FOURIER SERIES AND ANALYSIS AND APPLICATIONS

by

Rihab M. I. Al _ Zeer

Supervisor : Dr. Naji Qatanani

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Approved by:

Dr.Naji Qatanani (Supervisor)

N. Qatanani

Dr.Fathi Allan (External examiner)

Dr. Taha Abu - Kaff (Internal examiner)

T. Abu-Kaff

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Dedication

This thesis is respectfully dedicated to my mother, my father, my brothers and my sisters.

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Finally, I pray to Allah to keep me healthy and well.

Summary

Because of the important role that the *Fourier* series and *Fourier* transforms play in physics and engineering, we have therefore focussed our attention in this thesis on the theory of *Fourier* series and its applications.

Convolution theory and its relation to the transform methods has been investigated. Orthogonal and trigonometric system in two variables together with double *Fourier* series for a function with different periods have also been widely discussed.

Solutions to some boundary value problems in the field of heat flow and wave propagation have been obtained using the separation of variables method and the Eigenfunction_ expansion technique.

The *Fourier* expansion with respect to the Bessel's functions and Bessel's inequality have been used in the solutions of the boundary value problems.

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Chapter one

Fourier series on T

In this chapter we discuss the *Fourier* coefficients in section (1), the summability in norm and homogenous Banach space on T in section (2), the order of magnitude of *Fourier* series of square summable functions and absolutely convergent *Fourier* series in section (3), *Fourier* series of square summable functions and orthogonal and trigonometric system in two variables in section (4), double *Fourier* series for a function with different periods in x and y in section (5), absolutely convergent *Fourier* series in section (6) and *Fourier* coefficients of linear series in section (7).

1.1 Fourier coefficients

Let T be defined as the quotient $R/2\pi Z$ group, where $2\pi Z$ is the group of the integral multiples of 2π and we denote by $L^1(T)$ the space of all complex-valued Lebegue integrable functions on T.

For $f \in L^1(T)$ we define the norm of f by $\| f \|_{L^1(T)} = \frac{1}{2\pi}\int_T | f(t) | \, dt$.

Note: If $f \in L^1(T)$ then $\int f(t) \; dt$ is defined on T.

Definition (1.1.1):[20] A trigonometric polynomial on T is an expression of the form

$$P(t) = \sum_{n=-N}^{N} a_n e^{\text{int}} \quad . \qquad (1.1.1)$$

The numbers (n) appearing in (1.1.1) are called the frequencies of P.
The largest integer (n) such that (a_n)+(a_{-n}) $\neq 0$ is called the degree of p.
The values assumed by the numbers (n) are integers so that each of the summands in (1.1.1) is a function on T.

We can compute the coefficients (a_n) by the formula

$$a_n = \frac{1}{2\pi} \int_T p(t) \; e^{-\text{int}} dt \qquad (1.1.2)$$

which follows immediately from the fact that for each integer J we have

$$\frac{1}{2\pi}\int e^{iJt}\,dt=\begin{cases}1 & , \text{ if } \quad J=0\\ 0 & , \text{ if } \quad J\neq 0\end{cases} \tag{1.1.3}$$

Definition(1.1.2):[8], [9], [10] A trigonometric series on T is an expression of the form

$$S=\sum_{n=-\infty}^{\infty}a_n e^{\text{int}} \tag{1.1.4}$$

where n assumes integral values; however, the number of terms in (1.1.4) may be finite and there is no assumption whatsoever about the size of the coefficients or about convergence .

The conjugate $\overline{S}$ of (1.1.4) is the series

$$\overline{S}=\sum_{n=-\infty}^{\infty}-i\,\text{sgn}(n)a_n e^{int} \tag{1.1.5}$$

where $\text{sgn}\,(n)=\begin{cases}0 & , \text{ if } \quad n=0\\ \dfrac{n}{|n|} & , \text{ if } \quad n\neq 0 \end{cases}$.

Let $f\in L^1(T)$ be motivated by (1.1.3) we define the nth *Fourier* coefficient of f by

$$\hat{f}(n)=\frac{1}{2\pi}\int_T f(t)\,e^{-\text{int}}\,dt \ . \tag{1.1.6}$$

Definition (1.1.3): [9], [20] The *Fourier* series $S\,[f]$ of a function $f\in L^1(T)$ is the trigonometric series

$$S[f]=\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{\text{int}}=f(t) \tag{1.1.7}$$

where $\hat{f}(n)$ is the *Fourier* coefficients of f.

The series conjugate to $S\,[f]$ will be denoted by $\overline{S}\,[f]$ and is given in the form

$$\overline{S}[f]=\sum_{n=-\infty}^{\infty}-i\,\text{sgn}(n)\hat{f}(n)\,e^{\text{int}} \tag{1.1.8}$$

This is also referred to as the conjugate *Fourier* series of f.

Theorem (1.1.4): **[12]** Let $f, g \in L^1(T)$ then

(a) $(f+g\hat{)}(n)=\hat{f}(n)+\hat{g}(n)$

(b) For any complex number k, $(kf\hat{)}(n)=k\hat{f}(n)$

(c) If $\overline{f}$ is the complex conjegate of f then $(\overline{f}\hat{)}=\overline{\hat{f}(-n)}$

(d) Denote $f_a(t)=f(t-a),\ a\in T$ then $\hat{f}_a(n)=\hat{f}(n)\ e^{-ina}$

(e) $\left|\hat{f}(n)\right| \le \frac{1}{2\pi}\int |f(t)|\,dt=\|f\|_{L^1(T)}$

Proof:

(a) By definition $(f+g\hat{)}(n)=\frac{1}{2\pi}\int(f+g)(t)e^{-int}\,dt$

$$=\frac{1}{2\pi}\int(f(t)+g(t))e^{-int}\,dt=\frac{1}{2\pi}\int f(t)e^{-int}\,dt+\frac{1}{2\pi}\int g(t)e^{-int}\,dt$$

$$=\hat{f}(n)+\hat{g}(n)$$

hence $(f+g\hat{)}(n)=\hat{f}(n)+\hat{g}(n)$.

(b) Let k be any complex number then

$$(kf\hat{)}(n)=\frac{1}{2\pi}\int kf(t)e^{-int}\,dt=k\frac{1}{2\pi}\int f(t)e^{-int}\,dt=k\hat{f}(n)$$

hence $(kf\hat{)}(n)=k\ \hat{f}(n)$.

(c) $$(f^{-}\hat{)}(n)=\frac{1}{2\pi}\int f\overline{(n)}\ e^{-int}\,dt$$

$$\overline{\hat{f}}(-n)=\frac{1}{2\pi}\int\overline{f(t)e^{--int}}\,dt=\frac{1}{2\pi}\int\overline{f}(t)e^{-int}\,dt=(f^{-}\hat{)}(n)$$

hence $(f^{-}\hat{)}(n)=\overline{\hat{f}}(-n)$.

(d) $$\hat{f}_a(n)=\frac{1}{2\pi}\int f_a(t)e^{-int}\,dt=\frac{1}{2\pi}\int f(t-a)e^{-int}\,dt$$

let $u=t-a$ then $du=dt$ and $t=u+a$ thus

$$\hat{f}_a(n) = \frac{1}{2\pi}\int f(u)e^{-in(u+a)}du = \frac{1}{2\pi}e^{-ina}\int f(u)e^{-inu}du = e^{-ina}\hat{f}(n)$$

hence $\hat{f}_a(n) = e^{-ina}\hat{f}(n)$.

(e) $$|\hat{f}(n)| = \left|\frac{1}{2\pi}\int f(t)e^{-int}dt\right| = \frac{1}{2\pi}\left|\int f(t)e^{-int}dt\right| \le \frac{1}{2\pi}\int |f(t)e^{-int}|\,dt$$

$$= \frac{1}{2\pi}\int |f(t)|\,|e^{-int}|\,dt = \frac{1}{2\pi}\int |f(t)|\,dt = \|f\|_{L^1(T)},$$

since $|e^{-int}| = 1$, hence $|\hat{f}(n)| \le \|f\|_{L^1(T)}$.

Lemma(1.1.5):[12] Assume $f_J \in L^1(T)$, $J=0,1,2,\ldots$ and $\|f_J - f_0\|_{L^1(T)} \to 0$ then $\hat{f}_J(n)$ converges uniformly to $\hat{f}_0(n)$

Proof :

$$\hat{f}_J(n) = \frac{1}{2\pi}\int f_J(t)e^{-int}dt,\ \hat{f}_0(n) = \frac{1}{2\pi}\int f_0(t)e^{-int}dt,$$

and

$$\|f_J - f_0\|_{L^1(T)} = \frac{1}{2\pi}\int |f_J(t) - f_0(t)|\,dt \to 0 \quad .$$

$$\|\hat{f}_J(n) - \hat{f}_0(n)\| = \frac{1}{2\pi}\int_T |\hat{f}_J(t) - \hat{f}_0(t)|\,dt$$

$$= \frac{1}{2\pi}\int_T \left|\frac{1}{2\pi}\int (f_J(t) - f_0(t))e^{-int}dt\right| \le \frac{1}{2\pi}\int_T \frac{1}{2\pi}\int_T |f_J(t) - f_0(t)|\,dt\,dt \le \int 0\,dt = 0$$

then $\|\hat{f}_J(n) - \hat{f}_0(n)\| \to 0$.

hence $\hat{f}_J(n) \to \hat{f}_0(n)$ uniformly .

Theorem (1.1.6): [12] Let $f \in L^1(T)$, assume $\hat{f}(0) = 0$ and define

$$F(t) = \int_0^t f(u)\,du$$

then F is continuous, 2π periodic function and $\hat{F}(n) = \frac{1}{in}\hat{f}(n),\ n \ne 0$.

Proof :

To prove the continuity, let $t_0, t_1 \in T$, then

$$|F(t_0) - F(t_1)| = \left| \int_0^{t_0} f(u)du - \int_0^{t_1} f(u)\ du \right| = \left| \int_0^{t_0} f(u)\ du + \int_{t_1}^{0} f(u)\ du \right|$$

$$= \left| \int_{t_1}^{t_0} f(u)du \right| \le \int_{t_1}^{t_0} | f(u) |\, du \to 0 \ \text{, as } \ t_1 \to t_0.$$

Hence $F(t)$ is continuous and the periodicity follows from the fact that

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(u)du = 2\pi \hat{f}(0) = 0 \ ,$$

therefore $F(t+2\pi) = F(t)$.

and $\hat{F}(n) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} F(t)\ e^{-\text{int}}\, dt$

if we let $u = F(t)$, $dv = e^{-int}$ and using integration by parts formula, we obtain

$$\hat{F}(n) = \frac{-1}{2\pi} \int_0^{2\pi} F'(t) \frac{1}{-in} e^{-\text{int}} dt = \frac{1}{in} \hat{f}(n) \ .$$

Definition (1.1.7):[7] The *Fourier* transform of a function $f \in L^1(R)$ is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\ e^{-inx} dx$$

for $x \in R$.

Some of the basic properties of $\hat{f}(n)$ for every $f \in L^1(R)$ are summarized in the following theorem :

Theorem(1.1.8):**[7]** Let $f \in L^1(R)$, then the *Fourier* transform $\hat{f}(n)$ satisfies:

(1) $\hat{f} \in L^\infty(R)$, with $\| \hat{f} \|_\infty \le \| f \|_{L^1(R)}$ where $\| f \|_\infty = \sup\limits_{-\infty<x<\infty} | f(x) |$

(2) $\hat{f}(n)$ is uniformly continuous on R

(3) $\hat{f}(n) \to 0$ as $n \to \pm\infty$.

Proof:

(1) To prove the first property, we have by definition

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\ e^{-inx} dx$$

by taking the norm

$$\| \hat{f} \|_{\infty} = \sup_{-\infty<n<\infty} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\ e^{-inx} dx \right| \leq \sup_{-\infty<n<\infty} \int_{-\infty}^{\infty} | f(x)\ e^{-inx} dx |$$

$$= \sup_{-\infty<n<\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} | f(x) |\ dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} | f(x) |\ dx = \| f \|_{L^1(R)}$$

$$\Rightarrow \| \hat{f} \|_{\infty} \leq \| f \|_{L^1(R)} .$$

(2) To prove the second property, let δ be chosen arbitrary and consider

$$\sup_{n} | \hat{f}(n+\delta) - \hat{f}(n) | = \sup_{n} \left| \int_{-\infty}^{\infty} e^{-inx} (e^{-i\delta x} - 1) f(x)\ dx \right|$$

$$\leq \int_{-\infty}^{\infty} | e^{-i\delta x} - 1 |\ | f(x) |\ dx .$$

Now, since

$$| e^{-i\delta x} - 1 |\ | f(x) | \leq 2 | f(x) | \in L^1(R)$$

and $| e^{-i\delta x} - 1 | \to 0$ as $\delta \to 0$,

then when $\delta \to 0$, the last integral $\to 0$. Therefore $\hat{f}$ is uniformly continuous on R.

(3) Let $n \to \pm\infty$, then for any $\varepsilon > 0$ we can find g such that $g, g' \in L^1(R)$ and

$$\| f - g \|_{L^1(R)} < \varepsilon$$

therefore from (1) we have

$$| \hat{f}(n) | \leq | \hat{f}(n) - \hat{g}(n) | + | \hat{g}(n) | \leq \| f - g \|_{L^1(R)} + | \hat{g}(n) |$$

$$< \varepsilon + | \hat{g}(n) | < \varepsilon + \frac{1}{in} | g'(n) | \to 0 \quad \text{as} \quad n \to \pm\infty .$$

$$\Rightarrow \lim_{n \to \pm\infty} \hat{f}(n) = 0 \quad \text{and hence} \quad f \in L^1(R) .$$

Remark: It is important to note that in the last part of theorem (1.7) $\hat{f}(n) \to 0$ as $n\to\pm\infty$ is not necessary that $\hat{f}(n)\in L^1$(R). This can be proved by the following example

$$u_a(x)=\begin{cases}1 & , \quad x\geq a\\ 0 & , \quad x<a\end{cases}$$

where $a\in R$.
Let

$$f(x)=e^{-x}u_0(x) \in L^1(R) \quad \text{then} \quad \hat{f}(n)=\frac{1}{1+in} \notin L^1(R) \; .$$

Definition (1.1.9):[7] Let $\hat{f}(n) \in L^1(R)$ be the *Fourier* transform of some function $f\in L^1(R)$. Then the inverse *Fourier* transform of $\hat{f}$ is defined by

$$(F^{-1}\hat{f})(x)=\int_{-\infty}^{\infty} e^{inx}\hat{f}(n)\, dn \; .$$

1.2 Summmability in Norm and Homogenous Banach spaces on T

In this section we want to establish some of the main facts of the *Fourier* transforms. We shall see that $\hat{f}$ determines f uniquely and we show how we can find f if we know $\hat{f}$.

Two very important properties of the Banach Space $L^1(T)$ are the following:

(a) If $f\in L^1(T)$ and $a\in T$ then $f_a(t)=f(t-a) \in L^1(T)$ and $\|f_a\|_{L^1(T)}=\|f\|_{L^1(T)}$.

(b) The $L^1(T)$ valued function $a\to f_a$ is continuous on T, that is for $f\in L^1(T)$ and $a_0\in T$, we have

$$\lim_{a\to a_0} \|f_a-f_{a_0}\|_{L^1(T)} = 0 \; . \tag{1.2.1}$$

We shall refer to (a) as the translation invariance of $L^1(T)$; it's an immediate consequence of the translation invariance of the measure dt , (where the translation invariance is $\forall t_0 \in T$ and f defined on T.

