Coefficient Estimates of Univalent Functions

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Dedication

To my wounded home land. To our martyrs whose blood watered our thirsty land. To all people loving peace, freedom and justice around the world. To my parent's who stood and stand beside me and back me in each step of my life one word success.

Nofan Majed Alsaid Salah –Aldein.

Declaration:

I Certify that this submitted for the degree of Master is the result of my own research, except where otherwise acknowledge, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed……………………

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Abstract

Let S be the set of all normalized univalent function f in the open unit disk Δ with $f(z) = z + a_2 z^2 + \ldots$. We study the Biberbach's conjecture $|a_n| \leq n$, $\forall n \in N$ and the coefficient estimate of several subfamilies of S.

The Biberbach's conjecture about the coefficients of univalent functions of the unit disk was formulated by Luding Biberbach in 1916. The Biberbach's conjecture was quite a difficult problem, and it was proved by Louis de Branges in 1984, when some mathematician was rather trying to disprove it.

In this survey, we describe the historical development of the conjecture and the main ideas that led to the proof.

الخلاصة

افرض أن S هي مجموعة الإقترانات الأحادية والمعرفة على قرص الوحدة Δ على الشكل التالي $f(z) = z + a_2 z^2 + ...$

سنقوم بدراسة المعضلة التي طرحها العالم بايبرخ حول معاملات الإقترانات في المجموعة S , أي أن $|a_n| \le n, \ \forall n \in N$ وكذلك سندرس التقديرات لهذه المعاملات لبعض المجموعات الجزئية من S. .

إن معضلة بايبراخ وجدت عام 1916 وكانت من المسائل المعقدة التي شغلت كثير من العلماء حيث حاولوا كثيرا أن يثبتوا عكسها ولكن بدون نتيجة حتى وجد العالم ديبرانج طريقا لحل هذه المعضلة عام 1984.

في هذه الرسالة, سوف نسرد التطور التاريخي لهذه المعضلة والمحاولات لإثباتها ومدى صحتها على بعض المجموعات الجزئية من S وكذلك سنتعرض للأفكار الأساسية التي أدت إلى برهانها.

Chapter 1

Elementary Theory of Univalent Functions

In this chapter we present basic results about univalent functions. These include the elementary distortion theorems and the coefficient estimate $|a_n| \leq en$, for the functions f which are analytic and univalent in the unit disk and normalized by $f(0)=0$ and $f'(0)=1$, which is the class S of univalent functions.

A number of basic questions are answered by an elementary method. Most of the elementary results concerning the class S are direct consequences of the area theorem, which may be regarded as the cornerstone of the entire subject.

1.1 Introduction:

Definition 1.1.1:

The function defined on a set D is called a univalent (or Schlicht) in D if it is one-to-one in D.

Definition 1.1.2:

A complex function f is analytic at a point z_0 if it is differentiable at every point in some neighborhood of z_0.

We are interested in functions which are analytic and univalent in a domain D and we will concentrate on the case where D the open unit disk is $\Delta = \{z \in C : |z| < 1\}$.

The choice of Δ as the domain of the function is critical in terms of both the kind of problems considered and the techniques used. For example, the only functions that are analytic and univalent in the complex plane C are the form $f(z) = az + b$ with $a \neq 0$.

Suppose that f is analytic and univalent in Δ, the local behavior of an analytic function implies that $f'(z) \neq 0$ when $|z| < 1$ and so, in particular, $f'(0) \neq 0$. Let g be defined by $g(z) = \left[f(z) - f(0)\right] / f'(0)$; then g is analytic and univalent in Δ, $g(0) = 0$

and $g'(0)=1$. Properties of the function f correspond to properties of the function g and vice-versa and so we generally need to consider functions with the normalization $f(0)=0$ and $f'(0)=1$.

Definition 1.1.3:

Let S denote the set of functions that are analytic and univalent in the unit disk $\Delta=\{z\in C:|z|<1\}$ and satisfy the conditions $f(0)=0$ and $f'(0)=1$.

Thus each $f\in S$ has a Taylor series expansion of the form:

$$f(z)=z+a_2z^2+a_3z^3+.....\qquad |z|<1. \tag{1.1}$$

as $a_0=f(0)=0$ and $a_1=f'(0)=1$.

Example1.1.1:

The function $k(z)=\dfrac{z}{(1-z)^2}$, is analytic and univalent in Δ, the Taylor series expansion of k is

$$k(z)=z+2z^2+3z^3+...+nz^n+...$$

This function is known as (Koebe function).

The Koebe function maps the disk Δ onto the entire plane minus the part of the negative real axis from $-\frac{1}{4}$ to minus infinity. This is best seen by writing:

$$k(z)=\frac{1}{4}\left(\frac{1+z}{1-z}\right)^2-\frac{1}{4}.$$

We introduce a family of univalent functions where $D=\{z:0<|z|<1\}$ is the domain of definition. If f is analytic and univalent in D and $z=0$ is not a removable singularity, the singularity of f at $z=0$ must be a simple pole. There is no loss of generality in assuming that the residue at $z=0$ equals 1, since this may be achieved by multiplication by a constant. Thus, we come to the following definition.

Definition 1.1.3:

Let Σ denote the set of functions that are analytic and univalent in $D = \{z : 0 < |z| < 1\}$ and have a simple pole at $z = 0$ with the residue 1. The Laurent series expansion of a function in Σ has the form:

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n , \qquad (0 < |z| < 1) \qquad (1.2)$$

We shall study properties of functions in S, Σ and special subsets of S. The next result is of fundamental importance in the initial development of these facts.

1.2 The area Theorem:

The univalence of a function:

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n , \qquad (0 < |z| < 1)$$

places a strong restriction on the size of the Laurent coefficients b_n, $n = 1,2,...$ this is expressed by the area theorem, which is fundamental to the theory of univalent functions. The reason for the name will be apparent from the proof. Gronwall discovered the theorem in 1914.

Theorem 1.2.1(Area Theorem):

If $g \in \Sigma$ and g has the representation (1.2) then:

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

Proof: See [28, pages 2, 3].

Without knowing Gronwall's work, in 1916 Bieberbach proved the same relation, and received the first coefficient result within S.

Corollary 1.2.1:

If $g \in \Sigma$ with

$$g(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots\ldots + b_n z^n + \ldots, \qquad \text{then} \qquad |b_1| \le 1.$$

Moreover, if $|b_1| = 1$, then $g(z) = \frac{1}{z} + b_0 + b_1 z.$

and any function of this form belongs to Σ when $|b_1| \le 1$.

Proof:

If $g \in \Sigma$, then by the area theorem

$$\sum_{n=1}^{\infty} n|b_n|^2 \le 1.$$

But $|b_1|^2 \le \sum_{n=1}^{\infty} n|b_n|^2$, which implies that $|b_1| \le 1$.

Also, if $|b_1| = 1$ then $b_n = 0$ for $n = 2,3,....$ that is, g has the form

$$g(z) = \frac{1}{z} + b_0 + b_1 z.$$

It is easy to verify that such a function is univalent in $\{z : 0 < |z| < 1\}$ if $|b_1| \le 1$.

Lemma 1.2.1:

If $f \in S$ and $g(z) = \left[f(z^n)\right]^{1/n}$ then $g \in S$ for $n = 2,3,....$

Proof:

If $f \in S$ then $f(z) \ne 0$ for $z \ne 0$ and thus $h(z) = \left[f(z^n)/z^n\right]^{1/n}$ is analytic in Δ. Moreover, if x is any n^th^ root of 1, then $h(xz) = h(z)$. Thus $g(z) = zh(z)$ is analytic in Δ, $g(0) = 0, g'(0) = 1$ and $g(xz) = xg(z)$. Suppose that $|z_1| < 1, |z_2| < 1$ and $g(z_1) = g(z_2)$. Then, $g^n(z_1) = g^n(z_2)$, $f(z_1^n) = f(z_2^n)$ and so $z_1^n = z_2^n$,that is, $z_2 = yz_1$ with $y^n = 1$. Since $g(yz_1) = yg(z_1)$, we conclude that $y = 1$. Therefore, $z_2 = z_1$ and g is univalent in Δ.

Definition 1.2.1:

Every odd function in S is the square-root transform of some function in S. And the set of all odd functions in S is denoted by S^2.

Example1.2.1:

If $f \in S$ and $g(z) = \sqrt{f(z^2)}$ then g is odd function.

Solution:

If $f \in S$, then f has a Taylor series of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

$$\Rightarrow \qquad f(z^2) = z^2 + a_2 z^4 + a_3 z^6 + ...$$

But $$g(z) = \sqrt{f(z^2)} = \left[z^2 + a_2 z^4 + a_3 z^6 + ...\right]^{1/2}$$

By the last lemma $g \in S$, so g has the series of the form:

$$g(z) = z + c_2 z^2 + c_3 z^3 + c_4 z^4 + ...$$

$$\Rightarrow \quad f(z^2) = z^2 + a_2 z^4 + a_3 z^6 + ... = (g(z))^2 = (z + c_2 z^2 + c_3 z^3 + ...)^2$$

$$\Rightarrow \quad (z^2 + a_2 z^4 + a_3 z^6 + ...) = (z + c_2 z^2 + c_3 z^3 + ...)^2$$

$$\Rightarrow (z^2 + a_2 z^4 + a_3 z^6 + ...) = \begin{pmatrix} z^2 + c_2 z^3 + c_3 z^4 + ... \\ + c_2 z^3 + c_2^2 z^4 + c_2 c_3 z^5 + ... \\ + c_3 z^4 + c_3 c_2 z^5 + c_3^2 z^6 + ... \\ \\ + c_n z^{n+1} + c_n c_{n-1} z^{n+2} + ... + c_n^2 z^{2n} + ... \end{pmatrix}$$

$$= z^2 + 2c_2 z^3 + (c_2^2 + 2c_3) z^4 + ...$$

By comparing the coefficient we can get

$$c_n = 0 \quad , \forall n = 2,4,6,.......$$

Then g has the series of the form:

$$g(z) = z + c_3 z^3 + c_5 z^5 + ... \qquad \text{(Odd function)}$$

And a simple calculation gives

$$c_3 = \frac{a_2}{2} \ , \qquad c_5 = \frac{1}{2}(a_3 - \frac{1}{4}a_2^2)$$

From the corollary (1.2.1) it is the short step to a theorem of Bieberbach estimating the second coefficient a_2 of a function of class S .

Theorem 1.2.2(Bieberbach's Theorem):

If $f \in S$ and f has a Taylor series of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \text{ , then} |a_2| \le 2 .$$

The only functions in S with $|a_2| = 2$ are given by $f(z) = \dfrac{z}{(1 - xz)^2}$ where $|x| = 1$.

Proof:

Suppose that $g(z) = \sqrt{f(z^2)}$

$\Rightarrow \qquad g(z) = z + c_3 z^3 + c_5 z^5 + \ldots$

then $g(z) \in S$ (by lemma 1.2.1).

Let $h(z) = \dfrac{1}{g(z)} \qquad ,0 < |z| < 1$.

$\Rightarrow \quad h(z) = \dfrac{1}{z} - c_3 z + (c_3^2 - c_5) z^3 + \ldots$ and $h(z) \in \Sigma$.

But the Laurent series expansion of a function in Σ has the form:

$$g(z) = \frac{1}{z} + b_0 + b_1 z + \ldots$$

So, if we compare the coefficient of $h(z)$ with $g(z)$, we see that

$$b_0 = 0$$
$$b_1 = -c_3 = -\frac{1}{2} a_2$$

Using the corollary (1.2.1), $|b_1| \le 1$, therefore

$$\left| -\frac{1}{2} a_2 \right| \le 1 \qquad \Rightarrow \qquad |a_2| \le 2 .$$

Also the equality $|a_2| = 2$ requires that $h(z) = \dfrac{1}{z} + b_1 z$ with $|b_1| = 1$,which is the same as $f(z) = z/(1 + b_1 z)^2$.

To do this, let

$h(z)=(1+b_1z^2)/z$

But $g(z)=\dfrac{1}{h(z)}$, then $g(z)=\dfrac{z}{1+b_1z^2}$

But $g(z)=\sqrt{f(z^2)}$, so $(g(z))^2 = f(z^2)$

$\Rightarrow \quad f(z)=(g(\sqrt{z}))^2$

$$\Rightarrow \quad f(z)=\left(\frac{\sqrt{z}}{1+b_1z}\right)^2$$

$$=\frac{z}{(1+b_1z)^2}$$

We have seen that each function:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n = z+a_2z^2+a_3z^3+.....$$

of class S has the property $|a_2|\le 2$ with equality occurring only for rotations of the Koebe function:

$$k(z)=z(1-z)^{-2}=\sum_{n=1}^{\infty}nz^n\ .$$

The suggestion of the general problem to find:

$$A_n=\sup_{f\in S}|a_n| \qquad n=2,3.4...$$

Because the Koebe function plays the external role in so many problems for the class S, it is natural to suspect that it maximizes $|a_n|$ for every n. This is the famous conjecture of Bieberbach, first proposed in 1916.

Bieberbach Conjecture:

The coefficients of each function $f\in S$ satisfy $|a_n|\le n$ for $n=2,3.4...$ strict inequality holds for all n unless f is the Koebe function or one of its rotations.

For many years this conjecture has stood as a challenge to all mathematicians and has inspired the development of important new methods in complex analysis.

As a first application of Bieberbach's theorem, we shall now prove a famous covering theorem due to koebe. Each function $f\in S$ is an open mapping with $f(0)=0$,so its range contains some disk centered at the origin. As early as 1907, Koebe discovered that the ranges of all functions in S contain a common disk $|w| < \rho$, where ρ is an absolute constant. The "Koebe function" shows that $\rho \le \frac{1}{4}$, and Bieberbach later established Koebe's conjecture that ρ may be taken to be $\frac{1}{4}$.

Theorem 1.2.3(Koebe Covering Theorem):

The range of every function of class S contains the disk $\left\{ w : |w| < \frac{1}{4} \right\}$

Proof:

Suppose that $f(z) \ne c$ for $|z| < 1$.

The function $g = cf/(c - f) \in S$, and so has the power series beginning

$$g(z) = z + (a_2 + \frac{1}{c})z^2 + (\frac{2a_2}{c} + a_3 + \frac{1}{c^2})z^3 + \ldots$$

Bieberbach's theorem implies that

$$\left| a_2 + \frac{1}{c} \right| \le 2, \text{ and } |a_2| \le 2$$

Therefore,

$$2 \ge \left| a_2 + \frac{1}{c} \right| = \left| \frac{1}{c} - (-a_2) \right| \ge \left| \frac{1}{c} \right| - |a_2|$$

$$\Rightarrow \qquad \frac{1}{|c|} \le |a_2| + 2 \le 2 + 2 = 4$$

$$\Rightarrow \qquad |c| \ge \frac{1}{4}.$$

The proof actually shows that the koebe function and its rotations are the only functions in S which omit a value of modulus $1/4$. Thus the range of every other function in S covers a disk of larger radius.

