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ANALYSIS OF MAGNETOHYDRODYNAMIC
FLOW THROUGH POROUS MEDIA

By

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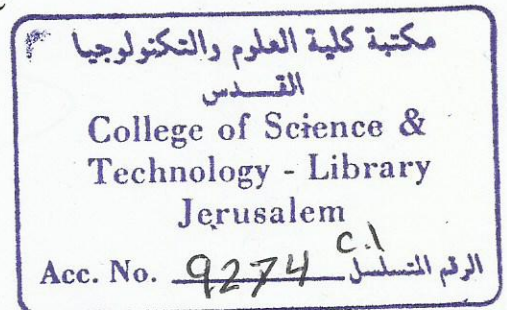
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Abstract

Analytical solutions to the two – dimensional, viscous fluid flows through porous media in the presence of a magnetic field are obtained for some flow situations. The governing equations are based on the Darcy – Lapwood – Brinkman model of flow through porous media .The medium is assumed to be traversed and aligned by a magnetic field .

Solutions are obtained for Riabouchinsky-type flows, however with a modified solution algorithm that is developed in this work to handle the type of flow considered while reducing the number of arbitrary constants arising. Solutions obtained are then classified into different types.

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Chapter one

Introduction

1.1 Importance of flow through porous media

The study of flow through porous media has many applications in the basic and applied sciences, including the study of groundwater flow movement in the porous earth layer, irrigation problems, the prediction of oil reservoir behavior, and the biophysical sciences, where the human lungs, for example, are modeled as porous layers [14].

One of the particular importance to the current study is the flow through porous media in the presence of a magnetic field. This type of flow may find some industrial applications in the design of filtration systems, in addition to the study of lubrication mechanisms wherein the load carrying mechanism is enhanced through the introduction of porous lining into the mechanism and the imposition of a magnetic field in a transverse direction to the flow. Other applications to this type of flow include the occurrence of this type of flow in nature. Typically, the magnetic field is available everywhere on earth; hence, the study of any natural flow phenomenon mandates taking into account the magnetic effects in the flow equations.

The above and various other applications emphasize the importance of undertaking studies in the field of flow through porous media in the presence of a magnetic field. This mandates a need to properly model the flow at hand,

taking into account the effects of the porous material and the magnetic field on the flowing phase, and provide solutions to the resulting models. It is the intention of this work to adopt a model that has received some success in lubrication theory, and seek analytical solutions to this model.

1.2 Overview of Previous Work

In this study, there are four interdependent fields of fluid dynamics namely: general fluid flows, flow through porous media, magnetohydrodynamic flow, and analytical approaches to flow problems.

In the field of general fluid flow, an enormous amount of work has been carried out over the past two centuries, and many aspects of fluid flow theory are well-developed [6] and in their study of inviscid aligned flows, by complex technique. However, they assumed $F(x,y)$ to be constant throughout the flow.

Various aspects of magnetohydrodynamic flow have been studied, and many models have been developed to study the interactions between the magnetic field, the electric field, and how they interact with a flowing fluid. Analyses of this type of flow have been provided by many famous authors [8]. Some rotational and irrotational viscous incompressible aligned plane flows were discussed, [9].

As per fluid flow through porous media, interest in the field dates as far back as 1856 by Darcy and currently one finds various models governing different types of fluid flow in various porous structures. The main types of single-phase flow models have been developed and reviewed by various authors [18]. The crux of the problem in this field is to properly model the interactions between the solid matrix and the flowing fluid, in addition to finding solutions to initial and boundary value problems, and experiments (including numerical experiments and simulation) to validate the models.