$\int f(t-t_0)dt=\int f(t)dt$, the integrals are taken over T) such that

$$\| f_a \| = \frac{1}{2\pi} \int | f(t-a) | dt$$

let $u = t - a$, $dt = du$ then

$$\| f_a \| = \frac{1}{2\pi} \int | f(u) | du = \| f \|_{L^1(T)} \quad .$$

In order to establish (b), we note that (1.2.1) is valid if f is continuous (the inverse is not true), and the continuous function is dense in $L^1(T)$. (Where dense here means for an arbitrary $\varepsilon > 0$ and for every continuous function f there exist $g \in L^1(T)$ with $\|f - g\| \le \varepsilon$).

Let f be arbitrary function such that $f \in L^1(T)$ and $\varepsilon > 0$ be given, furthermore let g be continuous function on T such that $\| g - f \| < \varepsilon/2$ then

$$\begin{aligned} \| f_a - f_{a_0} \|_{L^1(T)} &\le \| f_a - g_a \|_{L^1(T)} + \| g_a - g_{a_0} \|_{L^1(T)} + \| g_{a_0} - f_{a_0} \|_{L^1(T)} \\ &= \| (f-g)_a \|_{L^1(a)} + \| g_a - g_{a_0} \| + \| (g-f)_{a_0} \| \\ &< \varepsilon + \| g_a - g_{a_0} \|_{L^1(T)} . \end{aligned}$$

Hence $\overline{Lim} \| f_a - f_{a_0} \| < \varepsilon$ and ε being an arbitrary positive number. This proves (b).

Definition(1.2.1): [12] A summability kernel is a sequence $\{k_n\}$ of 2π -periodic continuous functions satisfying :

(1) $\frac{1}{2\pi} \int k_n(t) dt = 1, \; n = 1,2,3,...$

(2) $\frac{1}{2\pi} \int | k_n(t) | dt \le \text{constant}$

(3) For all $0 < \delta < \pi, \lim\limits_{n \to \infty} \int_{\delta}^{2\pi - \delta} | k_n(t) | \, dt = 0$.

A positive summability kernel is a kernel in which $k_n(t) \ge 0$ for all t and n. For positive kernels the assumption (2) is redundant .

Lemma (1.2.2): [12] Let B be a Banach space, Q is a continuous B-valued function on T and $\{k_n\}$ a summability Kernel then

$$\lim_{n \to \infty} \frac{1}{2\pi} \int k_n(T) Q(T) dT = Q(0)$$

Proof: (see **[12]** page(10)).

Theorem (1.2.3):[6] (Uniqueness theorem) Let $f \in L^1(T)$ and assume $\hat{f}(n) = 0$ for all n, then $f = 0$.
This theorem can also be formulated as:

Theorem: (Uniqueness theorem): let f, g $\in L^1(T)$ and assume $\hat{f}(n) = \hat{g}(n)$ for all n, then $f = g$.

Proof: Let $\hat{f}(n) = \hat{g}(n)$, then

$$\hat{f}(n) = \frac{1}{2\pi}\int f(t)\ e^{-\text{int}}dt = \frac{1}{2\pi}\int g(t)\ e^{-\text{int}}dt = \hat{g}(n).$$

From the last equation we have $\int (f(t) - g(t))\ e^{-\text{int}}dt = 0$, $\forall n = 0, 1, \ldots$

we have that $\int (f(t) - g(t))dt = \int (f(t) - g(t))e^{-it}dt = 0$ and we obtain

$$\int\ (f(t) - g(t))\ dt = \int\ (f(t) - g(t))\cos t\ dt = 0\ ,\ \forall t,$$

hence $f(t) = g(t)$.

Lemma (1.2.4): **[6]** (The Riemann _Lebegue lemma) Let $f \in L^1(T)$ then

$$\underset{|n| \to \infty}{Lim} \hat{f}(n) = 0.$$

Proof: By definition

$$\hat{f}(n) = \frac{1}{2\pi}\int f(t)\ e^{-\text{int}}dt \text{ then}$$

$$\underset{|n| \to \infty}{Lim} \hat{f}(n) = \underset{|n| \to \infty}{Lim} \frac{1}{2\pi}\int f(t)\ e^{-\text{int}}dt = \frac{1}{2\pi}\int f(t)\ \underset{|n| \to \infty}{Lim} e^{-\text{int}}dt$$

$$= \frac{1}{2\pi}\int f(t)\ (0)dt = 0$$

since we integrate with respect to t not to n.

Hence $\underset{|n| \to \infty}{Lim} \hat{f}(n) = 0$.

Remark: If K is a compact set in $L^1(T)$ and $\varepsilon > 0$, then there exist a finite number of trigonometric polynomials $p_1, p_2, \ldots, p_N$ such that for every $f \in K$ there exist J , $1 \le J \le N$ such that $\| f - P_J \|_{L^1} < \varepsilon$.

If $|n| > \max\limits_{1\le J\le N}$ (degree of p_J) then $|\hat{f}(n)| < \varepsilon$ for all $f\in K$ and the

Riemann_Lebesgue Lemma holds uniformly on a compact subset of $L^1(T)$.

Definition (1.2.5): [20] For $f\in L^1(T)$ we denote by $S_n(f)$ the nth partial sum of $S[f]$, given by:

$$S_n(f)(t) = S_n(f,t) = \sum_{-n}^{n} \hat{f}(J)e^{iJt}, \text{ then}$$

$$\sigma_n(f) = \frac{1}{n+1}(S_0(f) + S_1(f) + \ldots + S_n(f))$$

is the arithmetic mean of $S_n(f)$, and $S[f]$ converges in $L^1(T)$ and the limit is f.

Definition (1.2.6): [12] A homogenous Banach space on T is a linear subspace B of $L^1(T)$ having a norm $\| \;\|_B \ge \| \;\|_{L^1}$, under which is a Banach space, and having the following properties :

(a`) If $f\in B$ and $T\in T$, then $f_T\in B$ and $\|f_T\|_B = \|f\|_B$ (where $f_T(t) = f(t-T)$).

(b`) For all $f\in B$, $T, T_0\in T$, $\lim\limits_{T\to T_0} \| f_T - f_{T_0} \|_B = 0$.

We note that condition (a`) is referred to as translation invariance and (b`) as continuity of translation.

We could simplify (b`) some what by requiring continuity at one specific $T_0\in T$, say $T_0=0$, rather than at every $T_0\in T$, since by (a`)

$$\|f_T(t) - f_{T_0}(t)\|_B = \|f_{(T-T_0)}(t) - f(t)\|_B \to 0 .$$

Examples of Homogenous Banach spaces on T:

(1) C (T) is the space of all continuous 2π -periodic functions with the norm $\| f \|_\infty = \max_t |f(t)|$.

(2) $C^n(T)$ is the subspace of $C(T)$ of all n-times continuously differentiable functions (n being a partial integer), with the norm

$$\| f \|_{C^n} = \sum_{J=0}^{n} \frac{1}{J!} \max_t | f^{(J)}(t) | .$$

(3) $L^P(T)$, $1 \le p < \infty$, the subspace of $L^1(T)$ consisting of all the functions f for which $\int | f(t) |^p \, dt < \infty$ with the norm

$$\| f \|_{L^P} = (\frac{1}{2\pi} \int | f(t) |^p \, dt)^{\frac{1}{p}} .$$

Proof:

To prove (a`) and (b`) in definition (2.6) for the first example by taking the norm

$\| f \|_\infty = \max_t | f(t) |$

(a`) First we want $\| f_T \|_\infty = \| f \|_\infty$

$$\| f_T \|_\infty = \max_t | f_T | = \| f_T \|_\infty = \max_t | f(t - T) |$$

let $u = t\text{-}T$, this means that $\| f_T \|_\infty = \max_u | f(u) | = \| f \|_\infty$

(b`) We want to prove $\lim_{T \to T_0} \| f_T - f_{T_0} \|_B = 0$.

$$\| f_T - f_{T_0} \| = \max_t | f_T - f_{T_0} | = \max_t | f(t - T) - f(t - T_0) |$$

thus $\lim_{T \to T_0} \max_t | f(t - T) - f(t - T_0) | = 0.$

(2) To prove (a`) and (b`) for the second example, by taking the norm

$$\| f_T \|_{c^n} = \sum_{J=0}^{n} \frac{1}{J!} \max_t | f_T^{(J)}(t) | = \sum_{J=0}^{n} \frac{1}{J!} \max_u | f^{(J)}(u) |$$

$= \| f \|_{c^n}$, where $u = t\text{-}T$.

To prove (b`), we take

$$\| f_T - f_{T_0} \|_{C^n} = \sum_{J=0}^{n} \frac{1}{J!} \max_t | (f_T - f_{T_0})^{(J)} |$$

It is clear that $\underset{T \to T_0}{Lim} \| f_T - f_{T_0} \| = 0.$

(3) To prove (a`) and (b`) for the third example, by definition

$$\| f_T \|_{L^p} = (\frac{1}{2\pi} \int | f_T(t) |^p \, dt)^{\frac{1}{p}} = (\frac{1}{2\pi} \int | f(t-T) |^p \, dt)^{\frac{1}{p}},$$

let $u = t$-T. Thus we have

$$\| f_T \|_{L^p} = (\frac{1}{2\pi} \int | f(u) |^p \, dt)^{\frac{1}{p}} = \| f \|_{L^p}$$

(1) To prove (b`)

$$\| f_T - f_{T_0} \|_{L^p} = (\frac{1}{2\pi} \int | f_T(t) - f_{T_0}(t) |^p \, dt)^{\frac{1}{p}}.$$

If we take the limit, we obtain

$$\underset{T \to T_0}{Lim} \| f_T - f_{T_0} \|_{L^p} = (\frac{1}{2\pi} \int | 0 |^p \, dt)^{\frac{1}{p}} = 0.$$

1.3 *The order of magnitude of Fourier coefficients*

Definition (1.3.1): A sequence $\{a_n\}_{n=-\infty}^{\infty}$ is said to be an even sequence if and only if $a_n = a_{-n}$, $\forall\, n = 0, 1, 2, \ldots$.

The only things we know so far about the size of *Fourier* coefficients $\{ \hat{f}(n) \}$ of a function $f \in L^1(T)$ is that they are bounded by $\|f\|_{L^1}$ and

that $\underset{|n| \to \infty}{Lim} \hat{f}(n) = 0$, (The Riemann _Lebesgue Lemma).

Theorem (1.3.2): [12] Let $\{a_n\}_{n=-\infty}^{\infty}$ be an even sequence of nonnegative

numbers tending to zero at infinity. Assume that for

$n > 0$, $a_{n-1} + a_{n+1} - 2a_n \geq 0$ (1.3.1)

then there exists a nonnegative function $f \in L^1(T)$ such that $\hat{f}(n) = a_n$.

Proof: From the convexity condition in (1.3.1) implies that $(a_n - a_{n+1})$ is monotonically decreasing with n, hence $\lim\limits_{n\to\infty} n(a_n - a_{n+1}) = 0$ and

$$\sum_{n=1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) = a_0 - a_N - N(a_N - a_{N+1})$$

converges to a_0 as $N\to\infty$.

Let

$$f(t) = \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)k_{n-1}(t) \qquad (1.3.2)$$

where $\{k_n(t)\}$ is positive summability kernel which has the following properties :

(1) For $0 < \delta < \pi$, where

$$k_n(t) = \sum_{J=-n}^{n} (1 - \frac{|J|}{n+1})e^{iJt} \quad ,$$

$\lim\limits_{n\to\infty} (\sup\ k_n(t)) = 0,\ \delta < t < 2\pi - \delta$ and $k_n(t) = k_n(-t)$.

(2) Since $\|k_n\|_{L^1} = 1$, the series (1.3.2) converges in $L^1(T)$ and all it's terms being nonnegative, it's limit f is nonnegative and

$$\begin{aligned}\hat{f}(J) &= \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)\hat{k}_{n-1}(J) \\ &= \sum_{n=|J|+1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)\left(1 - \frac{|J|}{n}\right) \\ &= a_{|J|}.\end{aligned}$$

We want now to discuss the basic difference between sine_ series $(a_{-n} = -a_n)$ and cosine_ series $(a_{-n} = a_n)$.

Theorem(1.3.3): [12] (Fejer's theorem: Let $f \in L^1(T)$, assume that $\lim_{h\to 0} (f(t_0+h)+f(t_0-h))$ exists (we allow the values $-\infty$ and $+\infty$); then $\sigma_n(f(t_0)) \to (1/2) \lim_{h\to 0} (f(t_0+h)+f(t-h))$.

In particular, if t_0 is a point of continuity of f, then $\sigma_n(f(t_0) \to f(t_0)$.

Theorem (1.3.4): **[12]** Let $f \in L^1(T)$ and assume that $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0$ then

$$\sum_{n\neq 0} \frac{1}{n}\hat{f}(n) < \infty \quad .$$

Proof: Without loss of generality we assume $\hat{f}(0)=0$. Let

$$F(T)=\int_0^T f(u)du$$

with $F\in C(T)$. We can show that

$$\hat{F}(n)=\frac{1}{in}\hat{f}(n),\ n\neq 0\ ,\quad \text{from}\quad \hat{F}(n)=\frac{1}{2\pi}\int\int_0^T f(u)du\ e^{-int}dt$$

and integrating by parts letting

$$u=\int_0^T f(u)\ du,\ dv=e^{-int}$$

we can find

$$\hat{F}(n)=\frac{1}{2\pi}\frac{e^{-int}}{-in}\int_0^T f(u)du\ \Big]_0^{2\pi}+\frac{1}{in}\frac{1}{2\pi}\int e^{-int}f(t)dt$$

$$=0+\frac{1}{in}\hat{f}(n),\ n\neq 0\ .$$

Since F is continuous we can apply Fejer's theorem(stated below) and obtain

$$\mathop{Lim}_{N\to\infty}\ 2\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)\frac{\hat{f}(n)}{n}=i(F(0)-\hat{F}(0))\ ,\quad \text{since}\ \frac{\hat{f}(n)}{n}\geq 0\ . \qquad (1.3.3)$$

Corollary (1.3.5): [12] If $a_n>0$ and $\sum_{n=1}^{\infty}\frac{a_n}{n}=\infty$ then $\sum_{n=1}^{\infty}a_n\sin nt$ is not a *Fourier* series .

Hence there exist a trigonometric series with coefficients tending to zero which are not *Fourier* series .

<u>Examples</u>: The series $\sum_{n=2}^{\infty}\frac{\cos nt}{\log n}=\sum_{|n|\geq 2}\frac{e^{int}}{2\log|n|}$ is a *Fourier* series

while it's conjugate series

$$\sum_{n=2}^{\infty}\frac{\sin nt}{\log n}=-i\sum_{|n|\geq 2}\frac{\operatorname{sgn}(n)e^{int}}{2\log|n|}\qquad \text{is not .}$$

Theorem (1.3.6): **[12]** If $f\in L^1(T)$ is absolutely continuous, then $\hat{f}(n)=O\left(\frac{1}{n}\right)$

Proof: By theorem (1.3.2) we have $\hat{f}(n)=\frac{1}{in}\hat{f}'(n)$ and from Riemann_ Lebesgue lemma we have

$$\hat{f}'(n) \to 0 .$$

Note: If f is k-times differentiable function and $f^{(k)} \in L^1(T)$, then

$$\hat{f}(n)=O\left(\frac{1}{n^k}\right) \quad \text{as } |n| \to \infty . \tag{1.3.4}$$

Theorem (1.3.7):[13] If f is k-times differentiable and $f^{(k)} \in L^1(T)$ then

$$|\hat{f}(n)| \le \min_{0 \le J \le k} \frac{\| f^{(J)} \|_{L^1(T)}}{|n|^J} ,$$

if f is infinitely differentiable function, then

$$|\hat{f}(n)| \le \min_{0 \le J} \frac{\| f^{(J)} \|}{|n|^J} .$$

Proof: To prove this theorem, we use the mathematical induction method.

