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Numerical Treatment of Fredholm Integral Equation of the Second Kind

By

Nahed Salah Mohammad Salem

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Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ..N.ahed..salem

Nahed Salah Mohammad Salem

Date : 7 / 9 / 2003

Dedication

To my parents. To my Family and to my colleagues.

Acknowledgment

I would like to express my thanks to all those who helped me to prepare and complete this work.

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Abstract

During the period of 1960-1990, there has been much work on developing and analyzing numerical methods for solving linear Fredholm integral equation of the second kind with the integral operator K being compact on a suitable function space.

In this thesis we present some numerical methods for solving Fredholm integral equation of the second kind, these include the degenerate kernel approximations such as interpolatory degenerate schemes, and the projection methods where the main aspects of the collocation and Galerkin methods are investigated including recent work on solving Fredholm integral equations on surfaces in the Euclidean plane. Convergence, error analysis, and stability of these methods are also given great attention.

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Introduction

Over the past 20 years there has been a substantial increase in the use of integral equation in the formulation of solution strategies for scientific and engineering problems. In large measure this has been due to the work in the engineering and mathematics communities in using integral equation techniques to solve boundary value problems for partial differential equations as an alternative to domain-based methods such as the finite element and finite difference methods.

Parallel to this, there have been many important mathematical developments in establishing the convergence and stability of many numerical methods and in the discovery of new solution techniques. We believe these have considerable potential in increasing the efficiency of many current procedures.

In this thesis, some numerical methods are presented and analyzed for the solution of Fredholm integral equations of the second kind.During the period of 1960-1990,there has been much work on developing and analyzing numerical methods in this direction.

Integral equations are most frequently solved by collocation or Galerkin methods. Mathematically, these techniques are particular cases of projection methods.

Although the theory of projection methods has been understood for many years,the analysis of discrete projection methods is relatively of recent origin. Most what is known has been published in the past 10 years.

In chapter one, the main aspects of the theory of compact operators is presented, including integral equations with compact integral operators, the Fredholm

Glossary of Symbols

Spaces

(a,b)	open interval $(a,b)=\{x\in R: a<x<b\}$
$[a,b]$	closed interval $[a,b]=\{x\in R: a\leq x\leq b\}$
R^n	n-dimensional real Euclidean space
$C[a,b]$	space of real- or complex –valued continuous functions on the interval $[a,b]$
$C^m[a,b]$	space of m-times continuously differentiable functions
$L_2[a,b]$	space of real-or complex-valued square- integrable functions
ℓ^∞	space of bounded sequences of complex numbers with $\| x \| = \sup x$
ℓ^2	the vector space over C of all complex sequence $x=(x_n)_{n=1}^{\infty}$ which are square summable, i.e. $\sum_{n=1}^{\infty} \| x_n \|^2 < \infty$ with inner product $(x,y)=\sum_{n=1}^{\infty} x_n \overline{y}_n$

Norms

$\|\cdot\|$	norm on a linear space
$\|\cdot\|_1$	ℓ_1 norm of a vector, L_1 norm of a function
$\|\cdot\|_2$	ℓ_2 norm of a vector, L_2 norm of a function
$\|\cdot\|_\infty$	maximum norm of a vector or a function
$(.,.)$	scalar product on a linear space

alternative theorem (A key theorem concerning conditions for the existence and uniqueness of a solution) is also investigated.

Numerical methods for solving Fredholm integral equation of the second kind are considered in Chapter two, these include the degenerate kernel approximations such as the separable/ degenerate schemes.

Chapter three is devoted to studying other numerical methods such as projection methods where the main aspects of the collocation and Galerkin methods are presented including recent work on solving integral equations on surfaces in the Euclidean plane. .

Construction of these methods , convergence and stability of the resulting linear systems as well as regularization of solutions are also presented.

We give some illustrations which include three worked examples to compare the performance of the methods which are presented in this work.

Chapter One

Integral Equation with L_2-Kernel

1.1 Introduction

1.1.1 Types of Integral Equations

In this thesis we are concerned primarily with the numerical solution for the Fredholm integral equation of the second kind, but we begin classifying integral equations, we say, very roughly, that those integral equations in which the integration domain varies with the independent variable in the equation are Volterra integral equations, and those in which the integration domain is fixed are Fredholm integral equations of the second kind.

1. Volterra integral equations of the second kind

The general form that is studied is

$$\phi(x) - \int_a^x k(x,y)\,\phi(y)dy = f(x), \qquad x \geq a \tag{1.1.1}$$

The functions $k(x,y)$ and $f(x)$ are given, and $\phi(x)$ is sought.

2. Volterra integral equations of the first kind

The general nonlinear Volterra integral equation of the first kind has the form

$$\int_a^x k(x,y)\phi(y)dy = f(x), \quad x \geq a \tag{1.1.2}$$

The functions $k(x,y)$ and $f(x)$ are given functions, and the unknown is $\phi(y)$.

3. Abel integral equations of the first kind

The general form of such an integral equation is

$$\int_0^x \frac{H(x,y)\phi(y)}{\left(x^p - y^p\right)^\alpha} dy = f(x), \quad x > 0 \tag{1.1.3}$$

Here $0 < \alpha < 1$ and $p > 0$, and particularly important cases are $p = 1$ and $p = 2$ (both with $\alpha = \frac{1}{2}$). The function $H(x,y)$ is assumed to be smooth (that is, several times continuously differentiable).

4. Fredholm integral equations of the second kind

The general form of such an integral equation is

$$\lambda\phi(x) - \int_a^b k(x,y)\phi(y)dy = f(x), \qquad a \leq x \leq b, \quad \lambda \neq 0 \tag{1.1.4}$$

with $[a,b]$ a closed bounded set in R^m , some $m \geq 1$. The kernel function $k(x,y)$ is assumed to be absolutely integrable, and it is assumed to satisfy other properties that are sufficient to imply the Fredholm Alternative Theorem (see Theorem 1.3.1 in subsection (1.3.2). For $f \neq 0$,we have λ and f given and we seek ϕ, this is the non homogeneous problem. For $f = 0$, equation (1.1.4) becomes an eigenvalue problem, and we seek both the eigenvalue λ and the eigenfunction ϕ. The principal focus of the numerical methods presented in the following chapters is the numerical solution of equation (1.1.4) with

$f \neq 0$. In the next two sections we present some theory for the integral operator in equation (1.1.4).We define λ to be an eigenvalue of the operator K defined as

$$K\phi(x)=\int_a^b k(x,y)\phi(y)dy \tag{1.1.5}$$

5. Fredholm integral equations of the first kind

These equations take the form

$$\int_a^b k(x,y)\phi(y)dy = f(x), \qquad a \le x \le b \tag{1.1.6}$$

with the assumptions on K and $[a,b]$ the same as in equation (1.1.4).

1.2 Compact Operators

Definition 1.2.1

Let X and Y be normed vector spaces, and let $K: X \to Y$ be linear. Then K is compact if for every bounded sequence $\{f_n\} \subset X$, the sequence $\{K f_n\}$ has a subsequence that is convergent to some point in Y.

Theorem 1.2.1

Let H be $L_2[a,b]$, and let K be a degenerate integral operator

$$K_i f = \sum_{i=1}^{m} a_i(x)\int_a^b b_i(y)f(y)dy$$

K is a compact operator, if for all i , $a_i(x)$ and $b_i(x)$ belong to $L_2[a,b]$.

Proof:

Consider a bounded set of functions $\{f_n\}$

$$K f_n(x) = \sum_{i=1}^{m} a_i(x)\int_a^b b_i(y)f_n(y)dy$$

Clearly

$$\left| \int_a^b b_i(y)f_n(y)dy \right| \le \| b_i \| \, \| f_n \| \le M \| b_i(y) \|$$

so that the sequence $\left\{ \int_a^b b_i(y)f_n(y)dy \right\}$ is a bounded sequence of complex numbers . It must therefore have a point of accumulation, and a suitable subsequence will converge to that point of accumulation.

Let $\{f_{n^{(1)}}\}$ be a subsequence of $\{f_n\}$ such that $\left\{ \int_a^b b_1(y)f_{n^{(1)}}(y)dy \right\}$ converges to some complex number, say c_1 . By the same argument we can now extract a subsequence $\{f_{n^{(2)}}\}$ of $\{f_{n^{(1)}}\}$ such that $\left\{ \int_a^b b_2(y)f_{n^{(2)}}(y)dy \right\}$ converges to, say c_2 . By extracting successive subsequences we finally arrive at $\{f_{n^{(m)}}\}$ with the property that

$$\lim_{n^{(m)}\to\infty}\int_a^b b_m(y)f_{n^{(m)}}(y)\,dy = c_m$$

Since we have the inclusion

$$\{f_n\}\supset\{f_{n^{(1)}}\}\supset\{f_{n^{(2)}}\}\supset\ldots\supset\{f_{n^{(m)}}\}$$

we see that the above limit exists for all i.

Finally we note that the sequence $\{Kf_{n^{(m)}}(x)\}$ is a subsequence of $\{Kf_n\}$, and is also a Cauchy sequence. In fact

$$\lim_{n^{(m)}\to\infty} Kf_{n^{(m)}} = \sum_{i=1}^{m} c_i a_i(x)$$

so that K is compact operator.

Theorem 1.2.2

Let K be a compact operator on the Hilbert space H , and L be a bounded operator. Then both KL and LK are compact operators.

Proof:

Let $\{f_n\}$ be a bounded sequence in H . Since K is compact then $\{f_n\}$ contains a subsequence $\{f_{n'}\}$ such that $\{K f_{n'}\}$ is a Cauchy sequence. Since L is bounded, we see that

$$\| LK f_{n'} - LK f_{m'} \| \le \| L \| \; \| K f_{n'} - K f_{m'} \|$$

so that $\{LK f_{n'}\}$ is also a Cauchy sequence and LK is compact .

Now we consider the operator KL . If $\{f_n\}$ is bounded, and L is bounded, then

$\{Lf_n\}$ is again bounded .Since K is compact there exists a subsequence $\{Lf_{n'}\}$such that $\{KLf_{n'}\}$ is a Cauchy sequence, so that KL is also compact.

Theorem 1.2.3

Let K and L be two compact operators on a Hilbert space H . then $K+L$ is again compact, and so is αK for any scalar α .

Proof:

Let $\{f_n\}$ be a bounded sequence in H . It certainly contains a subsequence $\{f_{n'}\}$ such that $\{Kf_{n'}\}$ is a Cauchy sequence. . If α is any scalar then clearly $\{\alpha Kf_{n'}\}$ will again be a Cauchy sequence so αK is compact .

Now we can extract from $\{f_{n'}\}$ a subsequence $\{f_{n''}\}$ such that both $\{Kf_{n''}\}$ and $\{Lf_{n''}\}$ are Cauchy sequences. In that case $\{(K+L)f_{n''}\}$ is a Cauchy sequence ,so that $K+L$ is also compact.

Corollary 1.2.1

Let $\{K_n\}$ be a finite sequence of n compact operators on a Hilbert space H, and $\{\alpha_n\}$ a set of scalars .Then $\sum_{n=1}^{N}\alpha_n K_n$ is compact .

Theorem 1.2. 4

Let K be a compact operator on a Hilbert space H, and let $\{f_n\}$ be a linearly independent sequence of eigenfunctions corresponding to some nonzero eigenvalue μ, that is $Kf_n = \mu f_n$ for all n. Then $\{f_n\}$ contains a finite number of elements .

Proof :

We shall replace the set $\{f_n\}$ by equivalent set $\{g_n\}$, where all g_n are again eigenfunctions, but are orthonormal, so that

$$(g_n, g_m) = \begin{cases} 0 & , \quad m \neq n \\ 1 & , \quad m = n \end{cases}.$$

To accomplish this we use the well-known Gram-Schmidt process. Let

$$g_1 = \frac{f_1}{\| f_1 \|}$$

$$g_2 = \frac{f_2 - (f_2, g_1) g_1}{\| f_2 - (f_2, g_1) g_1 \|}$$

$$\vdots$$

$$g_n = \frac{f_n - \sum_{k=1}^{n-1} (f_n, g_k) g_k}{\left\| f_n - \sum_{k=1}^{n-1} (f_n, g_k) g_k \right\|}$$

$$\vdots$$

and it is easy to verify that both sets $\{f_1, f_2, \ldots, f_n\}$ and $\{g_1, g_2, \ldots, g_n\}$ span the same n dimensional subspace of H . Furthermore, a simple inductive argument shows that if

$(g_1, g_2, \ldots, g_{j-1})$ is orthonormal, then for $i < j$

$$(g_j, g_i) = \frac{(f_j, g_i) - \sum_{k=1}^{j-1} (f_j, g_k)(g_k, g_i)}{\left\| f_i - \sum_{k=1}^{j-1} (f_i, g_k) g_k \right\|} = 0$$

so that the set $\{g_n\}$ is orthonormal . Note that non of the denominators can vanish, otherwise the $\{f_n\}$ would not be linearly independent . One can verify that

$$\| g_n - g_m \|^2 = \| g_n \|^2 - (g_n, g_m) - (g_m, g_n) + \| g_m \|^2 = 2 \qquad \text{if } n \neq m$$

Clearly, the set of eigenfunctions $\{g_n\}$ is bounded. If it were infinite we could select a subsequence such that $\{K g_{n'}\}$ is a Cauchy sequence .Then

$$\| K g_{n'} - K g_{m'} \| = \| \mu g_{n'} - \mu g_{m'} \| = | \mu | \sqrt{2},$$

because the set $\{g_n\}$ is orthonormal. Since $\mu \neq 0$ the above can not be a Cauchy sequence, otherwise $\| K g_{n'} - K g_{m'} \|$ could be made smaller than $| \mu | \sqrt{2}$ for sufficiently large n', m' . It follows that of linearly independent eigenfunctions must be finite. The requirement that $\mu \neq 0$ is clearly vital in the proof. For a degenerate operator it is easy to verify that $\mu = 0$ is an eigenvalue with infinite multiplicity.