Although many models of flow through porous media are available, and various others are continually being developed, a model of particular importance to this work is the Darcy-Lapwood-Brinkman model [15]. As will be discussed in chapter two, this accounts for both viscous shear and inertial

effects of flow through porous media. As such, it has received considerable attention in the literature due to its many applications and implications. We will use a modified version of model in our current analysis, with the exception that a magnetic field is imposed on the flow field

1.3 *Overview of analytical solution to fluid flow problems*

The conservation of mass and conservation of linear momentum principles govern the steady flow of an incompressible viscous fluid. In the absence of sources and sinks, conservation of mass principle takes the following form

$$\nabla \cdot \vec{V} = 0 \quad (1.1)$$

where \vec{V} is the velocity vector, while conservation of linear momentum is given by the Navier- Stokes equations of the form

$$(\vec{V} \cdot \nabla) \vec{V} = -\frac{\nabla p}{\rho} + \frac{\nu}{\rho} \nabla^2 \vec{V} \quad (1.2)$$

where p is the pressure, ν is the viscosity coefficient, and ρ is the fluid density,[22](see appendix 1). The nonlinearity of the Navier –Stokes equations render the superposition principle inapplicable, and introduces a large degree of difficulty in obtaining analytical solutions to it. This difficulty resulted in the majority of the analytical solutions being obtained for parallel laminar flow problems for which the Navier –stokes equations can be linearised . The source of this nonlinearity is the convective inertial terms which , as Taylor [1] noted , vanish when the vorticity is a function of the Stokes stream function , and the flow is two – dimensional . Taylor [1] obtained an exact solution for the Navier-Stokes equations by taking the vorticity proportional to the stream function .His solution represents a double infinite array of vortices decaying exponentially with time.

In an extension to the method proposed by Taylor was also Kovaszny [3], who linearised the Navier-Stokes equations by taking the vorticity to be proportional to the stream function perturbed by a uniform stream. The two-dimensional solution that Kovaszny obtained represents the flow downstream of a two-dimensional grid. Lin and Tobak [12] obtained further solutions by extending the approach of Kovaszny. Their two solutions represent the reverse flow over a flat plate with suction and blowing. Extension of the method followed by Line and Tobak [12] was implemented by Chandna [20] to obtain three exact solutions for nonparallel flow governed by the Navier-Stokes equation. Various other authors have also obtained exact solutions to the Navier-Stokes equations for special types of flow [17].

For the case of flow through porous media as governed by the Darcy-Lapwood-Brinkman (DLB) equation, exact solutions are rare and some of the methods that are applicable to the Navier-stokes equations are not readily applicable to the DLB model (as we shall see in this work). However, a few exact solutions to the DLB model have been obtained for special types of flow. These are summarized as follows:

1. When the vorticity distribution is given, three analytical solutions have been obtained for DLB model. The resulting flow fields have been identified as reversing flows; shifting stagnation point flows and flows over a flat plate with varying blowing or suction.
2. When the flow is a Beltrami-type, analytical solutions are possible, with the resulting flow pattern representing an impingement of an oblique uniform stream with an oblique rotational divergent flow a porous layer [16],[19].
3. An extension of the method used by [3] has been employed by [19] to obtain a solution to the DLB model and the Brinkman equation, and analyzed the effect of permeability on the resulting flow. As anticipated by

Kovaszny, the resulting solution represents the flow behind a two dimensional grid.

A compatibility equation is obtained for steady plane transverse MHD flows and, as an example, solutions are obtained for radial flows. The geometric implication, when the compatibility equation does not hold, is also determined,[7].

1.4 *Scope of the current work*

In this work we consider single – phase fluid flow through porous media in the presence of a magnetic field. The flow model adopted is that of the Darcy-Lapwood –Brinkman model; however, modified to better handle the presence of the magnetic field. The aim here is to analyze the nonlinear model equation in an attempt to find possible solutions corresponding to a particular form of the stream function. The choice of the stream function in this work will be one that is linear with respect to one of the independent variables. This type of flow is referred to as the Riabouchinsky flow. Riabouchinsky flow has received considerable attention in the literature due to its application in boundary layer studies [4].

In this case, the two dimensional Navier-Stokes equations, written as a fourth order partial differential equation in terms of the stream function, may be replaced by two coupled fourth order ordinary differential equations in two unknown functions of a single variable. Solutions to the coupled set are then obtained based on the knowledge of particular integrals of one of the equations. A different type of flow may then be studied with the knowledge of one of the functions.