Thus we start by showing that it's true for $k=1$, in order to do so, we have to prove that

$\hat{f}(n)=\frac{1}{in}\hat{f}'(n)$, or $\hat{f}'(n)=(in)\hat{f}(n)$, so we have

$$\hat{f}'(n)=\frac{1}{2\pi}\int f'(t)\, e^{-\text{int}}\, dt$$

$$= \frac{1}{2\pi} f(t)\, e^{-\text{int}} + in\frac{1}{2\pi}\int f(t)\, e^{-\text{int}}\, dt = (in)\hat{f}(n)$$

thus $|\hat{f}(n)| \le \frac{\| f' \|_{L^1(T)}}{|n|}$.

Suppose that

$\hat{f}(n)=\frac{1}{(in)^J}\hat{f}^{(J)}(n)$ for $k=J$, we want to show now it's true for $k=J+1$

using

$\hat{f}(n)=\frac{1}{(in)^{J}}\hat{f}^{(J)}(n)$ we have $\hat{f}(n)=\frac{1}{(in)^{J}}(\hat{f}^{(J)}(n))'=\frac{1}{(in)^{j}}\frac{1}{(in)}\hat{f}^{(J+1)}(n)$

$=\frac{1}{(in)^{J+1}}\hat{f}^{(j+1)}(n)$ and $|\hat{f}(n)|=\frac{|f^{(k)}|}{|in|^{k}}\leq\frac{\|\hat{f}\|_{L^1(T)}}{|n|^{k}}$, $\forall k=1,2,3,\ldots$

hence $|\hat{f}(n)|\leq \min\limits_{0\leq J\leq k}\frac{\|f^{(J)}\|_{L^1(T)}}{|n|^{J}}$.

Definition (1.3.8): For a periodic function f define the total variation

$\mathrm{V}(f)=\sup\limits_{x_k}|\sum_{k=1}^{m}|f(x_k)-f(x_{k-1})||$

with respect to all sequences $(x_k)_{k=0}^{m}$ such that $x_0<x_1<\ldots\ldots<x_0+2\pi$ then f is of bounded variation if and only if $\mathrm{V}(f)<\infty$.

Theorem (1.3.9): [6] If f is of bounded variation on T, then

$|\hat{f}(n)|\leq\frac{\operatorname{var}(f)}{2\pi|n|}$, where $\operatorname{var}(f)=\sup\limits_{x_k}|\sum_{k=1}^{m}|f(x_k)-f(x_{k-1})||$

and (x_k) is the sequence given in definition (3.7).

Proof:

We integrate (1.1.6) by parts to obtain

$$|\hat{f}(n)| = |\frac{1}{2\pi}\int e^{-\mathrm{int}}f(t)dt| \leq |\frac{1}{2\pi in}\int e^{-\mathrm{int}}df(t)| \leq \frac{\operatorname{var}(f)}{2\pi|n|}.$$

Definition (1.3.10): **[6]** Let $f\in C(T)$, let $W(f,h)=\max|f(t+h)-f(t)|$ for $f\in L^1(T)$ and $\Omega(f,h)$ is the integral modulus of continuity of f, such that

$$\Omega(f,h)=\|f(t+h)-f(t)\|_{L^1(T)}$$

then it's clear that $\Omega(f,h)\leq W(f,h)$.

Theorem (1.3.11): **[6]** For $n\neq 0$, $|\hat{f}(n)|\leq\frac{1}{2}\Omega\left(f,\frac{\pi}{|n|}\right)$

Proof: By definition (1.1.6) we have

$$\hat{f}(n)=\frac{1}{2\pi}\int f(t)\ e^{-\mathrm{int}}dt=\frac{-1}{2\pi}\int f(t)e^{-in(t+\frac{\pi}{n})}dt\ ,$$

it's clear that

$$|\hat{f}(n)| = \left| \frac{1}{4\pi}\int (f\left(t+\frac{\pi}{n}\right)-f(t))e^{-\text{int}}dt \right|$$

hence

$$|\hat{f}(n)|<\frac{1}{2}\Omega\left(f,\frac{\pi}{|n|}\right) .$$

1.4 *Fourier series of square summable Functions*

Definition (1.4.1):[8] A Hilbert space is a complete inner product space. Define $L^2(T)$ as a Hilbert space, it's inner product being defined by

$$\langle f,g\rangle=\frac{1}{2\pi}\int f(t)\overline{g(t)}\,dt \qquad (1.4.1)$$

where f and g are complex_valued functions .

Definition (1.4.2): **[8]** Let H be a complex Hilbert space:
(1) Let f, $g\in$H then f is orthogonal to g if $\langle f,g\rangle=0$
(2) If E is a subset of H, we say that $f\in$H is orthogonal to E if f is orthogonal to each element of E
(3) If E is a subset of H then E is said to be orthogonal if any two vectors in E are orthogonal to each other
(4) A set E subset of H is said to be an orthonormal system if it's orthogonal and the norm of each vector in E equal 1. That is for every f, $g\in$E, $\langle f,g\rangle=0$,$f\neq g$ and $\langle f,f\rangle=1$.

Lemma (1.4.3): [17] Let $\{Q_n\}$ be a finite orthonormal system, let further $a_1,a_2,a_3,\ldots,a_n$, be complex numbers, then

$$\left\|\sum_{n=1}^{N}a_nQ_n\right\|^2=\sum_{n=1}^{N}|a_n|^2 .$$

Proof: Take a finite orthonormal system $\{Q_n\}$, then by definition of the norm

$$\left\|\sum_{n=1}^{N}a_nQ_n\right\|^2= \left\langle \sum_{n=1}^{N}a_nQ_n,\sum_{n=1}^{N}a_nQ_n\right\rangle=\sum_{n=1}^{N}a_n\left\langle Q_n,\sum_{m=1}^{N}a_mQ_m\right\rangle=\sum_{n=1}^{N}a_n\overline{a_n}=\sum_{n=1}^{N}|a_n|^2.$$

Corollary (1.4.4):[12] Let $\{Q_n\}_{n=1}^{\infty}$ be an orthonormal system in H and let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty}|a_n|^2<\infty$, then $\sum_{n=1}^{\infty}a_nQ_n$ converges in H .

Proof : Since H is a complete Hilbert space, we have to show that the partial sums

$S_N=\sum_{n=1}^{N}a_nQ_n$ form a Cauchy sequence in H.

Now for $N>m$, such that N, m are integers, we have

$$\| S_N-S_m\|^2=\|\sum_{m+1}^{N}a_nQ_n\|^2=\sum_{m+1}^{N}|a_n|^2\to 0 \quad \text{as} \quad m\to\infty .$$

Lemma(1.4.5):[17] Let H be a Hilbert space and let $\{Q_n\}_{n=1}^{N}$ be a finite orthonormal system in H. For $f\in$H let $a_n=\langle f,Q_n\rangle$,

then

$$\| f-\sum_{n=1}^{N}a_nQ_n\|^2=\| f\|^2-\sum_{n=1}^{N}|a_n|^2. \tag{1.4.2}$$

Proof : Let $\{Q_n\}$ be a finite orthonormal system in H, then

$$\| f-\sum_{n=1}^{N}a_nQ_n\|^2=\langle f-\sum_{n=1}^{N}a_nQ_n, f-\sum_{n=1}^{N}a_nQ_n\rangle$$

$$=\| f\|^2-\sum_{n=1}^{N}\overline{a_n}\langle f,Q_n\rangle-\sum_{n=1}^{N}a_n\langle Q_n,f\rangle+\sum_{n=1}^{N}|a_n|^2$$

$$=\| f\|^2-\sum_{n=1}^{N}|a_n|^2 \text{ since } \quad a_n=\langle f,Q_n\rangle \quad \text{and} \quad \overline{a_n}=\langle Q_n,f\rangle.$$

Corollary(1.4.6):[17](Bessel's inequality) Let H be a Hilbert space and $\{Q_n\}$ be an orthonormal system in H. For $f\in$H take $a_n=\langle f,Q_n\rangle$ then

$$\sum_{k=1}^{N}|a_n|^2\le\| f\|^2 . \tag{1.4.3}$$

Proof: Let $f\in$H and $f=\sum_{k=1}^{N}a_kQ_k$ then

$f=\sum\langle f,Q_k\rangle Q_k$, where $\| f\|^2=\langle f,f\rangle=\langle\sum\langle f,Q_k\rangle Q_k,f\rangle$

$\sum|a_k|^2=\sum\langle f,Q_k\rangle\langle Q_k,f\rangle=\sum|\langle f,Q_k\rangle|^2\le\| f\|^2\| Q_k\|^2=\| f\|^2$,

hence $\sum|\langle f,Q_k\rangle|^2=\sum|a_k|^2\le\| f\|^2$.

Definition(1.4.7): A complete orthonormal system in H is an orthonormal system having the additional property that the only vector in H orthogonal to it is the zero vector .

Lemma(1.4.8):[12] Let $\{Q_n\}$ be an orthonormal system in H, then the following statements are equivalent :

(a) $\{Q_n\}$ is complete

(b) For every $f \in$ H we have $\| f \|^2 = |\sum \langle f, Q_n \rangle|^2$

(c) $f = \sum \langle f, Q_n \rangle Q_n$.

Proof: [see [12]] .

Lemma(1.4.9) :[12] Let $\{Q_n\}$ be a complete orthonormal system in H and let f, $g \in$H, then

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, Q_n \rangle \langle Q_n, g \rangle .$$

Proof : If f is a finite linear combination of $\{Q_n\}$ then

$$f = \sum_{n=1}^{N} \langle f, Q_n \rangle Q_n, \langle f, g \rangle = \langle \sum_{n=1}^{N} \langle f, Q_n \rangle Q_n, g \rangle$$

$$= \sum_{n=1}^{N} \langle f, Q_n \rangle \langle Q_n, g \rangle .$$

In general case if $\{Q_n\}$ is a finite orthonormal system in H then

$$\langle f, g \rangle = \lim_{N \to \infty} \langle \sum_{n=1}^{N} \langle f, Q_n \rangle Q_n, g \rangle = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, Q_n \rangle \langle Q_n, g \rangle .$$

Remarks : For H=$L^2(T)$ the exponential $\{ e^{\text{int}} \}_{n=-\infty}^{\infty}$ form a complete orthonormal system with

$$\langle e^{\text{int}}, e^{imt} \rangle = \frac{1}{2\pi} \int e^{i(n-m)t} dt = \delta_{nm}, \quad \text{where}$$

$$\delta_{nm} = \begin{cases} 1 \ , & \text{for} \quad n = m \\ 0 \ , & \text{for} \quad n \neq m \end{cases}$$

and

$$\langle f, e^{\text{int}} \rangle = \frac{1}{2\pi}\int f(t)\overline{e^{\text{int}}}\,dt = \frac{1}{2\pi}\int f(t)\ e^{-\text{int}}\,dt = \hat{f}(n)\ .$$

We want to introduce some information about orthogonal and trigonometric system in two variables .

Let R be a rectangle in the xy-plane described by $a \le x \le b\ ,\ c \le y \le d$. Further more, let $\phi_n\ (x,y)$ be a system of continuous functions defined on R, $n = 0,1,2,\ldots,$. Non of $\phi_n(x,y) = 0$ is identical. (1.4.5)

The system (1.4.5) is orthogonal if $\iint\limits_R \phi_n(x,y)\phi_m(x,y)dxdy = 0, \forall n \ne m.$

The norm of ϕ_n is given by

$$\|\,\phi_n\,\| = \sqrt{\iint\limits_R {\phi_n}^2(x,y)dxdy}\ \ . \tag{1.4.6}$$

The system is said to be normalized if $\|\phi_n\| = 1$, $\forall n = 0,1,2,3,\ldots$ or

$$\iint\limits_R {\phi_n}^2(x,y) = 1\quad ,\quad n = 0,1,2,\ldots,\qquad .$$

Orthogonal system can often be normalized by multiplying each of ϕ_n , $n = 0,1,2,\ldots,$ by

$$M_n = \frac{1}{\|\,\phi_n\,\|}\ \ .$$

To this extend we can associate the *Fourier* series with every absolutely integrable function $f\,(x,$y) defined on R, this means that:

$$f(x,y) \sim c_0\phi_0(x,y) + c_1\phi_1(x,y) + c_2\phi_2(x,y) + \ldots.. + c_n\phi_n(x,y) + \ldots., \tag{1.4.7}$$

where c_n is the *Fourier* coefficients of $f\,(x,y)$ and is given by

$$c_n = \frac{\iint\limits_R f(x,y)\phi_n(x,y)dxdy}{\iint\limits_R {\phi_n}^2(x,y)dxdy} = \frac{\iint\limits_R f(x,y)\phi_n(x,y)dxdy}{\|\,\phi_n\,\|^2}\qquad . \tag{1.4.8}$$

The basic trigonometric system in two variables are the functions 1,cosmx,sinmx,cos$n$y,sin$n$y ,…, cosmx cos$n$y, sinmx cos$n$y,…, cosmxsin$n$y, sinmxsin$n$y,…. ($m = 1,2\ldots;\ n = 1,2,\ldots$) (1.4.9)

where $-\pi \le x \le \pi$,$-\pi \le$ y $\le \pi$,or $a \le x \le a+2\pi$, $b \le$ y $\le b+2\pi$ and

$$\iint 1.\cos mx dx dy = \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} \cos mx dx = 0,$$

$$\iint 1.\sin mx dx dy = \iint 1.\cos ny dx dy = \iint 1.\sin ny dx dy = 0 \quad \text{and}$$

$$\iint_{k} (\cos mx \cos ny)(\cos rx \cos sy) dy dx = (\int_{-\pi}^{\pi} \cos mx \cos rx dx)(\int_{-\pi}^{\pi} \cos ny \cos sy dy) = 0.$$

$\forall m \neq r$, or $n \neq s$.

