.8)

$$l!(n-l-d_l-1)! d_l! \geq \int_{g(t)}^t [u-g(t)]^{n-l-d_l-1} [g(t)-g(u)]^{d_l} [g(u)-t_1]^l p(u) du$$

For t sufficiently large we get a contradiction to (4.5.4). The proof is complete •

Example 4.5.1. : Consider the equation

$$x'''(t) + x(t - \frac{3}{2}\pi) = 0 \quad (4.5.9)$$

Choose $d_l = 0$ for $l = 0, 2$.

$$\limsup_{t \rightarrow \infty} \int_{t-\frac{3}{2}\pi}^t [s - (t - \frac{3}{2}\pi)]^0 [t-s]^0 [s - \frac{3}{2}\pi]^2 ds > 2$$

Then every solution of (4.5.9) is oscillatory.

Now we shall discuss the oscillation of linear, non-homogenous n -th order delay differential equation of the form.

$$x^{(n)}(t) + p(t)x(g(t)) = h(t) \quad (4.5.10)$$

where $p(t), g(t), h(t)$ are continuous functions on $[t_0, \infty)$, $p(t) > 0, g(t) < t, g(t)$ is non-decreasing and $\lim_{t \rightarrow \infty} g(t) = \infty$.

Theorem 4.5.2. Assume that

a) There exists an oscillatory function $\rho(t)$, such that $\rho^{(n)}(t) = h(t)$ and

$$\lim_{t \rightarrow \infty} \rho^{(i)}(t) = 0 \text{ for } 0 \leq i \leq n-1;$$

b) For every $l \in \{0, 1, \dots, n-1\}$ such that $n+l$ is odd and for some

$$d_l \in \{0, 1, \dots, n-l-1\};$$

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t [u - g(t)]^{n-l-d_l-1} [g(t) - g(u)]^{d_l} [g(u)]^l p(u) du > (n-l-d_l-1)! d_l!.$$

Then every solution of (4.5.10) is oscillatory.

Proof: Let $x(t)$ be a non-oscillatory solution of (4.5.10) without loss of generality we may assume that $x(t)$ is eventually positive. Now consider the function $y(t) = x(t) - \rho(t)$. Then from (4.5.10), we obtain $y^{(n)}(t) = -p(t)x(g(t))$. So that $y^{(n)}(t)$ is eventually of one-sign for $t \geq t_0$. Thus the lower derivatives $y^{(i)}(t)$, $0 \leq i \leq n-1$, are monotone and one-signed for all sufficiently large t , say $t \geq t_0$. If $y(t) < 0$ for $t \geq t_0$, then $x(t) < \rho(t)$, $t \geq t_0$ which shows that $x(t)$ takes on negative values for arbitrarily large t . But this contradicts the assumption that $x(t) > 0$, and we must have $y(t) > 0$ for $t \geq t_0$. Now $y^{(n)}(t) < 0$ for all sufficiently Large t , and by using

lemma (4.5.1) , there exist $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \dots, n-1\}$ such that $n+l$ is odd and (4.5.2) , (4.5.3) hold .For sufficiently large $t_2 \in [t_1, \infty)$ in view of (4.5.3) we have

$$\begin{aligned} |y^{(l-i)}(t)| &\geq \frac{t-t_1}{1+i} |y^{(l-i)}(t)|, \quad i = 0, 1, \dots, l-1, \quad 1 \leq l \leq n-1 \\ \Rightarrow y(t) &\geq \frac{(t-t_1)}{l!} y^{(l)}(t), t \in [t_2, \infty) \\ \Rightarrow y(g(t)) &\geq \frac{(g(t)-t_1)^l}{l!} y^{(l)}(g(t)) \end{aligned} \quad (4.5.11)$$