Theorem 1.2.4(Distortion Theorem):

If $f \in S$ and $|z| < 1$ then:

(1) $$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

(2) $$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}.$$

(3) $$\frac{1-|z|}{|z|(1+|z|)} \le \left|\frac{f'(z)}{f(z)}\right| \le \frac{1+|z|}{|z|(1-|z|)}.$$

Proof: See [29, pages 4, 5].

1.3 Prawitz Inequality and Littelwood's Theorem:

Theorem 1.3.1(Littelwood Theorem):

The coefficients of each function $f \in S$ satisfy $|a_n| \le en$ for $n = 2,3,\dots$.

Littlewood's theorem appeared in 1925. Since that time the constant e has been replaced by a succession of smaller constants, at the sacrifice of simplicity and elegance. Littlewood's proof is entirely elementary. It rests upon an estimate for integral means, which has independent interests, with the notation:

$$M_p(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p} \quad , \quad 0 < p < \infty \ .$$

This may be stated as follows.

Lemma 1.3.1 (Parawitz Inequality):

For each function $f \in S$,

$$M_1(r, f) \le \frac{r}{1-r} \qquad , \qquad 0 \le r < 1.$$

Proof of theorem (1.3.1):

Littlewood's theorem is an easy consequence of (lemma 1.3.1). The Cauchy's formula for a_n gives:

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz \qquad , \qquad 0 < r < 1.$$

Namely,

$$|a_n| = \left| \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz \right|$$

$$\leq \frac{1}{|2\pi i|}\int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}}\,|dz|.$$

Let $z = re^{i\theta}$ then $dz = rie^{i\theta}d\theta$ and $|dz| = rd\theta$.

$$\Rightarrow \quad |a_n| \leq \frac{1}{2\pi r^n}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

$$\leq \frac{1}{r^n}\frac{r}{1-r} = \frac{1}{r^{n-1}(1-r)}.$$

If $n \geq 2$, we may choose $r = 1 - \frac{1}{n}$ and thereby obtain:

$$|a_n| \leq \frac{1}{(1-\frac{1}{n})^{n-1}(1-1+\frac{1}{n})}.$$

$$= \frac{n}{\left(\frac{n-1}{n}\right)^{n-1}}.$$

$$= \left(\frac{n}{n-1}\right)^{n-1}.n.$$

$$= \left(1+\frac{1}{n-1}\right)^{n-1}.n < en$$

$$\Rightarrow |a_n| \leq (1+\frac{1}{n-1})^{n-1}.n < en.$$

This proves the theorem.

The crude estimate $|a_n| \le r^{-n} M_1(r, f)$ can lead to nothing better than $|a_n| < (e/2)n$, because the Koebe function has integral mean:

$$M_1(r,k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r}{1-2r\cos\theta + r^2} d\theta = \frac{r}{1-r^2} .$$

The sharp bound for $M_1(r, f)$ remained undetermined until 1973, when Baernstein was able to prove that $M_p(r, f) \le M_p(r,k)$ for all $f \in S$ and for $0 < p < \infty$. In particular, this proves $|a_n| < (e/2)n$.

Chapter 2

Special Families, Loewner' s method and Grunsky Inequalities

In this chapter we introduce the subclasses of S which are defined by natural geometric conditions. Among other things, we shall prove the Bieberbach conjecture for each of these subclasses. Also, we introduce Loewner's method to proof the third coefficient of class S, and the Grunsky inequalities to proof the fourth coefficient of class S.

2.1 Starlike Function.

Definition 2.1.1:

A set E is said to be starlike with respect to $w_0 \in E$ provided that

$w_0 + t(w - w_0) \in E$ whenever $w \in E$ and $0 \le t \le 1$.

Definition 2.1.2:

A function f belongs to S is called starlike if the image of f is starlike with respect to 0.

Definition 2.1.3 :

Let S^* be the set of all starlike function in S. In other words,

$S^* = \{f \in S : f(\Delta)$ is starlike with respect to $0\}$.

Example 2.1.1:

The function $f(z) = z$ is in S^*.

Definition 2.1.4:"Positive real part function ".

Let P denote the set of all functions p that are analytic in Δ and satisfy

Re $p(z) > 0$, $\quad |z| < 1$ and $p(0) = 1$.

This family is related to S*and to several other classes of univalent functions.

The following result provides the sharp coefficient estimate on functions in P.

This result in turn will be used to obtain coefficient estimates for function in S*.
An elementary estimate which goes back to Caratheodory [Caratheodory 1907], which is valid for function with positive real part.

Theorem 2.1.1 "Caratheodory theorem":

Let $p \in$ P and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (|z| < 1)$ then $|p_n| \le 2$ for $n = 1,2,...$

Proof:

The function $\phi = \dfrac{(p-1)}{(p+1)}$ is analytic in Δ, $|\phi(z)| < 1$, $\quad (|z| < 1)$ and

$$\phi(z) = \frac{1 + \sum_{n=1}^{\infty} p_n z^n - 1}{1 + \sum_{n=1}^{\infty} p_n z^n + 1} = \frac{p_1 z + p_2 z^2 + ...}{2 + p_1 z + p_2 z^2 + ...} \text{ , so } \phi(0) = 0 .$$

Therefore $\phi(z) = zw(z)$ where w is analytic in Δ, but ϕ is analytic in $\Delta, \phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \Delta$, by using Schwarz lemma,

$|\phi(z)| = |z||w(z)| \le |z|$, then $|w(z)| \le 1$. This implies that $|w(0)| \le 1$. And since

$$w(z) = \frac{\phi(z)}{z} = \frac{p(z) - 1}{z(p(z)+1)} = \frac{p_1 z + p_2 z^2 + ...}{2z + p_1 z^2 + p_2 z^3 + ...}$$

$$= \frac{p_1 + p_2 z + ...}{2 + p_1 z + p_2 z^2 + ...}$$

Then,

$w(0)=\frac{p_1}{2}$, $p_1 = 2w(0)$ which implies that $|p_1| = |2w(0)| \le 2$ is proved for $n = 1$.

The only functions with $|p_1| = 2$ are given by $p(z) = \frac{(1+xz)}{(1-xz)}$ where $|x| = 1$.

Now suppose that $n \ge 2$ and $y = e^{2\pi i/n}$, the function q defined by

$q(z) = \frac{1}{n}[p(z) + p(yz) + p(y^2 z) + ... + p(y^{n-1}z)]$ belongs to P and

$q(0) = \frac{1}{n}[1+1+1+...........] = \frac{n}{n} = 1$, $\operatorname{Re} q(z) > 0$.

Also, q has the form $q(z) = s(z^n)$ where s is analytic in Δ, since

$q(yz) = \frac{1}{n}[p(yz) + p(y^2 z) + p(y^3 z) + ... + p(y^n z)] = q(z)$

If we write $s(z) = 1 + s_1 z + s_2 z^2 + ...$ then

$s_1 = p_n$ Since $s \in$ P the inequality $|s_1| \le 2 \to |p_n| \le 2$.

The only function with $|p_n| = 2$ are given by $p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n$.

We now turn to the Biberbach conjecture for the class S^* of starlike functions.

The proof of Bieberbach conjecture for starlike functions is given by an analytic representation. First one proves with the aid of Schwarz' lemma that the starlikeness of $f(\Delta)$ implies the starlikeness of $f(\Delta_r)$ where $\Delta_r := \{z : |z| < r\}$ for all $r \in (0,1)$.

Next, by geometrical considerations it follows that $|\arg zf'(z) - \arg f(z)| < \frac{\pi}{2}$, hence

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \Delta).$$

Caratheodory' s theorem for functions with positive real part yields the result. Again, One can prove that above equality implies the univalence and starlikeness of f .

Lemma 2.1.1: "Schwarz's theorem".

If f is analytic on Δ with $f(0)=0$ and $|f(z)|<1$ for all $z\in\Delta$ then $|f'(0)|\le 1$ and $|f(z)|\le|z|$ In Δ, also $|f'(0)|=1$ and $|f(z)|=|z|$ if $f(z)=e^{i\theta}z$ for some θ.

Proof: See [12, page 3].

Lemma 2.1.2:

If $f\in S^*$ and $0<r<1$ then $f(\{z:|z|<r\})$ is starlike with respect to 0.

Proof: See [28, page 11].

The following theorem gives an analytic description of starlike functions.

Theorem 2.1.2:

Let f be analytic in Δ, with $f(0)=0$ and $f'(0)=1$. Then $f\in S^*$ if and only if $zf'(z)/f(z)\in P$.

Proof: See [12, pages 41, 42].

Example 2.1.2:

The koebe function $k(z)=\dfrac{z}{(1-z)^2}$ is in S^*.

By theorem (2.1.2) it is enough to show that $\operatorname{Re}\left(\dfrac{zf'(z)}{f(z)}\right)>0$.

The next step towards the Bieberbach conjecture was first proved by R. Nevanlinna in 1920. He considered univalent functions with an image domain that is starlike with respect to the origin.

Theorem 2.1.3 "Nevanlinna theorem":

Let $f\in S^*$ and $f(z)=z+\sum_{n=2}^{\infty}a_n z^n \quad (|z|<1)$ then $|a_n|\le n$ for $n=2,3,\ldots\ldots\ldots$

Proof:

If $f \in S^*$ then $p \in P$ where $p(z) = \dfrac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (|z| < 1)$

By equating coefficient of the power series in the relation $zf'(z) = f(z)p(z)$, where

$zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n = z + 2a_2 z^2 + 3a_3 z^3 + ...$ and

$$f(z)p(z) = (1 + p_1 z + p_2 z^2 + ...)(z + a_2 z^2 + a_3 z^3 + ...)$$

$$= z + (a_2 + p_1)z^2 + (a_3 + p_1 a_2 + p_2)z^3 + ...$$

We conclude that

$2a_2 = a_2 + p_1$, so $a_2 = p_1$ and, in general, for $n \geq 2$,

$(n-1)a_n = a_{n-1}p_1 + a_{n-2}p_2 + ... + a_2 p_{n-2} + p_{n-1}$.

With $n = 1$, Caratheodory theorem implies that $|a_2| \leq 2$.

Caratheodory theorem and an inductive argument complete the proof, since

$(n-1)|a_n| \leq (n-1)2 + (n-2)2 + ... + 2.2 + 2 = (n-1)n$

Hence $|a_n| \leq n$.

We also note that if $|a_n| = n$ for a given n the argument shows that $|a_2| = 2$ from which we conclude that $f(z) = z/(1 - xz)^2 \qquad (|x| = 1)$.

From Nevanlinna's result, it became clear that there was some chance for the Bieberbach conjecture to be true since it was true for a rather large subset of S. Nevertheless, it was not yet known whether the Bieberbach conjecture was true for a single $n > 2$. This step was finished by K.loewner, 1923, again. Maybe Loewner's article was the most important step towards the proof of the Bieberbach Conjecture; at least it was the first decisive one.

2.2 Convex Function.

Definition 2.2.1:

The set E is said to be convex if it is starlike with respect to each of its points; that is, if the linear segment joining any two points of E lies entirely in E.

Definition 2.2.2:

A function f is convex on Δ if and only if $f(\Delta)$ is convex.

Definition 2.2.3:

Let C be the set of all functions which are convex on Δ, in the other words,

$$C = \{ f \in S : f(\Delta) \text{ is convex}\}$$

It is clear that every convex function is starlike function, and so $C \subset S^*$.

Theorem 2.2.1:

Let f be analytic in Δ, with $f(0)=0$ and $f'(0)=1$. then $f \in C$ if and only if $[1+zf''(z)/f'(z)] \in \mathrm{P}$.

Proof: See [12, pages 42, 43].

The next result provides a useful correspondence between the classes S^* and C.

Theorem 2.2.2:(Alexander's Theorem).

Let f be analytic in Δ, with $f(0)=0$ and $f'(0)=1$, then $f \in C$ if and only if $g(z) = zf'(z) \in S^*$.

Proof: See [28, page 15].

Theorem 2.2.3:

If $f \in C$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|z|<1)$, then $|a_n| \le 1, \qquad n = 2,3,\ldots$.

Proof:

By Alexander's Theorem, $g \in S^*$ where

$$g(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n. \quad (|z| < 1).$$

Nevanlinna theorem implies that $|na_n| \le n$ and thus $|a_n| \le 1$. Moreover, it is easy to see that $|a_n| = 1$ for a given n only for functions of the form $f(z) = z/(1 - xz)$ where $|x| = 1$.

2.3 Loewner theorem:

Definition 2.3.1:

A Jordan arc is an arc without self-intersections.

Definition 2.3.2:

A slit mapping is a function which maps a domain conformally onto the complex plane minus a set of Jordan arcs.

Definition 2.3.3:

A single slit mapping is a slit mapping whose range is the complement of a single Jordan arc.

Definition 2.3.4:

The class S_1 is dense in S if S_1 is subclass of S and if every function $f(z)$ in S can be approximated by a sequence of functions $(f_n(z))$ in S_1 so that $f_n(z) \to f(z)$ uniformaly on every compact subset of Δ as $n \to \infty$.

The point of departure for Loewner's method is the observation that the single-slit mappings are dense in the class S. In other words, each function in S can be approximated uniformly on compact subsets of Δ by single-slit mapping. The following theorem states this more precisely.

Theorem 2.3.1:

To each function $f \in \mathrm{S}$ there corresponds a sequence of single-slit mappings $f_n \in \mathrm{S}$ such that $f_n \to f$ uniformly on each compact subset of Δ.

Proof: See [12, pages 80, 81].

Theorem 2.3.2: (Loewner theory).

Let $k(t)$ be measurable and complex valued for $0 \le t < \infty$ and satisfy $|k(t)| \le 1$.

Then if $|z| <1$, there exists a unique function $w = f(z,t)$ absolutely continuous in t for $0 \le t < \infty$, and satisfying for almost all t Loewner's differential equation

$$\frac{\partial f}{\partial t} = -f\,\frac{1+k(t)f}{1-k(t)f}, \tag{2.1}$$

with the initial condition $f(z,0) = z$. Also

$$g(z,t) = e^t f(z,t) \in S \;,\;\; 0 \le t < \infty \tag{2.2}$$

Finally there is a dense subclass S_1 of S such that if $f(z) \in S_1$, there exists $k(t)$ continuous and with $|k(t)| = 1$, such that the associated function $g(z,t)$ satisfies

$$f(z) = \lim_{t\to\infty} g(z,t) = \lim_{t\to\infty} e^t f(z,t), \quad |z| <1 \tag{2.3}$$

and the convergence is uniform on each compact subset of Δ.

Proof: See [29, pages 197-215].