It can easily be shown that $\|1\| = 2\pi$, $\|\cos mx\| = \|\sin mx\| = \|\cos ny\| = |\sin ny\| = (2)^{1/2}\pi$, and

$$\| \cos mx \cos ny \| = \| \sin mx \cos ny \| = \| \cos mx \sin ny \| = \| \sin mx \sin ny \| = \pi \ .$$

And the *Fourier* coefficients of $f(x, y)$are :

$$A_{00} = \frac{\iint_{R} f(x,y) dx dy}{\| 1 \|^2} = \frac{1}{4\pi^2} \iint_{R} f(x,y) dx dy$$

$$A_{m0} = \frac{\iint_{R} f(x,y) \cos mx dx dy}{\| \cos mx \|^2} = \frac{1}{2\pi^2} \iint_{R} f(x,y) \cos mx dx dy \ ,(m = 1,2,3,....)$$

$$A_{0n} = \frac{\iint_{R} f(x,y) \cos ny dx dy}{\| \cos ny \|^2} = \frac{1}{2\pi^2} \iint_{R} f(x,y) \cos ny dx dy, n = 1,2,3,...$$

$$B_{m0} = \frac{\iint_{R} f(x,y) \sin mx dx dy}{\| \sin mx \|^2} = \frac{1}{2\pi^2} \iint_{R} f(x,y) \sin mx dx dy, m = 1,2,3,...$$

$$B_{0n} = \frac{\iint_{R} f(x,y) \sin ny \ dy \ dx}{\| \sin ny \|^2} = \frac{1}{2\pi^2} \iint_{R} f(x,y) \sin ny \ dy \ dx \ , \ n = 1,2,.. \qquad (1.4.10)$$

and

$$a_{mn} = \frac{1}{\pi^2} \iint_{R} f(x,y) \cos mx \cos ny dx dy$$

$$b_{mn} = \frac{1}{\pi^2} \iint_{R} f(x,y) \sin mx \cos ny dx dy$$

$$c_{mn} = \frac{1}{\pi^2} \iint_{R} f(x,y) \cos mx \sin ny dx dy$$

$$d_{mn} = \frac{1}{\pi^2} \iint_{R} f(x,y) \sin mx \sin ny dx dy, \ m = 1,2,...; n = 1,2,,.... \qquad (1.4.11)$$

The *Fourier* series of $f(x,y)$ can be written more compactly in the complex form

$$f(x,y) \sim \sum_{m,n=-\infty}^{\infty} c_{mn} e^{i(mx+ny)}, \text{where} \quad c_{mn} = \frac{1}{4\pi^2} \iint_R f(x,y)\ e^{-i(mx+ny)} dx\ dy \quad ,$$

$$m = 0, \pm 1, \pm 2, \ldots, n = 0, \pm 1, \pm 2, \ldots,$$

$$f(x,y) \sim \sum_{m,n=0}^{\infty} \lambda_{mn} [a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + c_{mn} \cos mx \sin ny +$$

$$d_{mn} \sin mx \sin ny].$$

$f(x, y)$ is a periodic function of period 2π both in x and y and the partial sums of the double *Fourier* series is $s_{mn}(x,y)$, $m=0,1,2,\ldots$; $n=0,1,2,\ldots$ where

$$s_{mn}(x,y) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \lambda_{\mu\nu} (a_{\mu\nu} \cos \mu x \cos \nu y + b_{\mu\nu} \sin \mu x \sin \nu y + c_{\mu\nu} \cos \mu x \sin \nu y +$$

$$d_{\mu\nu} \sin \mu x \sin \nu y \ )$$

$$\lambda_{mn} = \begin{cases} \dfrac{1}{4} & , \quad \text{for} \quad m = n = 0. \\ \dfrac{1}{2} & , \quad \text{for} \quad m > 0, n = 0, \text{or} \quad m = 0,\ n > 0 \\ 1 & , \quad \text{for} \quad m > 0, n > 0 \ . \end{cases}$$

Where instead of A_{00} we write $(1/4)a_{00}$, $m = 0 = n$ and A_{m0}, A_{0n}, B_{m0} and B_{0n} equals $(1/2)a_{m0}, (1/2)a_{0n}, (1/2)b_{m0}$ and$(1/2)c_{0n}$

respectively

$$s_{mn}(x,y) = \frac{1}{\pi^2} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \lambda_{\mu\nu} \iint_R f(s,t) \cos \mu(s-x) \cos \nu(t-y)\ dsdt$$

$$= \frac{1}{\pi^2} \iint_R f(s,t) \left[\frac{1}{2} + \sum_{\mu=1}^{m} \cos \mu(s-x) \right] \left[\frac{1}{2} + \sum_{\nu=1}^{n} \cos(t-y) \right] dsdt\ . \qquad (1.4.12)$$

We can obtain this formula by substituting (1.4.10) in equation (1.4.12) to obtain

$$s_{mn}(x,y) = \frac{1}{\pi^2} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \lambda_{\mu\nu} (\iint f(s,t)\ \cos \mu s \cos \nu t \cos \mu x \cos \nu y\ dsdt +$$

$$\iint f(s,t) \sin \mu s \cos \nu t \sin \mu x \cos \nu y\ dsdt + \iint f(s,t) \cos \mu s \sin \nu t \cos \mu x \sin \nu y dsdt$$

$$+ \iint f(s,t) \sin \mu s \sin \nu t \sin \mu x \sin \nu y dsdt$$

which equals

$$\frac{1}{\pi^2}\sum_{\mu=0}^{m}\ \sum_{\nu=0}^{n}\lambda_{\mu\nu}(\iint f(s,t)\ \ (\cos\mu s\cos\nu t\cos\mu x\cos\nu y+\sin\mu s\cos\nu t\sin\mu x\cos\nu y$$

$$+\cos\mu s\sin\nu t\cos\mu x\sin\nu y+\sin\mu s\sin\nu t\sin\mu x\sin\nu y)dsdt.$$

By taking a common factor we obtain

$$\frac{1}{\pi^2}\sum_{\mu=0}^{m}\ \sum_{\nu=0}^{n}\lambda_{\mu\nu}(\iint f(s,t)(\cos\mu s\cos\mu x(\cos\nu t\cos\nu y+\sin\nu t\sin\nu y)+$$

$$\sin\mu s\sin\mu x(\cos\nu t\cos\nu y+\sin\nu t\sin\nu y))dsdt.$$

Which equals

$$\frac{1}{\pi^2}\sum_{\mu=0}^{m}\ \sum_{\nu=0}^{n}\lambda_{\mu\nu}(\iint(f(s,t)(\cos\mu s\cos\mu x+\sin\mu s\sin\mu x)(\cos\nu t\cos\nu y+$$

$$\sin\nu t\sin\nu y))dsdt=\frac{1}{\pi^2}\sum_{\mu=0}^{m}\ \sum_{\nu=0}^{n}\lambda_{\mu\nu}(\iint f(s,t)\cos\mu\ (s-x)\cos\nu\ (t-y)dsdt$$

$$=\frac{1}{\pi^2}\iint(f(s,t)\left[\frac{1}{2}+\sum_{\mu=0}^{m}\ \cos\mu(s-x)\ \right]\left[\frac{1}{2}+\sum_{\nu=0}^{n}\cos\nu(t-y))\ \right]dsdt.$$

1.5 *Double Fourier series for a function with different periods in x and y*

Let $f(x,y)$ be periodic in both x and y with periods $2l$ and $2h$ then if we let $u=(\pi x/l)$ and $v=(\pi y/h)$ then $f(x,y)=f((lu/\pi),(hv/\pi))=\phi(u,v)$ is periodic
If we have

$$\phi(u,v)\sim\sum_{m,n=0}^{\infty}\lambda_{mn}[a_{mn}\cos mu\cos nv+b_{mn}\sin mu\cos nv+c_{mn}\cos mu\sin nv+$$

$$d_{mn}\sin mu\sin nv]$$

then we obtain :

$$f(x,y)=\sum_{m,n=0}^{\infty}\lambda_{mn}[a_{mn}\cos\frac{\pi}{l}mx\cos\frac{\pi}{h}ny+b_{mn}\sin\frac{\pi}{l}mx\cos\frac{\pi}{h}ny$$

$$+c_{mn}\cos\frac{\pi}{l}mx\sin\frac{\pi}{h}ny+d_{mn}\sin\frac{\pi}{l}mx\sin\frac{\pi}{h}ny]$$

where $$a_{mn} = \frac{1}{lh}\iint_R f(x,y)\cos\frac{m\pi}{l}x\cos\frac{n\pi}{h}y \; dx\; dy$$

Thus the complex form of the *Fourier* series becomes:

$$f(x,y) \sim \sum_{m,n=-\infty}^{\infty} c_{mn} e^{i\pi[(\frac{mx}{l})+(\frac{ny}{h})]} \quad \text{with}$$

$$c_{mn} = \frac{1}{4lh}\iint_R f(x,y) e^{-i\pi[(\frac{mx}{l})+(\frac{ny}{h})]}\, dxdy, \; (m = 0,\pm1,\pm2,\ldots\ldots, \quad n = 0,\pm1,\pm2,\ldots)$$

1.6 *Absolutely convergent Fourier Series*

Definition (1.6.1):[13] A Banach space is complete with respect to its defined metric space.
Let A(T) be the space of all functions on T having an absolutely convergent *Fourier* series, that is, for the function f for which

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

and the mapping $f \to \{\hat{f}(n)\}_{n\in \mathbb{Z}}$ of A(T) into l^1 (the Banach space of absolutely convergent sequences)is a one to one function and linear.

If $\sum_{-\infty}^{\infty} |a_n| < \infty$ and $g = \sum_{-\infty}^{\infty} a_n e^{\text{int}}$ be uniformly convergent on T

then $a_n = \hat{g}(n)$.
If this condition is satisfied then the mapping $f \to \{\hat{f}(n)\}$ of A(T) into l^1 is isomorphism (one to one and onto) .

Let the norm of f on A(T) be :

$$\| f \|_{A(T)} = \sum_{-\infty}^{\infty} |\hat{f}(n)| \; .$$

Lemma(1.6.2):[12] Assume that $f, g \in$ A(T) , then $fg \in$ A(T) and $\|fg\|_{A(T)} \leq \|f\|_{A(T)} \cdot \|g\|_{A(T)}$.

Proof : Let $f, g \in A(T)$ then by using

$f(t) = \sum \hat{f}(n)\ e^{\mathrm{int}}$ and $g(t) = \sum \hat{g}(n)\ e^{\mathrm{int}}$ where both series converge absolutely, then

$$f(t)g(t) = \sum_k \sum_m \hat{f}(k)\ e^{ikt} \hat{g}(m) e^{imt} = \sum_k \sum_m \hat{f}(k)\hat{g}(m) e^{i(k+m)t},$$

collecting the terms for which $k + m = n$ to obtain

$$f(t)g(t) = \sum_n \sum_k \hat{f}(k)\hat{g}(n-k)\ e^{\mathrm{int}}$$

so that $\quad (fg\hat{)}(n) = \sum_k \hat{f}(k)\hat{g}(n-k)$

hence

$$\sum |(fg\hat{)}(n)| \leq \sum_n \sum_k \ |\hat{f}(k)|\ |\hat{g}(n-k)| = \sum_k |\hat{f}(k)| \sum_n |\hat{g}(n)|\ .$$

1.7 *Fourier coefficients of linear functionals*

Definition(1.7.1): [8] the dual space of the space B is the set of all linear forms on the space B and is denoted by B^*.

Let B be a homogenous Banach space on T, let $e^{int} \in B$ for all n and let further B^* be the dual space of B, then we have

Definition(1.7.2): The *Fourier* coefficients of a functional $M \in B^*$ are:

$\hat{M}(n) = \overline{\langle e^{\mathrm{int}}, M \rangle}, \quad n \in Z$

and the *Fourier* series of M is the trigonometric series

$S[M] \sim \sum \hat{M}(n)\ e^{\mathrm{int}}$ and $|\hat{M}(n)| \leq \| M \|_{B^*} \| e^{\mathrm{int}} \|_B$.

If $B = L^p(T)$, $1 < p < \infty$, B^* is canonically identified with $L^q(T)$, where $q = p/(p-1)$ and $g \in L^q(T)$ then the following function is a linear functional

$$f \to \langle f, g \rangle = \frac{1}{2\pi} \int f(t)\overline{g(t)}\, dt, f \in L^P(T) \quad \text{and}$$

$$\overline{\langle e^{\mathrm{int}}, g \rangle} = \frac{1}{2\pi} \int \overline{e^{\mathrm{int}}\, \overline{g(t)}}\ dt = \frac{1}{2\pi} \int e^{-\mathrm{int}} g(t) dt = \hat{g}(n)\ .$$

Thus $\hat{g}(n)$ coincides with the nth *Fourier* coefficient of the function g .

Theorem(1.7.3):[12] (Parseval's formula) Let $f \in B, M \in B^*$ then

$$\langle f, M \rangle = \underset{N\to\infty}{Lim} \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n)\overline{\hat{M}(n)}.$$

Proof:

For polynomials $p(t) = \sum_{-N}^{N} \hat{p}(n)\, e^{\text{int}}$, we have

$$\langle p, M \rangle = \langle \sum_{-N}^{N} \hat{p}(n)\, e^{\text{int}}, M(t) \rangle$$

$$= \frac{1}{2\pi} \int \sum_{-N}^{N} \hat{p}(n) \overline{e^{-\text{int}} M(t)}\, dt$$

$$= \sum_{-N}^{N} \hat{p}(n) \frac{1}{2\pi} \int \overline{M(t) e^{-\text{int}}}\, dt = \sum_{-N}^{N} \hat{p}(n)\overline{\hat{M}(n)} .$$

In general case we can obtain by using theorem(2.11) in **[12]** that

$$f = \underset{N\to\infty}{Lim}\, \sigma_N(f)$$

in the B norm and since M is continuous we have

$$\langle f, M \rangle = \underset{N\to\infty}{Lim} \langle \sigma_N(f), M \rangle = \underset{N\to\infty}{Lim} \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n)\overline{\hat{M}(n)} .$$

Corollary(1.7.4):[6] (Uniqueness theorem) If $\hat{M}(n) = 0, \forall n$ then $M = 0$.

Proof:

Let $\hat{M}(n) = 0,$

then by definition $\hat{M}(n) = \frac{1}{2\pi} \int M(t) e^{-\text{int}} dt, \forall n,\ 0 = \int M(t)\, dt$

$= \int M(t) e^{-it} dt = \int \cos t\ M(t)\ dt.$

First $n = 0$ and then $n = 1$ and from the fact that the imaginary part equal to zero we have

$\int M(t)dt = \int \cos t\ M(t)\ dt,$ then $M(t)\cos t = M(t), \forall t$

Hence $M(t) = 0$.

Chapter two
Convolution of Functions

We will discuss here some properties of convolution function in section one, approximate identities for convolution property and autocorrelation function will be discussed in section two.

2.1 *Definition and some properties of convolution*

The convolution function $(f*g)(x)$ is defined by the operation

$$\left(f * g)(x)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-y)\ g(y)\ dy \qquad (2.1.1)$$

where f is bounded function and g is absolutely integrable function.

Note: All the integrals for convolution is defined on $-\infty$ to ∞.

Lemma(2.1.1):[6] Let $e_n(x) = e^{inx}$ $(n \in \mathrm{Z})$, then by the orthogonality property, we obtain:

$$e_n(x) * e_m(x) = \begin{cases} e^{inx} & , \text{ if } \quad m = n \\ 0 & , \text{ if } \quad m \neq n \end{cases}$$

Proof : By definition we have

$$\begin{aligned} e_n(x) * e_m(x) &= \frac{1}{2\pi}\int e_m(x-y) e_n(y)\ dy \\ &= \frac{1}{2\pi}\int e^{im(x-y)} e^{iny}\, dy = \frac{1}{2\pi}\int e^{imx+iy(n-m)} dy \\ &= \frac{1}{2\pi} e^{imx}\int e^{iy(n-m)} dy = \begin{cases} e^{imx} & , \quad m = n \\ 0 & , \quad m \neq n \end{cases} \end{aligned}$$

Corollary(2.1.2):[6] If $((e_m * e_n)(n)\hat{)} = \hat{e}_m . \hat{e}_n$, then $f*g$ and $(\hat{f}\ \hat{g}\ \ )$ are bilinear in the pair (f, g).

Proof: It's clear that, if there exist $f,\ g,\ f_1,\ f_2,\ g_1,\ g_2 \in L^1$, α and β are constants then the following statements are true:

(1) $(f_1 + f_2)*g = f_1*g + f_2*g$

(2) $f*(g_1+g_2) = f*g_1 + f*g_2$

(3) $(\alpha f)*g = \alpha\,(f*g)$

(4) $f*(\beta\, g)=\beta\,(f*g)$.

Properties of convolution: If $f,\, g\in L^1(T)$ and $f*g\in L^1(T)$ then

(1) $\|f*g\|_1 \le \|f\|_1\|g\|_1$. (2.1.2)

Proof:

By definition we have:

$$f*g=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad \text{then} \quad \|f*g\|_1=\|\frac{1}{2\pi}\int f(x-y)g(y)dy\|$$

$$=\frac{1}{4\pi^2}\int|\int f(x-y)g(y)dy|\,dy \le \frac{1}{4\pi^2}\int\int|f(x-y)|\,|g(y)|\,dydy$$

$$\le \frac{1}{2\pi}\int|f(x-y)|\,dy.\frac{1}{2\pi}\int|g(y)|\,dy$$

hence $\|f*g\|_1 \le \|f\|_1 .\|g\|_1$.