From the equality

$$y^{(l+j)}(t) = \sum_{i=j}^{n-l-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(l+i)}(s) + \frac{1}{(n-l-j-1)!} \int_s^t (t-u)^{n-l-j-1} y^{(n)}(u) du$$

it follows that

$$\begin{aligned} y^{(l+j)}(t) &= \sum_{i=j}^{n-l-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(l+i)}(s) + \\ &\frac{(-1)^{n-l-j}}{(n-l-j-1)!} \int_t^s (u-t)^{n-l-j-1} y^{(n)}(u) du \end{aligned} \quad (4.5.12)$$

Now for $j = d_l$, $d_l \in \{0, 1, \dots, n-l-1\}$, and from (4.5.12) with regard to (4.5.2) we obtain

$$\left| y^{(l+d_l)}(g(t)) \right| \geq \frac{1}{(n-l-d_l-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_l-1} |y^{(n)}(u)| du \quad (4.5.13)$$

Also, from (4.5.12) for $u \in [g(t), t]$ and $d_l \in \{0, 1, \dots, n-l-1\}$, $j = 0$, we get

$$\left| y^{(l)}(g(u)) \right| \geq \frac{[g(t)-g(u)]^{d_l}}{d_l!} \left| y^{(l+d_l)}(g(t)) \right| \quad (4.5.14)$$

For $t \geq u \geq g(t)$ since $\left| y^{(l+d_l)}(t) \right|$ is decreasing and concave up for $d_l \in \{0, 1, \dots, n-l-1\}$. From (4.5.13) we obtain

$$|y^{(l+d_i)}(g(t))| \geq \frac{1}{(n-l-d_i-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_i-1} p(u)x(g(u))du \quad (4.5.15)$$

Since $y(t)$ is positive, increasing and $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$, therefore, we have $x(g(t)) \geq \lambda y(g(t))$, $t \geq t_0$, where λ is a positive constant such that $0 < \lambda < 1$.

Hence (4.5.15) becomes

$$|y^{(l+d_i)}(g(t))| \geq \frac{\lambda}{(n-l-d_i-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_i-1} p(u)y(g(u))du \quad (4.5.16)$$

Using (4.5.11) and (4.5.14) in (4.5.16) we get

$$\begin{aligned} |y^{(l+d_i)}(g(t))| &\geq \frac{\lambda}{(n-l-d_i-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_i-1} p(u) \times \frac{[g(u)-t_1]^l}{l!} y^{(l)}(g(u))du \geq \\ &\frac{\lambda}{(n-l-d_i-1)!} \int_{g(t)}^t [u-g(t)]^{n-l-d_i-1} \times \frac{[g(u)-t_1]^l}{l!} \frac{[g(t)-g(u)]^{d_i}}{d_i!} \times p(u) |y^{(l+d_i)}(g(t))| du \\ &\Rightarrow (n-l-d_i-1)! d_i! l! \geq \lambda \int_{g(t)}^t [u-g(t)]^{n+l-d_i-1} [g(u)-t_1]^l [g(t)-g(u)]^{d_i} p(u) du. \end{aligned}$$

So for sufficiently large t , we get a contradiction to (b). The proof is complete •

Example 4.5.2 Consider the equation

$$x'''(t) + e^{\frac{t}{2}-\pi} x(t-\pi) = 2e^{-t} \sin t \quad (4.5.17)$$

Now

$$h(t) = 2e^{-t} \sin t, \quad \rho(t) = -e^{-t} (\cos t + \sin t)$$

$$\rho^{(i)}(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad i = 0, 1, 2$$

Choose $d_i = 0$ for $l = 0, 2$ ($l+n$ is odd)

$$\limsup_{t \rightarrow \infty} \left[\int_{t-\pi}^t (u-t+\pi)^0 [t-\pi-u+\pi]^0 [u-\pi]^2 e^{\frac{u}{2}-\pi} du \right] =$$

$$\limsup_{t \rightarrow \infty} \left[\int_{t-\pi}^t (u-t+\pi)^0 (t-u)^0 (u-\pi)^2 e^{\frac{u}{2}-\pi} du \right] > 2 \text{ as } t \rightarrow \infty.$$

Then every solution of (4.5.17) is oscillatory.