Loewner gave an analytic representation for a subset of S which is dense with respect to the topology of locally uniform convergence. Loewner's theorem states that for every f of this dense subset of S there is a Loewner chain, i.e. a family of functions $\{f(z,t) : t \ge 0\}$ with

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \qquad (z \in \Delta, t \ge 0, a_n(t) \in C(n \ge 2)),$$

This starts with f

$$f(z,0) = f(z),$$

Theorem 2.3.3:

If $f(z) \in S_1$ so that $w = f(z) = \beta(z + a_2 z^2 + ...)$ with $\beta = 1$, if further

$g_t(z) = \beta e^t (z + a_2(t) z^2 + ...)$ where $\beta = \beta(0) = f'(0)$. And

$f(z,t) = g_t^{-1}[f(z)], \quad 0 \le t < \infty$.

Then

$w = f(z,t)$ satisfies the Loewner's differential equation (2.1) and $f(z,0) = z$. Also $g(z,t) = e^t f(z,t)$ satisfies (2.2) and (2.3).

Proof: See [29, pages 210,211].

2.4 The Third Coefficient.

As a first application of Loewner's method, we now propose to prove the Bieberbach conjecture for the third coefficient. The problem is to show that $|a_3| \le 3$ for all functions: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

of class S. Because S is preserved under rotation, it is equivalent to show that $\operatorname{Re}\{a_3\} \le 3$. The Loewner theory reduces the problem to functions of the form

$$f(z) = \lim_{t\to\infty} g(z,t) = \lim_{t\to\infty} e^t f(z,t),$$

where $f(z,t)$ is the solution to some Loewner differential equation

$$\frac{\partial f}{\partial t} = -f \frac{1+kf}{1-kf}, \qquad f(z,0) = z,$$

corresponding to a continuous function k of unit modulus. As before, let $f(z,t) = e^{-t}\left(z + a_2(t) z^2 + a_3(t) z^3 + ...\right)$.

Then it is clear that $a_n(0) = 0$ and

$$\lim_{t\to\infty} a_n(t) = a_n, \qquad n = 2,3,......$$

$$\frac{\partial f}{\partial t} = -f\,\frac{1+kf}{1-kf} = (-e^{-t}z - a_2(t)e^{-t}z^2 - \ldots)\left(\frac{1+k(t)e^{-t}z + a_2(t)k(t)e^{-t}z^2 + \ldots}{1-k(t)e^{-t}z - a_2(t)k(t)e^{-t}z^2 - \ldots}\right)$$

$$= (-e^{-t}z - a_2(t)e^{-t}z^2 - a_3(t)e^{-t}z^3 ..)(1+2k(t)e^{-t}z + (2k(t)^2e^{-2t} + 2k(t)a_2(t)e^{-t})z^2 + ..)$$

$$= -e^{-t}z - 2k(t)e^{-2t}z^2 - (2k(t)^2e^{-3t} + 2k(t)a_2(t)e^{-2t})z^3 - \ldots$$

$$- a_2(t)e^{-t}z^2 - 2k(t)a_2(t)e^{-2t}z^3 - \ldots$$

$$- a_3(t)e^{-t}z^3 - 2k(t)a_3(t)e^{-2t}z^4 - \ldots$$

$$= -e^{-t}z - (2k(t)e^{-2t} + a_2(t)e^{-t})z^2$$

$$-(a_3(t)e^{-t} + 2k(t)a_2(t)e^{-2t} + 2k(t)^2e^{-3t} + 2k(t)a_2(t)e^{-2t})z^3 - \ldots$$

But $\dfrac{\partial f}{\partial t} = e^{-t}\left(-z - (a_2(t) - a_2'(t))z^2 - (a_3(t) - a_3'(t))z^3 - \ldots\right)$

Equating the coefficients of z^2 in the Loewner differential equation, we find

$$e^{-t}(a_2'(t) - a_2(t)) = -(2k(t)e^{-2t} + a_2(t)e^{-t})$$

$$\Rightarrow \quad e^{-t}a_2'(t) - e^{-t}a_2(t) + e^{-t}a_2(t) = -2k(t)e^{-2t}$$

$$\Rightarrow \quad e^{-t}a_2'(t) = -2k(t)e^{-2t},$$

$$\Rightarrow \quad a_2'(t) = -2e^{-t}k(t), \tag{2.4}$$

Also, equating the coefficients of z^3 in the Loewner differential equation, we find

$$e^{-t}a_3'(t) - e^{-t}a_3(t) = -a_3(t)e^{-t} - 4k(t)a_2(t)e^{-2t} - 2k(t)^2e^{-3t}$$

$$\Rightarrow \quad e^{-t}a_3'(t) = e^{-t}a_3(t) - e^{-t}a_3(t) - 2k(t)^2e^{-3t} - 4k(t)a_2(t)e^{-2t}$$

$$\Rightarrow \quad e^{-t}a_3'(t) = -2k^2(t)e^{-3t} - 4k(t)a_2(t)$$

$$\Rightarrow \quad a_3'(t) = -2e^{-2t}[k(t)]^2 - 4e^{-t}k(t)a_2(t)\,. \tag{2.5}$$

Integration of (2.4) gives

$$a_2(t) = \int_0^\infty a_2'(t)dt = -2\int_0^\infty e^{-t}k(t)dt.$$

Since $|k(t)| \equiv 1$, it follows that

$$|a_2(t)| \le 2\int_0^\infty e^{-t}dt = 2,$$

In view of (2.4), the equation (2.5) has the form

$$a_3'(t) = -2e^{-2t}[k(t)]^2 + 2a_2(t)a_2'(t),$$

so that integration gives

$$a_3(t) = -2\int_0^\infty e^{-2t}[k(t)]^2 dt + 4\left\{\int_0^\infty e^{-t}k(t)dt\right\}^2.$$

Setting $k(t) = e^{i\theta(t)}$, we obtain

$$\operatorname{Re}\{a_3\} \le 2\int_0^\infty e^{-2t}[1-2\cos^2\theta(t)]dt + 4\left\{\int_0^\infty e^{-t}\cos\theta(t)dt\right\}^2.$$

Applying the Schwarz inequality, we conclude that

$$\operatorname{Re}\{a_3\} \le 1 - 4\int_0^\infty e^{-2t}\cos^2\theta(t)dt + 4\left\{\int_0^\infty e^{-t}dt\right\}\left\{\int_0^\infty e^{-t}\cos^2\theta(t)dt\right\}$$

$$= 1 + 4\int_0^\infty (e^{-t} - e^{-2t})\cos^2\theta(t)dt$$

$$\le 1 + 4\int_0^\infty (e^{-t} - e^{-2t})dt = 3.$$

2.5 Typically Real Part Functions.

As next step the Bieberbach conjecture could be verified for another subclass of S : for univalent functions with real Taylor coefficients, more details can be found in [11], [52]. These are characterized by domains that are symmetric with respect to the real axis.

Definition 2.5.1:

Let S_R denote the class of univalent functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots\ldots. \qquad |z| < 1$$

whose coefficients a_n are all real.

If $f \in S_R$, then $f(z)$ is real on the real axis and is not real any where else in the disk. A more general class than S_R is the set of typically real functions which we now define.

Definition 2.5.2:

Let T denote the set of functions that are analytic in Δ, normalized by $f(0) = 0$ and $f'(0) = 1$ and satisfy $f(z)$ is real if and only if z is real, ($|z| < 1$).

We define another class which is directly related to T .

Definition 2.5.3:

Let P_R denote the subset of P of functions p such that $p(z)$ is real when z is real, ($|z| < 1$).

The following theorem shows that the relation between the families Typically real part and positive real part.

Theorem 2.5.1(Rogosinski's Theorem):

If $f \in T$ then, $p(z) = \frac{1-z^2}{z} f(z) \in P_R$.

Conversely , if $p \in P_R$, then $f(z) = \frac{z}{1-z^2} p(z) \in T$.

Proof: See [12, pages 56, 57].

Theorem 2.5.2:

If $f(z)=\sum_{0}^{\infty}a_nz^n \in T$, then $\left|a_{n+2}-a_n\right|\le 2, \qquad n=0,1,2,...$

Proof:

By Rogosinski's theorem, the function

$$p(z)=\frac{1-z^2}{z}f(z)=1+\sum_{0}^{\infty}(a_{n+2}-a_n)z^{n+1}.$$

is of class P_R. In particular, $p\in P$ and so Caratheodory's theorem implies that $\left|p_n\right|\le 2$, (n =1, 2….)

By equating coefficient of the power series in the relation

$p(z)=(1-z^2)\frac{f(z)}{z}$ where $p(z)=\sum_{0}^{\infty}p_nz^n$ and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$.

We conclude that

$$p(z)=(1-z^2)(1+\sum_{0}^{\infty}a_nz^{n-1}).$$

$$\Rightarrow \qquad p(z)=1+a_2z+(a_3-1)z^2+(a_4-a_2)z^3+.....+(a_n-a_{n-2})z^{n-1}+...$$

$$\Rightarrow \qquad \left|p_{n-1}\right|=\left|a_n-a_{n-2}\right|\le 2.$$

Theorem 2.5.3:

If $f\in T$ and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (\left|z\right|<1)$, then $\left|a_n\right|\le n$, $\qquad (n=2,3,......)$.

Proof:

If $\quad p(z)=(1-z^2)\frac{f(z)}{z} \;=\; \sum_{0}^{\infty}p_nz^n$

Then by Rogosinski's theorem $p\in P_R$. In particular, $p\in P$ and so Caratheodory's theorem implies that $\left|p_n\right|\le 2$, (n=1,2….). Theorem (2.5.2) proves that $\left|a_{n+2}-a_n\right|\le 2$, to show that $\left|a_n\right|\le n$, we use mathematical induction.

Since $a_1 = 1$ then $|a_1| \le 1$. Now suppose $|a_k| \le k$, then

$|a_{k+1}| = |a_{k+1} - a_{k-1} + a_{k-1}| \le |a_{k+1} - a_{k-1}| + |a_{k-1}| \le 2 + k - 1 = k + 1$

Hence, $|a_n| \le n, \forall n$.

2.6 The Littlewood-Paley and the Robertson Conjecture.

Theorem 2.6.1. (Littlewood-Paley Theorem):

The coefficients of every odd univalent function

$h(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + ...$ satisfy $|c_n| \le A$, $\qquad$ ($n = 3,5,7,......$)

where A is an absolute constant.

Proof: See [12, pages 64, 65].

In 1932 Littlewood and Paley remarked in a footnote: "No doubt the true bound is given by A=1". This has become known as the Littlewood-Paley Conjecture.

The Littlewood-Paley conjecture would have implied the Bieberbach conjecture. Assume $h(z) = \sum_{n=1}^{\infty} c_n z^n$ is the square root transform of $f(z) = \sum_{n=1}^{\infty} a_n z^n$,

Then $\{h(z)\}^2 = f(z^2)$.

Now compare the coefficients of z^{2n} to obtain the relation

$a_n = c_1 c_{2n-1} + c_3 c_{2n-3} + ... + c_{2n-1} c_1, \qquad c_1 = 1$

or

$$a_n = \sum_{k=1}^{n} c_{2(n-k)+1} . c_{2k-1}, \qquad c_1 = 1.$$

Since there are n terms on the right-hand side, the uniform bound $|c_k| \le 1$would imply $|a_n| \le n.$

Because the square-root transform of the Koebe function is

$$\frac{z}{1-z^2} = z + z^3 + z^5 + ...$$

Littlewood and Paley conjectured that $|c_1| \leq 1$ for all $h \in S^2$.   .

Fekete and Szegö, in 1933 disproved the Littlewood-Paley Conjecture by using Loewner's method.

Each $h \in S^2$ has the form $h(z) = \sqrt{f(z^2)}$ for some function

$f(z) = z + a_2 z^2 + a_3 z^3 + ...$

of class S. An easy calculation gives

$$c_3 = \frac{1}{2} a_2, \qquad c_5 = \frac{1}{2}(a_3 - \frac{1}{4} a_2^2)$$

Because $|a_2| \leq 2$, it is clear that $|c_3| \leq 1$ for all $h \in S^2$.

However, Fekete and Szegö found that the maximum of $|c_5|$ is larger than 1. In fact, they obtained the sharp bound for $|a_3 - \alpha a_2^2|$ in S, for each fixed α in the interval $0 \leq \alpha \leq 1$. Their result may be viewed as an interpolation between Loewner's inequality $|a_3| \leq 3$ and the elementary inequality $|a_3 - a_2^2| \leq 1$. This was the first relapse in the history of the Bieberbach conjecture.

Theorem 2.6.2. (Fekete and Szegö Theorem).

For each $f \in$ S,

$$|a_3 - \alpha a_2^2| \leq 1 + 2e^{-2\alpha/(1-\alpha)}, \qquad 0 < \alpha < 1.$$

This bound is sharp for each α .

The choice $\alpha = \frac{1}{4}$ demonstrates the existence of an odd univalent function with $|c_5| > 1$.

Proof: See [29, pages 217-221].

Theorem 2.6.3:

For each $h \in$ S^2,

$$|c_5| \leq \frac{1}{2} + e^{-2/3} = 1.0134...$$

Proof:

If $h \in \mathrm{S}^2$, then

$$h(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + ...$$

where

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

belongs to the class S. A simple calculation gives

$$c_5 = \frac{1}{2}(a_3 - \frac{1}{4}a_2^2).$$

But for each α in the interval $0 \leq \alpha < 1$, Fekete and Szeg$\ddot{o}$ theorem gives the sharp inequality

$$\left|a_3 - \alpha a_2^2\right| \leq 1 + 2e^{-2\alpha/(1-\alpha)},$$

Choosing $\alpha = \dfrac{1}{4}$, we obtain the desired estimate $\left|c_5\right| \leq \dfrac{1}{2} + e^{-2/3} = 1.0134....$

Theorem 2.6.4:

To each odd integer $n \geq 5$ there corresponds a function $h \in S^2$ with all of its coefficients real and with $\left|c_n\right| > 1$.

Proof: See [12, pages 107-110].

Robertson in 1936 conjectured that the coefficients of odd univalent functions

$$h(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + ...$$

satisfy the inequality

$$\sum_{k=1}^{n} \left|c_{2k-1}\right|^2 \leq n, \qquad\qquad n = 1,2,...$$

where $c_1 = 1$. This is a weaker version of the Littlewood-Paley Conjecture which, nevertheless, implies the Bieberbach conjecture. The inequality clearly holds for $n = 2$, since

$$\sum_{k=1}^{2} \left|c_{2k-1}\right|^2 = \left|c_1\right|^2 + \left|c_3\right|^2 \leq 2,$$

where $c_1 = 1$, and $\left|c_3\right| \leq 1$.

We shall now give Robertson's proof, which uses the Loewner method and is quite similar to the Fekete and Szegö estimation of $|c_5|$.