Theorem 1.2.5

Let $\{K_n\}$ be a sequence of compact operators on a Hilbert space H, such that for some operator K we have

$$\lim_{n \to \infty} \| K - K_n \| = 0$$

Then K is also compact.

Proof:

Let $\{f_n\}$ be any infinite, bounded sequence. We can select a subsequence $\{f_{n^{(1)}}\}$ such that $\{K_1 f_{n^{(1)}}\}$ is a Cauchy sequence. From the sequence $\{f_{n^{(1)}}\}$ we now extract a subsequence $\{f_{n^{(2)}}\}$ such that $\{K_2 f_{n^{(2)}}\}$ is a Cauchy sequence. Proceeding in this fashion we obtain a succession of subsequences

$$\{f_n\} \supset \{f_{n^{(1)}}\} \supset \{f_{n^{(2)}}\} \supset \ldots \supset \{f_{n^{(k)}}\} \supset \ldots$$

such that $\{K_k f_{n^{(i)}}\}$ is a Cauchy sequence for $k = 1,2,\ldots,i$.

Finally we select the sequence $\{f_{n^{(n)}}\}$, which is a subsequence of every $\{f_{n^{(k)}}\}$, except possibly for a finite number of terms so that $\{K_k f_{n^{(n)}}\}$ will be a Cauchy sequence for all k.

We shall now show that $\{K f_{n^{(n)}}\}$ is also a Cauchy sequence from which we can conclude that K is compact. Now

$$\begin{aligned} &\| K f_{n^{(n)}} - K f_{m^{(m)}} \| \\ &\quad = \| (K - K_k) f_{n^{(n)}} + K_k f_{n^{(n)}} - K_k f_{m^{(m)}} + (K_k - K) f_{m^{(m)}} \| \\ &\quad \leq \| K - K_k \| \, \| f_{n^{(n)}} \| + \| K_k f_{n^{(n)}} - K_k f_{m^{(m)}} \| + \| K_k - K \| \, \| f_{m^{(m)}} \|. \end{aligned}$$

The first and the third term can be made smaller than $M\varepsilon$, if $\| f_{n^{(n)}} \| \leq M$, and $k > k(\varepsilon)$, and the middle term will also be smaller than ε if $n^{(n)}$ and $m^{(m)}$ are larger than $N(\varepsilon)$, so that

$$\| Kf_{n^{(n)}} - Kf_{m^{(m)}} \| \leq (2M+1)\varepsilon, \qquad k > k(\varepsilon), \quad n^{(n)}, m^{(m)} > N(\varepsilon)$$

It follows that $\{Kf_{n^{(n)}}\}$ is a Cauchy sequence so that K is compact.

Example 1.2.1

Prove that if $k(x,y)$ is continuous for $0 \leq x, y \leq 1$, then the associated integral operator on $L_2[0,1]$ is compact.

Solution

The space $L_2[0,1]$ can be viewed as the completion of $C[0,1]$, with respect to appropriate norm. In a similar manner we can view L_2 kernels as suitable limits. Then we have

$$\lim_{n\to\infty} \int_0^1 \int_0^1 | k(x,y) - k_n(x,y) |^2 dx\, dy = 0,$$

where $k(x,y)$ is an L_2 kernel and all $k_n(x,y)$ are continuous. If we knew that the associated integral operators K_n are compact, then the above statement is equivalent to

$$\lim_{n\to\infty} \| K - K_n \| = 0$$

so that K would also be compact.

Arzela Ascoli Theorem 1.2.6

Each sequence from a subset $U \subset C[a,b]$ contains uniformly convergent subsequence i.e., U is relatively sequentially compact, if and only if it is bounded and equicontinuous i.e., if there exists a constant M such that

$$\left| g(x) \right| \leq M \qquad \text{for all } x \in [a,b]$$

and all $g \in U$, and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| g(x_1) - g(x_2) \right| < \varepsilon$$

for all $x_1, x_2 \in [a,b]$ with $| x_1 - x_2 | < \delta$ for all $g \in U$.

Proof: [[Kr2] chap. 1]

Theorem 1.2.7

The integral operator with continuous kernel is a compact operator on $C[a,b]$.

Proof:

Let $U \subset C[a,b]$ be bounded : $\| \varphi \|_\infty \leq C$ for all $\varphi \in U$. Then

$$| (K\varphi)(x) | \leq C(b-a) \max_{a \leq x, y \leq b} | k(x,y) |$$

for all $x \in C[a,b]$ and all $\varphi \in U$,i.e., $K(U)$ is bounded .Since k is uniformly continuous on the compact set $[a,b] \times [a,b]$ for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$| k(x,z) - k(y,z) | < \frac{\varepsilon}{(b-a)}$$

for all $x, y, z \in [a,b]$ with $| x - y | < \delta$ .Then

$$| (K\varphi)(x) - (AK\varphi)(y) | < \varepsilon$$

for all $x, y \in [a,b]$ with $|x-y| < \delta$ and all $\varphi \in U$,i.e., $K(U)$ is equicontinuous .

Hence K is compact by the Arzela- Ascoli Theorem .

Theorem 1.2.8

Let $k(x,y)$ be such that

$$\int_0^1\int_0^1 |\ k(x,y)\ |^2\, dx\,dy < \infty.$$

Then the operator

$$Kf(x) = \int_0^1 k(x,y) f(y) dy$$

is a compact operator on $L_2[0,1]$.

Proof :

As was remarked earlier, we consider an L_2 kernel $k(x,y)$ as a limit of continuous kernels, such that,

$$\lim_{n\to\infty} \int_0^1\int_0^1 |\ k(x,y) - k_n(x,y)\ |^2\, dx\,dy = 0$$

and we define

$$\int_0^1\int_0^1 |\ k(x,y)\ |^2\, dxdy = \lim_{n\to\infty} \int_0^1\int_0^1 |\ k_n(x,y)\ |^2\, dxdy.$$

using the Cauchy -Schwarz inequality we see that

$$\| (K-K_n)f \| = \left\| \int_0^1 [k(x,y)-k_n(x,y)]f(y)dy \right\|$$

$$\leq \left\| \left\{ \int_0^1 | k(x,y)-k_n(x,y) |^2 dy \int_0^1 | f(y) |^2 dy \right\}^{1/2} \right\|$$

$$\leq \left\{ \int_0^1 \int_0^1 | k(x,y)-k_n(x,y) |^2 dx\, dy \right\}^{1/2} \|f\|$$

so that

$$\| K-K_n \| \leq \left\{ \int_0^1 \int_0^1 | k(x,y)-k_n(x,y) |^2 dx\, dy \right\}^{1/2}$$

and

$$\lim_{n\to\infty} \| K-K_n \| = 0$$

Since each K_n is compact operator, by theorem (1.2.6) ,it follows from theorem (1.2.5) That K is also compact.

Theorem 1.2.9

Let $k(x,y)$ be an L_2 kernel on $L_2[0,\infty)$. Then the operator

$$Kf(x) = \int_0^\infty k(x,y)f(y)dy$$

is compact.

Proof:

In order for $k(x, y)$ to be an L_2 kernel we require that

$$\int_0^\infty \int_0^\infty | k(x,y) |^2 dx\,dy < \infty .$$

So far we discussed integrals over finite domains. Integrals over infinite domains have to be treated as limits. We require that

$$\lim_{n\to\infty} \int_0^n \int_0^n | k(x,y) |^2 dx\,dy < \infty$$

and then this limit defines the integral over an infinite square.

Now we define the kernels

$$k_n(x,y) = k(x,y), \qquad 0 \le x, y \le n$$

$$= 0 \qquad \text{otherwise}$$

Evidently

$$\lim_{n\to\infty} \| K - K_n \|^2 \le \lim_{n\to\infty} \int_0^\infty \int_0^\infty | k(x,y) - k_n(x,y) |^2 dx\,dy = 0$$

Inspection of K_n shows that we can consider it as an operator on $L_2[0,n]$,and by theorem (1.2.8) it will be compact. Consequently, it will also be compact on $L_2[0,\infty)$, and by theorem (1.2.5) K will be compact on $L_2[0,\infty)$.

1.3 Integral Equations with Compact Integral Operator

Most integral operators defined on C[a, b] or $L_2[a,b]$ fall into a class known as compact operators.

Lemma 1.3.1

If K is a bounded ,finite rank operator on X into Y ,then K is compact.

Proof:

Let $\{x_n\}$ be a bounded sequence in X and let K be finite rank and bounded. Then $\{K\,x_n\}$ is a bounded set in the (complete) finite dimensional space $Range(K)$. By the standard Hein-Borel theorem ,there is a convergent subsequence.

1.3.1` Integral Operators on Closed Intervals

For $X = Y = [a, b]$ the operator

$$K\,\phi(x) = \int_a^b k(x,y)\phi(y)dy \qquad ,a \le x \le b \qquad ,\phi \in C[a,b] \qquad (1.3.1)$$

is compact if it satisfies the following two assumptions:

(1) $\sup_{a \le x \le b} \int_a^b |\, k(x,y) \,| dy < \infty$

(2) $$\lim_{\delta \to 0} \int_a^b \left| k(x+\delta, y) - k(x,y) \right| dy = 0 \qquad , a \le x \le b . \tag{1.3.2}$$

Using these assumptions, the set

$$S = \left\{ K\phi \,\middle|\, \| \phi \| \le 1 \right\}$$

is bounded, equicontinuous family of functions on the interval [a, b] . By the Arzela Ascoli theorem, for every sequence in S , there is a uniformly convergent subsequence. thus K is compact.

An alternative approach is to use the lemma.(1.3.1), if $k(x,y)$ is continuous and can be written as

$$k(x,y) = \sum_{j=1}^{r} \alpha_j(x) \beta_j(y) \tag{1.3.3}$$

with $\alpha_1, \dots, \alpha_r$ continuous, then the operator K is finite rank, and thus compact.

$$K\phi(x) = \int_a^b k(x,y)\phi(y)dy = \sum_{j=1}^{r} \alpha_j(x) \int_a^b \beta_j(y)\phi(y)dy$$

and $\{\alpha_1, \dots, \alpha_r\}$ spans $Range(K)$.Kernels of the form (1.3.3) are called degenerate or separable kernels. If $k(x,y)$ is continuous ,then there exists a sequence of degenerate kernels $k_n(x,y)$ such that

$$\left| k(x,y) - k_n(x,y) \right| \le \frac{1}{n}, \qquad a \le x, y \le b$$

use the Weierstrass theorem to obtain such a sequence with $k_n(x,y)$ being a polynomial in x and y. Then the associated integral operators satisfy

$$\| K - K_n \| \le \frac{b-a}{n},$$

by the lemma (1.3.1), K is compact.

For a singular kernel like $k(x, y) = \frac{1}{|x-y|^{\alpha}}$, $0 < \alpha < 1$,define a sequence of continuous kernels $k_n(x, y)$ by

$$k_n(x, y) = \begin{cases} \frac{1}{|x-y|^{\alpha}} , & |x-y| \geq \frac{1}{n} \\ n^{\alpha} , & |x-y| \leq \frac{1}{n} \end{cases}$$

Then the associated integral operators ,

$$\| K - K_n \| \leq \frac{2\alpha}{1-\alpha} n^{\alpha-1} .$$

which converges to zero as $n \to \infty$. By lemma (1.3.1), K is compact operator on $C[a,b]$.

1.3.2 The Fredholm Alternative Theorem

In early 1900s, Ivar Fredholm gave necessary and sufficient conditions for the solvability of a large class of Fredholm integral equations of the second kind and with these results, he then was able to give much more general existence theorems for the solution of boundary value problems. Fredholm alternative theorem is the most important results that he got .

Theorem 1.3.1

Let X be a Banach space, let $K: X \to X$ be compact . Then the equation $(\lambda - K)\phi = f$, $\lambda \neq 0$ has a unique solution $\phi \in X$ if and only if the homogeneous equation $(\lambda - K)z = 0$ has only the trivial solution $z = 0$. In such a case, the operator $\lambda - K: X \underset{onto}{\overset{1-1}{\longrightarrow}} X$ has a bounded inverse $(\lambda - K)^{-1}$.

Proof: [[Kr2] ,chap. 4]

Chapter Two

Degenerate Kernel Methods

The degenerate kernel methods are well-known classical method for solving Fredholm integral equations of the second kind, and it is one of easiest numerical methods to define and analyze.

In this chapter we first consider again the reduction of a degenerate kernel integral equation to an equivalent linear system, and we reduce the assumptions made in the earlier presentation in the proof of theorem 1.3.1 in Chapter One. Following this, we consider various ways of producing degenerate kernel approximations of more general Fredholm integral equations of the second kind. We consider the integral equation

$$\lambda \phi(x) - \int_a^b k(x,y)\phi(y)dy = f(x), \qquad a \le x \le b \tag{2.1}$$

One of the most popular spaces for the Fredholm integral equation is space of real –or complex – valued square- integrable functions $L_2[a,b]$. The integral operator K of equation (2.1) is assumed to be a compact operator on X.

The kernel function k is to be approximated by a sequence of degenerate kernel functions

$$k_n(x,y) = \sum_{j=1}^{n} \alpha_{j,n}(x)\beta_{j,n}(y) \quad , \quad n \ge 1 \tag{2.2}$$

In such a way that the associated integral operators K_n satisfy

$$\lim_{n\to\infty} \| K - K_n \| = 0 \tag{2.3}$$

Generally, we want this convergence to be rapid to obtain rapid convergence of ϕ_n to ϕ, where ϕ_n is the solution of the approximating equation

$$\lambda \phi_n(x) - \int_a^b k_n(x,y)\phi_n(y)\, dy = f(x), \qquad a \le x \le b \tag{2.4}$$

The two equations (2.1) and (2.4) can be written symbolically as

$$(\lambda - K)\phi = f \quad , \qquad (\lambda - K_n)\phi_n = f \quad .$$

Equation (2.4) will be reduced to an equivalent finite linear system.

We now give a general error analysis.

Theorem 2.1

Assume $\lambda - K : X \underset{onto}{\overset{1-1}{\longrightarrow}} X$, with X is a Banach space and K is a bounded operator.