Theorem 4.5.3. Assume that

(a) $\rho(t)$ is an oscillatory function such that

$$\rho^{(n)}(t) = h(t) \text{ and } \lim_{t \rightarrow \infty} \rho^{(i)}(t) = 0 \text{ for } 0 \leq i \leq n-1,$$

(b) the second-order linear differential equation

$$z''(t) + \frac{\lambda}{(n-1)!} [g(t) - T]^{n-2} p(t) z(g(t)) = 0 \text{ is oscillatory for some } 0 < \lambda < 1 \text{ and}$$

$$T > 0,$$

$$(c) \int_0^{\infty} [g(t)]^{n-2} p(t) dt = \infty.$$

(A) If n is odd, then every solution $x(t)$ of (4.5.10) is either oscillatory or

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0, \quad 0 \leq i \leq n-1.$$

(B) If n is even, then every solution $x(t)$ of (4.5.10) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (4.5.10). We may assume without loss of generality that $x(t)$ is eventually positive. Consider the function $x(t) = y(t) + \rho(t)$, then from (4.5.10) $y^{(n)}(t) = -p(t)x(g(t))$. So that $y^{(n)}(t)$ is eventually negative for $t \geq t_0$. Hence $y^{(i)}(t), 0 \leq i \leq n-1$, are monotone and one signed for all sufficiently large t , say, $t \geq t_0$. If $y(t) < 0$ for $t \geq t_0$, then $x(t) < \rho(t)$ for $t \geq t_0$, which shows that $x(t)$ have negative values for arbitrarily

large t . But this contradicts the assumption that $x(t) > 0$, so we must have $y(t) > 0$ for $t \geq t_0$. Hence for sufficiently large t , $y(t)y^{(n)}(t) < 0$ and by using lemma 4.5.1, there exists $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \dots, n-1\}$ such that $n-l$ odd and (4.5.2) holds.

Suppose $l > 0$ and $n-l$ odd. Then by applying Taylor's formula with remainder applied to $y^{(l)}(t)$, we get

$$\begin{aligned} y^{(l)}(t) &= \sum_{j=0}^{n-l-1} \frac{y^{(l+j)}(\tau)}{j!} (t-\tau)^j + \frac{1}{(n-l-1)!} \int_{\tau}^t (t-s)^{n-l-1} y^{(n)}(s) ds \\ &= \sum_{j=0}^{n-l-1} (-1)^j \frac{y^{(l+j)}(\tau)}{j!} (\tau-t)^j + \frac{1}{(n-l-1)!} \int_t^{\tau} (s-t)^{n-l-1} p(s)x(g(s)) ds \end{aligned}$$

Now using (4.5.2), we get

$$y^{(l)}(t) \geq \int_t^{\tau} \frac{(s-t)^{n-l-1}}{(n-l-1)!} p(s)x(g(s)) ds \quad \text{for } T \leq t \leq \tau.$$

As $\tau \rightarrow \infty$, we obtain

$$y^{(l)}(t) \geq \int_t^{\infty} \frac{(s-t)^{n-l-1}}{(n-l-1)!} p(s)x(g(s)) ds, \quad t \geq T. \tag{4.5.18}$$