Theorem 2.6.5:

For each function $h \in S^2$.

$$\sum_{k=1}^{3} |c_{2k-1}|^2 = |c_1|^2 + |c_3|^2 + |c_5|^2 \leq 3.$$

Proof: See [12, pages 111,112].

2.7 The Polynomial Area Theorem and the Grunsky Inequalities.

The area theorem can be generalized to produce a system of inequalities called the Grunsky inequalities which are necessary and sufficient for the univalence of the associated function. These inequalities contain a wealth of useful information about the coefficients of univalent functions.

For example, they lead to an elementary proof of the Bieberbach conjecture for the fourth coefficient.

We now turn to a direct generalization of the area theorem which may be considered a primitive form of the Grunsky inequalities. For a better name, we shall call it the polynomial area theorem.

Theorem 2.7.1:(Polynomial Area Theorem).

Let $g \in \Sigma,$ let P be an arbitrary non constant polynomial, and let

$$P(g(z)) = \sum_{k=-N}^{\infty} c_k z^{-k}, \qquad |z| > 1,$$

where N is the degree of P .Then

$$\sum_{k=-N}^{\infty} k|c_k|^2 \leq 0,$$

Equivalently, the theorem states that

$$\sum_{k=-N}^{\infty} k|c_k|^2 \leq \sum_{k=1}^{N} k|c_{-k}|^2 .$$

Proof: See [12, pages 120,121].

For all $g \in \Sigma$, where $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + ...$, consider the expansion

$$\frac{\zeta\, g'(\zeta)}{g(\zeta) - w} = \sum_{n=o}^{\infty} F_n(w)\zeta^{-n} ,$$

But $\dfrac{\zeta\, g'(\zeta)}{g(\zeta) - w} = \dfrac{\zeta - b_1\zeta^{-1} - 2b_2\zeta^{-2} - ....}{\zeta + (b_0 - w) + b_1\zeta^{-1} + b_2\zeta^{-2} + ...}$

$$= 1 - (b_0 - w)\zeta^{-1} + [(b_0 - w)^2 - 2b_1]\zeta^{-2} + ...$$

$$\Rightarrow \frac{\zeta\, g'(\zeta)}{g(\zeta) - w} = \sum_{n=o}^{\infty} F_n(w)\zeta^{-n}$$

The function $F_n(w) = w^n + \sum_{k=1}^{n} a_{nk} w^{n-k}$

is a monic polynomial of degree n, called *the nth Faber polynomial of the function* g .

In particular, straightforward calculations produce the formulas

$F_0(w) = 1,$

$F_1(w) = w - b_0,$

$F_2(w) = w^2 - 2b_0 w + (b_0^2 - 2b_1).$

$F_3(w) = w^3 - 3b_0 w^2 + (3b_0^2 - 3b_1)w + (-b_0^3 + 3b_1 b_0 - 3b_2).$

Thus

$$\sum_{n=o}^{\infty} F_n(w)\zeta^{-n} = 1 + (w - b_0)\zeta^{-1} + (w^2 - 2b_0 w + (b_0^2 - 2b_1))\zeta^{-2} + ...$$

Now observe that since g is univalent, the function

$$\frac{\zeta\, g'(\zeta)}{g(\zeta) - g(z)} - \frac{\zeta}{\zeta - z}$$

$$= \left(\frac{\zeta - b_1\zeta^{-1} - 2b_2\zeta^{-2} - 3b_3\zeta^{-3} - ...}{(\zeta - z) + b_1(\zeta^{-1} - z^{-1}) + b_2(\zeta^{-2} - z^{-2}) + ...} \right) - \frac{\zeta}{\zeta - z}$$

$$= \left(\frac{\zeta}{\zeta - z} + b_1\zeta(\zeta - z)^{-2}(\zeta^{-1} - z^{-1}) + ...\right) - \frac{\zeta}{\zeta - z}$$

$$= b_1\zeta(\zeta - z)^{-2}(\zeta^{-1} - z^{-1}) + ...$$

$$=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \beta_{nk} z^{-k}\zeta^{-n}$$

is analytic for $|z|>1$and $|\zeta|>1$.then

$$\sum_{n=o}^{\infty} F_n(g(z))\zeta^{-n} = 1+\sum_{n=1}^{\infty}\left\{z^n + \sum_{k=1}^{\infty}\beta_{nk} z^{-k}\right\}\zeta^{-n}.$$

Thus the Faber polynomial satisfies

$$F_n(g(z)) = z^n + \sum_{k=1}^{\infty}\beta_{nk} z^{-k}, \qquad n = 1,2,...$$

where,

$$F_1(g(z)) = g(z) - b_0 = z + b_1 z^{-1} + b_2 z^{-2} + ...$$

$$F_2(g(z)) = (g(z))^2 - 2b_0 g(z) + (b_0{}^2 - 2b_1)$$

$$\begin{aligned} &= z^2 + 2b_0 z + (2b_1 + b_0{}^2) + 2(b_2 + b_0 b_1)z^{-1} + (2b_3 + 2b_0 b_2 + b_1{}^2)z^{-2} + ... \\ &\quad - 2b_0 z - 2b_0{}^2 - 2b_0 b_1 z^{-1} - 2b_0 b_2 z^{-2} - ... \\ &\quad + b_0{}^2 - 2b_1 \end{aligned}$$

$$= z^2 + 2b_2 z^{-1} + (2b_3 + b_1{}^2)z^{-2} + ...$$

$$F_3(g(z)) = (g(z))^3 - 2b_0 g(z)^2 + (3b_0{}^2 - 3b_1)g(z) + (b_0{}^3 + 3b_1 b_0 - 3b_2).$$

$$= z^3 + 3b_3 z^{-1} + 3(b_4 + b_1 b_2)z^{-2} + 3(b_5 + b_1 b_3 + b_0{}^2 b_3 + b_2{}^2) + b_1{}^3)z^{-3}.$$

Thus $F_n(g(z)) = (z + b_1 z^{-1} + b_2 z^{-2} + ...) + (z^2 + 2b_2 z^{-1} + (2b_3 + b_1{}^2)z^{-2} + ...) + ...$

$$= z^n + \sum_{k=1}^{\infty}\beta_{nk} z^{-k}, \qquad n = 1,2,...$$

The coefficients $\beta_{nk}$are known as the Grunsky coefficients of$g$, and $\beta_{nk}$are polynomials in the coefficients b_n of g .where

$\beta_{11} = b_1, \qquad \beta_{12} = b_2, \qquad \beta_{13} = b_3,$
$\beta_{21} = 2b_2, \quad \beta_{22} = 2b_3 + b_1^2, \qquad \beta_{23} = 2(b_4 + b_1 b_2),$
$\beta_{31} = 3b_3, \quad \beta_{32} = 3(b_4 + b_1 b_2), \quad \beta_{33} = 3(b_5 + b_1 b_3 + b_0^2 b_3 + b_2^2) + b_1^3.$

These results suggest the symmetry property

$$k\beta_{nk} = n\beta_{kn}, \qquad k,n = 1,2,...$$

For all $g \in \Sigma$ let

$$\ln \frac{g(z)-g(\zeta)}{z-\zeta} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_{nk} z^{-n}\zeta^{-k} \qquad (z,\zeta \in \Delta).$$

where $\beta_{nk} = n\gamma_{nk}$,

Theorem 2.7.2: (Strong Grunsky Inequalities).

Let β_{nk} be the Grunnsky coefficients of a function $g \in \sum$. Then

$$\sum_{k=1}^{\infty} k\left|\sum_{n=1}^{N}\beta_{nk}\lambda_n\right|^2 \le \sum_{n=1}^{N} n|\lambda_n|^2 .$$

or,

$$\sum_{k=1}^{\infty} k\left|\sum_{n=1}^{N}\gamma_{nk}\lambda_n\right|^2 \le \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 .$$

For each positive integer N and for all complex numbers $\lambda_1, \lambda_2, \ldots\ldots \lambda_N$.

Proof:

Let

$$p(w) = \sum_{n=1}^{N}\lambda_n F_n(w)$$

be a polynomial of degree N, expressed in terms of the Faber polynomials of g

In view of the equation,

$$F_n(g(z)) = z^n + \sum_{k=1}^{\infty}\beta_{nk} z^{-k},\ n = 1,2,\ldots.$$

$$\Rightarrow \qquad p(g(z)) = \sum_{n=1}^{N}\lambda_n z^n + \sum_{k=1}^{\infty}\sum_{n=1}^{N}\lambda_n\beta_{nk} z^{-k} .$$

Thus the theorem follows directly from the polynomial area theorem.

Theorem 2.7.3: (Weak Grunsky Inequalities).

Let β_{nk} be the Grunsky coefficients of a function $g \in \sum$. Then

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N} k\beta_{nk}\lambda_n\lambda_k\right| \le \sum_{n=1}^{N} n|\lambda_n|^2 .$$

or,

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N} \gamma_{nk}\lambda_n\lambda_k\right| \le \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 .$$

For each positive integer N and for all complex numbers $\lambda_1, \lambda_2, \ldots \lambda_N$.

Proof:

We shall actually derive an important generalization which will be called the *generalized weak Grunsky inequalities,*

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N} k\beta_{nk}\lambda_n\mu_k\right|^2 \le \sum_{n=1}^{N} n|\lambda_n|^2 \sum_{k=1}^{N} k|\mu_k|^2 .$$

where λ_n and μ_k are arbitrary complex parameters. To prove the last inequality, let

$$v_k = \sum_{n=1}^{N} \beta_{nk}\lambda_n, \qquad k = 1, \ldots, N$$

and observe that the strong Grunsky inequalities imply

$$\sum_{k=1}^{N} k|v_k|^2 \le \sum_{n=1}^{N} n|\lambda_n|^2 .$$

Hence the Cauchy-Schwarz inequality gives

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N} k\beta_{nk}\lambda_n\mu_k\right|^2 = \left|\sum_{k=1}^{N} kv_k\mu_k\right|^2$$

$$\le \sum_{k=1}^{N} k|v_k|^2 \sum_{k=1}^{N} k|\mu_k|^2$$

$$\le \sum_{n=1}^{N} n|\lambda_n|^2 \sum_{k=1}^{N} k|\mu_k|^2 \ .$$

2.8 Close to Convex Functions

We now turn to an interesting subclass of S which contains S* and has a simple geometric description. This is the class of close-to-convex functions, introduced by Kaplan in 1952.

Definition 2.8.1:

A function f analytic in Δ is said to be close-to-convex if there is a convex function g and a real number α such that:

$$\operatorname{Re}\left\{\frac{f'(z)}{e^{i\alpha} g'(z)}\right\} > 0 \qquad (|z| < 1) \quad \text{and } |\alpha| < \frac{\pi}{2}\ .$$

Definition 2.8.2:

Let K be denot the class of close-to-convex functions f normalized by the usual conditions $f(0) = 0$ and $f'(0) = 1$.

Theorem 2.8.1:

Let $f \in K$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $\quad(|z| < 1)$ then $|a_n| < n$ for $n = 2,3,...$

Proof:

If $f \in K$ then there is a function g in C and a real number α such that:

$$\operatorname{Re}\left\{\frac{f'(z)}{e^{i\alpha} g'(z)}\right\} > 0\ , \ (|z| < 1)\ . \text{ If } \quad q(z) = \frac{f'(z)}{e^{i\alpha} g'(z)} = \sum_{n=0}^{\infty} q_n z^n\ .$$

Then the function

$$p(z) = \frac{q(z) + i\sin\alpha}{\cos\alpha} = 1 + \sum_{n=1}^{\infty} p_n$$

belongs to P, since

$$p(0) = \frac{e^{-i\alpha} + i\sin\alpha}{\cos\alpha} = \frac{\cos\alpha - i\sin\alpha + i\sin\alpha}{\cos\alpha} = \frac{\cos\alpha}{\cos\alpha} = 1,$$

and $\operatorname{Re} p(z) = \dfrac{q(z)}{\cos\alpha} > 0$.

By equating coefficient of the power series in the relation

$p(z)\cos\alpha = q(z) + i\sin\alpha$, where

$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $q(z) = \sum_{n=0}^{\infty} q_n z^n$

We conclude that:

$$q_n z^n = (\cos\alpha) p_n z^n$$

$$\Rightarrow \qquad q_n = (\cos\alpha) p_n \quad ,(n = 1,2,3,...).$$

Caratheodory's theorem implies that,

$|p_n| \leq 2$, so $|q_n| \leq |p_n| \leq 2$, $\quad (n = 1,2,3,...)$ (2.6)

If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (|z| < 1)$ then theorem (2.2.3) asserts that

$|b_n| \leq 1 \qquad (n = 1,2,3,...)$ (2.7)

As $q_0 = e^{-i\alpha}$ the relation

$f'(z) = e^{i\alpha} g'(z) q(z)$, where

$f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1}$, $g'(z) = 1 + \sum_{n=2}^{\infty} n b_n z^{n-1}$ and $q(z) = \sum_{n=0}^{\infty} q_n z^n$ implies that

$$n a_n = e^{i\alpha}[n b_n e^{-i\alpha} + (n-1) b_{n-1} q_1 + (n-2) b_{n-2} q_2 + ... + 2 b_2 q_{n-2} + q_{n-1}].$$

Apply (2.6) and (2.7) to this equality to get

$$|n a_n| \leq n|b_n| + (n-1)|b_{n-1}||q_1| + (n-2)|b_{n-2}||q_2| + ... + 2|b_2||q_{n-2}| + |q_{n-1}|.$$

$$\leq n + (n-1)2 + (n-2)2 + ... + 2.2 + 2 = n^2.$$

$|n a_n| \leq n^2$.

Therefore $|a_n| \leq n$ for $n = 2,3,...$

2.9 The Fourth Coefficient

Loewner proved the Bieberbach conjecture for the third coefficient in the year 1923. Over 30 years later, in 1955, the fourth coefficient finally yielded to the efforts of Garebedian and Schiffer. Their proof was involving a variational method, Loewner's equation, and a long series of laborious numerical estimates. Then in 1960 Charzynski and Schiffer astonished everyone with the discovery of an elementary proof based only upon the Grunsky inequalities. This striking development focused new attention on the Grunsky inequalities and inspired a number of further advances.

Theorem 2.9.1:

The fourth coefficient of each function $f \in S$ satisfies $|a_4| \leq 4,$ with equality only for a rotation of Koebe function.

