Deanship of Graduate Studies

AL-Quds University

# Support Points of the set of Univalent Functions

By

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## Declaration:

I Certify that this submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis(or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed.............

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Date: 31- 5 - 2004

# Dedication

To my father, to my mother who resembles the spring of giving and sacrifice and to my brothers and sisters.

# Acknowledgement

Praise Be To God; the Creator of the Universe. Peace Be Upon His Messenger Prophet Mohammad, and On the Prophet's Posterity and All his Companions.

Upon the completion of this thesis, I would like to extend my sincere appreciation, respect and thanks, to Dr. Ibrahim Al-Grouz (who supervised my thesis), for his efforts in giving me the best advice and counseling, and utilizing his practical experience, in order to reach at a correct, accurate, and successful study.

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Abstract

Our main goal in the thesis is to study the support points of normalized univalent functions .We begin with special classes of functions in the set of normalized univalent functions such as the family of starlike functions($S^*$); The family of convex functions(K) ; The family of closed convex functions(CL), then the support points of these families are studied by presenting some theorems. Later on, some families of univalent functions are discussed like Typically real functions(T) and positive real functions($\mathcal{P}$).

Considerable effort has been devoted to the study of support points of normalized univalent functions. In general many factors play a role in the determination of support points of a given class of functions. The support points of these classes($S^*$,K,CL,T,$\mathcal{P}$) are identified , but this is not the case of the class of normalized univalent functions.

Some facts and properties of a function to be a support point of the family of normalized univalent functions proved at the end of this thesis.

According to this study, one might think the support points and extreme points coincide. However there is an example of extreme points which is not a support points. On the other hand there is an example of support points which is not extreme points .

# الملخص

الهدف الأساسي من هذه الرسالة هو دراسة الاقترانات المساندة لمجموعة الاقترانات المركبة الأحادية والتي تحقق بعض الشروط الخاصة( S ) حيث أننا بدأنا بدراسة بعض المجموعات الجزئية لهذه المجموعة مثل الاقترنات شبه النجمية($S^*$) ، الاقترانات المقعرة(K)، الاقترانات القريبة من التقعر(CL) و ذلك عن طريق إثبات العديد من النظريات. كذلك درسنا مجموعات جزئية أخرى.

هنالك جهد عظيم تم تخصيصه لدراسة الاقترانات المساندة لمجموعة الاقترنات الأحادية حيث تبين أن هنالك العديد من العوامل التي تحدد كون الاقتران مساندا للمجموعة أم لا و بينا العديد من الاقترانات المساندة لبعض المجموعات الجزئية لهذه المجموعة ، و أخيرا درسنا الاقترانات المساندة للمجموعة( S ) حيث أنها وحتى الآن لم يتم تحديد الاقترانات المساندة لهذه المجموعة و إنما هنالك نتائج جزئيه تدل على ذلك.

و يعتقد البعض بان الاقترانات المساندة هي نفسها الاقترانات القصوى، و هذا غير صحيح،حيث أوردنا مثالا لاقتران مساند و ليس من الاقترانات القصوى و مثالا لأحد الاقترانات القصوى و ليس اقترانا مساندا.

# Contents

# Introduction

Let $\Delta$ be the unit disk consists of all point $z$ of the complex plane $C$ of modulus $|z| < 1$. In the first chapter , we introduce the definition of univalent functions and basic results about univalent functions.

The theory of univalent functions is a classical field beginning at least as early as 1907 with the paper P. Koebe [24].

A number of survey articles have been written about the general theory of univalent functions and more specific developments and we mention [3],[7],[12],[17],[18], and [36] .

The book [17] by A.W.Goodman almost provides an encyclopedia about univalent functions and describes a very large number of results in the field, even on quite specialized topics.

We shall need the material of this chapter throughout the following chapters.

In the second chapter, we present the special families of univalent functions, Riemann's Theorem enables many problems in general domains $D$ to be reduced to problems in $\Delta$. Thus the class of the corresponding conformal maps or functions univalent in $\Delta$, acquires a special importance. We may normalize so that $f(0) = 0,\ f'(0) = 1$. Otherwise we may consider $g(z) = (f(z) - f(0))/f'(0)$ instead of $f$, since $g$ is univalent if and only if $f$ is univalent . We accordingly denote by S the class of functions $f(z) = z + \sum_{n=0}^{\infty} a_n z^n$ univalent in $\Delta$.

We introduce the subsets of S consisting of starlike functions $S^*$, convex functions K, and close to convex functions CL, we also introduce the class of functions having a positive real functions $\mathcal{P}$, and the typically real functions T, and obtain similar results for these classes.