(2) $|(f*g\,(x))| \le |f(x)|*|g(x)|$ (2.1.3)

Proof :

By definition of convolution we have

$$|(f*g(x))|=\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-y)g(y)\;dy\right| \le \frac{1}{2\pi}\int|f(x-y)g(y)|\,dy$$

$$=\frac{1}{2\pi}\int|f(x-y)|\,|g(y)|\,dy=|f(x)|*|g(x)| .$$

(3) $f*g = g*f$ (2.1.4)

Proof :

To prove the commutative property

$$f*g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-y)g(y)dy,$$

let $x-y = u \Rightarrow$ at $y=\infty,\ u=-\infty$ and at $y=-\infty$, $u=\infty$, $du=-dy$

Then

$$f*g=\frac{1}{2\pi}\int_{\infty}^{-\infty} f(u)g(x-u)^{-}du=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-u)f(u)\;du=g*f .$$

Hence $f*g$ is commutative.

Theorem (2.1.3):[6] (Fubini_Tonelli Theorem) For any functions f and g, we have

$$(\frac{1}{2\pi}\int g(y)e^{-iny}\left\{\frac{1}{2\pi}\int f(x-y)e^{-in(x-y)}dx\right\}dy=\frac{1}{2\pi}\int g(y)e^{-iny}\hat{f}(n)dy$$
$$=\hat{f}(n)\ \hat{g}(n)\ ,\ \forall n\in Z\ .$$

(4) $(f*g\hat{)}(n)=\hat{f}(n).\hat{g}(n)$ (2.1.5)

Proof: To prove this property, by using the definition, we have

$$(f*g\hat{)}(n)=\frac{1}{2\pi}\int(f*g)(x)e^{-inx}dx$$
$$=\frac{1}{2\pi}\int e^{-inx}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x-y)g(y)\ dy\right\}\ dx$$
$$=\left(\frac{1}{2\pi}\right)^{2}\int\int_{-\infty}^{\infty}e^{-inx}f(x-y)g(y)\ dx\ dy$$
$$=\left(\frac{1}{2\pi}\right)^{2}\int\left\{\int_{-\infty}^{\infty}f(x-y)\ e^{-in(x-y)}g(y)e^{-iny}dx\right\}\ dy$$
$$=\frac{1}{2\pi}\int g(y)e^{-iny}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x-y)e^{-in(x-y)}dx\right\}\ dy$$

Using theorem (2.1.3) to obtain

$$\frac{1}{2\pi}\int g(y)e^{-iny}\ \hat{f}(n)dy=\frac{1}{2\pi}\int g(y)e^{-iny}\ \hat{f}(n)\ dy=\hat{f}(n)\ \hat{g}(n)\ .$$

(5) Convolution operation is associative, that is, $(f*g)*h = f*(g*h)$, for every $f, g, h\in L^1(T)$.

Proof: Trivial.

Theorem(2.1.4):**[18]** Let f and g be continuous functions then

i) If $\int_{-\infty}^{\infty}|f|\,dx$ converges and $|g|\le M$, then the integral $f*g$ converges,

$(f{*}g)$ is continuous and $\quad |f*g| \le M \int_{-\infty}^{\infty} |f|\, dx$.

(ii) If $\int_{-\infty}^{\infty} |f|^2$ and $\int_{-\infty}^{\infty} |g|^2$ converge, then $\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$ converges, $f*g$ is continuous and

$$|f*g|^2 \le \left(\int_{-\infty}^{\infty} |f|^2\right)\left(\int_{-\infty}^{\infty} |g|^2\right).$$

iii) If $\int_{-\infty}^{\infty} |f|$ and $\int_{-\infty}^{\infty} |g|$ converge and g is bounded, then

$\int_{-\infty}^{\infty} |f*g|$ converge to the value $(\int_{-\infty}^{\infty} |f| \int_{-\infty}^{\infty} |g|)$ and $(f*g)\hat{} = \hat{f}\, \hat{g}$.

(iv) If $\int_{-\infty}^{\infty} |f|, \int_{-\infty}^{\infty} |g|$ and $\int_{-\infty}^{\infty} |\hat{g}|$ all converges and g equals its *Fourier* integral then

$$(f\,g)\hat{} = 2\pi\ \hat{f} * \hat{g}, \text{ or } \quad \hat{f} * \hat{g} = \frac{1}{2\pi}(f\,g)\hat{} .$$

Proof:

(i) Since $\int_{-\infty}^{\infty} |f|\, dx$ converges and $|g| \le M$ then $\int (f*g)dx$

converges uniformly and $\quad |f*g| \le M \int_{-\infty}^{\infty} |f|\, dx$ and from the uniform convergence of $f{*}g$, the continuity follows from the following theorem:

Theorem(2.1.5):[18] ((Let $f(x, y)$ be continuous function in the region $a \le x \le \infty$, $\; b \le y \le c$ and if

$\int_{-\infty}^{\infty} f(x,y)dx$ converges uniformly for $b \le y \le c$ then the last integral is continuous function of y on that interval) see [18]for the proof) .

(ii)The uniform convergence of the integral for $f{*}$g follows from Schwarz inequality

$$\left|\int_{|t|>A} f(t)g(x-t)dt\right|^2 \leq \left(\int | f(t)|^2 \, dt\right) \left(\int_{-\infty}^{\infty} | g(x-t)|^2 \, dt\right)$$

$$= \left(\int_{|t|>A} | f(t)|^2 \, dt\right) \left(\int_{-\infty}^{\infty} | g(s)|^2 ds\right) \to 0, \text{ as } A \to \infty,$$

independently of x .

$$\text{(iii)} \int_{-A}^{B} | \int_{-\infty}^{\infty} f(t)g(x-t)dt \, | \, dx \leq \int_{-A}^{B} \int_{-\infty}^{\infty} | f(t)g(x-t) | \, dt \, dx$$

$$= \int_{-\infty}^{\infty} | f(t) | \int_{-A}^{B} | g(x-t) | \, dx \, dt \leq \int_{-\infty}^{\infty} | f(t) | \, dt \int_{-\infty}^{\infty} | g(x-t) | \, dx = \left(\int_{-\infty}^{\infty} | f |\right) \left(\int_{-\infty}^{\infty} | g |\right)$$

so $\int_{-\infty}^{\infty} | f * g |$ converges and bounded by $\left(\int_{-\infty}^{\infty} | f |\right) \left(\int_{-\infty}^{\infty} | g |\right)$.

The change of order of integration is justified by a complement of the theorem(2.1.5) in part (i) which is if $\int_{a}^{\infty} f(x,y)dx$ converges uniformly for $b \leq y \leq c$ then

$$\int_{a}^{\infty}\left[\int_{b}^{c} f(x,y)dy\right] dx = \int_{b}^{c}\left[\int_{a}^{\infty} f(x,y)dx\right] dy$$

and converges . Hence $(f * g\hat{)} = \hat{f} . \hat{g}$ is proved .

$$\text{(iv)} \hat{f} * \hat{g}(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(n-y)\hat{g}(y)dy = \frac{1}{2\pi}\frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-i(n-y)x} \hat{g}(y)dxd$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-inx}\hat{g}(y)e^{iyx}dydx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)g(x) \, e^{-inx}dx = \frac{1}{2\pi}(fg\hat{)}$$

since g equals its *Fourier* integral .

Theorem(2.1.6):**[6]** Suppose that $1 \leq p \leq \infty$ and p' is the conjugate exponent of p defined by $(1/p) + (1/p') = 1$ such that if $p' = 1$ then $p = \infty$ and if $p = 1$ then $p' = \infty$. If $f \in L^p$ and $g \in L^{p'}$, then $f*g$ is defined every where , continuous and

$$|| f*g||_\infty \leq || f ||_p \, || g||_{p'} \, . \qquad (2.1.6)$$

Proof: (Holders inequality ($\|fg\|_1 \le \|f\|_p \|g\|_{p'}$) where $f \in L^p$ and $g \in L^{p'}$)shows that the function $y \to f(x-y)g(y)$ is in this case integrable for each x, so that $f*g(x)$ is defined for all x and

$$|f*g(x)| = \left|\frac{1}{2\pi}\int f(x-y)g(y)dy\right| \le \|f(x-y)\|_p \|g\|_{p'} \ ,$$

where in the first factor $f(x-y)$ is regarded as a function of y . By translation invariance, $\|f(x-y)\|_p = \|f\|_p$ whence the stated inequality. To show that $f*g$ is continuous, we may suppose that $p < \infty$, then use

$$f*g = \frac{1}{2\pi}\int f(x-y)g(y)dy \text{ and } \|f*g\|_1 \le \|f\|_1 \|g\|_1$$

thus the inequality just established to obtain

$$\| T_a(f*g) - f*g\|_\infty = \|T_a f * g - f*g\|_\infty = \| (T_a f - f)*g\|_\infty$$
$$\le \|T_a f - f\|_p \|g\|_{p'}$$

where $T_a(f(x)) = f(-a+x)$ and $T_a(f*g) = (T_a f)*g$ since

$$T_a(f*g) = \frac{1}{2\pi}\int f(-a+x-y)g(y)dy = \frac{1}{2\pi}\int f(-a+x-y)g(y)dy$$
$$= T_a(f)*g \ .$$

Hence $\|T_a f - f\|_p \to 0$ as $a \to 0$ whenever $f \in L^p$ and $p < \infty$.

Theorem(2.1.7):[6] Suppose that $f \in L^1$ and that g is absolutely continuous, then $f*g$ is absolutely continuous .

Proof: For any two real numbers a and b we have

$$|(f*g)(b) - (f*g)(a)| \le \frac{1}{2\pi}\int |f(y)|\,|g(b-y) - g(a-y)|\,dy \tag{2.1.7}$$

and since g is absolutely continuous, then for any $\varepsilon > 0$ there exist a number $N = N(\varepsilon) > 0$, such that

$$\sum_{k=1}^{r} |g(b_k) - g(a_k)| \le \varepsilon \quad .$$

For any sequence ($[a_k, b_k]_{k=1}^{r}$) of non overlapping intervals $[a_k, b_k]$ for which $\sum_{k=1}^{r}(b_k - a_k) \le N$. But then under the same conditions on these intervals

$$\sum_{k=1}^{r} | g(b_k - y) - g(a_k - y) | \le \varepsilon.$$

For all y and so (2.1.7) shows that

$$\sum_{k=1}^{r} | f * g(b_k) - f * g(a_k) | \le \| f \|_1 (\varepsilon).$$

this shows that $f{*}g$ is absolutely continuous.

2.2 *Approximate identities for convolution*

We note that there is no identity element for convolution($*$) in L^1, C^n and L^p for $1 \le p \le \infty$.

Thus we are going to consider and seek the next best approximate identity. By approximate identity (for convolution) we shall mean a sequence $(k_n)_{n=1}^{\infty}$ of elements of L^1 with the following properties

(1) $\sup \| k_n \|_1 < \infty$, $\forall\, n = 1, 2, \ldots$ (2.2.1)

(2) $\lim_{n \to \infty} \frac{1}{2\pi} \int k_n(x)\, dx = 1$ (2.2.2)

(3) $\lim_{n \to \infty} \int | k_n(x) |\, dx = 0$ where $\delta \le | x | \le \pi$ (2.2.3)

for any sequence (k_n)of nonnegative integrable functions (2.2.2) and $\lim_{n \to \infty} \int k_n(x) dx = 0$, the integral on $\delta \le |x| \le \pi$ for each fixed δ satisfying $0 < \delta < \pi$, we might take for k_n ($n = 1,2,3,\ldots,$) which is defined on $[-\pi, \pi]$.

Theorem(2.2.1):**[6]** Let $(k_n)_{n=1}^{\infty}$ be an approximate identity then

$$\lim_{n \to \infty} \| k_n * f - f \|_\infty = 0 \quad (f \in C)$$

and

$$\lim_{n \to \infty} \| k_n * f - f \|_p = 0 \ , \ f \in L^p, 1 \le p < \infty \ .$$

Proof: The first statement will follow from the uniform continuity of f in the following way

$$k_n * f(x) - f(x)\frac{1}{2\pi}\int k_n(y)dy = \frac{1}{2\pi}\int k_n(y)[f(x-y) - f(x)]dy$$

putting $\alpha_n = \frac{1}{2\pi}\int k_n(y)dy$, gives

$$\| k_n * f - \alpha_n . f \|_\infty \leq \frac{1}{2\pi}\int \| k_n(y) \| \ \| T_y . f - f \|_\infty \ dy = I \ . \qquad (2.2.4)$$

Let $\varepsilon > 0$ be given, choose and fix δ satisfying $0 < \delta \leq \pi$, so that

$\|T_y f - f\|_\infty \leq \varepsilon$ for $|y| \leq \delta$ then

$$I \equiv \frac{1}{2\pi}\int | k_n(y) | \ \| T_y . f - f \|_\infty \ dy = \frac{1}{2\pi}\int_{|y|<\delta} | k_n(y) | \ \| T_y . f - f \|_\infty \ dy$$

$$+ \frac{1}{2\pi}\int_{\delta \leq |y| \leq \pi} | k_n(y) | \ \| T_y . f - f \|_\infty \ dy \qquad (2.2.5)$$

and we have

$$\frac{1}{2\pi}\int_{|y|<\delta} | k_n(y) \ | \ \| T_y . f - f \|_\infty \ dy \ \leq \varepsilon . \frac{1}{2\pi}\int | k_n | \, dy \leq M\varepsilon \qquad (2.2.6)$$

since $\sup \|k\|_1 < \infty$ and M is independent of n, then

$$\|T_y f - f\|_\infty \leq 2 \, \|f\|_\infty$$

for all y, the second integral is majored by

$$\frac{1}{2\pi}\int_{\delta \leq |y| \leq \pi} \| k_n(y) \| . \| T_y . f - f \|_\infty \ dy \leq 2 \| f \|_\infty \cdot \frac{1}{2\pi}\int_{\delta \leq |y| \leq \pi} | k_n(y) | \ dy \qquad (2.2.7)$$

keeping ε and δ fixed.

We find that (2.2.3) to (2.2.7) show that

$$\lim_{n \to \infty} \sup \| k_n * f - \alpha_n . f \|_\infty \leq M . \varepsilon \ .$$

Since $\varepsilon > 0$ and arbitrary and $\lim_{n \to \infty} \alpha_n = 1$ by (2.2.2) it follows that

$\underset{n\to\infty}{Lim} \| k_n * f - f \|_{\infty} = 0$ whenever f is continuous .

To prove the second statement, given $f \in L^{p}$ and $\varepsilon > 0$, choose $f^{+} \in C$ such that $\| f - f^{+} \|_{p} \le \varepsilon$, $\delta_n : f \to f^{\wedge}(n)$ is a continuous homomorphism of L^1 onto the continuous field and by (2.2.1) we have

$$\| k_n * f - k_n * f^{+} \|_{p} \le \| k_n \|_1 . \varepsilon \le M\varepsilon \qquad (2.2.8)$$

where M is independent of n, by what has been established, there exists $n_0 = n_0(\varepsilon)$ so that

$$\| k_n * f^{+} - f^{+} \|_{\infty} \le \varepsilon \quad \text{for } n > n_0$$

then $\| k_n * f^{+} - f^{+} \|_{p} \le \varepsilon$ for $n > n_0$

and therefore

$$\| k_n * f^{+} - f \|_{p} \le 2\varepsilon \text{ for } n > n_0 \qquad (2.2.9)$$

and $\| k_n * f - f \|_{p} \le M\varepsilon + 2\varepsilon$ for $n > n_0$ thus

$\underset{n\to\infty}{Lim} \| k_n * f - f \|_{p} = 0$, $f \in L^{p}$, which completes the proof .

we will discuss a very important tool in *Fourier* analysis. In approximate identity, if a function $d \in L^1(R)$ exists such that $f_* d = f$ holds, then from the fact that

$$(f * g)\hat{}(n) = \hat{f}(n)\hat{g}(n) .$$

We hav $\hat{f}(n)\ \hat{d}(n) = \hat{f}(n)$, where $f \in L^1(R)$. that is $\hat{d}(n) = 1$.