Proof:

Assume without loss of generality that $a_4 > 0$. Transform the function

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

to

$$g(z) = \left\{ f(1/z^2) \right\}^{-1/2} = (z^{-2} + a_2 z^{-4} + a_3 z^{-6} + ...)^{-1/2}$$
$$= z + b_1 z^{-1} + b_3 z^{-3} + ...$$

Then $g \in \Sigma$, and a calculation gives

$$b_1 = -\frac{1}{2} a_2, \quad b_3 = \frac{3}{8} a_2^2 - \frac{1}{2} a_3,$$
$$b_5 = \frac{3}{4} a_2 a_3 - \frac{5}{16} a_2^3 - \frac{1}{2} a_4.$$

Let γ_{nk} be the logarithmic Grunsky coefficients of g, as defined by section (2.7). Then

$$\gamma_{11} = b_1 = -\frac{1}{2} a_2, \qquad \gamma_{12} = \gamma_{21} = 0, \qquad \gamma_{13} = \gamma_{31} = b_3 = \frac{3}{8} a_2^2 - \frac{1}{2} a_3,$$
$$\gamma_{33} = \frac{1}{3} b_1^3 + b_1 b_3 + b_5 = -\frac{13}{24} a_2^3 + a_2 a_3 - \frac{1}{2} a_4.$$

with the choices $N = 3$, $\lambda_1 = \lambda$, $\lambda_2 = 0$, *and* $\lambda_3 = 1$, the weak Grunsky inequality is,

$$\left|\sum_{n=1}^{3}\sum_{k=1}^{3}\gamma_{nk}\lambda_{n}\lambda_{k}\right| = \left|\begin{array}{l}\gamma_{11}\lambda_{1}\lambda_{1}+\gamma_{12}\lambda_{1}\lambda_{2}+\gamma_{13}\lambda_{1}\lambda_{3}+\\ \gamma_{21}\lambda_{2}\lambda_{1}+\gamma_{22}\lambda_{2}\lambda_{2}+\gamma_{23}\lambda_{2}\lambda_{3}+\\ \gamma_{31}\lambda_{3}\lambda_{1}+\gamma_{32}\lambda_{3}\lambda_{2}+\gamma_{33}\lambda_{3}\lambda_{3}\end{array}\right|$$

$$= \left|\gamma_{11}\lambda_{1}\lambda_{1}+2\gamma_{13}\lambda_{1}\lambda_{3}+\gamma_{33}\lambda_{3}\lambda_{3}\right| \le \sum_{n=1}^{3}\frac{1}{n}\left|\lambda_{n}\right|^{2}$$

$$= \left|\lambda_{1}\right|^{2}+\frac{1}{2}\left|\lambda_{2}\right|^{2}+\frac{1}{3}\left|\lambda_{3}\right|^{2}$$

$$= \left|\lambda\right|^{2}+0+\frac{1}{3}$$

$$\Rightarrow \qquad \left|\gamma_{11}\lambda^{2}+2\gamma_{13}\lambda+\gamma_{33}\right| \le \left|\lambda\right|^{2}+\frac{1}{3}.$$

$$= \left|-\frac{1}{2}a_{2}\lambda^{2}+2(\frac{3}{8}a_{2}^{2}-\frac{1}{2}a_{3})\lambda+(-\frac{13}{24}a_{2}^{3}+a_{2}a_{3}-\frac{1}{2}a_{4})\right| \le \left|\lambda\right|^{2}+\frac{1}{3}.$$

$$= \left|a_{2}\lambda^{2}-4(\frac{3}{8}a_{2}^{2}-\frac{1}{2}a_{3})\lambda+\frac{26}{24}a_{2}^{3}-2a_{2}a_{3}+a_{4}\right| \le 2\left|\lambda\right|^{2}+\frac{2}{3}.$$

$$= \left|a_{2}\lambda^{2}-4b_{3}\lambda+\frac{18}{12}a_{2}^{3}-2a_{2}a_{3}-\frac{5}{12}a_{2}^{3}+a_{4}\right| \le 2\left|\lambda\right|^{2}+\frac{2}{3}.$$

$$= \left|a_{2}\lambda^{2}-4b_{3}\lambda+4a_{2}b_{3}-\frac{5}{12}a_{2}^{3}+a_{4}\right| \le 2\left|\lambda\right|^{2}+\frac{2}{3}.$$

$$= \left|a_{4}+4b_{3}(a_{2}-\lambda)-\frac{5}{12}a_{2}^{3}+a_{2}\lambda^{2}\right| \le 2\left|\lambda\right|^{2}+\frac{2}{3}.$$

But the area theorem applied to g gives

$$\left|b_{1}\right|^{2}+3\left|b_{3}\right|^{2} \le 1$$

$$\Rightarrow \qquad 3\left|b_{3}\right|^{2} \le 1-\left|b_{1}\right|^{2}$$

$$\Rightarrow \qquad 3\left|b_{3}\right|^{2} \le 1-\frac{1}{4}\left|a_{2}\right|^{2}$$

$$\Rightarrow \qquad \left|b_{3}\right|^{2} \le \frac{1}{3}-\frac{1}{12}\left|a_{2}\right|^{2}$$

$$\Rightarrow \qquad \left|b_{3}\right| \le \frac{1}{\sqrt{3}}(1-\frac{1}{4}\left|a_{2}\right|^{2})^{1/2}$$

$$\Rightarrow \qquad 4\left|b_{3}\right| \le \frac{2}{\sqrt{3}}(4-\left|a_{2}\right|^{2})^{1/2}$$

Thus

$$a_4 \le \frac{2}{\sqrt{3}}\left\{4-|a_2|^2\right\}^{1/2}|a_2-\lambda|+\mathrm{Re}\left\{\frac{5}{12}a_2^3-a_2\lambda^2\right\}+2|\lambda|^2+\frac{2}{3}. \tag{2.8}$$

Now let $a_2 = 2xe^{i\theta}$ $,0 \le x \le 1.$ Choose

$$\lambda = 2xe^{-i\theta/2}\cos(3\theta/2),$$

and let $y = |\sin(3\theta/2)|$, then

$$|a_2-\lambda| = |2xe^{i\theta}-2xe^{-i\theta/2}\cos(3\theta/2)|$$

$$\Rightarrow \qquad |a_2-\lambda| = 2x|1-e^{-i3\theta/2}\cos(3\theta/2)|$$

$$= 2x|1-(\cos 3\theta/2 - i\sin 3\theta/2)\cos 3\theta/2|$$

$$= 2x|1-\cos^2 3\theta/2 + i\sin 3\theta/2\cos 3\theta/2|$$

$$= 2x|\sin^2 3\theta/2 + i\sin 3\theta/2\cos 3\theta/2|$$

$$= 2x|\sin 3\theta/2(\sin 3\theta/2 + i\cos 3\theta/2)|$$

$$= 2x|\sin 3\theta/2||\cos 3\theta/2 + i\sin 3\theta/2|$$

$$\Rightarrow \qquad |a_2-\lambda| = 2xy\,.$$

and,

$$\mathrm{Re}\left\{\frac{5}{12}a_2^3-a_2\lambda^2\right\} = \mathrm{Re}\left\{\frac{5}{12}(2xe^{i\theta})^3-(2xe^{i\theta})(2xe^{-i\theta/2}\cos(3\theta/2))^2\right\}$$

$$= \mathrm{Re}\left\{\frac{5}{12}8x^3e^{i3\theta}-8x^3\cos^2 3\theta/2\right\}.$$

$$= \frac{40}{12}x^3\cos 3\theta - 4x^3(1+\cos 3\theta)$$

$$= \cos 3\theta(\frac{40}{12}x^3-4x^3)-4x^3$$

$$= \cos 3\theta(\frac{-2}{3}x^3)-4x^3$$

But $y = |\sin(3\theta/2)|$, then

$$y^2 = \sin^2 3\theta/2$$

$$= \frac{1-\cos 3\theta}{2}$$

$$\Rightarrow \cos 3\theta = 1-2y^2$$

$$\Rightarrow \cos 3\theta(\frac{-2}{3}x^3) - 4x^3 = \frac{-2}{3}x^3(1-2y^2) - 4x^3$$

$$= \frac{4}{3}y^2x^3 - 4x^3 - \frac{2}{3}x^3$$

$$= \frac{4}{3}y^2x^3 - \frac{14}{3}x^3$$

The inequality (2.8) therefore becomes

$$a_4 \leq \frac{2}{3} + 8x^2 - \frac{14}{3}x^3 + \frac{8}{\sqrt{3}}(1-x^2)^{1/2}xy - \frac{4}{3}x^2(6-x)y^2. \tag{2.9}$$

For each fixed x in the interval $0 < x \leq 1$, the right-hand side of (2.9) is a quadratic function of y which achieves its maximum for

$$y = \frac{\sqrt{3}(1-x^2)^{1/2}}{x(6-x)} \quad . \tag{2.10}$$

Substitute (2.10) into (2.9) to conclude that

$$a_4 \leq \frac{2}{3} + 8x^2 - \frac{14}{3}x^3 + \frac{4(1-x^2)}{6-x}. \tag{2.11}$$

it is now to be shown the right-hand side of (2.11) is not larger than 4 for $0 \leq x \leq 1$, which is equivalent to the inequality

$$0 \leq 24 - 5x - 66x^2 + 54x^3 - 7x^4 .$$

Or

$$0 \leq 3x^2(1-x) + (24 + 34x - 4x^2)(1-x)^2 .$$

This last inequality clearly holds for $0 \leq x \leq 1$, with equality only for $x = 1$.

Thus $|a_4| \leq 4$ for every $f \in S$, with equality only if $|a_2| = 2$.

Chapter 3

Milin Conjecture and de Branges' Theorem

The researchers in the former Soviet Union also worked on the last essential ingredient which made de Branges'proof possible. After some decades, one knew that it was very difficult to obtain informations about the coefficients $a_n(f)$ of univalent functions. The success of the Grunsky inequalities suggested that it should be simplier to obtain results about the logarithmic coefficients γ_n of the function

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n \ , \qquad |z| < 1 .$$

Lebedev and Milin worked on the question how to turn information about the logarithmic coefficients into information about the coefficients of f itself. They discovered that it was easier to lift information about certain quadratic means of coefficients. This could be combined with the Robertson conjecture.

3.1 Exponentiation of Power Series.

When a power series is exponentiatied, the coefficients of the new series have a complicated dependence upon those of the original series.

This section is devoted to three general inequalities, due to Lebedev and Milin, which estimate the new coefficients in terms of the old. Although these inequalities have nothing to do *per se* with univalent functions, they are the basic tool in Milin's exponentiation of the Grunsky inequalities.

Let

$$\varphi(z) = \sum_{k=1}^{\infty} \alpha_k z^k$$

be an arbitrary power series with a positive radius of convergence, normalized so that $\varphi(0) = 0$ denote the exponentiated power series by

$$\psi(z) = e^{\varphi(z)} = \sum_{k=0}^{\infty} \beta_k z^k \ .$$

The Lebedev-Milin inequalities may be stated as follows.

First Lebedev-Milin Inequality:

If $\sum_{k=1}^{\infty} k|\alpha_k|^2 < \infty$, then $\sum_{k=0}^{\infty} |\beta_k|^2 \le \exp\left\{\sum_{k=1}^{\infty} k|\alpha_k|^2\right\}$,

with equality if and only if $\alpha_k = \gamma^k / k$, $k = 1,2,...,$ for some complex constant γ with $|\gamma| < 1$.

Second Lebedev-Milin Inequality:

For $n = 1,2,...,$

$$\sum_{k=0}^{n} |\beta_k|^2 \le (n+1)\exp\left\{\frac{1}{n+1}\sum_{m=1}^{n}\sum_{k=1}^{m}(k|\alpha_k|^2 - \frac{1}{k})\right\}.$$

$$= (n+1)\exp\left\{\frac{1}{n+1}\sum_{k=1}^{n}(n+1-k)\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right\}$$

Equality occurs for a given integer n if and only if $\alpha_k = \gamma^k / k$, $k = 1,2,...,n,$ for some complex constant γ with $|\gamma| = 1$.

Third Lebedev-Milin Inequality:

For $n = 1,2,...,$

$$|\beta_n|^2 \le \exp\left\{\sum_{k=1}^{n}\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right\},$$

with equality if and only if $\alpha_k = \gamma^k / k$, $k = 1,2,...,n,$ for some complex constant γ with $|\gamma| = 1$.

Note that the Lebedev-Milin inequalities are completely independent of univalent functions. Hence it was possible to use them to solve quite different function theoretic problems. More details can be found in [1], [44], [38], and [39].

We now return to the problem of finding the sharp bound for the *n*th coefficient a_n among all functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

of class S. Littlewood's proof that $|a_n| < en$, relies on the Cauchy integral representation of a_n and the crude estimate

$$M_1(r, f) \le r(1-r)^{-1} \tag{2.12}$$

of the integral mean. We have already noted that even if (2.12) is replaced by the sharp estimate

$$M_1(r, f) \le M_1(r, k) = r(1-r^2)^{-1},$$

where k is the Koebe function, the method can give nothing better than $|a_n| < (e/2)n$. In 1965, Milin became the first to penetrate the $e/2$-barrier. His point of departure is not the Cauchy formula, but a new representation for the *n*th coefficient of a function $f \in S$ in terms of Faber polynomials.

Theorem 3.1.1 (Milin's Theorem):

For each $f \in S$, $|a_n| < 1.243n$, $\qquad n = 2,3,...$

Proof: See [12, page 150].

3.2 Logarithmic Coefficients.

Associated with each function f in S are its logarithmic coefficients γ_n defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n , \qquad |z| < 1.$$

The Grunsky inequalities provide a natural method for estimating the logarithmic coefficients by means of the Lebedev-Milin inequalities. These bounds can then be transferred to bounds on the coefficients of f and related functions.

Lemma 3.2.1:

The logarithmic coefficients of the Koebe function are $\gamma_n = 1/n$.

