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Numerical Treatment of Fredholm Integral Equation of the Second Kind

By

Nahed Salah Mohammad Salem

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Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ..N.ahed..Salem

Nahed Salah Mohammad Salem

Date : 7 / 9 / 2003

Dedication

To my parents. To my Family and to my colleagues.

Acknowledgment

I would like to express my thanks to all those who helped me to prepare and complete this work.

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Abstract

During the period of 1960-1990, there has been much work on developing and analyzing numerical methods for solving linear Fredholm integral equation of the second kind with the integral operator K being compact on a suitable function space.

In this thesis we present some numerical methods for solving Fredholm integral equation of the second kind, these include the degenerate kernel approximations such as interpolatory degenerate schemes, and the projection methods where the main aspects of the collocation and Galerkin methods are investigated including recent work on solving Fredholm integral equations on surfaces in the Euclidean plane. Convergence, error analysis, and stability of these methods are also given great attention.

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Introduction

Over the past 20 years there has been a substantial increase in the use of integral equation in the formulation of solution strategies for scientific and engineering problems. In large measure this has been due to the work in the engineering and mathematics communities in using integral equation techniques to solve boundary value problems for partial differential equations as an alternative to domain-based methods such as the finite element and finite difference methods.

Parallel to this, there have been many important mathematical developments in establishing the convergence and stability of many numerical methods and in the discovery of new solution techniques. We believe these have considerable potential in increasing the efficiency of many current procedures.

In this thesis, some numerical methods are presented and analyzed for the solution of Fredholm integral equations of the second kind.During the period of 1960-1990,there has been much work on developing and analyzing numerical methods in this direction.

Integral equations are most frequently solved by collocation or Galerkin methods. Mathematically, these techniques are particular cases of projection methods.

Although the theory of projection methods has been understood for many years,the analysis of discrete projection methods is relatively of recent origin. Most what is known has been published in the past 10 years.

In chapter one, the main aspects of the theory of compact operators is presented, including integral equations with compact integral operators, the Fredholm

Glossary of Symbols

Spaces

(a,b)	open interval $(a,b)=\{x\in R: a<x<b\}$
$[a,b]$	closed interval $[a,b]=\{x\in R: a\le x\le b\}$
R^n	n-dimensional real Euclidean space
$C[a,b]$	space of real- or complex –valued continuous functions on the interval $[a,b]$
$C^m[a,b]$	space of m-times continuously differentiable functions
$L_2[a,b]$	space of real-or complex-valued square- integrable functions
ℓ^∞	space of bounded sequences of complex numbers with $\| x \| = \sup x$
ℓ^2	the vector space over C of all complex sequence $x=(x_n)_{n=1}^{\infty}$ which are square summable, i.e. $\sum_{n=1}^{\infty} \| x_n \|^2 <\infty$ with inner product $(x,y)=\sum_{n=1}^{\infty} x_n \overline{y}_n$

Norms

$\|\cdot\|$	norm on a linear space
$\|\cdot\|_1$	ℓ_1 norm of a vector, L_1 norm of a function
$\|\cdot\|_2$	ℓ_2 norm of a vector, L_2 norm of a function
$\|\cdot\|_\infty$	maximum norm of a vector or a function
$(.,.)$	scalar product on a linear space

alternative theorem (A key theorem concerning conditions for the existence and uniqueness of a solution) is also investigated.

Numerical methods for solving Fredholm integral equation of the second kind are considered in Chapter two, these include the degenerate kernel approximations such as the separable/ degenerate schemes.

Chapter three is devoted to studying other numerical methods such as projection methods where the main aspects of the collocation and Galerkin methods are presented including recent work on solving integral equations on surfaces in the Euclidean plane. .

Construction of these methods , convergence and stability of the resulting linear systems as well as regularization of solutions are also presented.

We give some illustrations which include three worked examples to compare the performance of the methods which are presented in this work.

Chapter One

Integral Equation with L_2-Kernel

1.1 Introduction

1.1.1 Types of Integral Equations

In this thesis we are concerned primarily with the numerical solution for the Fredholm integral equation of the second kind, but we begin classifying integral equations, we say, very roughly, that those integral equations in which the integration domain varies with the independent variable in the equation are Volterra integral equations, and those in which the integration domain is fixed are Fredholm integral equations of the second kind.