We see that for such a family $\{d_\alpha\} \subset L^1(R)$ that seeks to approximate the identity is $\hat{d}_\alpha(n) \approx 1$, $n \in R$, as $\alpha \to 0$. And we may use the normalization

$$\hat{d}_\alpha(0) = 1, \text{ or } \int_{-\infty}^{\infty} d_\alpha(x)\, dx = 1. \qquad (2.2.10)$$

Let us define a family of Gaussian functions

$$g_\alpha(x) := \frac{1}{2\sqrt{\pi\alpha}}\, e^{-\frac{x^2}{4\alpha}} , \quad \alpha > 0 \qquad (2.2.11)$$

Theorem(2.2.2):[7] Let $f \in L^1(R)$, then

$$\underset{\alpha\to 0^+}{Lim} (f * g_\alpha)(x) = f(x) \qquad (2.2.12)$$

at every point of x where f is continuous.

Proof: Let f be continuous function at x and arbitrary given $\varepsilon > 0$. We select $a > 0$ such that

$$|f(x-y)-f(x)| < \varepsilon \,, \ \forall y \in R \text{ ,with } \ |y| < a,$$

then in view of (2.2.10) with $d_\alpha = g_\alpha$, we have

$$|(f * g_\alpha)(x) - f(x)| = \left| \int_{-\infty}^{\infty} [f(x-y) - f(x)] \ g_\alpha(y) \ dy \right|$$

$$\leq \int_{-a}^{a} |f(x-y) - f(x)| \ g_\alpha(y) \ dy + \int_{|y| \geq a} |f(x-y) - f(x)| \, g_\alpha(y) dy$$

$$\leq \ \varepsilon \int_{-a}^{a} g_\alpha(y) \ dy + \| f \|_1 \ \max_{|y| \geq a} g_\alpha(y) + |f(x)| \int_{|y| \geq a} g_\alpha(y) \ dy$$

$$\leq \ \varepsilon \int_{-\infty}^{\infty} g_\alpha(y) \ dy + \| f \|_1 \ g_\alpha(a) + |f(x)| \int_{|y| \geq a / \sqrt{\alpha}} g_1(y) \ dy$$

$$= \varepsilon + \| f \|_1 \ g_\alpha(a) + |f(x)| \int_{|y| \geq a / \sqrt{\alpha}} g_1(y) \ dy.$$

Since both $g_\alpha(a)$ and the last term obviously converge to zero as $\alpha \to 0^+$, this completes the proof of the theorem.

We will introduce the definition of the *Fourier* transform of functions in $L^2(R)$ and we need to define the autocorrelation function.

Definition (2.2.3):[7] The autocorrelation of an $f \in L^2(R)$ is defined by

$$F(x) = \int_{-\infty}^{\infty} f(x+y) \overline{f(y)} \ dy. \qquad (2.2.13)$$

Lemma (2.2.4):[7] Let F be the autocorrelation function of $f \in L^2(R)$. Then

(1) $|F(x)| \leq \| f \|_2^2 \,, \ \forall x \in R$

(2) F is uniformly continuous on R.

Proof: (1) To prove the first statement we use the Schwarz Inequality ($\| fg \|_1 \leq \| f \|_2 \| g \|_2$). Thus we have

$$|F(x)| \le \int_{-\infty}^{\infty} |f(x+y)|\,|\overline{f(y)}|\,dy \le \{\int_{-\infty}^{\infty} |f(x+y)|^2\,dy\}^{1/2}\,\|f\|_2$$

$$= \|f\|_2^2 .$$

(2) To prove the second statement, we consider an arbitrary real number a and apply the Schwarz Inequality to obtain

$$|F(x+a)-F(x)| = |\int_{-\infty}^{\infty}\{f(x+a+y)-f(x+y)\}\overline{f(y)}\,dy\,|$$

$$\le \{\int_{-\infty}^{\infty} |f(x+a+y)-f(x+y)|^2\,dy\}^{1/2}\,\|f\|_2$$

$$= \{\int_{-\infty}^{\infty} |f(y+a)-f(y)|^2\,dy\}^{1/2}\,\|f\|_2 .$$

Since $f \in L^2(R)$, the integral inside the braces, which is independent of x, tends to zero as $a \to 0$.

Chapter three

Applications

Dirichlet's problem and Poisson's theorem

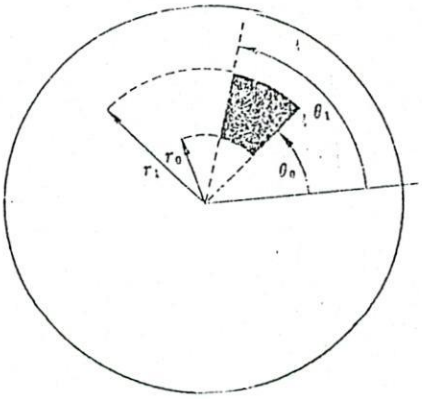
In this chapter we want to consider some applications of the *Fourier* series. To this end, we discuss the differential equation concerned with disk, the Poisson's theorem and its proof in section (1), some assumption about Poisson's theorem in section (2) and some applications about Poisson's theorem in section (3).

3.1 Dirichlets problem and Poisson's theorem:

We want first to find a differential equation governing the solution of the heat problems.

Since we are concerned with a disk, the natural coordinates are polar and the temperature at each point (r,θ) of the disk is $u(r,\theta)$, to find the heat equation, let

$$0<r_0<r_1<r<1,\ \theta_0<\theta<\theta_1 \qquad (3.1.1)$$



figure(3.1)

(3.1) is a circular disk with radius r.

Since we are considering steady state temperature distribution, the rate at which heat flows into this section must be zero, otherwise the average temperature would change with time. Thus the rate at which the heat crosses a curve ***C*** is proportional to the integral along ***C*** of the normal derivative $\partial u/\partial n$ of the temperature distribution, where $\partial u/\partial n$ is the derivative of u with respect to arc length along any curve perpendicular to ***C*** . When ***C*** is the side $\theta=\theta_1$ of the portion given in (3.1.1), we can take these perpendicular curves to be given by $r=$ constant. Then since the length of a circular arc is the angle times the radius, hence the normal derivative along $\theta=\theta_1$ is

$$\frac{\partial u}{\partial n} = \underset{h\to 0}{Lim}\frac{u(r,\theta_1+h)-u(r,\theta_1)}{rh} = r^{-1}u_\theta(r,\theta_1)$$

and the rate at which heat flows into the cross section in figure (3.1) along the boundary $\theta = \theta_1$ is

$$k\int_{r_0}^{r_1} r^{-1}u_\theta(r,\theta_1)dr \quad , k \text{ is the conductivity.}$$

By adding corresponding expressions for the other three boundaries and setting the net flow equal to zero, we have

$$\int_{r_0}^{r_1} r^{-1}\left[u_\theta(r,\theta_1)-u_\theta(r,\theta_0)\right]dr + \int_{\theta_0}^{\theta_1}\left[r_1u_r(r_1,\theta)-r_0u_r(r_0,\theta)\right]d\theta = 0 \qquad (3.1.2)$$

Dividing equation (3.1.2) by θ_1-θ_0 and taking the limit when $\theta_1\to\theta_0$, we obtain

$$\int_{r_0}^{r_1} r^{-1}u_{\theta\theta}(r,\theta_0)dr + r_1u_r(r_1,\theta_0)-r_0u_r(r_0,\theta_0)=0 \qquad (3.1.3)$$

if we divide equation (3.1.3) by r_1-r_0 and find the limit when $r_1\to r_0$, we get

$$\frac{1}{r_0}u_{\theta\theta}(r_0,\theta_0)+(ru_r)_r(r_0,\theta_0)=0 \quad .$$

Since r_0 and θ_0 are the coordinates of any point inside the disk, except the center, we have the equation

$$\frac{\partial(r\frac{\partial u}{\partial r})}{\partial r}+\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}=0, \qquad (r\neq 0) \qquad (3.1.4)$$

The solutions to this equation are called harmonic functions .

Let $u(r,\theta)$ be a continuous solution for $0\le r\le 1$ and

$$u(r,\theta+2\pi)=u(r,\theta)\ . \qquad (3.1.5)$$

To have a unique solution we complete the last equation by specifying the temperature distribution on the boundary $r=1$, given by

$$u(1,\theta)=f(\theta)=f(\theta+2\pi). \qquad (3.1.6)$$

The problem of finding a function u satisfying (3.1.4 __3.1.6) for a given function f is called a Dirichlet's problem. The solution of this problem can be found by the separation of variables method, let $u(r,\theta) = R(r)\theta(\theta)$, then we have

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\theta''}{\theta}$$

we then obtain

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta} \tag{3.1.7}$$

and

$$u(1,\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta} = f(\theta) \tag{3.1.8}$$

which yields

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \tag{3.1.9}$$

$$f(\theta)\ e^{-iN\theta} = \sum_{-\infty}^{\infty} A_n e^{i(n-N)\theta} \tag{3.1.10}$$

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta} d\theta \quad . \tag{3.1.11}$$

Then

$$\begin{aligned} u(r,\theta) &= \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int} dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi} r^{|n|} e^{in(\theta-t)} f(t)dt \ . \end{aligned}$$

For $r<1$, the series $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}$ converges uniformly in t and if we integrate it term by term we obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} f(t)\ dt \quad .$$

For $r<1$, (we have a geometric series) and we can find

$$\sum_{n=0}^{\infty} r^n\ e^{in(\theta-t)} = \frac{1}{1-re^{i(\theta-t)}} \quad \text{and} \quad \sum_{n=-\infty}^{-1} r^{-n}\ e^{in(\theta-t)} = \frac{re^{-i(\theta-t)}}{1-re^{-i(\theta-t)}}$$

thus

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} = \frac{1}{1-re^{i(\theta-t)}} + \frac{re^{-i(\theta-t)}}{1-re^{-i(\theta-t)}}$$

$$= \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

by substituting in $u(r,\theta)$ we obtain

$$u(r,\theta) = \frac{1}{2\pi} \int_{\pi}^{\pi} \left(\frac{1-r^2}{1-2r\cos(\theta-t)+r^2} \right) f(t)dt, \quad r<1 \qquad (3.1.12)$$

this is the Poisson's integral with the Poisson's kernel

$$P(r,\phi) = \frac{1-r^2}{2\pi\ (1-2\cos\phi+r^2)} \quad .$$

3.2 Assumption of boundary values Poisson's theorem

Theorem (3.2.1): [18] If f is a periodic, continuous function of period 2π and

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2\cos(\theta-t)+r^2} f(t)dt,$$

then

$$\lim_{r\to 1} u(r,\theta) = f(\theta) \quad .$$

Proof: Given $\varepsilon > 0$, We must find $r_0 < 1$ so that, for all r in $r_0 < r < 1$, $|u(r,\theta)-f(\theta)| < \varepsilon$ holds and

$$\left| \ u(r,\theta) - f(\theta) \ \right| = \left| \int_{\theta-\pi}^{\theta+\pi} P(r,\theta-t)[f(t)-f(\theta)]\ dt \right|$$

$$\leq \int_{\theta-\pi}^{\theta+\pi} P(r,\theta-t)\,|\,f(t)-f(\theta)\,|\,dt.$$

This inequality above uses the fact that $p \geq 0$. Since f is continuous, we can make $|f(t)-f(\theta)|$ small whenever t is near θ and formulate $P(r,\theta-t)$, we can also make P small by taking r close to 1 when t is not near 0. This suggests breaking the integral above into two parts and using the two effects separately to have for any number δ, with $0 < \delta < \pi$, that

$$|u(r,\theta) - f(\theta)| \leq \int_{\theta-\pi}^{\theta+\pi} P(r,\theta-t)\,|f(t) - f(\theta)|\,dt$$

$$= \int_{|\theta-t|\leq\delta} + \int_{\delta<|\theta-t|<\pi} = I_1 + I_2 \ ,$$

where I_1 is the integral for $|\theta - t| \leq \delta$ and I_2 is the integral for $\delta < |\theta - t| < \pi$. Since f is continuous there, then we can choose δ such that $|f(t)-f(\theta)| < \varepsilon/2$, whenever $|\theta - t| \leq \delta$,

thus

$$I_1 \leq \int_{|t-\theta|\leq\delta} P(r,\theta-t)\frac{\varepsilon}{2}\ dt < \frac{\varepsilon}{2}\int_{\theta-\pi}^{\theta+\pi} P(r,\theta-t)dt = \frac{\varepsilon}{2}$$

considering the remaining term, we have

$$I_2 < \frac{1-r^2}{2\pi(1-2r\cos\delta + r^2)} \quad \int_{\delta<|t-\theta|<\pi} (f(t) - f(\theta))\ dt, \text{ since}$$

$$\max_{\delta < |t-\theta| < \pi} P(r,\theta-t) = \frac{1-r^2}{2\pi(1-2r\cos\delta + r^2)}.$$

Now for a fixed δ between 0 and π, we obtain

$$\lim_{r\to 1} \ \frac{1-r^2}{2\pi(1-2r\cos\delta + r^2)} \int_{-\pi}^{\pi} |f(t) - f(\theta)|\,dt = 0$$

So that there is an $r_0 < 1$ such that when $r_0 < r < 1$ this expression is less than $\varepsilon/2$, making $I_2 < \varepsilon/2$, then finally for $r_0 < r < 1$ we get

$$|u(r,\theta) - f(\theta)| \leq \varepsilon \quad \text{and} \quad \lim_{r\to 1} u(r,\theta) = f(\theta)$$

hence

$$|u(r,\theta) - f(\theta)| \leq I_1 + I_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Theorem (3.2.2): [16] If f and g are continuous functions of period 2π and if they have the same *Fourier* coefficients, then $f = g$.

Proof: By assumption we have

$$\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta = \int_{-\pi}^{\pi} g(\theta)e^{-in\theta}\, d\theta = a_n.$$

Then by theorem (3.2.1), the function

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

converge to $f(\theta)$ and to $g(\theta)$ as $r \to 1$, thus $f(\theta) = g(\theta)$.

Corollary (3.2.3): [18] If f is continuous function of period 2π and if the *Fourier* series of f converges uniformly to some function g, then $g = f$.

Proof: Let a_n be the *Fourier* coefficients of f and $g = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$,

since this series is uniformly convergent by assumption, then we can calculate the *Fourier* coefficients b_m of g by

$$\begin{aligned} b_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta} \sum_{n=-\infty}^{\infty} a_n e^{in\theta}\, d\theta \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n \int_{-\pi}^{\pi} e^{i(n-m)\theta}\, d\theta = a_m \end{aligned} .$$

The function g is the sum of a uniformly convergent series; thus g is continuous and by theorem (3.2.2) we find that $f = g$.