Further, assume $\{K_n\}$is a sequence of bounded linear operators on X with

$$\lim_{n\to\infty} \| K - K_n \| = 0$$

Then the operators $(\lambda - K_n)^{-1}$ exist from X onto X for all sufficiently large n, say $n \ge N$, and

$$\left\| (\lambda - K_n)^{-1} \right\| \le \frac{\left\| (\lambda - K)^{-1} \right\|}{1 - \left\| (\lambda - K)^{-1} \right\| \, \| K - K_n \|}, \qquad n \ge N \tag{2.5}$$

For the equation $(\lambda - K)\phi = f$ and $(\lambda - K_n)\phi_n = f$, $n \ge N$, we have

$$\| \phi - \phi_n \| \le \| (\lambda - K_n)^{-1} \| \; \| (K - K_n)\phi \| \quad , \qquad n \ge N \tag{2.6}$$

Proof :

Use the identity

$$\begin{aligned} \lambda - K_n &= \lambda - K + (K - K_n) \\ &= (\lambda - K)\left[I + (\lambda - K)^{-1}(K - K_n)\right] \end{aligned} \tag{2.7}$$

Choose N so that

$$\| K - K_n \| < \frac{1}{\| (\lambda - K_n)^{-1} \|} \quad , \qquad n \ge N \tag{2.8}$$

By the geometric series theorem, the quantity $I + (\lambda - K)^{-1}(K - K_n)$ has bounded inverse, with

$$\left\| \left[I + (\lambda - K)^{-1}(K - K_n)\right]^{-1} \right\| \le \frac{1}{1 - \| (\lambda - K)^{-1} \| \; \| K - K_n \|}$$

Using equation (2.7), this yields the existence of $(\lambda - K_n)^{-1}$ and its bound equation (2.5).

For the error bound equation (2.6), use the identity

$$\begin{aligned} \phi - \phi_n &= (\lambda - K)^{-1} f - (\lambda - K_n)^{-1} f \\ &= (\lambda - K_n)^{-1} [K - K_n](\lambda - K)^{-1} f \\ &= (\lambda - K_n)^{-1} [K\phi - K_n \phi] \end{aligned}$$

The error bound follows immediately.

A modification of the above also yields

$$\| (\lambda - K)^{-1} - (\lambda - K_n)^{-1} \| \le \| (\lambda - K_n)^{-1} \| \; \| (\lambda - K)^{-1} \| \; \| K - K_n \| \tag{2.9}$$

From (2.3) and (2.5), this shows $(\lambda - K_n)^{-1} \to (\lambda - K)^{-1}$ in $L[X, X]$.

Also, $\| (\lambda - K_n)^{-1} \| \to \| (\lambda - K)^{-1} \|$.

□

An important consequence of the convergence theorem (2.1) is that the speed of convergence need not depend on the differentiability of the unknown ϕ , since equation (2.6) implies

$$\| \phi - \phi_n \| \le \| (\lambda - K_n)^{-1} \| \; \| (K - K_n) \| \| \phi \| \tag{2.10}$$

If $\| K - K_n \|$ converges rapidly to zero, then the same is true of $\| \phi - \phi_n \|$, independent of the differentiability of ϕ.

With $X = C[a,b]$ we choose the degenerate kernel equation (2.2) so that the functions $\alpha_i(x)$ are all continuous and the functions $\beta_i(y)$ are all absolutely integrable.

To apply the convergence theorem (2.1), note that

$$\| K - K_n \| = \underset{a \le x \le b}{Max} \int_a^b | k(x,y) - k_n(x,y) | \, dy \tag{2.11}$$

With $X = L_2[a,b]$, we require that all $\alpha_i, \beta_i \in L_2[a,b]$. To apply the convergence theorem (2.1), we can use

$$\| K - K_n \| \le \left[\int_a^b \int_a^b | k(x,y) - k_n(x,y) |^2 \, dy \, dx \right]^{\frac{1}{2}} \tag{2.12}$$

2.1 Solution of Degenerate Kernel Equation

Using the formula (2.2) for $k_n(x,y)$, the integral equation $(\lambda - K_n)\phi_n = f$ is uniquely solvable for all $f \in C[a,b]$ becomes

$$\lambda\,\phi_n(x) - \sum_{j=1}^{n} \alpha_j(x) \int_a^b \beta_j(y)\,\phi_n(y)\,dy = f(x), \qquad a \le x \le b \tag{2.13}$$

Then the solution ϕ_n is given by

$$\phi_n(x) = \frac{1}{\lambda} f(x) + \frac{1}{\lambda}\left[c_1\alpha_1(x) + \ldots + c_n\alpha_n(x)\right] \tag{2.14}$$

with $$c_i = (\beta_i, \phi_n) = \int_a^b \beta_i(y)\phi_n(y)\,dy$$

To determine $\{c_j\}$, multiply equation (2.13) by $\beta_i(x)$ and integrate over [a, b]. This yields the system .

$$\lambda c_i - \sum_{j=1}^{n} c_j(\alpha_j, \beta_i) = (f, \beta_i)\ , \qquad i = 1,\ldots,n\ . \tag{2.15}$$

with

$$(\alpha_j, \beta_i) = \int_a^b \beta_i(x)\alpha_j(x)\,dx \tag{2.16}$$

Denote this linear system by $A\breve{c} = \breve{y}$ with

$$\breve{c} = (c_1, c_2, \ldots, c_n)^T\ .$$

Thus we solve equation (2.4) by solving the linear system (2.15) and using equation (2.14) which gives ϕ_n.

It remains to show that equation (2.15) is uniquely solvable .

Theorem 2.2

Assume $\lambda - K : X \underset{onto}{\overset{1\text{-}1}{\longrightarrow}} X$, with $\lambda \neq 0$ and with $X = C[a,b]$ or $L_2[a,b]$, and let K_n have the degenerate kernel (2.2). Then the linear system (2.15) is nonsingular .

Proof :

The following proof is divided into two cases, depending on whether $\{\beta_1, \cdots, \beta_n\}$ is an independent or dependent set of functions .

Case (1)

Let $\{\beta_1, \cdots, \beta_n\}$ is an independent set of functions . From the assumption that $(\lambda - K_n)^{-1}$ exists, we know that the linear system (2.15) is solvable for all right-hand sides of the form

$$\gamma \equiv \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} = \begin{bmatrix} (f, \beta_1) \\ \vdots \\ (f, \beta_n) \end{bmatrix} \quad \text{for some } f \in L_2(a,b) \tag{2.17}$$

A solution of equation (2.15) in this case is furnished by the coefficients in the expansion equation (2.14), with $\phi_n = \left(\lambda - K_n\right)^{-1} f$. Given any $\gamma \in R^n$, we will show that there exists $f \in L_2\left(a,b\right)$for which equation (2.17) is valid.

Given $\gamma \in R^n$, define

$$f\left(x\right)=\sum_{j=1}^{n} \delta_j \beta_j\left(x\right) \tag{2.18}$$

for some $\left(\delta_1,\cdots,\delta_n\right)$. These coefficients are to be chosen so as to have equation (2.17) be true, i.e. $\gamma_i = \left(f,\beta_i\right), i=1,\cdots,n$. This requires $\left(\delta_1,\cdots,\delta_n\right)$to be the solution of the linear system

$$\sum_{j=1}^{n} \delta_j \left(\beta_j,\beta_i\right)=\gamma_i\,, \quad i=1,\cdots,n \tag{2.19}$$

The matrix for this system is called a Gram matrix, and we show below that the linear independence of $\left\{\beta_i\right\}$ implies that equation (2.19) is nonsingular, thus making it uniquely solvable for $\left(\delta_1,\cdots,\delta_n\right)$.

To show equation (2.19) is nonsingular, we consider the homogeneous system

$$\sum_{j=1}^{n} \delta_j \left(\beta_j,\beta_i\right)= 0\,, \quad i=1,\cdots,n \tag{2.20}$$

Multiply the ith equation by δ_i, then summing over all i we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i\, \delta_j \left(\beta_j,\beta_i\right)=0$$

or equivalently,

$$\left(g,g\right)=0\,, \qquad g=\sum_{i=1}^{n} \delta_i\, \beta_i$$

This implies $g=0$, and by the linear independence of $\{\beta_i\}$, $g=0$ implies $\delta_i=0$, $i=1,\cdots,n$. Since the homogeneous system (2.20) has only the zero solution, the nonhomogeneous system (2.19) has a unique solution for every right-hand side δ . This shows that for each $\delta\in R^n$,there is $f\in L_2(a,b)$ for which equation (2.16) is valid. Consequently, the linear system (2.15) has a solution for every right-hand side δ . By standard results of linear algebra, this proves the system (2.15) is nonsingular. $\square$

Case (2) :

Assume $\{\beta_1,\cdots,\beta_n\}$ is a dependent set of functions . We reduce this case to that of case(1), and then we cite its conclusion to obtain a similar conclusion for the present case. Rather than proving the nonsingular of equation (2.15) for general $n\geq 1$, we do a simple case to illustrate the general idea of the proof.

Let $n=3$, let β_1 and β_2 be independent, and let $\beta_3=c_1\beta_1+c_2\beta_2$. The matrix of coefficients for equation (2.15) is

$$A=\begin{bmatrix} \lambda-(\alpha_1,\beta_1) & -(\alpha_2,\beta_1) & -(\alpha_3,\beta_1) \\ -(\alpha_1,\beta_2) & \lambda-(\alpha_2,\beta_2) & -(\alpha_3,\beta_1) \\ -(\alpha_1,\beta_3) & -(\alpha_2,\beta_3) & \lambda-(\alpha_3,\beta_3) \end{bmatrix}$$

Based on the dependence of β_3 on β_1 and β_2, introduce the matrices

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix} \qquad P^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c_1 & -c_2 & 1 \end{bmatrix}$$

to carry out elementary row and column operations on A . Doing this, we obtain the following.

Add c_1 times column 3 to column 1, and add c_2 times column 3 to column 2 :

$$AP=\begin{bmatrix} \lambda-(\alpha_1+c_1\alpha_3,\beta_1) & -(\alpha_2+c_2\alpha_3,\beta_1) & -(\alpha_3,\beta_1) \\ -(\alpha_1+c_1\alpha_3,\beta_2) & \lambda-(\alpha_2+c_2\alpha_3,\beta_2) & -(\alpha_3,\beta_2) \\ c_1\lambda-(\alpha_1+c_1\alpha_3,\beta_3) & c_2\lambda-(\alpha_2+c_2\alpha_3,\beta_3) & \lambda-(\alpha_3,\beta_3) \end{bmatrix}$$

Then subtract c_1 times row 1 and c_2 times row 2 from row 3, and use $\beta_3=c_1\beta_1+c_2\beta_2$ to obtain

$$P^{-1}AP=\begin{bmatrix} \lambda-(\alpha_1+c_1\alpha_3,\beta_1) & -(\alpha_2+c_2\alpha_3,\beta_1) & -(\alpha_3,\beta_1) \\ -(\alpha_1+c_1\alpha_3,\beta_2) & \lambda-(\alpha_2+c_2\alpha_3,\beta_2) & -(\alpha_3,\beta_2) \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\equiv\begin{bmatrix} \hat{A} & * \\ 0 & \lambda \end{bmatrix}$$

with $\hat{A}$ of order 2×2. Thus det $(A)=\lambda$ det$(\hat{A})$. This matrix $\hat{A}$ is the matrix of coefficients associated with the degenerate kernel

$$\hat{k}_3(x,y)=\left[\alpha_1(x)+c_1\alpha_3(x)\right]\beta_1(y)+\left[\alpha_2(x)+c_2\alpha_3(x)\right]\beta_2(y)$$

and this equals the original degenerate kernel k_3 when β_3 has been replaced by $c_1\beta_1+c_2\beta_2$. By case (1), $\hat{A}$ is nonsingular, and thus A is nonsingular. This proof can be generalized to handle any dependent set $\{\beta_i\}$, by reducing the matrix of coefficients A for equation (2.2) to a lower order matrix $\hat{A}$ for a rewritten form $\hat{k}_n$ of k_n, one that fits under case (1) . ☐

In section (3.4) we will examine the conditioning of the matrix A .

Now we obtain a bound for $\left\| (\lambda - K_n)^{-1} \right\|$. Let

$$A^{-1} = \left[c_{ij} \right].$$

Using $\bar{c} = A^{-1}\bar{y}$ in equation (2.15)

$$\phi_n(x) = \frac{1}{\lambda} . f(x) + \frac{1}{\lambda} \int_a^b R_n(x, y, \lambda) f(y) dy, \qquad a \le x \le y \quad ,$$

$$R_n(x, y, \lambda) = \sum_{j=1}^{n} \sum_{i=1}^{n} c_i \, \beta_i(y) \alpha_i(x) \quad . \tag{2.21}$$

Thus in $C[a,b]$,

$$\| (\lambda - K_n)^{-1} \| \le \frac{1}{|\lambda|} \left[1 + \max_{a \le x \le b} \int_a^b | R(x, y, \lambda) | dy \right] \tag{2.22}$$

2.2 Construction of Degenerate Kernel Approximation

Degenerate kernel methods are now reduced to finding good degenerate kernel approximations to $k(x, y)$. In this section we consider three methods for constructing degenerate kernel approximations, and we consider the calculation of their coefficients.