Riabouchinsky [2] assumed one of the functions to be zero and studied the resulting flow which represents a plane flow in which the flow is

separated in the two symmetrical regions by a vertical or a horizontal plane . In addition to the study of Navier-Stokes flows and their applications [1], Riabouchinsky flows have also received considerable attention in the study of non –Newtonian flows [10] and in magnetohydrodynamics [13].

The approach used to solve the resulting coupled set of ordinary differential equations suffers from some limitations, among which is its dependence on the knowledge of the particular solutions of one of the equations . In addition , the resulting solutions involve a number of arbitrary constants , the determination of which usually involves making many restrictive assumptions on the flow. A major disadvantage of the previous traditional approach is its inapplicability to the analysis of the DLB model.

In this work, a modest modification of the previous approach is employed, and a methodology that is capable of handling a wider class of flow problems is developed. The developed methodology overcomes some of the disadvantages of the traditional approach used for the Navier-Stokes equations , and is also a powerful technique in handling Brinkman –type flow problems in porous media.

1.5 Organization of the thesis

In this chapter, a general introduction has been given to provide an overview of the scope of the current thesis . In chapter two, we will consider in details the model equation used , and provide a brief overview of the method of solution. Chapter three will provide details of the solution on methodology as carried out to solve the current problem , while chapter four we consider the plane aligned or parallel flow ,we also consider infinitely electrically fluids and in this chapter the governing equations in chapter two are transform into more convenient form for this work by

employing the definition of aligned flows, we use an alternative approach to the Riabouchinsky method conducting and give some analysis of the solutions obtained and we will summarize what has been accomplished in this work.

Chapter Two

Problem formulation

2.1 The (1950) Woodward - Bradman Model (P.L.R.)
The study flow of an incompressible viscous fluid through a porous media is governed by the conservation of mass and conservation of linear momentum. Assuming a steady state flow, the conservation of mass principle can be written as follows:

2.2 The (1950) Woodward - Bradman Model (P.L.R.)
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Chapter Four

Plane Aligned Flows

4.1 Equation of motion

The modification introduced in Chapter three is capable of handling a wider class of flow problems. The developed methodology overcomes some of the disadvantages of the traditional approach, and is also a powerful technique in handling Brinkman – type flow problems in porous media. In this Chapter, we will employ the modified methodology to analyze aligned fluid flow through porous media in the presence of a magnetic .

Magnetohydrodynamics (MHD) plane flows are said to be aligned or parallel flows if the magnetic field (\vec{H}) is everywhere parallel to the velocity vector field (\vec{V}) and all the flow variables are functions of x, y only.

From the definition of aligned flows \vec{H} and \vec{V} are related by

$$\vec{H}(x,y) = F(x,y) \vec{V} \quad (4.1)$$

Where

$$(\vec{V} \cdot \nabla)F = 0 \quad (4.2)$$

The steady flow of a viscous, incompressible, electrically conducting fluid through porous media in the presence of a magnetic field is governed by the following system of partial differential equations:

$$\nabla \cdot \vec{V} = 0 \quad (4.3)$$

$$\rho(\vec{V} \cdot \nabla)\vec{V} = -\nabla p + \mu_1 \nabla^2 \vec{V} - \frac{\mu_2}{k_1} \vec{V} + k_2 (\nabla \times \vec{H}) \times \vec{H} \quad (4.4)$$

$$\nabla \times (\vec{V} \times \vec{H}) - \frac{1}{k_2 \sigma} \nabla \times (\nabla \times \vec{H}) = 0 \quad (4.5)$$

$$\nabla \cdot \vec{H} = 0 \quad (4.6)$$

where \vec{V} is the velocity vector field, \vec{H} is the magnetic vector field, p is the pressure, σ is the electrical conductivity, k_1 is the medium permeability to the fluid, k_2 is the magnetic permeability, μ_1 is the fluid viscosity, μ_2 is the viscosity of fluid in the porous medium and ρ is the fluid density. It is required to solve this system for the unknowns \vec{V} , \vec{H} and p .