Theorem (3.2.4): [18] If $f(\theta)$ is a continuous periodic function and if $v(r,\theta)$ satisfies the conditions (3.1.4 …3.1.6) and $v(r,\theta) \to f(\theta)$ uniformly as $r \to 1$, then $v(r,\theta)$ is identical with the solution given by the Poisson's kernel

$$P(r,\phi) = \frac{1-r^2}{2\pi(1-2r\cos\phi + r^2)} .$$

Proof: For each fixed $r < 1$, $v(r,\theta)$ has a *Fourier* series given by

$$\sum_{n=-\infty}^{\infty} a_n(r)\, e^{in\theta} \qquad (3.2.1)$$

with $$a_n(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta)e^{-in\theta}\,d\theta. \qquad (3.2.2)$$

Taking into account that $v(r,\theta)$ satisfies the equation

$$\frac{\partial\left(r\dfrac{\partial u}{\partial r}\right)}{\partial r}+\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}=0\ , \qquad r\neq 0\ , \qquad (3.2.3)$$

differentiating equation (3.2.2) to obtain

$$(a_n(r))'=\frac{1}{2\pi}\int_{\pi}^{\pi} v_r(r,\theta)\ e^{-in\theta}d\theta \qquad (3.2.4)$$

multiplying (3.2.4) by r and differentiating with respect to r to obtain

$$\begin{aligned}(ra_n')'&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial r}(rv_r(r,\theta))e^{-in\theta}d\theta\\ &=\frac{-1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{r}\ v_{\theta\theta}e^{-in\theta}d\theta\end{aligned} \qquad (3.2.5)$$

Integrating by parts and remembering that $v(r,\theta-\pi)=v(r,\theta+\pi)$, we obtain

$$\begin{aligned}(ra_n')'&=\left[\frac{-1}{2\pi r}v_\theta(r,\theta)e^{-in\theta}\right]_{-\pi}^{\pi}-\frac{in}{2\pi}\int_{-\pi}^{\pi}\frac{1}{r}v_\theta e^{-in\theta}d\theta\\ &=0-\frac{in}{2\pi}\int_{-\pi}^{\pi}\frac{1}{r}v_\theta e^{-in\theta}d\theta\\ &=\left[\frac{-in}{2\pi r}v(r,\theta)e^{-in\theta}\right]_{-\pi}^{\pi}+\frac{n^2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{r}v(r,\theta)e^{-in\theta}d\theta\\ &=\frac{n^2}{r}a_n\ .\end{aligned} \qquad (3.2.6)$$

From (3.2.6) a_n satisfies the ordinary differential equation

$$(ra_n')'=(\frac{n^2}{r})a_n \qquad (3.2.7)$$

which leads to $a_n=A_n r^n+B_n r^{-n}$ for some constants A_n and B_n ($n\neq 0$), since $v(r,\theta)$ is assumed to be bounded, we see that

$$|a_n(r)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta) e^{-in\theta} d\theta \right| \le 2 \max |v| \qquad (3.2.8)$$

so that the coefficients B_n above must be 0 and $a_n(r) = A_n r^n$, for A_n constant and $a_0(r) = A_0$, to evaluate A_n by letting $r \to 1$ as follows

$$A_n r^n = \frac{1}{2\pi} \int_{\pi}^{\pi} v(r,\theta) e^{-in\theta} d\theta \qquad (3.2.9)$$

when

$$r \to 1, \ v(r,\theta) \to f(\theta)$$

uniformly, so that the integral in (3.2.9) converges to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta} d\theta$$

as $r \to 1$ and we find

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

for each fixed $r < 1$, the *Fourier* series of $v(r,\theta)$ is precisely the series for $u(r,\theta)$, the solution of the problem constructed through the Poisson's kernel, since this series converges uniformly to $u(r,\theta)$.
The last corollary asserts that $u(r,\theta) = v(r,\theta)$, which is exactly what we wanted to show.
Thus we can say that the temperature at the center of a disk with a steady state distribution is the average of the temperatures on the boundary, since

$$u(0,\theta) = A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

and $u(r,\theta)$ is the only solution of the heat problem.

3.3 Some applications of Poisson's theorem

Theorem (3.3.1): [18] Let f be continuous function of period 2π, then for every $\varepsilon > 0$ there exists a trigonometric polynomial

$$P = \sum_{n=-N}^{N} a_n r^{|n|} e^{in\theta}$$

such that

$$|P(\theta)-f(\theta)| < \varepsilon \text{ for all } \theta \quad .$$

Proof: By Poisson's theorem (3.2.1) there is an $r < 1$ such that $|u(r,\theta)-f(\theta)| < \varepsilon/2$,

where

$$u(r,\theta)=\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$$

and

$$a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ e^{-in\theta}\ d\theta$$

since $|a_n| \leq \max|f|$ we have

$$\left|\sum_{|n|\geq N} a_n r^{|n|} e^{in\theta}\right| \leq 2\max|f|\frac{r^N}{1-r} \quad . \tag{3.3.1}$$

If we choose N very large we obtain

$$2\max|f|\frac{r^N}{1-r} < \varepsilon/2$$

from (3.3.1) we have

$$\left|f(\theta)-\sum_{|n|\leq N} a_n r^{|n|} e^{in\theta}\right| < |f(\theta)-u(r,\theta)|+\frac{\varepsilon}{2} < \varepsilon \quad .$$

Theorem (3.3.2): [18] Let a_n be the *Fourier* coefficients of a continuous function f of period 2π and $s_N(\theta)=\sum_{-N}^{N} a_n e^{in\theta}$ then

(1) $\int_{-\pi}^{\pi} |s_N(\theta)-f(\theta)|^2\, d\theta \to 0 \quad \text{as} \quad N\to\infty$ (3.3.2)

(2) If $t_N(\theta)=\sum_{-N}^{N} b_n e^{in\theta}$ (3.3.3)

with arbitrary coefficients b_n, then

$$\int_{-\pi}^{\pi} |t_N(\theta) - f(\theta)|^2 \, d\theta \; > \int_{-\pi}^{\pi} |s_N(\theta) - f(\theta)|^2 \, d\theta \qquad (3.3.4)$$

unless $b_n = a_n$ for $-N \le n \le N$.

Proof: We begin the proof of (3.3.4) by letting

$$\int_{-\pi}^{\pi} |t_N(\theta) - f(\theta)|^2 d\theta = \int_{-\pi}^{\pi} \left(\left[\sum_{n=-N}^{N} b_n e^{in\theta} - f(\theta) \right] \left[\sum_{m=-N}^{N} \overline{b_m} e^{-im\theta} - \overline{f}(\theta) \right] \right) d\theta$$

$$= 2\pi \sum |b_n|^2 - 2\pi \sum b_n \overline{a_n} - 2\pi \sum \overline{b_m} a_m + \int |f|^2 d\theta$$

then substituting a_n for b_n yields

$$\int_{-\pi}^{\pi} |s_N(\theta) - f(\theta)|^2 \, d\theta = \int |f|^2 \, d\theta - 2\pi \sum_{n=-N}^{N} |a_n|^2$$

and

$$\int |t_N(\theta) - f(\theta)|^2 \, d\theta - \int |s_N(\theta) - f(\theta)|^2 \, d\theta$$

$$= 2\pi \left(\sum |b_n|^2 - \sum b_n \bar{a}_n - \sum \overline{b_m} a_m + \sum |a_n|^2 \right)$$

$$= 2\pi \sum (b_n - a_n)(\overline{b_n} - \overline{a_n}) > 0$$

unless $b_n = a_n$ for $-N \le n \le N$.

In the last proof we use the fact that $\int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta = 0$, when $n \ne m$.

(2) To prove (3.3.3) let $\varepsilon > 0$ be given and choose a trigonometric polynomial

$$P(\theta) = \sum_{n=-N}^{N} b_n e^{in\theta},$$

with

$$|P - f|^2 < \frac{\varepsilon}{2\pi}$$

then by theorem (3.3.1) and from (3.3.4), we obtain

$$\int |s_N(\theta) - f(\theta)|^2 \, d\theta \le \int |P(\theta) - f(\theta)|^2 \, d\theta < \varepsilon .$$

Chapter four

The eigenfunction method and its application to *Fourier* series

In this chapter we will discuss method of solving boundary value problems in particular solving partial differential equations in two variables. The boundary value problems and a method of solutions will be discussed in section (1), the eigenfunctions and their orthogonality in section (2), examples on heat equation and vibrating on membrane are discussed in section (3), the sign of the eigenvalues and *Fourier* series with respect to the eigenfunctions in section (4) and the eigenfunction expansion method in section (5).

4.1 *The boundary value problem and the method of solutions*

In this section we will consider the linear partial differential equations **[19], [21]**

$$p\frac{\partial^2 u}{\partial x^2}+R\frac{\partial u}{\partial x}+Qu=\frac{\partial^2 u}{\partial t^2} \qquad (4.1.1)$$

$$p\frac{\partial^2 u}{\partial x^2}+R\frac{\partial u}{\partial x}+Qu=\frac{\partial u}{\partial t} \qquad (4.1.2)$$

where p, R and Q are continuous functions of the variable x and $u(x,t)$ is the solution of the problem.
(4.1.1) is the equation of mechanical vibrations of the string and (4.1.2) is the equation governs the heat flow in a metallic rod.
Let $u(x,t)$ be the solution of (4.1.1), defined on $a \le x \le b$ and $t \ge 0$ which satisfy the boundary conditions

$$\alpha\ u(a,t)+\beta\frac{\partial u(a,t)}{\partial x}=0$$

$$\gamma\ u(b,t)+\delta\frac{\partial u(b,t)}{\partial x}=0, \qquad (4.1.3)$$

with α, β, γ and δ are constants and $u(x,t)$ satisfies the initial conditions

$$u(x,0)=f(x), \qquad \frac{\partial u(x,0)}{\partial t}=g(x) \qquad (4.1.4)$$

where $f(x)$ and $g(x)$ are continuous functions and neither α, β nor γ, δ vanish simultaneously. This mean

$$\alpha^2+\beta^2 \neq 0 \tag{4.1.5}$$
$$\gamma^2+\delta^2 \neq 0 .$$

To solve (4.1.1) by separation of variable method, let $u(x,t) = \phi(x)T(t)$ and obtain

$$\frac{\mathrm{p}\,\phi'' + R\phi' + \varphi\phi}{\phi} = \frac{T''}{T} = -\lambda , \tag{4.1.6}$$

Thus we have

$$\mathrm{p}\,\phi'' + R\,\phi' + \varphi\phi = -\lambda\phi \tag{4.1.7}$$

$$T'' + \lambda T = 0 \tag{4.1.8}$$

To have $u \neq 0$ and u satisfies the boundary conditions, we obtain

$$\alpha\phi(a) + \beta\phi'(a) = 0$$
$$\gamma\phi(b) + \delta\phi'(b) = 0 \tag{4.1.9}$$

then there exist an infinite set of values $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, for which the boundary value problem has a solution, provided only that $\mathrm{p} \neq 0$. Every value of λ for which the boundary value problem has a solution $\phi \neq 0$ is called an eigenvalue and the solution ϕ corresponding to λ is called an eigenfunction.

We have an infinite set of eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, with corresponding eigenfunctions

$$\phi_0(x), \phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots, \tag{4.1.10}$$

(4.1.10) form an orthogonal system on [a, b] with a certain weight r.

If $\lambda_n > 0$, $n = 0,1,2,\ldots$, then we have

$$T_n(t) = A_n \cos\sqrt{\lambda_n}\, t + B_n \sin\sqrt{\lambda_n}\, t \tag{4.1.11}$$

where A_n and B_n are arbitrary constants. Then each function $u_n(x,t) = \phi_n(x)T_n(t)$, $(n = 0,1,2,\ldots)$ will be a solution of (4.1.1) satisfying the boundary conditions (4.1.3).

Because of linearity and homogeneity (with respect to u and its derivatives) of equation(4.1.1), every finite sum of solutions of (4.1.1) is also a solution.

The same is also true of an infinite series

$$u = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} T_n(t)\phi_n(x) \tag{4.1.12}$$

If u is uniformly convergent series and twice differentiable with respect to both x and t, then we have

$$\begin{aligned} \mathrm{p}\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Qu &= \frac{\partial^2 u}{\partial t^2} \\ &= \mathrm{p}\sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2} + R\sum_{n=0}^{\infty} \frac{\partial u_n}{\partial x} + Q\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial t^2} \\ &= \sum_{n=0}^{\infty} (\mathrm{p}\frac{\partial^2 u_n}{\partial x^2} + R\frac{\partial u_n}{\partial x} + Qu_n - \frac{\partial^2 u_n}{\partial t^2}) = \sum_{n=0}^{\infty} (0) = 0. \end{aligned}$$

With the initial condition

$$\begin{aligned} u(x,0) &= f(x) = \sum_{n=0}^{\infty} \phi_n(x)T_n(0), \\ \frac{\partial u(x,0)}{\partial t} &= g(x) = \sum_{n=0}^{\infty} \phi_n(x)T_n'(0) \quad . \end{aligned} \tag{4.1.13}$$

From (4.1.13) we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \phi_n(x)T_n(0) = \sum_{n=0}^{\infty} c_n\phi_n(x) \\ g(x) &= \sum_{n=0}^{\infty} d_n\phi_n(x) \end{aligned} \tag{4.1.14}$$

$$T_n(0) = \mathrm{c}_n,\ T_n'(0) = d_n,\ n=0,1,2,\ldots \tag{4.1.15}$$

In order to find the constants in (4.1.15) we apply the initial conditions

$$\begin{aligned} u(x,0) &= f(x) = \sum_{n=0}^{\infty} A_n\phi_n(x) \\ \frac{\partial u(x,0)}{\partial t} &= g(x) = \sum_{n=0}^{\infty} B_n\sqrt{\lambda_n}\phi_n(x) \end{aligned} \tag{4.1.16}$$

we can easily see that

$$A_n = c_n, \; B_n = \frac{d_n}{\sqrt{\lambda_n}}, \; n = 0,1,2,....$$

Lemma (4.1.1):[21] If we assume $p \neq 0$ in (4.1.1) and if we multiply (4.1.7) by non vanishing function then (4.1.7) neither loses nor gains eigenvalues and eigenfunctions and we can transform (4.1.7) to

$$(p\phi')' + q\phi = -\lambda\, r\phi \tag{4.1.17}$$

where p is a positive function with continuous derivative, q is a positive continuous functions on [a, b] and r is a positive function .

Proof: First we assume that $\mathrm{P} > 0$ (this can be done without loss of generality) since otherwise we need only to multiply all the terms of (4.1.7) by –1 and replace -λ by λ~. Then we solve the system

$$p = r\mathrm{P}, \quad p' = r\,R \tag{4.1.18}$$

and we obtain

$$\frac{p'}{p} = \frac{R}{\mathrm{P}}, \; \ln p = \int_{x_0}^{x} \frac{R}{\mathrm{P}}\, dx\,, \quad p = \exp\left\{ \int_{x_0}^{x} \frac{R}{\mathrm{P}}\, dx \right\}, \quad r = \frac{p}{\mathrm{P}}$$

where x_0 is some point belong to the interval $[a, b]$. (we take the constant of integration to be zero). Since $p > 0$, p' is continuous and $r > 0$, we need only to consider the differential equation

$$r\mathrm{P}\,\phi'' + rR\,\phi' + rQ\,\phi = -\lambda\, r\phi \tag{4.1.19}$$

and set

$$q = r\,Q$$

then, according to (4.1.18) we obtain

$$p\phi'' + p'\phi' + q\phi = -\lambda r\phi$$

which is the desired equation (4.1.17) .

4.2 Eigenfunctions and their orthogonality

Theorem (4.2.1):[3], [21] If λ in (4.1.7) is an eigenvalue then there exist unique eigenfunction $\phi(x)$.

Proof: Suppose that $\phi(x)$ and $\psi(x)$ belong to the same eigenvalue λ, then by a familiar property of linearly independent solutions of differential equation we would have

$$\begin{vmatrix} \phi(x) & \phi'(x) \\ \psi(x) & \psi'(x) \end{vmatrix} \neq 0 \tag{4.2.1}$$

every where on $[a, b]$ and in particular at $x = a$, but according to the first conditions (4.1.9)

$$\alpha\phi(a) + \beta\phi'(a) = 0$$
$$\alpha\psi(a) + \beta\psi'(a) = 0$$

(4.2.1) would imply that $\alpha = 0$ and $\beta = 0$, which contradicts the hypothesis of (4.1.5), then there exist only one eigenfunction corresponds to each eigenvalue .

Lemma (4.2.2): [1], [21] Let

$$L(\phi) = \frac{d}{dx}(p\frac{d\phi}{dx}) + q\phi \tag{4.2.2}$$

where ϕ is a function depending on x (if ϕ also depends on other variables such that t, we write the partial derivative with respect to x)
then the identity

$$\phi L(\psi) - \psi L(\phi) = \frac{d}{dx}[p(\phi\psi' - \phi'\psi)] \tag{4.2.3}$$

holds for any twice differentiable functions ϕ and ψ .