Recall that the nth Faber polynomial F_n of a function $g \in \Sigma$ is defined by the generating relation

$$\frac{g'(z)}{g(z)-w} = \sum_{n=0}^{\infty} F_n(w) z^{-n-1}.$$

Integration with respect to z gives

$$\log \frac{z}{g(z)-w} = \sum_{n=1}^{\infty} \frac{1}{n} F_n(w) z^{-n}.$$

$$\Rightarrow \qquad \log \frac{z}{g(z)-0} = \sum_{n=1}^{\infty} \frac{1}{n} F_n(0) z^{-n}.$$

$$\Rightarrow \qquad \log \frac{z}{g(z)} = \log \frac{z}{\left[f(1/z)\right]^{-1}} = \log zf(1/z)$$

But $\dfrac{z}{g(z)} = zf(1/z) = 1 + a_2 z^{-1} + a_3 z^{-2} + \ldots$

Since,

$$\log \frac{f(z)}{z} = \log\left(1 + a_2 z + a_3 z^2 + \ldots\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n \ .$$

$$\Rightarrow \qquad \log \frac{z}{g(z)} = \log(1 + a_2 z^{-1} + a_3 z^{-2} + \ldots) = 2\sum_{n=1}^{\infty} \gamma_n z^{-n} \ .$$

But $\qquad \log \dfrac{z}{g(z)} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} F_n(0) z^{-n}.$

$$\Rightarrow \qquad 2\sum_{n=1}^{\infty} \gamma_n z^{-n} = \sum_{n=1}^{\infty} \frac{1}{n} F_n(0) z^{-n}.$$

$$\Rightarrow \qquad 2\gamma_n = \frac{1}{n} F_n(0).$$

Proof of lemma:

Let $\quad g(z) = 1/f(1/z) = z - a_2 + (a_2^2 - a_3)z^{-1} + (-a_2^3 - a_4 + 2a_2 a_3)z^{-2} + \ldots$

But $g \in \Sigma$, and has the form

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots.$$

Then

$b_0 = -a_2 = -2$

$b_1 = (a_2^2 - a_3) = (4-3) = 1$

$b_2 = (-a_2^3 - a_4 + 2a_2 a_3) = 0$

From the expansion

$$\frac{\zeta g'(\zeta)}{g(\zeta)-w}=\sum_{n=0}^{\infty}F_n(w)\zeta^{-n}$$

$\Rightarrow$

$$\frac{\zeta g'(\zeta)}{g(\zeta)-0}=\sum_{n=0}^{\infty}F_n(0)\zeta^{-n}=1-b_0\zeta^{-1}+(b_0^2-2b_1)\zeta^{-2}+(-b_0^3+3b_0b_1-3b_2)\zeta^{-3}+\ldots$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty}F_n(0)=1-b_0+(b_0^2-2b_1)+(-b_0^3+3b_0b_1-3b_2)+\ldots$$

$$\Rightarrow \qquad \begin{aligned} &F_1(0)=-b_0=2\\ &F_2(0)=b_0^2-2b_1=2\\ &F_3(0)=-b_0^3+3b_0b_1-3b_2=2 \end{aligned}$$

Then

$$F_n(0)=2 \qquad , \qquad \text{for all } n=1,2,3,\ldots$$

In view of the inequality

$$2\gamma_n=\frac{1}{n}F_n(0).$$

where F_n is n th Faber polynomial of a function $g(z)=1/f(1/z)$.

$$\Rightarrow \qquad \begin{aligned} &\gamma_1=1.\\ &\gamma_2=1/2.\\ &\gamma_3=1/3.\\ &\vdots\\ &\gamma_n=1/n. \end{aligned}$$

Lemma 3.2.2: (Milin's lemma)

For some constant $\delta<0.312$,

$$\sum_{k=1}^{n}\frac{1}{k}\le\sup_{f\in S}\sum_{k=1}^{n}k|\gamma_k|^2\le\sum_{k=1}^{n}\frac{1}{k}+\delta \qquad ,n=1,2,3,\ldots$$

Proof: See [12, pages 151,152,153].

Theorem 3.2.1

For each odd function $h\in S$,

$$|c_n|<e^{\delta/2}<1.17 \qquad , \qquad n=3,5,7,\ldots$$

where δ is Milin's constant.

Proof:

Each odd univalent function h has the form

$$h(z)=\sqrt{f(z^2)}=z+c_3z^3+c_5z^5+\ldots, \qquad |z|<1.$$

Thus

$$\log\frac{h(\sqrt{z})}{\sqrt{z}}=\log\left(\frac{f(z)}{z}\right)^{1/2}=\frac{1}{2}\log\frac{f(z)}{z}=\sum_{n=1}^{\infty}\gamma_n z^n.$$

But $\dfrac{h(\sqrt{z})}{\sqrt{z}}=1+c_3z+c_5z^2+c_7z^3+\ldots$

In other words,

$$\sum_{n=0}^{\infty}c_{2n+1}z^n=\exp\left\{\sum_{n=1}^{\infty}\gamma_n z^n\right\} \qquad , \quad c_1=1.$$

Hence by the third Lebedev-Milin inequalities,

$$|c_{2n+1}|^2\le\exp\left\{\sum_{k=1}^{n}\left(k|\gamma_k|^2-\frac{1}{k}\right)\right\}=\exp\left\{\sum_{k=1}^{n}k|\gamma_k|^2-\sum_{k=1}^{n}\frac{1}{k}\right\}.$$

Now Milin's lemma gives

$$\sum_{k=1}^{n}\frac{1}{k}\le\sup_{f\in S}\sum_{k=1}^{n}k|\gamma_k|^2\le\sum_{k=1}^{n}\frac{1}{k}+\delta$$

$$\Rightarrow \qquad 0\le\sup_{f\in S}\sum_{k=1}^{n}k|\gamma_k|^2-\sum_{k=1}^{n}\frac{1}{k}\le\delta$$

$$\Rightarrow \qquad |c_{2n+1}|^2\le\exp\left\{\sum_{k=1}^{n}\left(k|\gamma_k|^2-\frac{1}{k}\right)\right\}=\exp\left\{\sum_{k=1}^{n}k|\gamma_k|^2-\sum_{k=1}^{n}\frac{1}{k}\right\}=e^{\delta}.$$

$$\Rightarrow \qquad |c_{2n+1}|^2\le e^{\delta}.$$

$$\Rightarrow \qquad |c_{2n+1}|\le e^{\delta/2}<e^{0.156}<1.17 \ , \qquad n=1,2,\ldots$$

It is an open problem of some interest to find the best possible value of the Milin constant δ. Observe that δ cannot be reduced to zero, since by the last theorem this would imply the Littlewood-Paley conjecture $|c_n|\le 1$, which is known to be false. Nevertheless, Milin has proposed the following conjecture, which asserts that $\delta=0$ in an average sense.

Milin Conjecture. For each $f \in S$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2-\frac{1}{k}\right)\le 0, \qquad n=1,2,...$$

Or

$$\sum_{k=1}^{n}(n+1-k)\left(k|\gamma_k|^2-\frac{1}{k}\right)\le 0.$$

In view of Theorem 3.2.2 and the second Lebedev-Milin inequality, Milin's conjecture implies Robertson's conjecture,

$$\sum_{k=0}^{n}|c_{2k+1}|^2 \le n+1 ,$$

This implies the Bieberbach conjecture $|a_n|\le n$.

Theoreme 3.2.2:

The Milin's conjecture implies Robertson's conjecture, and Robertson's conjecture implies Bieberbach's conjecture.

Proof:

Assume that Milin's conjecture is true. Let

$$g(z)=c_1z+c_2z^3+...$$

be an odd element of S define $f(z)=z+a_2z^2+a_3z^3+...$ so that $f(z^2)=g(z)^2$.

If $w_1,w_2\in\Delta$ and $f(w_1)=f(w_2)$, then $w_1=z_1^2$ and $w_2=z_2^2$ for some $z_1,z_2\in\Delta$ such that

$$g(z_1)^2=g(z_2)^2$$

Thus $f(z)$ is in S ,writing

$$\log\frac{f(z)}{zf'(0)}=2\sum_{n=1}^{\infty}\gamma_n z^n ,$$

we obtain

$$\left[\frac{g(z)}{zc_1}\right]^2=\frac{f(z^2)}{z^2f'(0)}=\exp\left(2\sum_{n=1}^{\infty}\gamma_n z^{2n}\right) ,$$

since $f'(0)=c_1^2$. Therefore

$$\frac{1}{c_1}(c_1+c_2z^2+c_3z^4+...)=\exp\left(\sum_{n=1}^{\infty}\gamma_n z^{2n}\right) \quad ,$$

and so

$$\frac{1}{c_1}(c_1+c_2z+c_3z^2+...)=\exp\left(\sum_{n=1}^{\infty}\gamma_n z^{n}\right).$$

By the second Lebedev-Milin inequality, for $n\geq 1$,

$$\frac{1}{n+1}\frac{1}{c_1^2}(|c_1|^2+|c_2|^2+.....+|c_{n+1}|^2)\leq \exp\left(\frac{1}{n+1}\sum_{k=1}^{n}k(n+1-k)|\gamma_k|^2-\sum_{k=1}^{n}\frac{1}{k+1}\right).$$

Milin's conjecture gives

$$\sum_{k=1}^{n}k(n+1-k)|\gamma_k|^2\leq \frac{n}{1}+\frac{n-1}{2}+\frac{n-2}{3}+.....+\frac{1}{n}$$

$$=n+\frac{n+1-2}{2}+\frac{n+1-3}{3}+.....+\frac{n+1-n}{n}$$

$$=1+\frac{n+1}{2}+\frac{n+1}{3}+.....+\frac{n+1}{n}$$

$$=(n+1)\sum_{k=1}^{n}\frac{1}{k+1}$$

Hence

$$\frac{1}{n+1}\frac{1}{c_1^2}(|c_1|^2+|c_2|^2+...+|c_{n+1}|^2)\leq 1.$$

$$\Rightarrow \qquad \frac{1}{c_1^2}(|c_1|^2+|c_2|^2+...+|c_{n+1}|^2)\leq n+1.$$

Thus the Milin's conjecture implies Robertson's conjecture.

Assume Robertson's conjecture is true. Let

$f(z)=a_1z+a_2z^2+...$ be in S. We may choose an odd function $g(z)=c_1z+c_2z^3+...$ of S such that $f(z^2)=g(z)^2$.

For all $n\geq 2$,

$$a_n = c_1 c_n + c_2 c_{n-1} + \ldots + c_n c_1,$$

Or $$a_n = \sum_{k=1}^{n} c_k c_{n-k+1}$$

and so

$$|a_n|^2 = \left|\sum_{k=1}^{n} c_k c_{n-k+1}\right|^2 \le (|c_1|^2 + |c_2|^2 + \ldots + |c_n|^2)^2 \le (nc_1^2)^2 = n^2 a_1^2 .$$

Thus the Robertson’s conjecture implies Bieberbach’s conjecture.

3.3 De Branges’ Theorem.

In this section we prove de Branges’ theorem [1985], conjectured by Bieberbach [1916] that, if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S,$$

we have $|a_n| \le n,\ n = 2,3,\ldots$ with equality only for the Koebe functions.

This had previously been proved for $n = 4$ by Garabedian and Schiffer [1955], for $n = 5$ by Pederson and Schiffer [1972] and for $n = 6$ by Pederson [1968] and Ozawa [1969].

De Branges proved his theorem by first establishing a conjecture of Milin [1971], which Milin had shown to imply Bieberbach conjecture.

The proof of de Branges has been simplified successively by Milin [1984] and Emelyanov, by Fitzgerald and Pommerenke [1985] and Weinstein [1991]. Inevitably these simpler proofs however miss the operator theory basis of de Brange’s subtle ideas. All the proofs rely on Loewner theory, and a positivity result for the coefficients of certain special functions. The earlier proofs used an inequality of Askey Gasper [1976] concerning Jacobi polynomials. Weinstein’s proof, which we shall follow here, uses instead the addition formula for Legendre polynomials which goes back to Legendre himself and seems simplier to establish. However Wilf [1993] has now shown rather surprisingly that the two results are equivalent. We start off by proving Legendre’s formula, then proving Milin’s conjecture, Milin’s inequalities and de Branges’ theorem, which in fact generalizes to give sharp bounds for the coefficients of $(f(z)/z)^{\lambda}$, when $\lambda \geq 1$. The analogous result fails for $0 < \lambda < 1$. De Branges' theorem, both the result itself and the subtlety of its proof, represents a milestone of twentieth century analysis.

Definition 3.3.1 (Legendre polynomials):

The Legendre polynomials $P_n(z)$ are defined by the expansion

$$\frac{1}{(1-2zh+h^2)^{\frac{1}{2}}} = 1 + \sum_{1}^{\infty} h^n P_n(z),$$

valid when $|h|$ is sufficiently small depending on z .

Lemma 3.3.1:

Suppose that $t \geq 0$, and that $w = w_t(z)$ is defined by

$$\frac{e^t w}{(1-w)^2} = \frac{z}{(1-z)^2}, \quad w_t(0) = 0. \tag{3.1}$$

Then we have for $|z| < 1$

$$\frac{e^t w^{k+1}}{1-w^2} = \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1}, \tag{3.2}$$

where $\Lambda_k^n(t) \geq 0, \ \textit{for } t \geq 0 \textit{ and } 0 \leq k \leq n < \infty$.

Proof: See [29, pages 235,236].

Theoreme 3.3.1:

If $f(z) \in S$ *and*

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} \gamma_k z^k, \tag{3.3}$$

then we have for, $n = 1,2,3,.....$

$$\sum_{k=1}^{n} (n+1-k)\left(\frac{4}{k} - k|\gamma_k|^2\right) \geq 0. \tag{3.4}$$

Theorem (3.3.1) is the key step in de Branges' proof. Weinstein's proof actually shows that equality holds in (3.4) only for the Koebe functions $f_\phi(z)=\dfrac{z}{(1-ze^{i\phi})^2}$.

To prove Theorem (3.3.1), it is sufficient by Lowner theorem to consider the function $f(z)$ in the subclass S_1 of that theorem (see definition 2.3.4). Thus

$$f(z)=\lim_{t\to\infty} e^t f(z,t),$$

where $f(z,t)$ satisfies the Loewner's differential equation (2.1), $k(t)$ is continuous and $|k(t)|=1$.