4.2 Simplifying the governing equation

In this section we will consider the case of two-dimensional aligned flow x and y with the velocity field $\vec{V}=(u,v,0)$, a magnetic field $\vec{H}=(0,0,H)$ and $\frac{\partial}{\partial z}=0$. Using equation (4.1) into the governing equation gives

$$\begin{aligned} (\nabla \times \vec{H}) \times \vec{H} = \\ [-F^2 v(v_x - u_y) - v^2 F F_x + uv F F_y, F^2 u(v_x - u_y) + uv F F_x - u^2 F F_y, 0] \end{aligned} \quad (4.7)$$

substitute (4.1) into (4.5), we obtain

$$\nabla \times (\vec{V} \times \vec{H}) = 0 \quad (4.8)$$

and if $\frac{1}{k_2 \sigma} \neq 0$ then

$$\nabla \times (\nabla \times \vec{H}) = \nabla^2 \vec{H} = [\nabla^2(Fu), \nabla^2(Fv), 0] = 0$$

or

$$\nabla^2(Fu) = \nabla^2(Fv) = 0, \quad (4.9)$$

Employing equation (4.1) into equation (4.6), we obtain

$$u F_x + v F_y = 0 \quad (4.10)$$

using the results (4.7) – (4.9) into the equation (4.4) yields

$$\rho \left\{ \left[\frac{1}{2} q^2 \right]_x - v(v_x - u_y) \right\} = -p_x + \mu_1 \nabla^2 u - \frac{\mu_2}{k_1} u + k_2 \left[-F^2 v(v_x - u_y) - v^2 F F_x + u v F F_y \right] \quad (4.11)$$

$$\rho \left\{ \left[\frac{1}{2} q^2 \right]_y + u(v_x - u_y) \right\} = -p_y + \mu_1 \nabla^2 v - \frac{\mu_2}{k_1} v + k_2 \left[F^2 u(v_x - u_y) + u v F F_x - u^2 F F_y \right] \quad (4.12)$$

where $q^2 = u^2 + v^2$ is the square of the velocity field.

Substitute equation (4.10) into the equation (4.11) and (4.12) yield the two equations

$$\rho \left\{ \left[\frac{1}{2} q^2 \right]_x - v(v_x - u_y) \right\} = -p_x + \mu_1 \nabla^2 u - \frac{\mu_2}{k_1} u + k_2 \left[-F^2 v(v_x - u_y) - (u^2 + v^2) F F_x \right] \quad (4.11)$$

$$\rho \left\{ \left[\frac{1}{2} q^2 \right]_y + u(v_x - u_y) \right\} = -p_y + \mu_1 \nabla^2 v - \frac{\mu_2}{k_1} v + k_2 \left[F^2 u(v_x - u_y) - (v^2 + u^2) F F_y \right] \quad (4.12)$$

and equation (4.3) take the form $u_x + v_y = 0$. (4.13)

then the system of equations (4.3) to (4.6), we find that the flow equations governing the motion of a steady plane aligned flow of a viscous incompressible fluid of finite electrical conductivity, in the presence of a magnetic field, are given by the equations (4.13), (4.11), (4.12), (4.9), (4.10) and (4.1).

In the case of infinitely electrical conductivity flow ($\frac{1}{k_2 \sigma} \rightarrow 0$) the diffusion equation (4.9) is identically satisfied. However, for finitely electrical conductivity flows, the functions F , u and v must satisfy this additional conditions (4.2).

4.3 Vorticity – Stream function form of the equations

To derive the compatibility or integrability equation for aligned flows by employing the flow equation for the flow in section 4.2, where the stream function is linear with respect to x or y.

