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Algebraic Approach To The Fractional Derivatives

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Abstract

Many concepts of mathematics can be generalized. In this thesis, we discuss the generalization of the concept of derivatives to include the derivatives of fractional derivatives. Two approaches to the definition of fractional derivatives are given and proved that are equal.

We introduce an approach to the fractional derivatives of the functions using the Taylor series of analytic functions. In order to calculate the fractional derivatives of *f*, it is not sufficient to know the Taylor expansion of *f*, but we should also know the constants of all consecutive integrations of *f*. The method of calculating the fractional derivatives very often requires a summation of divergent series, and thus in this note, we first introduce a method of such summation of series via analytical continuation of functions.

A derivative of a function of order α, for any real number α(called a fractional derivative) is the subject of this thesis. Here the definition will depend on the formal power series summation. We used this definition to find the fractional derivative of the constant functions and the polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved properties of the fractional derivative. And we proved that the fractional derivative of order $\alpha \in R$ of the exponential function e^x is the exponential function e^x and this help us in finding the fractional derivatives of the trigonometric functions and hyperbolic functions.

Finally, we introduce a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives.

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Introduction

he traditional integral and derivative are, to say the least, a staple for the technology ofessional, essential as a means of understanding and working with natural and artificial stems. Fractional Calculus is a field of mathematic study that grows out of the traditional finitions of the calculus integral and derivative operators in much the same way fractional ponents is an outgrowth of exponents with integer value. Consider the physical meaning of the ponent. According to our primary school teachers exponents provide a short notation for what essentially a repeated multiplication of a numerical value. This concept in itself is easy to asp and straight forward. However, this mathematical definition can clearly become confused hen considering exponents of non integer value. While almost anyone can verify that $x^3 =$ $x.x$, how might one describe the mathematical meaning of $x^{3/4}$, or moreover the anscendental exponent $x^{¼}$. One cannot conceive what it might be like to multiply a number or iantity by itself 3.4 times, or ¼ times, and yet these expressions have a definite value for any ilue x, verifiable by infinite series expansion, or more practically, by calculator.

ow, in the same way consider the integral and derivative. Although they are indeed concepts of higher complexity by nature, it is still fairly easy to physically represent their meaning. Once astered the idea of completing numerous of these operations, integrations or differentiations llows naturally. Given the satisfaction of a very few restrictions (e.g. function continuity), mpleting n integrations can become as methodical as multiplication.

ut the curious mind can not be restrained from asking the question what if n were not restricted an integer value? Again, at first glance, the physical meaning can become convoluted, but as is report will show, fractional calculus flows quite naturally from our traditional definitions. nd just as fractional exponents such as the square root may find their way into innumerable juations and applications, it will become apparent that integrations of order $\frac{1}{2}$ and beyond can nd practical use in many modern problems.

he fractional derivative is natural a natural extension of the familiar derivative $\frac{d^n f(x)}{dx^n}$ where =0,1,2,...to arbitrary number α ((integral, rational, irrational or complex)). Fractional fferentiation is of use in the solution of ordinary, partial, and integral equations as well as in e contexts, a few of which are indicated in the bibliography although other methods of solution e available, the fractional derivative approach to these problems often suggests methods that e not obvious in a classical formulation. The fractional calculus forms a special chapter in the on-general "Operation Calculus" which considers functions of the differential operator "D" ore general that D^{α}.

ractional Calculus is the branch of calculus that generalizes the derivative of function to non-teger order allowing calculations such as deriving a function to $1/2$ order despite generalized ould be a better option, the name "Fractional" is used for denoting this kind of derivative, see].

The simplest approaches to the definition of fractional differentiation begin by looking at a few well-known functions, and try to find various derivatives by means of an intuitive approach. We will be making use of the usual notation for derivatives, see [2], and we get the following:

$$D^n f(x) = \lim_{h \to o} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x + (n-m)h)$$

Differentiation and integration are usually regarded as discrete operations, in the sense that we differentiate or integrate a function once, twice, or any whole number of times. However, in some circumstances it's useful to evaluate a *fractional derivative*. In a letter to L'Hospital in 1695, Leibniz raised the possibility of generalizing the operation of differentiation to non-integer orders, and L'Hospital asked what would be the result of half-differentiating x. Leibniz replied The paradoxical "It leads to a paradox, from which one day useful consequences will be drawn".