2.2.1 Taylor Series Approximations

Consider the one-dimensional integral equation

$$\lambda\,\phi(x)-\int_a^b k(x,y)\,\phi(y)\,dy=f(x),\qquad a\le x\le b \tag{2.23}$$

Often we can write k as a power series in y

$$k(x,y)=\sum_{i=0}^{\infty}\kappa_i(x)(y-a)^i\quad . \tag{2.24}$$

or in x

$$k(x,y)=\sum_{i=0}^{\infty}\kappa_i(x)(x-a)^i\quad . \tag{2.25}$$

Let k_n denote the partial sum of the first n terms on the right side of equation (2.24),

$$k_n(x,y)=\sum_{i=0}^{n-1}\kappa_i(x)(y-a)^i\quad . \tag{2.26}$$

Using the notation of equation (2.2), k_n is degenerate kernel with

$$\alpha_i(x)=\kappa_{i-1}(x),\qquad \beta_i(y)=(y-a)^{i-1},\qquad i=1,\ldots,n \tag{2.27}$$

The linear system (2.15) becomes

$$\lambda\,c_i-\sum_{j=1}^{n}c_j\int_a^b (y-a)^{i-1}\kappa_{j-1}(y)\,dy=\int_a^b f(y)(y-a)^{i-1}\,dy\qquad \text{for } i=1,\ldots,n\,. \tag{2.28}$$

and the solution ϕ_n is given by

$$\phi_n(x)=\frac{1}{\lambda}\,f(x)+\frac{1}{\lambda}\sum_{i=0}^{n-1}c_{i+1}\,\kappa_i(x)\quad , \tag{2.29}$$

The error analysis

$$\| K - K_n \| = \underset{a \le x \le b}{Max} \int_a^b \left| \sum_{i=n}^{\infty} \kappa_i(x)\,(y-a)^i \right| dy$$

and this can generally be bounded.

Example 2.1

Consider the integral equation

$$\lambda \phi(x) - \int_0^b e^{xy} \phi(y)\,dy = f(x) \quad , \quad 0 \le x \le b \tag{2.30}$$

Write

$$k(x,y) = e^{xy} = \sum_{i=0}^{\infty} \frac{(xy)^i}{i!}$$

and define k_n is the degenerate kernel

$$k_n(x,y) = \sum_{i=0}^{n-1} \frac{x^i y^i}{i!} \tag{2.31}$$

The linear system becomes

$$\begin{aligned} \lambda c_i - \sum_{j=1}^{n} c_j \int_0^b y^{i-1} \frac{y^{j-1}}{(j-1)!}\,dy &= \int_0^b f(y)\, y^{i-1}\,dy \quad , \quad i = 1, \ldots, n \\ = \lambda c_i - \sum_{j=1}^{n} \frac{b^{i+j-1}}{(j-1)!(i+j+1)} &= \int_0^b f(y)\, y^{i-1}\,dy \end{aligned} \tag{2.32}$$

and the solution ϕ_n of the degenerate kernel equation $(\lambda - K_n)\,\phi_n = f$ is given by

$$\phi_n(x) = \frac{1}{\lambda} \left[f(x) + \sum_{i=0}^{n-1} c_{i+1} \frac{x^i}{i!} \right]$$

For the error analysis, let $X = C[0,b]$ with $\|.\|_\infty$. Then

$$\| K - K_n \| = \max_{0\le x\le b} \int_0^b | k(x,y) - k_n(x,y) | \, dy$$

$$= \max_{0\le x\le b} \int_0^b \left| \sum_{i=0}^{\infty} \frac{(xy)^i}{i!} - \sum_{i=0}^{n-1} \frac{(xy)^i}{i!} \right| dy$$

$$= \max_{0\le x\le b} \int_0^b \left| \sum_{i=n}^{\infty} \frac{(xy)^i}{i!} \right| dy$$

$$= \max_{0\le x\le b} \int_0^b \frac{(xy)^n}{n!} e^{\varsigma} \, dy, \qquad \varsigma = \varsigma(x,y) \in [0,b]$$

$$\le \frac{b^{2n+1}}{(n+1)!} e^{b^2}$$

This converges to zero as $n \to \infty$. By theorem (2.1), we obtain convergence of ϕ_n to ϕ , along with the error bound (2.6) [or (2.10)] whenever $(\lambda - K)^{-1}$ exists.

It straightforward to calculate

$$\| K \| = \max_{a\le x\le b} \int_a^b | k(x,y) | dy$$

$$= \max_{0\le x\le b} \int_0^b | e^{xy} | dy$$

$$\le \frac{e^{b^2} - 1}{b}$$

If $|\lambda| > \| K \|$,then $(\lambda - K)^{-1}$ exists by the geometric series theorem, and

$$\left\| (\lambda - K)^{-1} \right\| \le \frac{1}{|\lambda| - \| K \|} \tag{2.33}$$

For values of n for which

$$\| K - K_n \| < |\lambda| - \| K \| \tag{2.34}$$

Table 2.1. Degenerate kernel example :

$b = 1$, $\lambda = 5$

n	E_1	B_1	E_2	B_2
1	6.37E-2	9.57E-1	1.31E-1	8.07E-1
2	1.74E-2	2.17E-1	4.33E-2	1.83E-1
3	3.72E-2	4.84E-2	1.05E-2	4.08E-2
4	6.59E-4	9.41E-3	2.03E-3	7.93E-3
5	1.00E-4	1.56E-3	3.30E-4	1.31E-3
6	1.33E-5	2.23E-4	4.61E-5	1.88E-4
7	1.57E-6	2.78E-5	5.67E-6	2.35E-5
8	1.67E-7	3.09E-6	6.22E-7	2.60E-6
9	1.61E-8	3.09E-7	6.15E-8	2.60E-7
10	1.42E-9	2.81E-8	5.54E-9	2.37E-8

Table 2.2. Degenerate kernel example

$b = 2, \ \lambda = 75$

n	E_1	B_1	E_2	B_2
4	1.75E-3		1.92E-1	
8	2.93E-4	1.07E+0	1.18E-2	1.11E+0
12	4.59E-6	9.48E-3	1.50E-4	9.87E-3
16	2.05E-8	4.22E-5	6.15E-7	4.40E-5
20	3.61E-11	7.53E-8	1.04E-9	7.84E-8

we can use the arguments of Theorem (2.1) to obtain the bound

$$\left\| (\lambda - K_n)^{-1} \right\| \leq \frac{1}{|\lambda| - \|K\| - \|K - K_n\|} \qquad (2.35)$$

Use $\phi = (\lambda - K)^{-1} f$ in equation (2.10) to obtain a computable error bound, with the needed norms of inverses obtained from equations (2.35) and (2.33).

We give numerical examples for several values of λ, b, and f in Tables 2.1 and 2.2.

For true solutions, we use

$$\phi^{(1)}(x) = e^{-1}\cos(x), \qquad \phi^{(2)}(x) = \sqrt{x} \qquad (2.36)$$

with the right-hand sides f defined accordingly. The tables contain the error

$$E_i = \left\| \phi^{(i)} - \phi_n^{(i)} \right\|_\infty, \quad i = 1, 2$$

and the corresponding error bound (2.10), labeled as B_i [provided equation (2.34) is satisfied]. The value λ is so chosen that $\|K\| \,/\, |\lambda|$ is approximately constant.

In Table 2.1, with $b = 1$, the error bounds are not too much larger than the actual errors, and Table 2.2, with $b = 2$, there is a greater disparity between the true error bound, but the bounds are still reasonable.

In Figure 2.1, we give the graph of the errors $\phi^{(i)} - \phi_n^{(i)}$ for $b = 1$ and $n = 10$, and in Figure 2.2, we graph the errors as n varies, again with $b = 1$. This second graph shows there is a regularity in the behavior of the error as n increases .

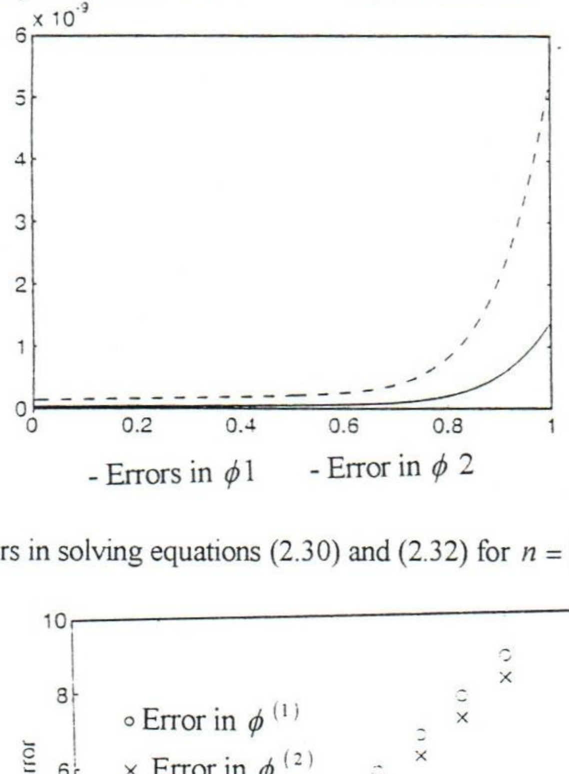

Figure 2.1. Errors in solving equations (2.30) and (2.32) for $n = 10$ and $b = 1$.

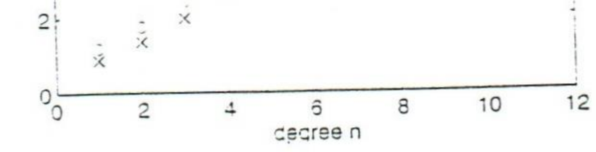

Figure 2.2. n vs. $-\log_{10} \| \phi - \phi_n^{(i)} \|_\infty$.

Table 2.3. Condition numbers of equation (2.32)

$b=1$ n	$\lambda=5$ $cond(A)$	$b=2$ n	$\lambda=75$ $cond(A)$
4	1.47	4	1.32
6	1.51	8	6.14
8	1.54	12	3.47E+2
10	1.56	16	4.68E+4
12	1.58	20	7.68E+6
14	1.59	24	1.37E+9

The rate of change in the error is also independent of the differentiability of the unknown ϕ.

2.2.2 Orthonormal Expansions

Let $X = L_2[a,b]$ and let $K : X \to X$ be a compact integral operator. A very popular degenerate kernel method for approximating K is based on using orthonormal expansions of the kernel function $k(x,y)$.

Let the space $L_2[a,b]$ have the inner product

$$(f,g)=\int_a^b w(x)f(x)g(x)dx$$

The weight function $w(x)$ is assumed to satisfy the usual properties

1) For most all $x \in [a,b]$, $w(x) \geq 0$.

2) For all $n \geq 0$,

$$\int_a^b w(x)|\ x\ |^n dx < \infty$$

3) If $f \in C[a,b]$ and is nonnegative on $[a,b]$, then

$$\int_a^b w(x) f(x) dx = 0 \Rightarrow f(x) \equiv 0$$

Let $\{\varphi_1, \varphi_2, \dots, \varphi_n, \dots\}$ be a complete orthonormal set in $L_2[a,b]$ with respect to a weight function $w(x)$. This means that

1. $(\varphi_n, \varphi_m) = \delta_{nm}$, for $1 \leq m, n < \infty$.

2. If $\phi \in L_2[a,b]$, we can write

$$\phi(x) = \sum_{i=1}^{\infty} (\phi, \varphi_i) \varphi_i(x)$$

This is the Fourier series of ϕ with respect to $\{\varphi_n\}$, and it converges in $L_2[a,b]$. We can apply this construction to the approximation of $k(x,y)$, with respect to either variable.

Expanding with respect to x, we have

$$k(x,y) = \sum_{i=1}^{\infty} \varphi_i(x) \beta_i(y) \tag{2.37}$$

with

$$\beta_i(y) = (k(.,y), \varphi_i) = \int_a^b w(x) k(x,y) \varphi_i(x) dx = K^* \varphi_i(y) \tag{2.38}$$

Define the approximating degenerate kernel by

$$k_n(x,y)=\sum_{i=1}^{n}\varphi_i(x)\beta_i(y)\ . \tag{2.39}$$

The equation $(\lambda-K_n)\phi_n=f$ is solved by

$$\phi_n(x)=\frac{1}{\lambda}f(x)+\frac{1}{\lambda}\sum_{i=1}^{n}c_i\varphi_i(x) \tag{2.40}$$

From equation (2.15), the coefficients $\{c_i\}$ satisfy the linear system

$$\lambda c_i-\sum_{j=1}^{n}c_j\int_a^b\varphi_j(y)\beta_i(y)\,dy=\int_a^b\beta_i(y)\,f(y)dy \qquad \text{for } i=1,\ldots,n \tag{2.41}$$

If we write out the needed integrals more completely, we have

$$\int_a^b\varphi_j(y)\beta_i(y)dy=\int_a^b\varphi_j(y)\int_a^b w(x)k(x,y)\varphi_i(x)dx\,dy$$

$$\int_a^b\beta_i(y)f(y)dy=\int_a^b f(y)\int_a^b w(x)k(x,y)\varphi_i(x)dx\,dy$$

For the error in using K_n, we have

$$\|K-K_n\|\leq\left\{\sum_{n+1}^{\infty}\|\beta_i\|_2^2\right\}^{\frac{1}{2}}$$

A bound for $\left\|(\lambda-K_n)^{-1}\right\|$ is quite complicated, and probably much too large.

2.2.3 Interpolatory Degenerate Kernel Approximations

Interpolation is a simple way to obtain degenerate kernel approximations. There are many kinds of interpolation, but we consider interpolation using only the values of $k(x, y)$. There are many candidates for interpolation functions (including spline functions), and others. We give a general framework for all of these, and then we illustrate these ideas with particular cases.

Let $\varphi_1(x), \cdots, \varphi_n(x)$ be a basis for the space of interpolation functions we are using .

For example, with polynomial interpolation of degree $< n$, we would use

$$\varphi_i(x) = x^{i-1}, \qquad 1 \le i \le n \tag{2.42}$$

Let $x_1, \ldots, x_n$ be interpolation nods in the integration region [a, b]. The interpolation problem as follows : Given data $f_1, \cdots, f_n$, find

$$z(x) = \sum_{j=1}^{n} c_j \varphi_j(x) \tag{2.43}$$

with

$$z(x_i) = f_i, \qquad i = 1, \ldots, n \tag{2.44}$$

Thus, we want to find the coefficients $c_1, \ldots, c_n$ by solving the linear system

$$\sum_{j=1}^{n} c_j \varphi_j(x_i) = f_i, \qquad i = 1, \ldots, n \tag{2.45}$$

In order for the interpolation problem to have a unique solution for all possible data $\{f_i\}$, it is necessary and sufficient that

$$\det(\Gamma_n) \neq 0, \quad \text{where } \Gamma_n = \left[\varphi_j(x_i)\right] \tag{2.46}$$

is the coefficient matrix of the linear system (2.45).