1. Volterra integral equations of the second kind

The general form that is studied is

$$\phi(x)-\int_a^x k(x,y)\,\phi(y)dy = f(x), \qquad x \geq a \tag{1.1.1}$$

The functions $k(x,y)$ and $f(x)$ are given, and $\phi(x)$ is sought.

2. Volterra integral equations of the first kind

The general nonlinear Volterra integral equation of the first kind has the form

$$\int_a^x k(x,y)\phi(y)dy = f(x), \quad x \geq a \tag{1.1.2}$$

The functions $k(x,y)$ and $f(x)$ are given functions, and the unknown is $\phi(y)$.

3. Abel integral equations of the first kind

The general form of such an integral equation is

$$\int_0^x \frac{H(x,y)\phi(y)}{\left(x^p - y^p\right)^\alpha} dy = f(x), \quad x > 0 \tag{1.1.3}$$

Here $0<\alpha<1$ and $p>0$, and particularly important cases are $p=1$ and $p=2$ (both with $\alpha=\frac{1}{2}$). The function $H(x,y)$ is assumed to be smooth (that is, several times continuously differentiable).

4. Fredholm integral equations of the second kind

The general form of such an integral equation is

$$\lambda\phi(x) - \int_a^b k(x,y)\phi(y)dy = f(x), \qquad a \leq x \leq b, \quad \lambda \neq 0 \tag{1.1.4}$$

with $[a,b]$ a closed bounded set in R^m , some $m \geq 1$. The kernel function $k(x,y)$ is assumed to be absolutely integrable, and it is assumed to satisfy other properties that are sufficient to imply the Fredholm Alternative Theorem (see Theorem 1.3.1 in subsection (1.3.2). For $f \neq 0$,we have λ and f given and we seek ϕ, this is the non homogeneous problem. For $f=0$, equation (1.1.4) becomes an eigenvalue problem, and we seek both the eigenvalue λ and the eigenfunction ϕ. The principal focus of the numerical methods presented in the following chapters is the numerical solution of equation (1.1.4) with

$f \neq 0$. In the next two sections we present some theory for the integral operator in equation (1.1.4).We define λ to be an eigenvalue of the operator K defined as

$$K\,\phi(x)=\int_a^b k(x,y)\phi(y)dy \qquad (1.1.5)$$

5. Fredholm integral equations of the first kind

These equations take the form

$$\int_a^b k(x,y)\phi(y)dy = f(x), \qquad a\leq x\leq b \qquad (1.1.6)$$

with the assumptions on K and $[a,b]$ the same as in equation (1.1.4) .

1.2 Compact Operators

Definition 1.2.1

Let X and Y be normed vector spaces, and let $K: X \to Y$ be linear. Then K is compact if for every bounded sequence $\{f_n\}\subset X$, the sequence $\{K f_n\}$ has a subsequence that is convergent to some point in Y .

Theorem 1.2.1

Let H be $L_2[a,b]$, and let K be a degenerate integral operator

$$K f = \sum_{i=1}^{m} a_i(x)\int_a^b b_i(y)f(y)dy$$

K is a compact operator, if for all i , $a_i(x)$ and $b_i(x)$ belong to $L_2[a,b]$.

Proof:

Consider a bounded set of functions $\{f_n\}$

$$K f_n(x) = \sum_{i=1}^{m} a_i(x)\int_a^b b_i(y)f_n(y)dy$$

Clearly

$$\left| \int_a^b b_i(y)f_n(y)dy \right| \leq \| b_i \| \| f_n \| \leq M \| b_i(y) \|$$

so that the sequence $\left\{ \int_a^b b_i(y)f_n(y)dy \right\}$ is a bounded sequence of complex numbers . It must therefore have a point of accumulation, and a suitable subsequence will converge to that point of accumulation.

Let $\{f_{n^{(1)}}\}$ be a subsequence of $\{f_n\}$ such that $\left\{ \int_a^b b_1(y)f_{n^{(1)}}(y)dy \right\}$ converges to some complex number, say c_1 . By the same argument we can now extract a subsequence $\{f_{n^{(2)}}\}$ of $\{f_{n^{(1)}}\}$ such that $\left\{ \int_a^b b_2(y)f_{n^{(2)}}(y)dy \right\}$ converges to, say c_2 . By extracting successive subsequences we finally arrive at $\{f_{n^{(m)}}\}$ with the property that