The idea of generalizing the concepts of differentiation and integration to non-integer (fractional) orders has a long mathematical history. It was first discussed in the correspondence of G.W. Leibniz around 1690. Over the centuries many famous mathematicians including Euler, Riemann, Liouville and Weyl have built up a body of mathematical knowledge on fractional integrals and derivatives that is known under the name of fractional calculus.

In chapter (2) an approach to the fractional derivative of order $\alpha \in R$ of a function is given. This definition will depend on the formal power series summation. We used this new definition to find the fractional derivative of the constant functions and polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved properties of the fractional derivative. Finally, a proof of the well known fact that fractional derivative of $e^{\lambda x}$ of order $\alpha \in R$ Is equal to $\lambda^\alpha e^{\lambda x}$. Also, we proved that $sin^{(\alpha)}(x) = \sin(x + \alpha\pi/2)$ and $cos^{(\alpha)}(x) = \cos(x + \alpha\pi/2)$.

In chapter (3) we introduce an alternative definition of the fractional derivatives and also a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives (also integrals). Further are found the expansions of the functions $\frac{xe^x}{e^x-1}, \frac{1}{\cos(x)}, x \tanh x$, and some other functions of the form $\sum_{k=-\infty}^{\infty} a_k \frac{x^k}{k!}$, which enables us to calculate any fractional derivative of these functions at $x = 0$. These calculations lead to representations of the Bernoulli and Euler numbers B_k and E_k for any complex number k via fractional derivatives of some functions at $x = 0$.

Chapter 1

Riemann-Liouville Operator

The concept of non-integral order of integration can be traced back to the genesis of differential calculus itself: the philosopher and creator of modern calculus G.W. Leibniz made some remarks on the meaning and possibility of fractional derivative of order $\alpha \in R$ in the late 17:th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since.

Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of the time; it has been said to have lead him to construct the Gamma function for fractional powers of the factorial [2, p. 243]. An early attempt by Liouville was later purified by the Swedish mathematician Holmgren, who in 1865 made important contributions to the growing study of fractional calculus. But it was Riemann [4] who reconstructed it to fit Abel's integral equation, and thus made it vastly more useful. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types [1, p. xxxi], but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed.

1.1 The Gamma function:

As will be clear later, the gamma function is intrinsically tied to fractional calculus. The simple interpretation of the gamma function is simply the generalization of the factorial for all real numbers. The definition of the gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-u}\, u^{z-1} du\,, \qquad \text{for all } z \in R \qquad (1)$$

The beauty of the gamma function can be found in its properties. First as seen in (2), this function is unique in that the value for any quantity is , by consequence of the form of the integral, equivalent to that quantity z minus one times the gamma of the quantity minus one,

$$\Gamma(z+1) = z\,\Gamma(z), \quad \text{also, when } z \in N^+, \quad \Gamma(z) = (z-1)! \qquad (2)$$

This can be shown through a simple integration by parts. The consequence of this relation for integer values of z is the definition for factorial. Note that at negative integer values, the gamma function goes to infinity, yet is defined at non-integer values.

Now, we give an example.

Example 1.1.1

Using equation (2), then we get

$$\Gamma\left(\frac{3}{2}\right)=\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

1.2 Beta function:

Also known as the Euler Integral of the First Kind, the Beta Function is in important relationship in fractional calculus. Equation (3) demonstrates the Beta Integral and its solution in terms of the Gamma function.

$\mathrm{B}(p,q):=\int_0^1(1-u)^{p-1}\,u^{q-1}du=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}=B(q,p)$, where $p,q\in R^{+}$ (3)

Now, we give an example.

Example 1.2.1

Using equation (3), then we get

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\pi.$$

1.3 Riemann-Liouville integral:

The fractional derivative of order $\alpha\in R$ of a function f is

$$\frac{d^{\alpha}f(x)}{dx^{\alpha}}=f^{(\alpha)}(x)=\frac{1}{\Gamma(-\alpha)}\int_0^x\frac{f(t)}{(x-t)^{\alpha+1}}dt$$

Where $\Gamma(n)$ is the Euler's Gamma function.