With polynomial interpolation and the basis of equation (2.42),

$$\Gamma_n = \left[x_i^{j-1}\right]_{i,j=1}^{n}$$

This called a Vandermonde matrix, and it is known that $\det(\Gamma_n) \neq 0$ for all distinct choices of $x_1, \ldots, x_n$.

To give an explicit formula for $k_n(x, y)$, we introduce a special basis for the interpolation method. Define $\ell_k(x)$ to be the interpolation function for which

$$\ell_k(x_i) = \delta_{ik}, \qquad i = 1, \ldots, n$$

Then the solution to the interpolation problem is given by

$$z(x) = \sum_{j=1}^{n} c_j \, \ell_j(x) \tag{2.47}$$

For polynomial interpolation, this is called Lagrange's form of the interpolation polynomial, and the functions $\ell_k(x)$ are usually called Lagrange basis functions. With polynomial interpolation,

$$\ell_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} \left(\frac{x - x_i}{x_k - x_i} \right)$$

Interpolation with respect to the variable x

Define

$$k_n(x, y) = \sum_{j=1}^{n} \underbrace{\ell_j(x)}_{\alpha_j(x)} \; \underbrace{k(x_j, y)}_{\beta_j(y)} \tag{2.48}$$

Then $k_n(x_i, y) = k(x_i, y)$, $i = 1, \ldots, n$, all $y \in [a, b]$. For the case $[a, b]$, with $k(x, y)$ being considered on the domain $[a, b] \times [a, b]$, we have that $k_n(x, y)$ equals $k(x, y)$ along all lines $x = x_i$.

The linear system $A_n c = r$ associated with degenerate kernel method $(\lambda - K_n)\phi_n = f$ is

$$\lambda c_i - \sum_{j=1}^{n} c_j \int_a^b k(x_i, y)\, \ell_j(y)\, dy = \int_a^b k(x_i, y) f(y) dy , \quad i = 1, \ldots, n \tag{2.49}$$

The solution ϕ_n is given by

$$\phi_n(x) = \frac{1}{\lambda}\left[f(x) + \sum_{j=1}^{n} c_j\, \ell_j(x) \right] \tag{2.50}$$

When analyzing this degenerate kernel method within the context of the space $C[a, b]$, the error depends on

$$\| K - K_n \| = \underset{a \le x \le b}{Max} \int_a^b | k(x, y) - k_n(x, y) |^2 dy \tag{2.51}$$

Piecewise Linear Interpolation

Consider the interval $[a, b]$, and let $h = (b - a)/n$, $x_i = a + i h$, $i = 0, 1, \ldots, n$.

Given a function $g \in C[a, b]$, we interpolate g at the node points $\{x_i\}$ using piecewise linear interpolation. For $i = 1, 2, \ldots, n$, define

$$P_n\, g(x) = \frac{(x_i - x) g(x_{i-1}) + (x - x_{i-1}) g(x_i)}{h}, \quad x_{i-1} \le x \le x_i \tag{2.52}$$

This defines a projection operator $P_n : C[a, b] \to C[a, b]$, with $\| P_n \| = 1$. Note that we

$$\left(f,\beta_i\right)=\int_a^b f(y)k\left(x_i,y\right)dy \tag{2.56}$$

These must be evaluated over the entire interval $\left[a,b\right]$.

For the error in this degenerate kernel method, we apply equation (2.49) to obtain

$$\| K-K_n \| \le \frac{h^2(b-a)}{8}\left[\int_a^b \max_{a\le x\le b}\left|\frac{\partial^2 k(x,y)}{\partial x^2}\right| dy\right] \tag{2.57}$$

Thus the error in convergence of $\phi_n=\left(\lambda-K_n\right)^{-1} f$ is bounded by $c\,h^2$ for some $c>0$.

The above bound assumes $k(x,y)$ is twice continuously differentiable with respect to x.

Finally, from equation (2.21) and equation (2.22),

$$\| (\lambda-K_n)^{-1} \| \le \frac{1}{\lambda}\left[1+\| A^{-1} \| \int_a^b \underset{a\le x\le b}{Max} | k(x,y) | dy\right] \tag{2.58}$$

with
$$\| A^{-1} \| = \underset{0\le i\le n}{Max} \sum_{j=0}^{n} | \alpha_{ij} | \ .$$

Example 2.2

Consider again the integral equation (2.23) with the unknown functions $\phi^{(i)}(x)$ of equation (2.36). Numerical results for varying b and n are given in Table 2.4 and 2.5. The columns labeled " Ratio" give the ratio of successive maximum errors. Recall the notation

$$E_i = \| \phi^{(i)}-\phi_n^{(i)} \|_\infty , \quad i=1,2$$

We give error bounds B_i that are based on equation (2.10), in the same manner as was done for the earlier degenerate kernel example in Tables 2.1 and 2.2.

In addition, we give the condition numbers for the coefficient matrix A of equation (2.49), and the reader should compare these results with those in Table 2.3.

Note that the condition numbers for $b = 2$ are comparable to those for $b = 1$, in contrast to the results in Table 2.3. Later, in Chapter Three, we give a rigorous justification that the size of b does not affect the size of $cond\left(A\right)$ for the linear system (2.49) when using piecewise polynomial interpolation.

Note that the errors $\left\| \phi^{(i)} - \phi_n^{(i)} \right\|_\infty$ are decreasing by a factor of approximately 4 whenever n is doubled, which is consistent with equation (2.57).

Table 2.4. Piecewise linear degenerate kernel

$b = 1, \quad \lambda = 5$

n	E_1	*Ratio*	B_1	E_2	*Ratio*	B_2	$cond\left(A_n\right)$
4	4.24E-4		2.32E-3	1.10E-3		1.95E-3	1.65
8	1.11E-4	3.8	5.79E-4	2.89E-4	3.8	4.88E-4	1.72
16	2.83E-5	3.9	1.45E-4	7.40E-5	3.9	1.22E-4	1.77
32	7.15E-6	4.0	3.62E-5	1.87E-5	4.0	3.05E-5	1.80
64	1.80E-6	4.0	9.05E-6	4.71E-6	4.0	7.62E-6	1.81

Table 2.5. Piecewise linear degenerate kernel

$b = 2, \quad \lambda = 75$

n	E_1	*Ratio*	B_1	E_2	*Ratio*	B_2	$cond(A_n)$
4	1.45E-4		8.41E-1	2.84E-2		8.76E-1	1.34
8	5.59E-5	2.6	1.49E-1	8.69E-3	3.3	1.55E-1	1.36
16	1.73E-5	3.2	3.48E-2	2.41E-3	3.6	3.62E-2	1.42
32	4.79E-6	3.6	8.55E-3	6.36E-4	3.8	8.91E-3	1.48
64	1.26E-6	3.8	2.13E-3	1.63E-4	3.9	2.22E-3	1.52

Piecewise linear interpolation does not converge rapidly, but it illustrates the general idea of the use of piecewise polynomial interpolation of any degree. For more rapidly convergent method, use piecewise polynomial interpolation with polynomials of degree $p \geq 0$. With such interpolation, it is straightforward to show that the error $\| \phi - \phi_n \|_\infty$ is $O(h^{p+1})$, provided and are sufficiently differentiable.

Chapter Three

Projection Methods

To solve approximately the integral equation

$$\lambda \phi(x) - \int_a^b k(x,y)\phi(y)dy = f(x), \qquad a \le x \le b \tag{3.1}$$

choose a finite dimensional family of functions that is believed to contain a function $\widetilde{\phi}(y)$ close to the true solution $\phi(y)$. The desired numerical solution $\widetilde{\phi}(y)$ is selected by having it satisfy equation (3.1) approximately. There are various senses in which $\widetilde{\phi}(y)$ can be said to " satisfy equation (3.1) approximately " , and these lead to different types of methods. The most popular of these are collocation methods are formulated in an abstract framework using functional analysis, they all make essential use of projection operators. Since the error analysis is most easily carried out within such a functional analysis framework, we refer collectively to all such methods as projection methods.

We write the integral equation (3.1) in the operator form

$$(\lambda - K)\phi = f$$

and the operator K is assumed to be compact on a Banach space X . In practice, we choose a sequence of finite dimensional subspaces $X_n \subset X$, $n \ge 1$, with X_n having dimension n . Let X_n have a basis $\{\varphi_1, \dots, \varphi_n\}$. We seek a function $\phi_n \in X_n$, and it can be written as

$$\phi_n(x)=\sum_{j=1}^{n} c_j\varphi_j(x), \qquad a\le x\le b \tag{3.2}$$

This is substituted into (3.1), and the coefficients $\{c_1,\dots,c_n\}$ are determined by forcing the equation to be almost exact in some sense to be specified below. Introduce

$$r_n(x)=\lambda\,\phi_n(x)-\int_a^b k(x,y)\phi_n(y)dy-f(y)$$

$$=\sum_{j=1}^{n} c_j\left\{\lambda\varphi_j(x)-\int_a^b k(x,y)\varphi_j(y)dy\right\}-f(x), \qquad a\le x\le b \tag{3.3}$$

This is called the residual in the approximation of the equation when using $\phi\approx\phi_n$.

Symbolically,

$$r_n=(\lambda-K)\phi_n-f$$

The coefficients $\{c_1,\dots,c_n\}$ are chosen by forcing $r_n(x)$ to be approximately zero in some sense to be specified later in this section.

Let X be a Banach space, and let $\{X_n \mid n\ge 1\}$ be a sequence of finite dimensional subspaces, say of dimension n. Let $P_n : X\to X_n$ be a bounded projection operator. i.e. P_n is a bounded linear operator from X to X_n with

$$P_n\,\phi=\phi \qquad \text{for all } \phi\in X_n .$$

Then $P_n^2=P_n$,

$$\| P_n \|=\| P_n^2 \|\le\| P_n \|^2$$

implies

$$\| P_n \| \geq 1 \quad . \tag{3.4}$$

The projection method for solving $(\lambda - K) \phi = f$ is: solve

$$(\lambda - P_n K) \phi_n = P_n f \quad . \tag{3.5}$$

An equivalent formulation is: pick ϕ_n such that

$$P_n (\lambda - K) \phi_n = P_n f , \quad \phi_n \in X_n \quad . \tag{3.6}$$

This says to pick that ϕ_n in X_n for which "the components in X_n of $(\lambda - K) \phi_n$ and f will agree " .

Theorem 3.1

Assume $K : X \rightarrow X$ is bounded, with X is a Banach space, and assume $\lambda - K : X \xrightarrow[onto]{1-1} X$.

Further assume

$$\| K - P_n K \| \rightarrow 0 \qquad \text{as} \qquad n \rightarrow \infty \tag{3.7}$$

Then for all sufficiently large n, say $n \geq N$, the operator $(\lambda - P_n K)^{-1}$ exists as a bounded operator from X to X . Moreover, it is uniformly bounded :

$$\sup_{n \geq N} \| (\lambda - P_n K)^{-1} \| < \infty \tag{3.8}$$

For the solutions of equation (3.5) and $(\lambda - K) \phi = f$,

$$\phi - \phi_n = \lambda (\lambda - P_n K)^{-1} (\phi - P_n \phi) \tag{3.9}$$

$$\frac{| \lambda |}{\| \lambda - P_n K \|} \| \phi - P_n \phi \| \leq \| \phi - \phi_n \| \leq | \lambda | \| (\lambda - P_n K)^{-1} \| \| \phi - P_n \phi \| \tag{3.10}$$

This leads to $\| \phi - \phi_n \|$ converging to zero at exactly the same speed as $\| \phi - P_n \phi \|$.

Proof :

(a) Pick N such that

$$\varepsilon_N \equiv \sup_{n \geq N} \| K - P_n K \| < \frac{1}{\| (\lambda - K)^{-1} \|} .$$

This inverse $[\ I + (\lambda - K)^{-1} (K - P_n K\ )\ ]^{-1}$ exists and is uniformly bounded by the geometric series theorem

$$\left\| \left[I + (\lambda - K)^{-1} (K - P_n K\ ) \right]^{-1} \right\| \leq \frac{1}{1 - \varepsilon_N \| (\lambda - K)^{-1} \|}$$

Using

$$\begin{aligned} \lambda - P_n K &= (\lambda - K) + (K - P_n K) \\ &= (\lambda - K) \left[I + (\lambda - K)^{-1} (K - P_n K) \right] \end{aligned}$$

$(\ \lambda - P_n K\ )^{-1}$ exists,

$$(\ \lambda - P_n K\ )^{-1} = [\ I + (\lambda - K)^{-1} (K - P_n K\ )\ ]^{-1} (\lambda - K)^{-1}$$

$$\| (\ \lambda - P_n K\ )^{-1} \| \leq \frac{\| (\lambda - K)^{-1} \|}{1 - \varepsilon_N \| (\lambda - K)^{-1} \|} \equiv M \qquad (3.11)$$

This shows equation (3.8).

(b) For the error formula (3.9), multiply $(\ \lambda - K\ )\phi = f$ by P_n, and then rearrange to obtain

$$(\ \lambda - P_n K\ )\phi = P_n f + \lambda\ (\ \phi - P_n \phi\ )$$

Subtract $(\ \lambda - P_n K\ )\phi_n = P_n f$ to get

$$(\lambda - P_n K)(\phi - \phi_n) = \lambda(\phi - P_n \phi) \tag{3.12}$$

$$\phi - \phi_n = \lambda(\lambda - P_n K)^{-1}(\phi - P_n \phi)$$

which is equation (3.9). Taking norms and using equation (3.11),

$$\| \phi - \phi_n \| \le |\lambda| M \| \phi - P_n \phi \| \quad . \tag{3.13}$$

Thus $P_n \phi \to \phi$, then $\phi_n \to \phi$ as $n \to \infty$.

(c) The upper bound in equation (3.10) follows directly from equation (3.9), as we have just seen. The lower bound follows by taking bounds in equation (3.12), to obtain

$$|\lambda| \| \phi - P_n \phi \| \le \| \lambda - P_n K \| \| \phi - \phi_n \|$$

This equivalent to the lower bound in equation (3.10) .