We define

$g_t(z)$ as in Theorem (2.3.3) by

$$g_t(z)=\beta e^t(z+a_2(t)z^2+...), \quad \text{where } \beta=\beta(0)=f'(0) \text{ and}$$

$$f(z,t)=g_t^{-1}[f(z)], \quad 0\le t<\infty, \text{then}$$

$$g_t\{f(z,t)\}=f(z). \tag{3.5}$$

Thus $g_0(z)=f(z)$. Also for large t,

$$g_t(z)=\frac{e^t z}{(1-ze^{i\phi})^2}.$$

We now write

$$h(z,t)=\log\frac{g_t(z)}{ze^t}=\sum_{k=1}^{\infty}\gamma_k(t)z^k \ . \tag{3.6}$$

Thus for $(t=0)$

$$h(z,0)=\log\frac{g_0(z)}{ze^0}=\log\frac{f(z)}{z}=\sum_{k=1}^{\infty}\gamma_k z^k \ .$$

and for $(t\ge t_0)$

$$h(z,t)=\log\frac{g_t(z)}{ze^t}=\log\frac{e^t z}{(1-ze^{i\phi})^2 e^t z}=\log\frac{1}{(1-ze^{i\phi})^2}$$

$$=2(e^{i\phi}z+\frac{1}{2}e^{2i\phi}z^2+\frac{1}{3}e^{3i\phi}z^3+......)$$

Then

$$\gamma_k(0)=\gamma_k, \ \mathit{and} \ \gamma_k(t)=\frac{2}{k}e^{ik\phi}, \quad t\ge t_0. \tag{3.7}$$

If we write

$$\zeta = f(z,t), \quad g = g_t(\zeta)$$

and differentiate the inequality (3.5) with respect to t. This gives

$$\frac{\partial g}{\partial \zeta}\frac{\partial \zeta}{\partial t} + \frac{\partial g}{\partial t} = 0.$$

Substituting in the Loewner's differential equation (2.1) we obtain

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial \zeta}\zeta\frac{1+k(t)\zeta}{1-k(t)\zeta} \ .$$

or writing z instead of $\zeta, g = g_t(z)$ we have

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial z}z\frac{1+k(t)z}{1-k(t)z} \ . \tag{3.8}$$

Using (3.6) we deduce

$$\frac{\partial}{\partial t}h(z,t) = \frac{\partial}{\partial t}\log\frac{g_t(z)}{ze^t} = \frac{\partial g/\partial t}{g} - 1$$

$$\frac{\partial}{\partial z}h(z,t) = \frac{\partial}{\partial z}\log\frac{g_t(z)}{ze^t} = \frac{\partial g/\partial z}{g} - \frac{1}{z}.$$

Thus (3.8) yields

$$1 + \frac{\partial h}{\partial t} = \left(\frac{1}{z} + \frac{\partial h}{\partial z}\right)z\frac{1+k(t)z}{1-k(t)z}.$$

Or

$$\frac{\partial h}{\partial t} = \left(1 + z\frac{\partial h}{\partial z}\right)\left(\frac{1+k(t)z}{1-k(t)z}\right) - 1.$$

But

$$h(z,t) = \sum_{k=1}^{\infty}\gamma_k(t)z^k \ .$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} = \sum_{k=1}^{\infty}\gamma_k'(t)z^k$$

$$\Rightarrow \quad \frac{\partial h}{\partial z} = \sum_{k=1}^{\infty}k\gamma_k(t)z^{k-1}$$

and

$$\frac{1+k(t)z}{1-k(t)z} = 1 + 2\sum_{k=1}^{\infty}k^k z^k$$

Then we substitute the last three inequalities in the equation

$$\frac{\partial h}{\partial t} = \left(1 + z\frac{\partial h}{\partial z}\right)\left(\frac{1+k(t)z}{1-k(t)z}\right) - 1 .$$

$$\Rightarrow \quad \sum_{k=1}^{\infty}\gamma_k'(t)z^k = \left(1+\sum_{k=1}^{\infty}k\gamma_k(t)z^k\right)\left(1+2\sum_{k=1}^{\infty}k^k z^k\right)-1$$

$$= 1+2\sum_{k=1}^{\infty}k^k(t)z^k + \sum_{k=1}^{\infty}k\gamma_k(t)z^k + 2\sum_{k=1}^{\infty}\sum_{r=1}^{k-1}r\gamma_r(t)k^{k-r}(t)z^k - 1 .$$

$$\Rightarrow \quad \gamma_k'(t) = k\gamma_k + 2k^k(t) + 2\sum_{r=1}^{k-1} r\gamma_r k^{k-r}(t) . \tag{3.9}$$

We need some crude bounds for $\gamma_k(t), \gamma_k'(t)$.

Lemma 3.3.2:

We have for $0 \le t < \infty$ and $1 \le k < \infty$,

$$|\gamma_k(t)| < 9, \quad |\gamma_k'(t)| < 11k^2 .$$

Proof

We deduce from (3.6) that

$$z\frac{\partial g/\partial z}{g} = 1 + \sum_{k=1}^{\infty}k\gamma_k(t)z^k .$$

Also $e^{-t}g_t(z) \in S$. Thus Destrotion theorem and Cauchy's inequality yield

$$k|\gamma_k(t)| \le \frac{1}{r^k}\sup_{|z|=r}\left\{r\left|\frac{\partial g/\partial z}{g}\right|\right\} \le \frac{1+r}{r^k(1-r)} .$$

Choosing $r = k/(k+1)$, we obtain

$$k|\gamma_k(t)| < \frac{1+\dfrac{k}{(k+1)}}{\left(\dfrac{k}{k+1}\right)^k\left(1-\dfrac{k}{k+1}\right)} .$$

$$= \frac{2k+1}{\left(\dfrac{k}{k+1}\right)^k} .$$

$$= (2k+1)\left(\frac{k+1}{k}\right)^k .$$

$$= (2k+1)\left(\frac{1}{k}+1\right)^k < 3ke < 9k$$

$$\Rightarrow \; k|\gamma_k(t)| < (2k+1)\left(1+\frac{1}{k}\right)^k < 3ke < 9k \, .$$

Now (3.9) gives

$$|\gamma_k'(t)| \le 2 + 9k + 2\sum_{r=1}^{k-1} 9r = 9k^2 + 2 < 11k^2 .$$

This proves lemma (3.3.2).

Weinstein proved (3.4) by showing that

$$I(z) = \sum_{n=1}^{\infty}\left\{\sum_{k=1}^{n}\left(\frac{4}{k} - k|\gamma_k(0)|^2\right)(n-k+1)\right\} z^{n+1}$$

$$= \sum_{n=1}^{\infty} z^{n+1}\int_0^{\infty} g_n(t)\,dt \, , \tag{3.10}$$

where

$$g_n(t) \ge 0, \quad \text{for} \quad t \ge 0 \quad \text{and} \quad n = 1,2,......$$

To prove (3.10) we fix z , such that $|z| < 1$, and define $w = w_t(z)$ by (3.1), i.e.

$$\frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2} \, . \tag{3.11}$$

We recall that by (3.11) $|z| < 1$ corresponds to $|w| < 1$cut along a segment of the negative real axis. So Schwarz's Lemma yields $|w_t(z)| \le |z|$ for $0 \le t < \infty$.

Also

$$I(z) = \sum_{n=1}^{\infty}\left\{\sum_{k=1}^{n}\left(\frac{4}{k} - k|\gamma_k(0)|^2\right)(n-k+1)\right\} z^{n+1}$$

$$= \frac{z}{(1-z)^2}\sum_{k=1}^{\infty}\left(\frac{4}{k} - k|\gamma_k(0)|^2\right) z^k$$

$$= \int_0^{t_0} -\frac{z}{(1-z)^2}\frac{d}{dt}\left\{\sum_{k=1}^{\infty}\left(\frac{4}{k} - k|\gamma_k(t)|^2\right) w^k\right\} dt \, . \tag{3.12}$$

For if

$$\psi(t)=\sum_{k=1}^{\infty}\left(\frac{4}{k}-k|\gamma_k(t)|^2\right)w_t(z)^k,$$

we have by (3.7)

$$\psi(t)=\sum_{k=1}^{\infty}\left(\frac{4}{k}-k\left|\frac{2}{k}e^{ik\phi}\right|^2\right)w_t(z)^k=0, \qquad t\ge t_0,$$

and

$$\psi(0)=\sum_{k=1}^{\infty}\left(\frac{4}{k}-k|\gamma_k(0)|^2\right)z^k.$$

We differentiate the series in (3.12) term by term and integrate the result term by term. To justify this, we need to show that the differentiated series converges uniformly in $[0,t_0]$. We write $x'=\partial x/\partial t$. Then (3.11) yields

$$w'=\frac{-(1-w)w}{(1+w)}.$$

Also by Lemma 3.3.2

$$\left|\frac{\partial}{\partial t}|\gamma_k(t)|^2\right|=\left|\frac{\partial}{\partial t}\left(\gamma_k(t)\overline{\gamma_k(t)}\right)\right|$$

$$=\left|\gamma_k(t)\overline{\gamma_k'(t)}+\overline{\gamma_k(t)}\gamma_k'(t)\right|<99k^2+99k^2=198k^2.$$

Thus

$$\frac{\partial}{\partial t}\left\{(\frac{4}{k}-k|\gamma_k(t)|^2)w^k\right\}$$

$$=\left(\frac{4}{k}-k|\gamma_k(t)|^2\right)kw^{k-1}\left(-\frac{(1-w)w}{1+w}\right)+w^k\left(-k\frac{\partial}{\partial t}|\gamma_k(t)|^2\right)$$

$$=-\frac{1-w}{1+w}(4-k^2|\gamma_k|^2)w^k-kw^k\frac{\partial}{\partial t}(\gamma_k(t)\overline{\gamma_k(t)})..$$

Since by (3.11) $|w|\le|z|<1$, the right-hand side is bounded by $Ak^3|z|^k$, where A is a constant depending only on z. This implies the required uniform convergence. Using also (3.11) and (3.12) we obtain

$$I(z)=\int_0^{\infty}\frac{e^t w}{1-w^2}\left\{\frac{1+w}{1-w}\sum_{k=1}^{\infty}k\left(\gamma_k(t)\overline{\gamma_k(t)}\right)'w^k+\sum_{k=1}^{\infty}\left(4-k^2|\gamma_k(t)|^2\right)w^k\right\}dt\ . \tag{3.13}$$

Next we write $z_1=r_1e^{i\theta}$, where $|z|<r_1<1$ and define

$$G(\theta)=\frac{1}{g(z_1,t)}\frac{\partial g(z_1,t)}{\partial t}=1+\frac{\partial h(z_1,t)}{\partial t}=1+\sum_{m=1}^{\infty}\gamma_k'(t)r_1^m e^{im\theta}\ . \tag{3.14}$$

Thus by Lemma 3.3.2, and since $r_1<1$ and $|w|<1$, we have

$$\sum_{k=1}^{\infty}k\overline{\gamma_k(t)}\gamma_k'(t)w^k=\sum_{k=1}^{\infty}k\overline{\gamma_k(t)}w^k\ \frac{1}{2\pi}\int_0^{2\pi}G(\theta)\overline{z_1}^{\,k}d\theta\,.$$

$$=\sum_{k=1}^{\infty}k\gamma_k'(t)\overline{\gamma_k(t)}r_1^{2k}w^k\ .$$

Hence

$$\frac{1+w}{1-w}\sum_{k=1}^{\infty}k\left(\gamma_k(t)\overline{\gamma_k(t)}\right)'w^k=\frac{1+w}{1-w}\left\{\sum_{k=1}^{\infty}\left(k\gamma_k'(t)\overline{\gamma_k(t)}(r_1^2w)^k+k\gamma_k(t)\overline{\gamma_k'(t)}(r_1^2w)^k\right)\right\}$$

But

$$\frac{1+w}{1-w}\sum_{k=1}^{\infty}k\gamma_k'(t)\overline{\gamma_k(t)}(r_1^2w)^k$$

$$=\left\{1+2\sum_{k=1}^{\infty}w^m\right\}\left\{\sum_{k=1}^{\infty}k\gamma_k'(t)\overline{\gamma_k(t)}(r_1^2w)^k\right\}$$

$$=\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}G(\theta)\left\{1+2\sum_{k=1}^{\infty}w^m\right\}\sum_{k=1}^{\infty}k\overline{\gamma_k(t)}w^kr_1^ke^{-ik\theta}d\theta$$

$$=\sum_{l=1}^{\infty}w^l\ \frac{1}{2\pi}\int_0^{2\pi}G(\theta)\overline{\left\{2\sum_{m=1}^{l}m\gamma_m(t)z_1^m-l\gamma_l(t)z_1^l\right\}}d\theta$$

$$=\sum_{l=1}^{\infty}w^l\left\{\frac{1}{2\pi}\int_0^{2\pi}G(\theta)\overline{\left\{2\left(1+\sum_{m=1}^{l}m\gamma_m(t)z_1^m\right)-l\gamma_l(t)z_1^l\right\}}d\theta-2\right\}. \tag{3.15}$$

Similarly

$$\frac{1+w}{1-w}\sum_{k=1}^{\infty}k\gamma_k(t)\overline{\gamma_k'(t)}(r_1^2w)^k$$

$$=\left\{1+2\sum_{k=1}^{\infty}w^m\right\}\left\{\sum_{k=1}^{\infty}k\gamma_k(t)\overline{\gamma_k'(t)}(r_1^2w)^k\right\}$$

$$=\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\overline{G(\theta)}\left\{1+2\sum_{k=1}^{\infty}w^m\right\}\sum_{k=1}^{\infty}k\gamma_k(t)w^kr_1^ke^{ik\theta}d\theta$$

$$=\sum_{l=1}^{\infty}w^l\ \frac{1}{2\pi}\int_0^{2\pi}\overline{G(\theta)}\left\{2\sum_{m=1}^{l}m\gamma_m(t)z_1^m-l\gamma_l(t)z_1^l\right\}d\theta$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{2\pi} \overline{G(\theta)} \left\{ 2\left(1+\sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_l(t) z_1^l \right\} d\theta - 2 \right\}. \tag{3.16}$$

Next we deduce from (3.6) that

$$1 + z_1 \frac{\partial h(z_1,t)}{\partial z_1} = z_1 \frac{\partial g(z_1,t)/\partial z_1}{g(z_1,t)} = 1 + \sum_{m=1}^{\infty} m\gamma_m(t) z_1^m .$$

In particular,

$$\left\{ 1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right\} \Bigg/ \left\{ \frac{z_1 \partial g(z_1,t)/\partial z_1}{g(z_1,t)} \right\} = 1 + P_{l+1}(z_1) ,$$

where $P_{l+1}(z_1)$ is a power series having a zero of order at least $(l+1)$ at $z_1 = 0$. Thus we may multiply the integrand in the coefficient of w^l in (3.15) by $1 + P_{l+1}(z_1)$ without affecting the value of the integral.