Now, we give an example.

Example 1.3.1

The $(1/2)$ th derivatives of the functions $f(x)=x$ and $g(x)=\sqrt{x}$ are

$$f^{(1/2)}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} \qquad \text{and} \qquad g^{(1/2)}(x) = \frac{\sqrt{\pi}}{2}$$

Solution:

Using this definition, the $(1/2)$ th derivative of the function $f(x) = x$, is given by

$$f^{(1/2)}(\mathrm{x}) = \frac{1}{\Gamma(-(1/2))}\int_0^{\mathrm{x}} \frac{\mathrm{t}}{(\mathrm{x}-\mathrm{t})^{\frac{1}{2}+1}}\mathrm{dt}$$

$$= \frac{\sqrt{x}}{\Gamma(-(1/2)}\int_0^1 \mathrm{u}(1-\mathrm{u})^{-\frac{1}{2}-1}\mathrm{du}$$

$$= \frac{\sqrt{x}}{\Gamma(-(1/2))}\frac{\Gamma(2)\Gamma(-(1/2))}{\Gamma((3/2))}$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

Also, using the same definition, the $(1/2)$ th derivatives of the functions $g(x) = \sqrt{x}$ is given by

$$g^{(1/2)}(x) = \frac{1}{\Gamma(-(1/2))}\int_0^{x} \frac{\sqrt{t}}{(x-t)^{\frac{1}{2}+1}}dt$$

$$= \frac{1}{\Gamma(-(1/2))}\int_0^1 \sqrt{\mathrm{u}}(1-\mathrm{u})^{-\frac{1}{2}-1}\mathrm{du}$$

$$= \frac{1}{\Gamma(-(1/2))}\frac{\Gamma(3/2)\Gamma(-(1/2))}{\Gamma(1)}$$

$$= \frac{\sqrt{\pi}}{2}. \qquad \blacksquare$$

Now we will give several properties of the fractional derivatives.

Theorem 1.3.2 :[4]

If f(x) = c, where c is constant, then $f^{(\alpha)}(x) = \frac{c}{\Gamma(1-\alpha)}x^{-\alpha}$

Proof:

Using the definition, the α th derivative of the function is

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{c}{(x-t)^{\alpha+1}} dt$$

And so,

$$\begin{aligned} f^{(\alpha)}(x) &= \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{c}{(x-t)^{\alpha+1}} dt \\ &= \frac{c}{x^{\alpha}\,\Gamma(-\alpha)} \int_0^1 (1-u)^{-\alpha-1} du \\ &= \frac{c}{x^{\alpha}\,\Gamma(-\alpha)} \frac{\Gamma(1)\Gamma(-\alpha)}{\Gamma(1-\alpha)} \\ &= \frac{c}{\Gamma(1-\alpha)} x^{-\alpha}. \end{aligned}$$

Theorem 1.3.3 :[4]

If $f(x) = x^n$, n $\in Z$, then $f^{(\alpha)}(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$

Proof:

Using the definition, the α th derivative of the function is

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha+1}} dt$$

And so

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{t^n}{(x-t)^{\alpha+1}} dt$$

$$= \frac{x^{n-\alpha}}{\Gamma(-\alpha)} \int_0^1 u^n (1-u)^{-\alpha-1} du$$

Conclusion

We have studied in this thesis the background of the concept of fractional derivatives, fractional differentiation and fractional integration, by providing two different approaches of the concept of fractional derivatives. In the beginning, we have introduced the concept of fractional derivatives as defined by the Riemann - Liouville which has been generalized to include also the real numbers, not integers only. After that we discussed in many of the characteristics of this definition. The second concept depends on the formal power series summation. We used this definition to find the fractional derivatives of the constant functions and polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved several properties of the fractional derivatives.

Finally, we preseuted introduce an alternative definition of the fractional derivatives and also a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives (also integrals). Further, we are found the expansions of the functions
$\frac{xe^x}{e^x-1}, \frac{1}{\cos(x)}, x\tanh x$, and some other functions of the form $\sum_{k=-\infty}^{\infty} a_k \frac{x^k}{k!}$, which enables us to calculate any fractional derivative of these functions at $x = 0$.