Introducing the vorticity function $\omega(x,y)$ and the pressure function $h(x,y)$ defined respectively by

$$\omega(x,y) = v_x - u_y \quad (4.14)$$

$$h(x,y) = p + \frac{1}{2} \rho q^2 \quad (4.15)$$

then equations (4.11) and (4.12) take the following forms respectively,

$$h_x - \rho v \omega = \mu_1 \nabla^2 u - \frac{\mu_2}{k_1} u + k_2 [-F^2 v \omega - (u^2 + v^2) F F_x] \quad (4.16)$$

$$h_y + \rho u \omega = \mu_1 \nabla^2 v - \frac{\mu_2}{k_1} v + k_2 [F^2 u \omega - (v^2 + u^2) F F_y] \quad (4.17)$$

from equations (4.13), (4.14) the last two equations respectively yields

$$h_x - \rho v \omega = \mu_1 \omega_y - \frac{\mu_2}{k_1} u + k_2 [-F^2 v \omega - (u^2 + v^2) F F_x] \quad (4.18)$$

$$h_y + \rho u \omega = \mu_1 \omega_x - \frac{\mu_2}{k_1} v + k_2 [F^2 u \omega - (v^2 + u^2) F F_y] \quad (4.19)$$

Now, letting $\psi(x,y)$ be the stream function defined in terms of the components as

$$\psi_y = u \text{ and } \psi_x = -v \quad (4.20)$$

Then we can see clearly that the continuity equation (4.13) is identically satisfied since

$$u_x + v_y = \psi_{xy} - \psi_{yx} = 0$$

and the vorticity equation (4.14) becomes

$$\omega = -\nabla^2 \psi \quad (4.21)$$

and equations (4.18), (4.19) will be, in terms of the stream function as follows

$$h_x = \rho \psi_x \nabla^2 \psi - \mu_1 (\nabla^2 \psi)_y - \frac{\mu_2}{k_1} \psi_y - k_2 [F^2 \psi_x \nabla^2 \psi - (\psi_y^2 + \psi_x^2) F F_x] \quad (4.22)$$

$$h_y = \rho \psi_y \nabla^2 \psi + \mu_1 (\nabla^2 \psi)_x - \frac{\mu_2}{k_1} \psi_x + k_2 [-F^2 \psi_y \nabla^2 \psi - (\psi_y^2 + \psi_x^2) F F_y] \quad (4.23)$$

A compatibility equation can be derived from equation (4.22) and (4.23) by using the integrability condition: $h_{xy} = h_{yx}$. Thus, differentiating equation (4.22) with respect to y , and equation (4.23) with respect to x and using the integrability condition we obtain the following compatibility equation

$$\rho \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} - k_1 F^2 \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} + k_1 F \frac{\partial(|\nabla \psi|^2, F)}{\partial(x, y)} - \frac{\mu_2}{k_1} \nabla^2 \psi + \mu_1 \nabla^4 \psi = 0 \quad (4.24)$$

where $\nabla^2 \equiv \partial_{xx} + \partial_{yy}$, $\nabla^4 \equiv \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$.

4.4 Reduction of the governing partial differential equations to ordinary differential equations

Thus, once equation (4.24) is solved for $\psi(x, y)$, the vorticity function can be calculated from equation (4.21), the pressure function $h(x, y)$ can be obtained from equations (4.18) and (4.19), while $p(x, y)$ from equations (4.15) and the velocity components can then be obtained from equation (4.20).

Since the compatibility equation (4.24) is very difficult to solve and order to provide a solution for equation (4.24), assume that the stream function $\psi(x, y)$ is linear and has the form

$$\psi(x, y) = yf(x) + g(x) \quad (4.25)$$

where $f(x)$, $g(x)$ are four times differentiable arbitrary functions of x , substituting equation (4.25) into (4.24) and equating the coefficients of similar power of y , lead to the following coupled set of fourth order ordinary differential equations

$$\mu_1 f^{(iv)} - \frac{\mu_2}{\rho k_1} f'' + (\rho - k_1 F^2) [f' f'' - f f'''] = 0 \quad (4.26)$$

$$\mu_1 g^{(iv)} - \frac{\mu_2}{\rho k_1} g'' + (\rho - k_1 F^2) [g' f'' - f g'''] = 0. \quad (4.27)$$

For the particular case of aligned flows, recalling equation (4.6) for the case

$$k_1 \rightarrow \infty$$

$$\frac{\partial(F, \psi)}{\partial(x, y)} = 0, \quad (4.28)$$

the solution of equation (4.28) is either

$$F(x, y) = c, \text{ where } c \text{ is arbitrary constant or } F(x, y) = F(\psi).$$