Generalization of the families $S^*$ and K were introduced by M.S .Robertson in [35].The geometric characterization mentioned for close to convex functions is due to Z. Lewandowski[26,27].The class T of typically real functions was introduced and studied by W.Rogosinski[40].

In chapter three , we first present the definition of subordination and prove some initial facts about subordination.

Subordination was more formally introduced and studied by J.E.Littlewood[28] and later by Rogosinski[39], we first prove some basic properties(Littlewood 1925).

In the second part of this chapter we discuss the linear topological structure of $\mathcal{A}$, where $\mathcal{A}$ denotes the set of all functions analytic in the unit disk $\Delta$. The topology on $\mathcal{A}$ is given by a metric which is equivalent to the topology of uniform convergence on compact subsets of $\Delta$. We study the topological and convexity properties of the various subsets of $\mathcal{A}$. We introduce the idea of a continuous linear functional and begin our study of one of the central themes in this chapter. We also present the Krein – Milman theorem as a fundamental tool in this development.

The book[11] by N.Dunford and J.T Schwarts is a source for information about locally convex linear topological spaces, including the Krein – Milman theorem. Finally in this chapter we identify closed convex hulls and the extreme points of some classes of univalent functions.

In chapter four, we introduce some of the relationships between support points and extreme points of a compact subset $\mathcal{F}$ of $\mathcal{A}$, described at the beginning of this chapter are contained in [6].

Subclasses of functions of $\mathcal{A}$ have been studied throughout the twentieth century . However the systematic application of linear methods to study the extreme points and support points of subclasses is more recent . In the field of geometric function theory it began to play an important role starting in the 1970. In this survey we hope to convey some of the substance and flavor of the use of the linear methods in this field.

For a given compact family $\mathcal{F}$ two basic problems in the application of linear methods are:

Problem 1. Determine support points.

Problem 2. Identify geometric analytic properties of the functions in the set of support points.

Families defined by geometric analytic conditions frequently lead to integral representation which in turn yield a complete answer to Problem 1," determine support points" , represented by some theorems, as we prove in section 2.

The family S itself has not yielded to such a tractable description . However some success has been achieved in answering Problem 2 for S. In section 3 we elaborate on these remarks.

# Chapter one

# Elementary properties of univalent functions

The aim of this introductory chapter is to review and gather for later reference some of the general principles of complex analysis which underlie the theory of univalent functions. In many instances the statements of theorems are supported by bare indications of a proof , or by no proof at all.

## 1.1 Basic Principles

Let $\Delta$ be the unit disk consists of all points $z \in C$ of modulus $|z| < 1$.Its boundary is the unit circle, is denoted by $\partial\Delta$.

A function $f$ is a complex function if its domain is a subset of a complex set. Let $z = x + iy$ .Then $f$ can be written as $f(z) = U(x, y) + iV(x, y)$, where $U(x, y) = \operatorname{Re} f(z)$ and $V(x, y) = \operatorname{Im} f(z)$

### Definition 1.1.1:

Let $f$ be a complex function and write $f(z) = U(x, y) + iV(x, y)$ ,where $z = x + iy$.

If $z_0 = x_0 + y_0$ is a complex number then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} U(x, y) + i \lim_{(x,y)\to(x_0,y_0)} V(x, y)$$

### Definition 1.1.2:

A function $f$ is differentiable at z if $\lim_{h\to 0} \dfrac{f(z+h) - f(z)}{h}$ exists. If so, we write

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

**Definition 1.1.3:**

A complex function $f$ is continuously differentiable on a domain $D$ if $f'(z)$ is continuous on $D$.

**Definition 1.1.4:**

A complex function $f$ is analytic on D if $f$ is continuously differentiable on D. In particular $f$ is analytic at a point $z_0$ if it is analytic in a neighborhood of $z_0$.

Let $\mathcal{A}$ be the set of all analytic functions on $D$.

The following theorem is a simple application of the definition of analytic functions.

**Theorem 1.1.1:**

Suppose $f$ and $g$ are analytic on $D$ , then

1. $f+g, f-g, f.g$ are analytic on $D$.
2. if $g(z) \neq 0 \, \forall z \in D$ , then $\dfrac{f}{g}$ is analytic on $D$ .

**Theorem 1.1.2:**

If $f$ is differentiable on a domain D and $f'(z) = 0, \forall z \in D$, then $f$ is constant.

Proof : see[9,page 37].

**Lemma 1.1.1(Schwarz's lemma):**

If $f(z)$ is analytic on $\Delta$ with $f(0)=0$ and $|f(z)| < 1, \forall z \in \Delta$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in $\Delta$. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \neq 0$ then there is a constant $c, |c| = 1$, such that $f(z) = cz$ for all $z$ in $\Delta$.

Proof: see [9,pages 130,131].