To obtain a lower bound that is uniform in n, note that for $n \ge N$,

$$\begin{aligned} \| \lambda - P_n K \| &\le \| \lambda - K \| + \| K - P_n K \| \\ &\le \| \lambda - K \| + \varepsilon_N \end{aligned}$$

The lower bound in equation (3.10), can now be replaced by

$$\frac{|\lambda|}{\| \lambda - K \| + \varepsilon_N} \| \phi - P_n \phi \| \le \| \phi - \phi_n \|$$

Combining this and equation (3.13), we have

$$\frac{|\lambda|}{\| \lambda - K \| + \varepsilon_N} \| \phi - P_n \phi \| \le \| \phi - \phi_n \| \le |\lambda| M \| \phi - P_n \phi \| \tag{3.14}$$

This shows that ϕ_n converges to ϕ if and only if $P_n \phi$ converges to ϕ .Moreover, if convergence does occur, then $\| \phi - P_n \phi \|$ and $\| \phi - \phi_n \|$ tend to zero with exactly the

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same speed .

To apply theorem (3.1),we need to know whether $\| K - P_n K \| \to 0$ as $n \to \infty$.

The following lemmas address this question.

Lemma 3.1

Let X , Y be Banach spaces, and let $A_n : X \to Y$, $n \geq 1$ be a sequence of bounded linear operators. Assume $\{ A_n \, \phi \}$ converges for all $\phi \in X$. Then the convergence is uniform on compact subsets of X .

Proof :

By the principle of uniform boundedness theorem , the operators A_n are uniformly bounded:

$$M \equiv \sup_{n \geq 1} \| A_n \| < \infty$$

The functions A_n are also equicontinuous :

$$\| A_n \phi - A_n f \| \leq M \| \phi - f \|$$

Let S be a compact subset of X . Then $\{ A_n \}$ is a uniformly bounded and equicontinuous family of functions on the compact set S ; it is then a standard result of analysis that $\{ A_n \, \phi \}$ is uniformly convergent for $\phi \in S$. □

Lemma 3.2

Let X be a Banach space, let $\{P_n\}$ be a family of bounded projections on X with

$$P_n \phi \to \phi \qquad \text{as} \quad n \to \infty, \quad \text{for all} \quad \phi \in X. \tag{3.15}$$

Let $K : X \to X$ be compact . Then

$$\| K - P_n K \| \to 0 \quad \text{as} \quad n \to \infty,$$

Proof :

From the definition of operator norm,

$$\| K - P_n K \| = \sup_{\| \phi \| \leq 1} \| K \phi - P_n K \phi \| = \sup_{z \in K(U)} \| z - P_n z \|$$

with $K(U) = \{ K \phi \mid \| \phi \| \leq 1 \}$. The set $K(U)$ is compact. Therefore, by the preceding Lemma (3.1) and the assumption (3.15)

$$\sup_{z \in K(U)} \| z - P_n z \| \to 0 \text{ as } n \to \infty$$

□

3.1 The Collocation Method

Pick distinct node points $x_1, \dots, x_n \in [a, b]$, and require

$$r(x_i) = 0, \qquad i = 1, 2, \dots, n \tag{3.16}$$

This leads to determining $\{c_1, \dots, c_n\}$ as the solution of the linear system

$$\sum_{j=1}^{n} c_j \left[\lambda \varphi_j(x_i) - \int_a^b k(x_i, y) \varphi_j(y) dy \right] = f(x_i), \qquad i = 1, 2, \dots, n \tag{3.17}$$

As a part of writing equation (3.17) in more abstract form, we introduce a projection operator P_n that maps $X = C[a,b]$ onto X_n. Given $\phi \in C[a,b]$,define $P_n \phi$ to be that element of X_n that interpolates ϕ at the nodes $\{x_1, \ldots, x_n\}$. This means writing

$$P_n \phi(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x)$$

with the coefficients $\{\alpha_j\}$ determined by solving the linear system

$$\phi(x_i) = \sum_{j=1}^{n} \alpha_j \varphi_j(x_i), \qquad i = 1, \ldots, n$$

This linear system has a unique solution if

$$\det\left[\varphi_j(x_i)\right] \neq 0 \tag{3.18}$$

By a simple argument, this condition also implies that the functions $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ are an independent set over $[a,b]$ .In the case of polynomial interpolation for functions of one variable, the determinant in equation (3.18) is referred to as the Vandermonde determinant.

To see more clearly that P_n is linear, and to give a more explicit formula, we introduce a new set of basis functions. For each i, $1 \le i \le n$, let $\ell_i \in X_n$ be that element that satisfies the interpolation conditions

$$\ell_i(x_j) = \delta_{ij}, \qquad j = 1, \ldots, n$$

By equation (3.18), there is a unique such ℓ_i , and the set $\{\ell_1, \ldots, \ell_n\}$ is a new basis for X_n. With polynomial interpolation, such functions ℓ_i are called Lagrange basis functions.

Piecewise Linear Interpolation

Recall the definition of piecewise linear interpolation given in and following

$$P_n g(x)=\frac{(x_i-x)\,g(x_{i-1})+(x-x_{i-1})\,g(x_i)}{h}\quad,\ x_{i-1}\le x\le x_i\ .$$

Let $[a,b]$ and $n\ge 1$,and define $h = b-a/n$,

$$x_j=a+jh\,,\quad j=0,1,\dots,n$$

The subspace X_n is the set of all functions that are piecewise linear on $[a,b]$, with breakpoints $\{x_0,x_1,\dots,x_n\}$. Its dimension is $n+1$.

Introduce the Lagrange basis functions for piecewise linear interpolation:

$$\ell_i(x)=\begin{cases}1-\dfrac{|x-x_i|}{h}, & x_{i-1}\le x x\le x_i\\[2ex] 0, & otherwise\end{cases}\tag{3.24}$$

with the obvious adjustment of the definition for $\ell_0(x)$and $\ell_n(x)$. The projection operator is defined by

$$P_n\phi(x)=\sum_{i=0}^{n}\phi(x_i)\,\ell_i(x)\tag{3.25}$$

For convergence of $P_n\phi$,

$$\|\phi-P_n\phi\|_\infty\le\begin{cases}\omega(\phi,h), & \phi\in C[a,b]\\ \dfrac{h^2}{8}\|\phi''\|_\infty, & \phi\in C^2[a,b]\end{cases}\tag{3.26}$$

This shows that $P_n\phi\to\phi$ for all $\phi\in C[a,b]$. For any compact operator

$K:C[a,b]\to C[a,b]$, Lemma (3.2) implies $\| K-P_n K \|\to 0$ as $n\to\infty$. Therefore the results of Theorem (3.1) can be applied directly for large n, say $n\geq N$, the equation $(\lambda-P_n K)\phi_n=P_n f$ has a unique solution ϕ_n for each $f\in C[a,b]$, and by equation (3.13),

$$\| \phi-\phi_n \|_\infty \leq |\lambda| M \| \phi-P_n \phi \|_\infty \quad .$$

For $\phi\in C^2[a,b]$,

$$\| \phi-\phi_n \|_\infty \leq |\lambda| M \frac{h^2}{8} \| \phi'' \|_\infty \quad . \tag{3.27}$$

The linear system (3.18) takes the simpler form

$$\lambda \phi_n(x_i)-\sum_{j=1}^{n} \phi_n(x_j)\int_a^b k(x_i,y)\ell_j(y)dy=f(x_i), \qquad i=0,1,\ldots,n \tag{3.28}$$

The integral can be simplified. For $j=1,\ldots,n-1$,

$$\int_a^b k(x_i,y)\ell_j(y)dy=\frac{1}{h}\int_{x_{j-1}}^{x_j} k(x_i,y)(y-x_{j-1})dy+\frac{1}{h}\int_{x_j}^{x_{j+1}} k(x_i,y)(x_j-y)\,dy \tag{3.29}$$

The integrals for $j=0$ and $j=n$ are modified accordingly.

There is some interest in looking more carefully at the operator $P_n K$. Using equation (3.25), we write

$$P_n K\phi(x)=\int_a^b k_n(x,y)\phi(y)dy \tag{3.30}$$

$$k_n(x,y)=\sum_{i=0}^{n} k(x_i,y)\ell_i(x) \tag{3.31}$$

This shows $P_n K$ is degenerate kernel integral operator, and in fact, it is the degenerate kernel introduce in equation (2.34) . Using equation (3.30),

$$\| K-P_n K \| = \underset{a \le x \le a}{Max} \int_a^b | k(x,y)-k_n(x,y) | \, dy \tag{3.32}$$

If $k(x,y)$ is twice continuously differentiable with respect to x, uniformly for $a \le y \le b$,

then

$$\| K-P_n K \| \le \frac{h^2}{8} \int_a^b \underset{a \le x \le b}{Max} \left| \frac{\partial^2 k(x,y)}{\partial x^2} \right| dy \tag{3.33}$$

Example 3.1

Recall the integral equation

$$\lambda \phi(x) - \int_0^b e^{xy} \phi(y) \, dy = f(x), \quad 0 \le x \le b \tag{3.34}$$

which was used as a numerical example in section 2.2 .1 and 2.2.3 in Chapter Two.

We use the same two unknowns as were used previously,

$$\phi^{(1)}(x) = e^{-x} \cos(x), \qquad \phi^{(2)}(x) = \sqrt{x}, \quad 0 \le x \le b \tag{3.35}$$

Table 3.1. Example of piecewise linear collocation for solving equation (3.34)

n	$E_n^{(1)}$	*Ratio*	$E_n^{(2)}$	*Ratio*
2	5.25E-3		2.32E-2	
4	1.31E-3	4.01	7.91E-3	2.93
8	3.27E-4	4.01	2.75E-3	2.88
16	8.18E-5	4.00	9.65E-4	2.85
32	2.04E-5	4.00	3.40E-4	2.84
64	5.11E-6	4.00	1.20E-4	2.83
128	1.28E-6	4.00	4.24E-5	2.83

The results of the use of piecewise linear collocation are given in Table 3.1.

The parameters are $b = 1$, $\lambda = 5$, as in Table 2.4 in Chapter Two. The errors given in the Table are the maximum errors on the collocation node points,

$$E_n^{(k)} = \max_{0 \le i \le n} \left| \phi^{(k)}(x_i) - \phi_n^{(k)}(x_i) \right|$$

The column labeled *Ratio* is the ratio of the successive of $E_n^{(k)}$ as n is doubled.

The function $\phi^{(2)}(x)$ is not continuously differentiable on $[0, b]$, and we have no reason to expect a rate of convergence of $O(h^2)$. Empirically, the errors $E_n^{(2)}$ appear to be $O(h^{1.5})$. From Theorem (3.1) we know that $\| \phi^{(2)} - \phi_n^{(2)} \|_\infty$ converges at exactly the same speed as $\| \phi^{(2)} - P_n \phi_n^{(2)} \|_\infty$, and it can be shown that the latter is only $O(h^{0.5})$. This apparent contradiction between the empirical and theoretical rates is due to $\phi_n(x)$ super convergent at the collocation node points : for the numerical solution $\phi_n^{(2)}$,

$$\lim_{n\to\infty} \frac{E_n^{(2)}}{\| \phi^{(2)} - \phi_n^{(2)} \|_\infty} = 0 \tag{3.36}$$

3.2 The Galerkin Method

Let $X = L_2[a,b]$ or some other Hilbert space, and let $(.,.)$ denote the inner product for X. Require r_n to satisfy

$$(r_n, \varphi_i) = 0, \quad i = 1, \dots, n \tag{3.37}$$

The left side is the Fourier coefficient of r_n associated with φ_i. If $\{\varphi_1, \dots, \varphi_n\}$ are the leading members of an orthonormal family $\Psi \equiv \{\varphi_1, \dots, \varphi_n, \dots\}$ that is complete in X, then equation (3.37) requires the leading terms to be zero in the Fourier expansion of r_n with respect to Ψ.

To find ϕ_n, apply equation (3.37) to equation (3.3). This yields the linear system

$$\sum_{j=1}^{n} c_j \left\{ \lambda (\varphi_j, \varphi_i) - (K\varphi_j, \varphi_i) \right\} = (f, \varphi_i), \quad i = 1, \dots, n \tag{3.38}$$

This is Galerkin's method for obtaining an approximate solution to equation (3.1).As a part of writing equation (3.38) in a more abstract form, we introduce a projection operator P_n that maps X onto X_n. For the general $\phi \in X$, define $P_n \phi$ to be the solution of the following minimization problem.

$$\| \phi - P_n \phi \| = \min_{z \in X_n} \| \phi - z \| \tag{3.39}$$

Since X_n is finite dimensional, it can be shown that this problem has a solution; and by X_n being an inner product space, the solution can be shown to be unique.

To obtain a better understanding of P_n, we give an explicit formula for $P_n\phi$. Introduce a new basis $\{\psi_1,\ldots,\psi_n\}$ for X_n by using the Gram-Shmidt process to create an orthonormal basis from $\{\varphi_1,\ldots,\varphi_n\}$. The element ψ_i is a linear combination of $\{\varphi_1,\ldots,\varphi_{in}\}$, and moreover

$$(\psi_i,\psi_j)=\delta_{ij}, \quad i,j=1,\ldots,n$$

With this new basis, it is straightforward to show that

$$P_n\phi=\sum_{i=1}^{n}(\phi,\psi_i)\psi_i \tag{3.40}$$

This shows immediately that P_n is a linear operator.

With this formula, we can show the following results.

$$\|\phi\|^2=\|P_n\phi\|^2+\|\phi-P_n\phi\|^2 \tag{3.41}$$

$$\|P_n\phi\|^2=\sum_{i=1}^{n}|(\phi,\psi_i)|^2$$

$$(P_n\phi,f)=(\phi,P_nf), \qquad \phi,f\in X \tag{3.42}$$

$$((I-P_n)\phi,P_nf)=0, \quad \phi,f\in X \tag{3.43}$$

Because of the result (3.43), $P_n\phi$ is called the orthogonal projection of ϕ onto X_n. The operator P_n is called an orthogonal projection operator. The first result (3.40) leads to

$$\|P_n\|=1 \tag{3.44}$$

Using equation (3.43), we can show

$$\| \phi - z \|^2 = \| \phi - P_n \phi \|^2 + \| P_n \phi - z \|^2 , \quad z \in X_n \tag{3.45}$$

This shows $P_n \phi$ is the unique solution to equation (3.39).