The case $F(x, y) = c$, equations (4.26), (4.27), (4.22) and (4.23) take the following forms respectively

$$\mu_1 f^{(iv)} + (\rho - k_1 c^2) [f' f'' - f f'''] = 0 \quad (4.29)$$

$$\mu_1 g^{(iv)} + (\rho - k_1 c^2) [g' f'' - f g'''] = 0 \quad (4.30)$$

$$h_x = \rho \psi_x \nabla^2 \psi - \mu_1 (\nabla^2 \psi)_y - k_2 [f^2 \psi_x \nabla^2 \psi] \quad (4.31)$$

$$h_y = \rho \psi_y \nabla^2 \psi + \mu_1 (\nabla^2 \psi)_x + k_2 [-f^2 \psi_y \nabla^2 \psi]. \quad (4.32)$$

4.5 An alternative approach to the Riabouchinsky method

Using the alternative approach present in chapter three, two choices for the function $g(x)$ can be considered.

4.5.1 The first choice

when $g(x) = g_1(x) = \alpha x^3$, where α is a constant.

Substituting in equation (4.30) we obtain

$$f(x) = c_1 x^{-1} + c_2 x^2 \quad (4.33)$$

where c_1 and c_2 are arbitrary constants that can be determined by substituting in equation (4.29) and then equating the equal powers of x , to give the constants

$$c_2 = 0 \text{ and } c_1 = -\frac{6\mu_1}{(\rho - k_1 c^2)}$$

Hence (4.33) gives

$$f(x) = -\frac{6\mu_1}{(\rho - k_1 c^2)} x^{-1}. \quad (4.34)$$

The corresponding stream function, vorticity, and velocity components take the following forms, respectively

$$\psi(x,y) = -\frac{6\mu_1}{(\rho - k_1 c^2)} x^{-1} y + \alpha x^3 \quad (4.35)$$

$$\omega(x,y) = \frac{12\mu_1}{(\rho - k_1 c^2)} x^{-3} y - 6\alpha x \quad (4.36)$$

$$u(x,y) = -\frac{6\mu_1}{(\rho - k_1 c^2)} x^{-1} \quad (4.37)$$

$$v(x,y) = -\frac{6\mu_1}{(\rho - k_1 c^2)} x^{-2} y - 3\alpha x^2. \quad (4.38)$$

We see clearly that this solution does not involve any arbitrary constants. However, depending on the choice of α , different flow patterns may be obtained.

The pressure function $h(x,y)$ can be obtained by using (4.31) and (4.32) to get :

$$h(x,y) = -\frac{6\mu_1^2}{(\rho - k_1 c^2)} x^{-2} + \frac{18\mu_1^2}{(\rho - k_1 c^2)} x^{-4} y^2 + \frac{9}{2} (\rho - k_1 c^2) \alpha^2 x^4 \\ + \frac{3\mu_1}{(\rho - k_1 c^2)} x^{-2} y^2 + 3\alpha x^2 y - 36\mu_1 \alpha y + \frac{36\mu_1^2}{(\rho - k_1 c^2)} x^{-4}. \quad (4.39)$$

and

$$q^2 = \frac{36\mu_1^2}{(\rho - k_1 c^2)^2} x^{-2} + \frac{36\mu_1^2}{(\rho - k_1 c^2)^2} x^{-4} y^2 + \frac{36\alpha\mu_1}{(\rho - k_1 c^2)} y + 9\alpha^2 x^4 \quad (4.40)$$

then

$$p(x,y) = -\frac{6\mu_1^2}{(\rho - k_1 c^2)} x^{-2} + \frac{18\mu_1^2}{(\rho - k_1 c^2)} x^{-4} y^2 + \frac{9}{2} (\rho - k_1 c^2) \alpha^2 x^4 \\ + \frac{3\mu_1}{(\rho - k_1 c^2)} x^{-2} y^2 + 3\alpha x^2 y - 36\mu_1 \alpha y + \frac{36\mu_1^2}{(\rho - k_1 c^2)} x^{-4} \\ - \frac{18\rho\mu_1^2}{(\rho - k_1 c^2)^2} x^{-2} - \frac{18\rho\mu_1^2}{(\rho - k_1 c^2)^2} x^{-4} y^2 - \frac{18\rho\alpha\mu_1}{(\rho - k_1 c^2)} y - \frac{9}{2} \rho\alpha^2 x^4 \quad (4.41)$$