Integrating (4.5.18) from T to t , it follows that

$$\begin{aligned} y^{(l-1)}(t) &\geq y^{(l-1)}(T) + \frac{1}{(n-l-1)!} \int_T^t \left[\int_s^{\infty} (r-s)^{n-l-1} p(r)x(g(r)) dr \right] ds \\ &= y^{(l-1)}(T) + \frac{1}{(n-l-1)!} \int_T^t \left[\int_T^r (r-s)^{n-l-1} ds \right] p(r)x(g(r)) dr + \\ &\quad \frac{1}{(n-l-1)!} \int_T^t \left[\int_T^t (r-s)^{n-l-1} ds \right] p(r)x(g(r)) dr. \end{aligned}$$

for $t \geq T$. Hence, by virtue of the inequality

$\int_T^t (r-s)^{n-l-1} ds \geq \frac{1}{(n-l)}(t-T)(r-T)^{n-l-1}, T \leq t \leq r$, we obtain

$$y^{(l-1)}(t) \geq y^{(l-1)}(T) + \int_T^t \frac{(r-T)^{n-l}}{(n-l)!} p(r)x(g(r))dr + (t-T) \int_t^\infty \frac{(r-T)^{n-l-1}}{(n-l)!} p(r)x(g(r))dr,$$

$$t \geq T. \tag{4.5.19}$$

Let us denote the right hand side of (4.5.19) by $z(t)$. And see that $z(t)$ is positive and satisfies

$$z''(t) + \frac{(t-T)^{n-l-1}}{(n-l)!} p(t)x(g(t)) = 0 \quad t \geq T. \tag{4.5.20}$$

Recall that $x(g(t)) = y(g(t)) + \rho(g(t))$. Since $y(t)$ is positive, increasing and

$\rho(t) \rightarrow 0$ as $t \rightarrow \infty$, hence for large enough T , we have

$$x(g(t)) \geq \lambda y(g(t)), \quad t \geq T \tag{4.5.21}$$

where λ is the constant appearing in (b). Moreover $y(t)$ satisfies the inequality

$$y(t) \geq \frac{(t-T)^{l-1} y^{(l-1)}(t)}{l!}, \quad t \geq T$$

So that for all large t , we have

$$y(g(t)) \geq \frac{1}{l!} (g(t)-T)^{l-1} y^{(l-1)}(g(t)) \geq \frac{1}{l!} (g(t)-T)^{l-1} z(g(t)) \tag{4.5.22}$$

Using (4.5.20),(4.5.21),(4.5.22), we obtain

$$z''(t) + \frac{\lambda}{(n-l)!} (g(t)-T)^{n-2} p(t)z(g(t)) \leq 0, \quad t \geq T \tag{4.5.23}$$

We see that the second -order equation (b) has an eventually positive solution. This, however contradicts the hypothesis of the theorem, so that the integer l associated with $y(t)$ must be zero. Note that the oscillation of (b) implies

$\int_0^{\infty} [g(t)]^{n-2} p(t) dt = \infty$, and if we take (a) into account, we see that

$\lim_{t \rightarrow \infty} |x^{(i)}(t)| = 0$ for $0 \leq i \leq n-1$ if $l = 0$. Also note that $l = 0$ is possible only if n

is odd. This completes the proof of the theorem •

Example 4.5.3. Consider the following delay differential equation

$$x^{(iv)}(t) + e^{\frac{t}{2}-\pi} x(t-\pi) = e^{-\frac{t}{2}} [1 - 4e^{-\frac{t}{2}}] \cos t \quad (4.5.24)$$

Now

$$h(t) = e^{-\frac{t}{2}} \cos t - 4e^{-t} \cos t$$

$$\rho^{(iv)}(t) = h(t)$$

$$\rho(t) = e^{-t} \cos t + \frac{112}{625} e^{-\frac{t}{2}} \cos t - \frac{384}{625} e^{-\frac{t}{2}} \sin t.$$

$$\int_0^{\infty} [g(t)]^{n-2} p(t) dt = \int_0^{\infty} (t-\pi)^2 e^{\frac{t}{2}-\pi} dt = e^{\frac{t}{2}-\pi} (t-\pi) - e^{\frac{t}{2}-\pi} = \infty, \text{ as } t \rightarrow \infty.$$

The conditions (a), (b) and (c) of theorem 4.5.3 are satisfied. And the solution

$x(t) = e^{-t} \cos t$ of (4.5.24) is oscillatory.