We note that

$$P_n z = 0 \quad \text{if and only if} \quad (z, \varphi_i) = 0 , \quad i = 1, \ldots, n \tag{3.46}$$

With P_n, we can rewrite equation (3.37) as

$$P_n r_n = 0$$

or equivalently,

$$P_n(\lambda - K)\phi_n = P_n f , \quad \phi_n \in X_n \tag{3.47}$$

3.3 Regularization of the Solution

Consider the regularity (that is, the differentiability) of the solutions of the integral equation

$$\lambda \phi(x) - \int_a^b k(x, y)\phi(y)dy = f(x), \qquad a \le x \le b \tag{3.48}$$

Let $k(x, y)$ be m-times continuously differentiable with respect to x, for all $y \in [a, b]$.

Then for $1 \le j \le m$, the solution $\phi \in C^j[a, b]$ if and only if $f \in C^j[a, b]$.

From the equation

$$\phi = \frac{1}{\lambda} f + \frac{1}{\lambda} K \phi \tag{3.49}$$

we consider $z = K\phi$ as a new and smoother unknown function. If

$$\phi = \frac{1}{\lambda}(f + z) \tag{3.50}$$

is substituted into the original equation $(\lambda - K)\phi = f$, then after simplication we have

$$(\lambda - K)z = K f. \tag{3.51}$$

but now the right –hand side of equation (3.51) is at least as smooth as the kernel function $k(x, y)$ when considered as a function of x .Then the formula (3.50) can be used to obtain an approximate solution of the original equation.

Apply a projection method to equation (3.51),

$$(\lambda - P_n K)z_n = P_n K f \tag{3.52}$$

and then define

$$\widetilde{\phi}_n = \frac{1}{\lambda}\{f + z_n\} \tag{3.53}$$

We begin by showing that

$$(\lambda - P_n K)\widetilde{\phi}_n = f \tag{3.54}$$

Substituting equation (3.53) into the left side of equation (3.54),

$$\begin{aligned}(\lambda - P_n K)\widetilde{\phi}_n &= (\lambda - P_n K)\frac{1}{\lambda}\{f + z_n\} \\ &= \frac{1}{\lambda}[\lambda f + \lambda z_n - P_n K f - P_n K z_n] \\ &= \frac{1}{\lambda}\left[\lambda f - P_n K f + \underbrace{\lambda z_n - P_n K z_n}_{= P_n K f}\right] \\ &= f\end{aligned}$$

Thus to analyze the convergence of $\widetilde{\phi}_n$,we can work directly with equation (3.54). We use the same hypotheses as assumed for the projection method, namely that $\| K - P_n K \| \to 0$ as $n \to \infty$. As in Theorem (3.1), we can then conclude the existence and uniform boundedness of $(\lambda - P_n K)^{-1}$. For the error,

$$\begin{aligned}\phi - \widetilde{\phi}_n &= (\lambda - K)^{-1} f - (\lambda - P_n K)^{-1} f \\ &= (\lambda - P_n K)^{-1} [(\lambda - P_n K) - (\lambda - K)] (\lambda - K)^{-1} f \\ &= (\lambda - P_n K)^{-1} [K - P_n K] \phi\end{aligned}$$

and

$$\| \phi - \widetilde{\phi}_n \| \leq \| (\lambda - P_n K)^{-1} \| \| K - P_n K \| \| \phi \|$$

In this, the speed of convergence is apparently independent of the smoothness of ϕ.

3.4 Stability Of Linear Systems

In this section we assume that the integral equation $(\lambda - K)\phi = f$ is uniquely solvable for all f in the appropriate Banach space X.

Letting $\{K_n\}$ denote a sequence of approximating integral operators for which $\| K - K_n \| \to 0$, we can include both projection and degenerate kernel methods. Let $N(\lambda)$ be a minimum value for which

$$\| K - K_n \| < \frac{1}{\| (\lambda - K)^{-1} \|}, \quad n \geq N(\lambda) \quad . \tag{3.55}$$

As in Theorem (2.1) and (2.2), this infers the existence of $(\lambda - K_n)^{-1}$. We also assume that $N(\lambda)$ is so chosen that .

$$\left\| (\lambda - K_n)^{-1} \right\| \le B, \quad n \ge N(\lambda), \tag{3.56}$$

for some finite B .

Both the degenerate kernel methods and the projection methods reduce to equivalent non-singular linear systems,

$$A_n \; \breve{\alpha} = \breve{\gamma} \tag{3.57}$$

with $\breve{\alpha}$ and $\breve{\gamma}$ column vectors of length n and A_n a matrix of order n. We want to know the effect on $\breve{\alpha}$ of small perturbations in $\breve{\gamma}$ and A_n. To do this, we will derive bounds

$$\left\| A_n^{-1} \right\| \le B_n, \quad n \ge N(\lambda). \tag{3.58}$$

The norm used will vary with each particular case.

To see that equation (3.58) is sufficient for the stability analysis, first consider the case

$$A_n \left(\breve{\alpha} + \breve{\delta} \, \breve{\alpha} \right) = \breve{\gamma} + \breve{\delta} \, \breve{\gamma}$$

in which γ has been perturbed. Then $\breve{\delta} \, \breve{\alpha} = A_n^{-1} \, \breve{\delta} \, \breve{\gamma}$,

$$\left\| \breve{\delta} \, \breve{\alpha} \right\| \le B_n \left\| \breve{\delta} \, \breve{\gamma} \right\|, \quad n \ge N(\lambda), \tag{3.59}$$

thus bounding the effect, of the perturbation in $\breve{\gamma}$. Now consider the system

$$\widetilde{A}_n \left(\breve{\alpha} + \breve{\delta} \, \breve{\alpha} \right) = \breve{\gamma} \tag{3.60}$$

in which $\widetilde{A}_n$ is a perturbed version of A_n . We further assume that

$$\left\| A_n - \widetilde{A}_n \right\| < \frac{1}{B_n} \quad . \tag{3.61}$$

Then $\widetilde{A}_n^{-1}$ exists since

$$\widetilde{A}_n = A_n \left[1 - A_n^{-1} \left(A_n - \widetilde{A}_n \right) \right]$$

is invertible and then

$$\| \widetilde{A}_n^{-1} \| \le \frac{B_n}{1 - B_n \| A_n - \widetilde{A}_n \|}, \quad n \ge N(\lambda). \tag{3.62}$$

The perturbation $\delta \breve{\alpha}$ of equation (3.60) satisfies

$$\delta \breve{\alpha} = \widetilde{A}_n^{-1} \breve{\gamma} - A_n^{-1} \breve{\gamma} = \widetilde{A}_n^{-1} \left(A_n - \widetilde{A}_n \right) A_n^{-1} \breve{\gamma} ,$$

$$\delta \breve{\alpha} = \widetilde{A}_n^{-1} \left(A_n - \widetilde{A}_n \right) \breve{\alpha} ,$$

$$\| \delta \breve{\alpha} \| \le \| \widetilde{A}_n^{-1} \| \| A_n - \widetilde{A}_n \| \| \breve{\alpha} \| ,$$

$$\| \delta \breve{\alpha} \| \le \frac{B_n \| A_n - \widetilde{A}_n \| \| \breve{\alpha} \|}{1 - B_n \| A_n - \widetilde{A}_n \|} \quad . \tag{3.63}$$

Obtaining equation (3.58) is therefore quite sufficient for saying a great deal, as in equation (3.59) and equation (3.63). In addition, we see that the size of B_n is important in obtaining small perturbations.

Let $d(\lambda)$ denote the minimum distance of λ from 0 and from the eigenvalues of K. Then

$$\| (\lambda - K)^{-1} \| \ge \frac{1}{d(\lambda)}. \tag{3.64}$$

To prove this, we assume the contrary and derive a contradiction. Having assumed

$$d(\lambda) < \frac{1}{\| (\lambda - K)^{-1} \|},$$

There is a value λ_0 for which $(\lambda_0 - K)$ does not have a bounded inverse on X and for which

$$d(\lambda) = | \lambda - \lambda_0 | < \frac{1}{\| (\lambda - K)^{-1} \|}.$$

If $\lambda_0 \neq 0$, then it is an eigenvalue, if $\lambda_0 = 0$, then we know K^{-1}, if it exists, is an unbounded operator on X. Also

$$\| (\lambda - K) - (\lambda_0 - K) \| = | \lambda - \lambda_0 | < \frac{1}{\| (\lambda - K)^{-1} \|},$$

and by using the following theorem (Let X and Y be normed linear spaces, at least one of which is complete. Let $S, T \in L[X, Y]$, and $S^{-1} \in L[X, Y]$. Also assume

$$\| S - T \| < \frac{1}{\| S^{-1} \|}.$$

Then T^{-1} exists as a bounded linear operator from Y onto X ,

$$\| T^{-1} \| \leq \frac{\| S^{-1} \|}{1 - \| S^{-1} \| \, \| S - T \|} \qquad ,$$

$$\| S^{-1} - T^{-1} \| \leq \| S^{-1} \| \, \| T - S \| \, \| T^{-1} \| \qquad)$$

$(\lambda_0 - K)^{-1}$ exists as a bounded operator on X and Y ,contrary to the above restriction.

Having proven equation (3.64), we see that as λ becomes close to zero or to an eigenvalue of K, the bound of $(\lambda - K)^{-1}$ becomes infinite, " i.e.

$$d(\lambda) \mapsto 0 \Rightarrow \| (\lambda - k)^{-1} \| \to \infty. \qquad (3.65)$$

Thus the index $N(\lambda)$, based on equation (3.23), must also become infinite as $d(\lambda) \to 0$,

$$d(\lambda) \to 0 \Rightarrow N(\lambda) \to \infty. \qquad (3.66)$$

Even when $n \geq N(\lambda)$, if $d(\lambda)$ is quite small, then the bound B in equation (3.56) will be quite large, we will find that the bound B_n in equation (3.58) must also become unbounded as $d(\lambda) \to 0$. Thus as $d(\lambda)$ becomes smaller, the system equation (3.57) will become increasingly ill-conditioned.

To obtain equation (3.58) we will illustrate the necessary techniques by means of particular cases .

3.4.1 Collocation Method

The linear system $A_n \breve{\alpha} = \breve{\gamma}$ takes the form

$$\sum_{j=1}^{n} \alpha_j \left\{ \lambda \varphi_j(y_i) - \int_a^b K(y_i, y) \varphi_j(y) dy \right\} = f(y_i), \qquad 1 \leq i \leq n$$

refer back to the section 3.1 . The space is $X = C[a,b]$, as before.

To bound A_n^{-1}, let $\gamma_1, \ldots, \gamma_n$ be an arbitrary set of constants. Let f be a function for which

$$f(y_i) = \gamma_i, \quad i = 1, \ldots, n \quad ,$$

and

$$\| f \|_\infty = \underset{1 \leq i \leq n}{Max} | \gamma_i | \equiv \| \breve{\gamma} \|_\infty \ .$$

Then the solution $\breve{\alpha}$ of $A_n \breve{\alpha} = \breve{\gamma}$ satisfies

$$\phi_n(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x), \quad a \leq x \leq b, \tag{3.67}$$

and

$$(\lambda - P_n K)\phi_n = P_n f.$$

Thus

$$\| \phi_n \|_\infty \le \| (\lambda - P_n K)^{-1} \| \, \| P_n \| \, \| f \|_\infty \tag{3.68}$$

$$\le B \| P_n \| \, \| \bar{\gamma} \|_\infty .$$

Denote

$$\Gamma_n = [\varphi_j(y_i)], \quad \Gamma_n^{-1} = [c_{ij}],$$

matrices of order n. Recall that Γ_n is non-singular by condition equation (3.5).

From equation (3.67),

$$\phi_n(y_i) = \sum_{j=1}^{n} \alpha_j \varphi_j(y_i), \quad 1 \le i \le n \quad ,$$

$$\breve{\alpha} = \Gamma_n^{-1} \breve{\phi}_n , \quad \breve{\phi}_n = [\phi_n(y_1), \ldots, \phi_n(y_n)]^T .$$

Thus

$$\| \breve{\alpha} \|_\infty \le \| \Gamma_n^{-1} \| \, \| \phi_n \|_\infty$$

with

$$\| \Gamma_n^{-1} \| = \underset{1 \le i \le n}{Max} \sum_{j=1}^{n} | c_{ij} | \quad . \tag{3.69}$$

Combining this with equation (3.68) ,

$$\| \breve{\alpha} \| \le B \| P_n \| \, \| \Gamma_n^{-1} \| \, \| \breve{\gamma} \|_\infty ,$$

and thus

$$\| A_n^{-1} \| \le B \| P_n \| \, \| \Gamma_n^{-1} \| \quad , \tag{3.70}$$

using for $\| A_n^{-1} \|$ the same norm as in equation (3.69).This matrix norm is the operator norm induced by the maximum vector norm.

The bound equation (3.70) suggests we should pick our basis functions $\{\varphi_j(x)\}$ with considerable caution, the number $\| \Gamma_n^{-1} \|$ can grow quite rapidly with a poor choice, e.g. $\varphi_j(x) = x^{j-1}, \quad 1 \le j \le n,$ in $C[0,1]$. If we use the cardinal functions for interpolation,

$$\varphi_i = \ell_i \ , \qquad i = 1,2,\dots,n \ ,$$

with $\ell_1,\dots,\ell_n$ defined as preceding equation (3.19), then Γ_n is the identity matrix and

$$\| A_n^{-1} \| \le B \, \| P_n \|. \tag{3.71}$$

For the polynomial interpolation in which $P_n\phi$ is the interpolating polynomial of degree $\le n-1$ on some particular set of nodes, it follows that

$$\| P_n \| \to \infty \quad as \quad n \to \infty .$$

The choice for which this sequence grows most slowly is to use as nodes the zeros of the n^{th} degree Chebyshev polynomial on [a, b],

$$T_n(x) = \cos\left[n \cos^{-1}\left(\frac{2x-a-b}{b-a} \right) \right], \qquad a \le x \le b \ . \tag{3.72}$$

These are

$$x_i = \cos\left[\frac{2i-1}{2n}\pi \right] \quad , \quad i = 1,\dots,n$$

Then

$$\| P_n \| = O\left(\log n \right) \ , \tag{3.73}$$

i.e. $\| P_n \|$ is directly proportional to log n.