Substituting for $G(\theta)$ from (3.14) and using (3.8) we obtain from (3.15)

$$\frac{1+w}{1-w} \sum_{k=1}^{\infty} k \overline{\gamma_k(t)} \gamma_k'(t) (r_1^2 w)^k + 2\sum_{k=1}^{\infty} w^k$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{2\pi} G(\theta) \overline{\left\{ 2\left(1+\sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_l(t) z_1^l \right\}} d\theta \right\}$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\frac{\partial g(z_1,t)}{\partial t}}{g(z_1,t)} \Bigg/ \frac{z_1 \frac{\partial g(z_1,t)}{\partial z_1}}{g(z_1,t)} \right\} \left(1+\sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) \overline{\left\{ 2\left(1+\sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_l(t) z_1^l \right\}} d\theta \right\}$$

Since

$$\frac{\frac{\partial g(z_1,t)}{\partial t}}{z_1 \frac{\partial g(z_1,t)}{\partial z_1}} = = \left(\frac{1+k(t) z_1}{1-k(t) z_1} \right) .$$

Then the last equation equal

$$= \sum_{l=1}^{\infty} w^l \, \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1+k(t) z_1}{1-k(t) z_1} \right) \left\{ 1+\sum_{m=1}^{l} m\gamma_m(t) z_1^m \right\}$$

$$\times (2(1 + ... + m\overline{\gamma_m(t) z_1^m}) - m\overline{\gamma_m(t) z_1^m}) d\theta$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{\infty} \frac{1+k(t)z_1}{1-k(t)z_1} 2\left| \left(1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - \frac{1}{2} l\gamma_l(t) z_1^l \right|^2 d\theta + \frac{1}{2} l^2 |\gamma_l(t)|^2 r_1^{2l} \right\}.$$

Similarly

$$\frac{1+w}{1-w} \sum_{k=1}^{\infty} k\gamma_k(t)\overline{\gamma_k'(t)}(r_1^2 w)^k + 2\sum_{k=1}^{\infty} w^k$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{2\pi} \overline{G(\theta)} \left\{ 2\left(1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_l(t) z_1^l \right\} d\theta \right\}$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\overline{\frac{\partial g(z_1,t)}{\partial t}}}{g(z_1,t)} \middle/ \frac{\overline{\frac{\partial g(z_1,t)}{\partial z_1}}}{g(z_1,t)} \right\} \left(1 + \sum_{m=1}^{l} \overline{m\gamma_m(t) z_1^m} \right) \left\{ 2\left(1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_l(t) z_1^l \right\} d\theta \right\}$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{1+k(t)z_1}{1-k(t)z_1} \left(1 + \sum_{m=1}^{l} \overline{m\gamma_m(t) z_1^m} \right) \left\{ 2\left(1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_l(t) z_1^l \right\} d\theta \right\}$$

$$= \sum_{l=1}^{\infty} w^l \left\{ \frac{1}{2\pi} \int_0^{\infty} \frac{1+k(t)z_1}{1-k(t)z_1} 2\left| \left(1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - \frac{1}{2} l\gamma_l(t) z_1^l \right|^2 d\theta + \frac{1}{2} l^2 |\gamma_l(t)|^2 r_1^{2l} \right\}$$

Let r_1 tend to 1, the series all converge uniformly and absolutely for fixed z, variable t and $|z| < r_1 < 1$. We also note that

$$\mathrm{Re}\left(\frac{1+k(t)z_1}{1-k(t)z_1} \right) = u(z_1,t)$$

where $u \geq 0$. Thus (3.13) yields finally

$$I(z) = \int_0^{\infty} \frac{e^t w dt}{1-w^2} \sum_{l=1}^{\infty} w^l$$

$$\times \lim_{r_1 \to 1} \frac{1}{2\pi} \int_0^{\infty} u(z_1,t) \left| 2\left(1 + \sum_{m=1}^{l} m\gamma_m(t) z_1^m \right) - l\gamma_m(t) z_1^l \right|^2 d\theta$$

$$= \int_0^{\infty} \frac{e^t w dt}{1-w^2} \sum_{l=1}^{\infty} A_l(t) w^l$$

where $A_l(t) \geq 0$. On combining this with lemma (3.3.1) we obtain

$$I(z) = \int_0^{\infty} dt \sum_{l=1}^{\infty} A_l(t) \sum_{n=1}^{\infty} \Lambda_l^n(t) z^{n+1}$$

$$= \sum_{n=1}^{\infty} z^{n+1} \int_{0}^{\infty} g_n(t)dt,$$

where

$$g_n(t) = \sum_{l=1}^{n} A_l(t)\Lambda_l^n(t) \geq 0.$$

This proves (3.10) and so theorem (3.3.1).

Theorem 3.3.2:

Suppose that $\sigma_1, \sigma_2, ..., \sigma_{N+1}$ is a sequence such that

$$\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_{N+1} = 0. \tag{3.17}$$

and

$$\sigma_k - \sigma_{k+1} \geq \sigma_{k+1} - \sigma_{k+2}, \qquad 1 \leq k \leq N-1. \tag{3.18}$$

Then

$$\sum_{k=1}^{N} \left(k|\gamma_k|^2 - \frac{4}{k} \right) \sigma_k \leq 0. \tag{3.19}$$

Proof:

We choose nonnegative numbers α_1 to α_N .

Multiply

$\sum_{k=1}^{n} (n+1-k)\left(\frac{4}{k} - k|\gamma_k|^2 \right) \geq 0$, by $-\alpha_n$, and add for $n = 1$ to N. We deduce that

$$\sum_{k=1}^{N} \left(k|\gamma_k|^2 - \frac{4}{k} \right) \sigma_k \leq 0,$$

where

$$\sigma_k = \sum_{n=k}^{N} \alpha_n (n-k+1) \tag{3.20}$$

It remains to show that if the σ_k satisfy (3.17) and (3.18) we can solve this system of equations for nonnegative α_n. In fact (3.20) yields

$$\alpha_N = \sigma_N$$

Now, if $k < N$, and α_{k+1} to α_N have been chosen we obtain from (3.20)

$$\sigma_k = \sum_{n=k}^{N} \alpha_n (n-k+1)$$

$$\sigma_k = \alpha_k + ... + \alpha_N$$

$$\sigma_{k+1} = \alpha_{k+1} + ... + \alpha_N$$

$$\sigma_{k+2} = \alpha_{k+2} + ... + \alpha_N$$

$$\Rightarrow \quad \sigma_k - 2\sigma_{k+1} + \sigma_{k+2} = \alpha_k .$$

Thus $\alpha_n \geq 0$ by (3.17) and (3.18).

Hence (3.19), i.e. Theorem 3.3.2 is proved.

Now by section (3.1) Milin and Lebedev proved some subtle inequalities involving coefficients of power series. Milin showed that by means of these inequalities Biberbach's conjecture could be deduced from Theorem 3.3.1. We proceed to develop these results.

We suppose that

$$\varphi(z) = \sum_{k=1}^{\infty} \alpha_k z^k \tag{3.21}$$

Write

$$\psi(z) = e^{\varphi(z)} = \sum_{k=0}^{\infty} \beta_k z^k \tag{3.22}$$

We also define the binomial coefficients $d_k(\lambda)$ by

$$(1-z)^{-\lambda} = \exp\left(\lambda\sum_{1}^{\infty}\frac{z^k}{k}\right) = \sum_{0}^{\infty} d_k(\lambda) z^k \qquad , d_k(\lambda) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)\Gamma(k+1)}. \tag{3.23}$$

Here $\Gamma(x)$ is the gamma function.

Theorem 3.3.3:

With the above notation we have for $n = 1,2,.....\mathit{and}\ \lambda > 0$

$$\sum_{k=0}^{n} \frac{|\beta_k|^2}{d_k(\lambda)} \le d_n(\lambda+1)\exp\left\{\frac{1}{d_n(\lambda+1)}\sum_{k=1}^{n}\left(\frac{k^2|\alpha_k|^2-\lambda^2}{k\lambda}\right)d_{n-k}(\lambda+1)\right\} \tag{3.24}$$

Equality holds if and only if

$$\alpha_k = \frac{\lambda}{k}\eta^k, \quad \mathit{for} \quad k = 1,2,.....,n, \quad \mathit{where} \quad |\eta| = 1 \tag{3.25}$$

Proof: See [29, pages 243,244,245].

Theoreme 3.3.4:

With the hypotheses of Theorem 3.3.3 we have

$$|\beta_n|^2 \le d_n(\lambda)\exp\left\{\frac{1}{2d_n(\lambda)}\sum_{k=1}^{n}\left(\frac{k^2|\alpha_k|^2-\lambda^2}{k\lambda}\right)d_{n-k}(\lambda)\right\} \tag{3.26}$$

Equality holds if and only if (3.25) holds.

Proof: See [29, page 246].

Proof of de Branges' Theorem we can prove Biberbach's conjecture and a little more.

Theorem 3.3.5:

Suppose that $f(z) \in S$ and that

$$\left(\frac{f(z)}{z}\right)^{\lambda} = \sum_{n=1}^{\infty} a_n(\lambda) z^{n-1}. \tag{3.27}$$

Then, if $\lambda \geq 1,$ and $n \geq 2,$ we have

$$|a_n(\lambda)| \leq d_{n-1}(2\lambda) = \frac{\Gamma(n-1+2\lambda)}{\Gamma(n)\Gamma(2\lambda)}. \tag{3.28}$$

Equality holds if and only if $f(z)$ is the Koebe function

$$f(z) = \frac{z}{(1-z)^2} \tag{3.29}$$

Proof:

If $\lambda = 1,\ a_n(\lambda) = a_n,$ we obtain Biberbach's conjecture.

$$\Rightarrow a_2(\lambda) = \lambda a_2 = a_2 \quad and \quad a_3(\lambda) = \lambda a_3 + \frac{1}{2}\lambda(\lambda-1)a_2^2 = a_3.$$

To prove (3.28) we deduce from Theorem 3.3.1 that (3.4) holds. Thus

$$\left(\frac{f(z)}{z}\right)^{\lambda} = \exp\left\{\sum_{k=1}^{\infty} \lambda \gamma_k z^k\right\},$$

where

$$\sum_{k=1}^{n}\left(k|\gamma_k|^2 - \frac{4}{k}\right)(n-k+1) \leq 0, \quad n = 1,2,....$$

We apply Theorems 3.3.2 and 3.3.4 with $\mu = 2\lambda$ instead of λ, and $\alpha_k = \lambda\gamma_k$, $\beta_k = a_{k+1}(\lambda)$.

In order to apply Theorem 3.3.2, we need to check that, if $\sigma_k = d_{N-k}(\mu),\ k = 1,2,....,N+1,\quad \mu \ge 2,$ (3.17) and (3.18) are satisfied.

By our convention

$$\sigma_{N+1} = d_{-1}(\mu) = \frac{\Gamma(\mu-1)}{\Gamma(\mu)\Gamma(0)} = 0.$$

Also $\sigma_k > 0$ for $1 \le k \le N$, and by (3.23) for $\mu > 1$.

$$\frac{\sigma_{k+1}}{\sigma_k} = \frac{d_{N-k-1}(\mu)}{d_{N-k}(\mu)} = \frac{\Gamma(\mu+N-k-1)}{\Gamma(\mu)\Gamma(N-k-1+1)}\frac{\Gamma(\mu)\Gamma(N-k+1)}{\Gamma(\mu+N-k)},$$

$$= \frac{\Gamma(\mu+N-k-1)\Gamma(N-k+1)}{\Gamma(N-k)\Gamma(\mu+N-k)}.$$

$$= \frac{\mu(\mu+1)...(\mu+N-k-1-1)\Gamma(\mu)(N-k)\Gamma(N-k)}{\Gamma(N-k)\mu(\mu+1)...(\mu+N-k-2)(\mu+N-k-1)\Gamma(\mu)}.$$

$$= \frac{N-k}{\mu+N-k-1}.$$

$$\Rightarrow \frac{\sigma_{k+1}}{\sigma_k} = \frac{d_{N-k-1}(\mu)}{d_{N-k}(\mu)} = \frac{N-k}{\mu+N-k-1} < 1.$$

Thus (3.17) holds.

Next

$$\sigma_{k+1} - \sigma_k = \left(\frac{N-k}{\mu+N-k-1} - 1\right)\sigma_k,$$

$$\sigma_k - \sigma_{k-1} = \left(1 - \frac{\mu+N-k}{1+N-k}\right)\sigma_k.$$

Thus

$$\sigma_{k+1} - 2\sigma_k + \sigma_{k-1} = \sigma_k\left\{\frac{1-\mu}{\mu+N-k-1} - \frac{1-\mu}{1+N-k}\right\}$$

$$= \frac{(\mu-1)(\mu-2)\sigma_k}{(N-k+\mu-1)(N-k+1)} \ge 0 \qquad \text{If } 1 \le k \le N.$$

Hence (3.17) and (3.18) are satisfied and we deduce that

$$\sum_{k=1}^{N}\left(k|\gamma_k|^2-\frac{4}{k}\right)d_{N-k}(\mu)=\sum_{k=1}^{N}\left(k\left|\frac{\alpha_k}{\lambda}\right|^2-\frac{4}{k}\right)d_{N-k}(2\lambda)\,.$$

$$=\frac{2}{\lambda}\sum_{k=1}^{N}\left(\frac{k|\alpha_k|^2-4\lambda^2}{2k\lambda}\right)d_{N-k}(2\lambda)\le 0$$

Now (3.26), with 2λ instead of λ, yields

$$|a_{N+1}(\lambda)|=|\beta_N|\le d_N(2\lambda),\quad \text{if } N\ge 1.$$

This is (3.28).

If $N\ge 1$, equality is only possible if (3.25) holds. In particular we have

$$|\alpha_1|=2\lambda,\quad |\gamma_1|=|a_2(1)|=2.$$

Now it follows from Biberbach theorem that $f(z)$ is given by (3.29).

Theorem 3.3.6:

If $f_2(z)=\sum_{n=1}^{\infty}a_{2n-1}z^{2n-1}\in S$, then

$$|a_1|^2+|a_3|^2+.......+|a_{2N-1}|^2\le N+1. \tag{3.30}$$

.

Proof:

The result was conjectured by Robertson [1936] and proved by de Branges.

Since $f_2(z)$ is odd and univalent, $f(z)=\{f_2(z^{1/2})\}^2$ is univalent. Thus

$$1+a_3z+a_5z^2+.....=\frac{f_2(z^{1/2})}{z^{1/2}}=\left(\frac{f(z)}{z}\right)^{1/2}.$$

Hence with the notation of (3.27) we can write (3.30) in the form

$$\sum_{k=1}^{N}\left|a_n\left(\frac{1}{2}\right)\right|^2 \le N . \tag{3.31}$$

To prove (3.31) we apply Theorem 3.3.3 with $\alpha_k = \frac{1}{2}\gamma_k$ and $\lambda = 1$.

$$\Rightarrow \sum_{k=0}^{n}\frac{|\beta_k|^2}{d_k(1)} \le d_n(1+1)\exp\left\{\frac{1}{d_n(1+1)}\sum_{k=1}^{n}\left(\frac{k^2\left|\frac{1}{2}\gamma_k\right|^2-1}{k}\right)d_{n-k}(1+1)\right\}.$$

Then it follows from Theorem 3.3.1 that

$$\sum_{k=1}^{n}\left(\frac{k^2|\alpha_k|^2-\lambda^2}{k}\right)d_{n-k}(\lambda+1) = \sum_{k=1}^{n}\left(\frac{k|\gamma_k|^2}{4}-\frac{1}{k}\right)(n-k+1) \le 0 .$$

$$\Rightarrow \sum_{k=0}^{n}\frac{|\beta_k|^2}{d_k(1)} \le d_n(2)\exp\left\{\frac{1}{d_n(2)}\sum_{k=1}^{n}\left(\frac{k|\gamma_k|^2}{4}-\frac{1}{k}\right)(n-k+1)\right\}.$$

But

$$d_k(1) = \frac{\Gamma(k+1)}{\Gamma(1)\Gamma(k+1)} = 1, \qquad \text{and} \qquad d_n(2) = \frac{\Gamma(n+2)}{\Gamma(2)\Gamma(n+1)} = n+1 .$$

$$\Rightarrow \quad \sum_{k=0}^{n}|\beta_k|^2 \le n+1 .$$

$$\Rightarrow \quad |a_1|^2 + |a_3|^2 + \ldots\ldots. + |a_{2N-1}|^2 \le N+1 .$$

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