4.3.2 The second choice

When $g(x) = g_2(x) = \alpha e^{\beta x}$ where α, β are constants, then substitute in equation (4.30) gives

$$f'' - \beta^2 f = -\frac{\mu_1 \beta^3}{(\rho - k_1 c^2)} \quad (4.42)$$

which has the following general solution

$$f(x) = c_1 e^{\beta x} + c_2 e^{-\beta x} + \frac{\mu_1 \beta}{(\rho - k_1 c^2)} \quad (4.43)$$

to find the constants c_1 and c_2 substituting equation (4.43) into equation (4.29) to give

$c_2 = 0$ and $c_1 \neq 0$ where c_1 is an arbitrary constant.

$$\text{Hence } f(x) = c_1 e^{\beta x} + \frac{\mu_1 \beta}{(\rho - k_1 c^2)} \quad (4.44)$$

The arbitrary constant c_1 appears in the above $f(x)$ can be determined by imposing one condition on the stream function ψ , or by assigning various values to produce different flow patterns.

The corresponding stream and vorticity functions are given respectively by

$$\psi(x, y) = \left(c_1 e^{\beta x} + \frac{\mu_1 \beta}{(\rho - k_1 c^2)} \right) y + \alpha e^{\beta x} \quad (4.45)$$

$$\omega(x, y) = -\beta^2 [\alpha + c_1 y] e^{\beta x} \quad (4.46)$$

The velocity field components $u(x, y)$, $v(x, y)$ and the pressure function $h(x, y)$ are given by:

$$u(x, y) = \psi_y(x, y) = c_1 e^{\beta x} + \frac{\mu_1 \beta}{(\rho - k_1 c^2)} \quad (4.47)$$

$$v(x, y) = -\psi_x(x, y) = -\beta [\alpha + c_1 y] e^{\beta x} \quad (4.48)$$

$$\begin{aligned}
 h(x,y) = & -c_1 \mu_1 \beta e^{\beta x} - \frac{1}{2}(\rho - k_1 c^2) \alpha^2 \beta^2 e^{2\beta x} + (\rho - k_1 c^2) c_1 \alpha \beta^2 e^{2\beta x} y \\
 & + (\rho - k_1 c^2) c_1^2 \beta^2 e^{2\beta x} y^2 .
 \end{aligned} \tag{4.49}$$

Then

$$\begin{aligned}
 q^2 = & c_1^2 e^{2\beta x} + \frac{2c_1 \mu_1 \beta}{(\rho - k_1 c^2)} e^{\beta x} + \frac{\mu_1^2 \beta^2}{(\rho - k_1 c^2)^2} + c_1^2 \beta^2 e^{2\beta x} y^2 \\
 & + 2c_1 \beta^2 \alpha e^{2\beta x} y + \alpha^2 \beta^2 e^{2\beta x}
 \end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
 p(x,y) = & -c_1 \mu_1 \beta e^{\beta x} - \frac{1}{2}(\rho - k_1 c^2) \alpha^2 \beta^2 e^{2\beta x} + (\rho - k_1 c^2) c_1 \alpha \beta^2 e^{2\beta x} y \\
 & + (\rho - k_1 c^2) c_1^2 \beta^2 e^{2\beta x} y^2 + \frac{2c_1 \mu_1 \beta}{(\rho - k_1 c^2)} e^{\beta x} \\
 & + \frac{\mu_1^2 \beta^2}{(\rho - k_1 c^2)^2} + c_1^2 \beta^2 e^{2\beta x} y^2 + 2c_1 \beta^2 \alpha e^{2\beta x} y + \alpha^2 \beta^2 e^{2\beta x} .
 \end{aligned}$$