Since system (2.49) for the degenerate kernel formed by interpolation coincides with the coefficient matrix of equation (3.19), we have for the system $A_n \alpha = \gamma$ of equation (2.49) the bound of equation (3.71). Thus the system is well-conditioned if $\| P_n \|$ is not large and $\| (\lambda - K)^{-1} \|$ is not large.

3.4.2 The Galerkin Method

We will first consider the linear system (3.38) with $\{\psi_1, \dots, \psi_n\}$ equal the orthonormal family $\{\vartheta_1, \dots, \vartheta_n\}$. In this case our linear system takes the simpler form

$$\lambda \beta_i - \sum_{j=1}^{n} \beta_j (\vartheta_i, K \vartheta_j) = (f, \vartheta_i), \qquad i = 1, \dots, n \ , \tag{3.74}$$

with

$$\phi_n(x) = \sum_{j=1}^{n} \beta_j \vartheta_j(x).$$

Symbolically, write the system (3.74) as $H_n \breve{\beta} = \breve{f}$ with

$$\tilde{f}^T \equiv [(f, \vartheta_1), \dots, (f, \vartheta_n)] \ .$$

The Banach space for solving $(\lambda - P_n K)\phi_n = P_n f$ is $X = L_2(a, b)$, also $X_n = Span\{\vartheta_1, \dots, \vartheta_n\}$. The appropriate norms are

$$\| \phi \|_2 = \sqrt{\int_a^b | \phi(y) |^2 dy} \quad , \quad \phi \in L_2(a, b),$$

$$\| \breve{\gamma} \|_2 = \sqrt{\sum_{j=1}^{n} | \gamma_j |^2} , \quad \breve{\gamma} \in \ell^2 (u) .$$

Define $Q : \ell^2 (n) \rightarrow X_n$ by

$$Q_n \breve{\gamma} = \sum_{j=1}^{n} \gamma_j \Psi_j , \quad \breve{\gamma} \in \ell^2 (n) \quad .$$

Then

$$\| Q_n \breve{\gamma} \|_2 = \| \breve{\gamma} \|_2 , \quad \breve{\gamma} \in \ell^2 (n),$$

and thus

$$\| Q_n \| = \| Q_n^{-1} \| = 1 \quad .$$

To bound $\| H_n^{1} \|$, we bound the solution $\breve{\beta}$ of $H_n \breve{\beta} = \breve{\gamma}$, for arbitrary $\breve{\gamma}$. Given $\breve{\gamma}$, let $\breve{\gamma} = Q_n \breve{\gamma}$. Then $\phi_n = (\lambda - P_n K)^{-1} f$, using $f = P_n f$, and $\breve{\beta}_n = Q_n^{-1} \phi_n$ implies

$$H_n^{-1} \breve{\gamma} = \breve{\beta} = Q_n^{-1} (\lambda - P_n K)^{-1} Q_n \breve{\gamma} . \tag{3.75}$$

Taking norms,

$$\| H_n^{-1} \| \leq \| (\lambda - P_n K)^{-1} \| \leq B , \tag{3.76}$$

which says that H_n is well- conditioned as long as $\| (\lambda - P_n K)^{-1} \|$ is not too large, this is true if $\| (\lambda - K)^{-1} \|$ is not too large and n is sufficiently large.

To deal with general case of equation (3.38) in which $\{ \psi_1 , \psi_2 , \ldots , \psi_n \}$ need not be orthonormal, we introduce some auxiliary matrices,

$$D_n = [(\psi_i , \vartheta_j)], \; \Gamma_n = [(\psi_i , \psi_j)], \; B_n = \Gamma^{-1} D \quad .$$

The matrices D_n and B_n are change of basis matrices for changing between $\{\psi_1, \psi_2, \ldots, \psi_n\}$ and the orthonormal basis $\{\vartheta_1, \ldots, \vartheta_n\}$.

$$\psi_i = \sum_{j=1}^{n} (\psi_i, \vartheta_j)\vartheta_j, \quad \vartheta_i = \sum_{j=1}^{n} B_{ji}\psi_j \quad .$$

The matrix Γ is well-known Gram matrix for $\{\psi_1, \psi_2, \ldots, \psi_n\}$, and Γ is positive definite. These matrices satisfy the following relations,

$$\Gamma_n B_n = D_n, \quad D_n^* B_n = I_n, \quad D_n D_n^* = \Gamma_n \, . \tag{3.77}$$

Let A_n be the matrix of system (3.16), we will bound the solution $\breve{\alpha}$ of $A_n \breve{\alpha} = \breve{\gamma}$ for arbitrary $\breve{\gamma}$. Using the above matrices, $A_n \breve{\alpha} = \breve{\gamma}$ converts to

$$H_n \breve{\beta} = B_n^* \breve{\gamma}, \quad \breve{\beta} = D_n^* \breve{\alpha} \quad .$$

Combining these with equation (3.75), we get

$$A_n^{-1}\breve{\gamma} = \breve{\alpha} = (D_n^*)^{-1} Q_n^{-1} (\lambda - P_n K)^{-1} Q_n B_n^* \breve{\gamma} \quad .$$

Using equation (3.77), $B_n^* = D_n^{-1}$ and

$$A_n^{-1} = (D_n^*)^{-1} Q_n^{-1} (\lambda - P_n K)^{-1} Q_n D_n^{-1} \, . \tag{3.78}$$

Before taking norms, we need additional information on the matrix norm induced by $\ell^2(n)$. For any square matrix M, the operator norm induced by $\ell^2(n)$ is

$$\| M \| = \sqrt{r_\sigma (M M^*)} = \sqrt{r_\sigma (M^* M)} \quad .$$

For any square matrix E, $r_\sigma(E)$ is the maximum magnitude of eigenvalues of E, is called the spectral radius of E. Using the relation $D D^* = \Gamma$, we obtain

$$\left\| \left(D_n^* \right)^{-1} \right\| = \| D_n^{-1} \| = \sqrt{r_\sigma\left(\Gamma_n^{-1} \right)} = \| \Gamma_n^{-1} \|^{\frac{1}{2}} .$$

Also,

$$r_\sigma\left(\Gamma_n^{-1} \right) = \frac{1}{Minimum\ eigenvalue\ of\ \Gamma_n} .$$

Combining these results with equation (3.78), we obtain

$$\| A_n^{-1} \| \le \| \Gamma_n^{-1} \| \left\| \left(\lambda - - P_n K \right)^{-1} \right\| \le \| \Gamma_n^{-1} \| B \quad . \tag{3.79}$$

This result is basically the same as that obtained for collocation, equation (3.70), in our present case, $\| P_n \| = 1$, which is the only difference in the two results. We will illustrate this stability result by letting X_n be space of piecewise linear functions on $[a, b]$. Let $n > 1$, $h = (b - a)/(n - 1)$, $y_j = a + (j - 1)h$ for $j = 1, \dots, n$. Define

$$\Lambda(x) = \begin{cases} \frac{1}{h}(x + h), & -h \le x \le 0 , \\ \frac{1}{h}(h - x), & 0 \le x \le h , \\ 0 \quad , & |x| \ge h . \end{cases}$$

Also let $c_1 = c_n = \sqrt{\frac{3}{h}}$ and $c_j = \sqrt{\frac{3}{2h}}$ for $j = 2, \dots, n - 1$. Define

$$\varphi_j(x) = c_j \Lambda(x - y_j), \quad a \le x \le b, \quad j = 1, \dots, n .$$

These trivially from a basis for the piecewise linear functions on $[a, b]$ with nodes $y_1, \dots, y_n$. The family is not orthonormal, although close to it. In particular,

$$\| \vartheta_i \| = 1, \qquad i = 1, 2, \dots, n ,$$

Appendix

Theorem A.1 (Weierstrass Approximation Theorem)

Suppose f is defined and continuous on $\left[a,b\right]$. For each $\varepsilon>0$, there exists a polynomial $P(x)$, defined on $\left[a,b\right]$, with the property that

$$\mid f(x)-P(x)\mid<\varepsilon, \quad \text{for all} \quad x \text{ in } \left[a,b\right].$$

Theorem A.2 (Heine- Borel Theorem)

Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering. Thus is, if C is a collection of open sets such that $F\subset\bigcup\left\{O:O\in C\right\}$, then there is a finite collection $\left\{O_1,\ldots,O_n\right\}$ of sets in C such that $F\subset\bigcup\limits_{i=1}^{n}O_i$.

Theorem A.3 (Geometric Series Theorem)

Let X be a Banach space, and let A be a bounded operator from X into X , with

$$\| A \|<1$$

Then $I - A: X \underset{onto}{\overset{1-1}{\rightarrow}} X$ is bounded linear operator, and

$$\left\| \left(I-A\right)^{-1} \right\| \leq \frac{1}{I-\| A \|}$$

The series $\left(I-A\right)^{-1}=\sum\limits_{j=0}^{\infty}A^j$

Is called the Neumann series, under the assumption $\| A \|<1$, it converges in the space of bounded operators on X to X .

Theorem A.4 (Gerschgorin Circle Theorem)

Let A be an $n \times n$ matrix and let R_i denote the circle in the complex plane with center a_{ii} and radius $\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \left| a_{ij} \right|$, that is

$$R_i = \left\{ z \in C : \left| z - a_{ii} \right| \le \sum_{\substack{j=i \\ j \neq i}}^{n} \left| a_{ij} \right| \right\},$$

where C is used to denote the complex plane.

The eigenvalues of A are contained with in $R = \bigcup\limits_{i=1}^{n} R_i$.

Theorem A.5 (Principle of uniform boundedness)

Let $\{ A_n \}$ be a sequence of bounded linear operators from a Banach space X to a normed space Y . Further, assume

$$\lim_{n \to \infty} A_n x$$

exists in Y, for every $x \in X$. Then

$$\sup_n \| A_n \| < \infty$$

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ملخص

تميزت المعادلات التكاملية منذ نشأتها إلى هذا اليوم بعدد من المزايا جعلتها تحتل مكانه بارزة بين سائر التطبيقات الرياضية، فقد وجدت المعادلات التكاملية بادىء ذي بدء من اجل غايات تطبيقية خالصة، ونعني بها صياغة استراتيجية الحلول للمسائل العلمية وخاصة الهندسية والفيزيائية منها. وقد سعيت في كتابة هذه الرسالة الى تجسيد المباديء التي قامت عليها المعادلات التكاملية وخصائصها والجديد في تطورها وحاولت ما وسعني الجهد تحقيق ذلك كله ضمن الاعتبارات التالية

١. توجيه صياغة كل فصل في هذه الرسالة بحسب الاهداف والمشكلات التي تعالجها، وكذلك عرض الطرق والحلول ذات الصلة.

٢. تقديم التعريفات والشروح الكافية لمن تعوزهم المعرفة الرياضية ذات الصلة.

٣. عرض بعض الامثلة التطبيقية في مختلف مجالات استخدام المعادلات التكاملية وطرق حلها.

٤. اضافة شروح مفصلة واضحة للقطاعات والطرق الجديد.

٥. الاهتمام بالجوانب الحديثة في هذا المجال

ومن اجل تلبية الحاجات المختلفة للمهتمين والباحثين في المعادلات التكاملية قمت بتوزيع المادة إلى ثلاثة فصول، اسوق فيما يلي ملخصا مختصرا لكل منها:

الفصل الأول: المعادلات التكاملية ضمن (L_2 Kernel)

يقع هذا الفصل في ثلاث أجزاء، الجزء الأول يشمل بعد التمهيد على أنواع المعادلات التكاملية وخاصة معادلات فريدهولم التكاملية الخطية والتي تم التركيز في هذه الدراسة على النوع الثاني منها.

تم عرض ال (compact operator) والمعادلات التكاملية الخطية ضمن (compact integral operator) والتعريفات والنظريات ذات الصلة في الجزء الثاني. أما الجزء الثالث فقد خصص للبحث في نظرية فريدهولم الاساسية التي تهتم في وجود ووحدانية الحل لمعادلات فريدهولم التكاملية من النوع الثاني.

الفصل الثاني: (Degenerate kernel methods)

يقع هذا الفصل في جزأين، حيث يعالج الطرق العددية لحل معادلة فريدهولم الخطية من النوع الثاني واهم ما تم عرضة في هذا الفصل هو طريق التقريب ل (degenerate kernel) والتي تتمثل في (separable and degenerate schemes)، حيث تم عرض عديد من الأمثلة العددية توضح هذه الطرق.

الفصل الثالث: (Projection methods)

تم التركيز في هذا الفصل على طرق عدديه اخرى لحل معادلة فريدهولم التكاملية الخطية من النوع الثاني من اهمها (Projection methods)، حيث يقع هذا الفصل في أربع أجزاء، كان الاهتمام في الجزء الاول على (collection method) وهي طريقة من خلالها يمكن تحويل المعادلة التكاملية الى نظام من المعادلات الخطية والتي يسهل حلها بطرق الجبر الخطي.

اما الجزء الثاني فكان جل اهتمامنا عرض طريقة (Galerkin) لحل معادلة فريدهولم التكاملية الخطية من النوع الثاني عدديا في سطوح ضمن مستوى اقليدي (Euclidian plane).

انشاء الحلول بالطرق السابقة، بحث ال (convergence) واستقرار الحلول المتمثلة في انظمة خطية تم بحثها في الجزأين الثالث والرابع من هذا الفصل.

فإنني آمل ان يكون الجهد الذي بذلتة في اعداد هذه الرسالة قد حقق اهدافه المتوخاه، والله اسأل ان تحقق هذه الرسالة منفعة لقارئها تعوضني عن كل مشقة عانيتها، انه نعم المولى ونعم النصير.

ناهد سالم