Proof:

By (4.2.2) $L(\phi)$ and $L(\psi)$ are equals, that is

$$L(\psi)=\frac{d}{dx}(p\frac{d\psi}{dx})+q\psi, L(\phi)=\frac{d}{dx}(p\frac{d\phi}{dx})+q\phi.$$

Substituting $L(\phi)$ and $L(\psi)$ in the left-hand side of (4.2.3) to obtain

$$\begin{aligned}\phi\ L(\psi)-\psi\ L(\phi)&=\phi\ (\frac{d}{dx}(p\frac{d\psi}{dx})+q\psi)-\psi\ (\frac{d}{dx}(p\frac{d\phi}{dx})+q\phi)\\ &=\phi\frac{d}{dx}(p\frac{d\psi}{dx})-\psi\frac{d}{dx}(p\frac{d\phi}{dx})\\ &=\phi\frac{d}{dx}(p\psi')-\psi\frac{d}{dx}(p\phi')\\ &=\phi\ (p\psi''+p'\psi')-\psi\ (p\phi''+p'\phi')\\ &=\phi\ p\psi''+\phi\ p'\psi'-\psi\ p\phi''-\psi\ p'\phi'\end{aligned}$$

and the right-hand side of (4.2.3) equal

$$\begin{aligned}\frac{d}{dx}[p(\phi\psi'-\phi'\psi)]&=p'\phi\psi'-p'\phi'\psi+p(\phi'\psi'+\phi\psi''-\phi''\psi-\phi'\psi')\\ &=p'\phi\psi'-p'\phi'\psi+p\phi'\psi'+p\phi\psi''-p\phi''\psi-p\phi'\psi'\\ &=p'\phi\psi'-p'\phi'\psi+p\phi\psi''+p\phi''\psi\\ &=\phi\quad L(\psi)-\psi\quad L(\phi)\quad .\end{aligned}$$

From the last simplification, we have the right-hand side of equation (4.2.3) equal the left- hand side.

Lemma (4.2.3): [1], [21] If ϕ and ψ satisfy the boundary conditions (4.1.9) then

$$[\phi\psi'-\phi'\psi]_{x=a}=[\phi\psi'-\phi'\psi]_{x=b}=0\quad . \tag{4.2.4}$$

Proof:

The numbers α and β which don't vanish simultaneously according to (4.1.5) satisfy the homogenous system

$$\begin{aligned}\alpha\phi(a)+\beta\phi'(a)&=0\\ \alpha\psi(a)+\beta\psi'(a)&=0\end{aligned}$$

this is valid only if the determinant of the next system vanish, this means

$$\begin{vmatrix} \phi(a) & \phi'(a) \\ \psi(a) & \psi'(a) \end{vmatrix} = [\phi\psi' - \phi'\psi]_{x=a} = 0$$

and

$$\alpha\phi(b) + \beta\phi'(b) = 0$$
$$\alpha\psi(b) + \beta\psi'(b) = 0$$

thus

$$\begin{vmatrix} \phi(b) & \phi'(b) \\ \psi(b) & \psi'(b) \end{vmatrix} = (\phi\psi' - \psi\phi')_{x=b} = 0 \quad .$$

Lemma (4.2.4): **[3], [5], [21]** Any two eigenfunctions ϕ (x) and ψ (x) corresponding to different eigenvalues λ_0 and λ_1 respectively are orthogonal on [a,b] with weight *r*.

Proof:
Let ϕ and ψ satisfy the equations

$$L(\phi) = -\lambda_0 r\phi \ ,$$
$$L(\psi) = -\lambda_1 r\psi$$

with the boundary conditions (4.1.9).
If we multiply the first equation by ψ and the second equation by ϕ to have

$$-\lambda_0 r\psi\phi = \psi L(\phi) \qquad (4.2.4)$$
$$-\lambda_1 r\psi\phi = \phi L(\psi) \qquad (4.2.5)$$

Subtracting equation (4.2.5) from (4.2.4) and using lemma (4.2.1) to obtain

$$[p(\phi\psi' - \phi'\psi)]' = (\lambda_0 - \lambda_1) r\phi\psi .$$

Therefore we have

$$[p(\phi\psi' - \phi'\psi)]_{x=a}^{x=b} = (\lambda_0 - \lambda_1)\int_a^b r\phi\psi \, dx \qquad (4.2.6)$$

by lemma (4.2.3) we get

$$[p(\phi\psi' - \phi'\psi)]_{x=a}^{x=b} = 0$$

so that (4.2.5) implies that

$$(\lambda_0 - \lambda_1)\int_a^b r\phi\psi\ dx = 0$$

since $\lambda_0 \neq \lambda_1$ it follows that

$$\int_a^b r\phi\psi dx = 0$$

then ϕ is orthogonal to ψ with weight r.

Theorem (4.2.5): [1], [2], [21] The eigenvalues of the boundary value problem (4.1.6) are real .

Proof: Let $\lambda = u + iv$, $(v \neq 0)$ is an eigenvalue for an eigenfunction $\phi(x) = Q(x) + i\psi(x)$ then by substituting in (4.1.17) we obtain

$$[p(Q' + i\psi')' + q(Q + i\psi) = -(u + iv)r(Q + i\psi)$$

which yields

$$[p(Q' - i\psi')]' + q(Q - i\psi) = -(u - iv)r(Q - i\psi) \quad .$$

This means that $\overline{\lambda} = u\text{-}iv$ is an eigenvalue and that the function

$\overline{\phi(x)} = Q(x) - i\psi(x)$ is an eigenfunction corresponding to $\overline{\lambda}$.

But this implies that

$$\int_a^b r\phi\overline{\phi}dx = \int_a^b r(Q^2 + \psi^2)dx \neq 0$$

which is impossible since ϕ ,$\overline{\phi}$ must be orthogonal since $\lambda \neq \overline{\lambda}$.

4.3 Applications to Fourier series and eigenfunctions method

we want to consider examples about the eigenfunctions and eigenvalues method, first we want to introduce some information about *Fourier* Bessel series.

Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \ldots$, be the positive roots of the equation $J_p(x) = 0$, (p >-1) where

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{p+2m}}{\Gamma(m+1)\Gamma(p+m+1)}, \tag{4.3.1}$$

with

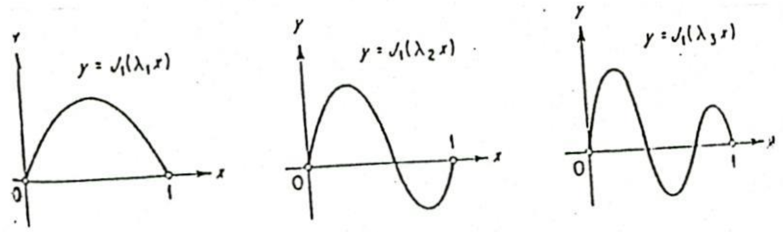
$$\Gamma(m+1) = m\Gamma(m) = m!$$

The functions

$$J_p(\lambda_1 x), J_p(\lambda_2 x), \ldots, J_p(\lambda_n x), \ldots$$

forms an orthogonal system on [0,1] with weight x. ([5]).

The graphs of the functions $J_1(\lambda_1 x)$, $J_1(\lambda_2 x)$, $J_1(\lambda_3 x)$ on the interval [0,1]are given in figure (4.1)



(a) (b) (c)

figure (4.1)

(a) Bessel function at λ_1 , (b) Bessel function at λ_2, (c) Bessel function at λ_3 . The graph of the functions $J_1(\lambda_n x)$, $n = 3,4,\ldots$, on [0,1] become more and more complicated.

For any function $f(x)$ which is absolutely integrable on [0,1], we can form it's *Fourier* Bessel series with respect to the system

$$J_p(\lambda_1 x), J_p(\lambda_2 x), \ldots, J_p(\lambda_n x), \ldots, \tag{4.3.2}$$

as

$$f(x) \sim c_1 J_p(\lambda_1 x) + c_2 J_p(\lambda_2 x) + \ldots, \tag{4.3.3}$$

where the *Fourier* –Bessel coefficients c_n of $f(x)$ is given by **[22]**

$$c_n = \frac{\int_0^1 x f(x) J_p(\lambda_n x)dx}{\int_0^1 x J_p^{\,2}(\lambda_n x)dx} = \frac{2}{J^2_{p+1}(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x)dx \tag{4.3.4}$$

In order to show that we write

$$f(x) = c_1 J_p(\lambda_1 x) + c_2 J_p(\lambda_2 x) + \ldots, \tag{4.3.5}$$

then if we multiply both sides of (4.3.5) by $xJ_p(\lambda_n x)$ and integrate over the interval [0,1] with respect to x, we get

$$\int_0^1 x.J_p(\lambda_n x) f(x) dx =$$

$$\int_0^1 x.J_p(\lambda_n x)[c_1 J_p(\lambda_1 x) + c_2 J_p(\lambda_2 x) + \ldots + c_n J_p(\lambda_n x) + \ldots] dx$$

then term by term integration and using orthogonality condition yields

$$\int_0^1 xf(x).J_p(\lambda_n x) dx = c_n \int_0^1 x.J^2{}_p(\lambda_n x) dx \quad ,$$

hence

$$c_n = \frac{\int_0^1 xf(x).J_p(\lambda_n x) dx}{\int_0^1 x.J^2{}_p(\lambda_n x) dx} = \frac{2}{J^2{}_{p+1}(\lambda_n)} \int_0^1 xf(x).J_p(\lambda_n x) dx \text{ [3].}$$

The orthogonality with weight x of the functions of the system (4.3.2) can be regarded as ordinary orthogonality of the functions

$$\sqrt{x}.J_p(\lambda_1 x), \sqrt{x}.J_p(\lambda_2 x), \ldots, \sqrt{x}.J_p(\lambda_n x), \ldots, \tag{4.3.6}$$

In order to make a series expansion of the function $f(x)$ with respect to the system (4.3.6), we can first make a series expansion of the function

$\sqrt{x} f(x)$ with respect to the orthogonal system (4.3.2) obtaining

$$\sqrt{x} f(x) \sim c_1 \sqrt{x}.J_p(\lambda_1 x) + c_2 \sqrt{x}.J_p(\lambda_2 x) + \ldots + \ldots, \tag{4.3.7}$$

It's clear that the coefficients of (4.3.3) and (4.3.7) are the same.

1. The first example is a special heat equation which is temperatures along cylinder

Let the lateral surface $\rho = c$ of an infinitely long cylinder, or a circular cylinder of finite length with insulated ends, be kept at temperature zero with initial temperature distribution be a given function of only the variable ρ, the distance from the axis of the cylinder.

We want to derive an expression for the temperature $u(\rho,t)$ in the cylinder, assuming that the material of that solid is homogenous.

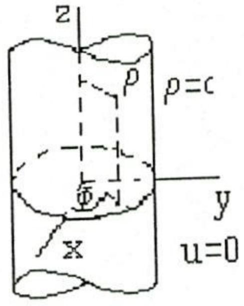


figure (4.2)

Figure (4.2) is a circular cylinder.

In the cylinder situated as shown in figure (4.2) the heat equation and the boundary conditions are

$$\frac{\partial u}{\partial t}=K\left(\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}\right) \quad ,(0<\rho<c,t>0), \qquad (4.3.8)$$

$$u(c,t)=0 \qquad (t>0), \qquad (4.3.9)$$

$$u(\rho,0)=f(\rho) \qquad (0<\rho<c). \qquad (4.3.10)$$

also, when the function $u(\rho,t)$ is to be continuous throughout the cylinder and on the axis $\rho=0$.

Any solution of the homogenous equations (4.3.8) and (4.3.9) that are of the type $u(\rho,t)=R(\rho)T(t)$ (separation of variable method) must satisfy the equation **[3], [4]**

$$RT'=KT\left(R''+\frac{1}{\rho}R'\right) \quad , \qquad R(c)T(t)=0 \quad .$$

and we have

$$\frac{T'}{KT}=\frac{1}{R}\left(R''+\frac{1}{\rho}R'\right)=-\lambda, \_\lambda$$ is the separation constant.

then we have

$$\rho R''(\rho)+R'(\rho)+\lambda\rho R(\rho)=0, \qquad R(c)=0 \qquad (0<\rho<c), \qquad (4.3.11)$$

and

$$T'(t)+\lambda K\, T(t)=0 \qquad (t>0) \tag{4.3.12}$$

(4.3.11) is Bessel's equation with parameter λ, it is a Sturm-Liouville problem with eigenvalues λ_j (where $\lambda_j=\alpha_j^2$ $(j=1,2,3,\ldots)$ and α_j are the positive roots of the equation

$$J_0(\alpha\, c)=0\ ; \tag{4.3.13}$$

and the corresponding eigenfunctions

$$R_j(\rho)\ =J_0(\alpha_j\rho\ )$$

the solution of equation (4.3.12) is

$$T_j(t)=e^{-\alpha_j^2Kt}.$$

So the desired products are

$$u_j=R_j(\rho)T_j(t)=J_0(\alpha_j\rho)e^{-\alpha_j^2Kt}, \qquad (j=1,2,3,\ldots) \tag{4.3.14}$$

and the generalized linear combination of these functions,

$$u(\rho,t)=\sum_{j=1}^{\infty}A_jJ_0(\alpha_j\rho)e^{-\alpha_j^2Kt}$$

which satisfies the homogenous conditions (4.3.8)and (4.3.9), it's also satisfies the non homogenous initial condition (4.3.10) and the coefficients A_j are such that

$$f(\rho)=\sum_{j=1}^{\infty}A_jJ_0(\alpha_j\rho) \qquad (0<\rho<c)$$

If the coefficients have the values

$$A_j=\frac{2}{c^2[J_1(\alpha_jc)]^2}\int_0^c\rho f(\rho)J_0(\alpha_j\rho)d\rho \qquad (j=1,2,3,\ldots) \tag{4.3.15}$$

then $f(\rho)$ is a *Fourier*-Bessel series representation.
The formal solution of the boundary value problem (4.3.8 _ 4.3.10) is given by equation (4.3.12) and the coefficients are given by (4.3.13).
The temperature formula $u(\rho,t)$ can be written as

$$u(\rho,t)=\frac{2}{c^2}\sum_{j=1}^{\infty}\frac{J_0(\alpha_j\rho)}{[J_1(\alpha_jc)]^2}e^{-\alpha_j^2Kt}\int_0^c sf(s)J_0(\alpha_js)\,ds\ .$$

2.The second example is the heat transfer at the surface of the cylinder.

Let us replace the condition that the surface of the infinite cylinder in example (1) be a temperature zero by the condition that heat transfer take place there into surrounding at temperature zero.
The equation we want to solve is

$$Ku_{\rho}(c,t)=-Hu(c,t) \qquad (K>0, H>0),$$

where K is the thermal conductivity of the material of the cylinder and H is its surface conductance.
The boundary value problem for the temperature function $u(\rho,t)$ is **[23]**

$$u_t(\rho,t)=k\left[u_{\rho\rho}(\rho,t)+\frac{1}{\rho}u_{\rho}(\rho,t)\right] \quad 0<\rho<c, t>0), \tag{4.3.16}$$

$$cu_{\rho}(c,t)=-hu(c,t) \qquad (t>0), \tag{4.3.17}$$

$$u(\rho,0)=f(\rho) \qquad (0<\rho<c). \tag{4.3.18}$$

We have written $h = c\, H/K$; and allow the possibility that the constant h be zero, in this case $\rho = c$ is insulated.
By using separation of variable method let $u = R(\rho)T(t)$ then we have

$$\rho R''(\rho)+R'(\rho)+\lambda\rho R(\rho)=0, \quad hR(c)+cR'(c)=0,\ 0<\rho<c) \tag{4.3.19}$$

if $h>0$ the eigenvalues are $\lambda_j=\alpha_j^2$, $(j=1,2,3,\ldots)$ where α_j are the positive roots of the equation

$$hJ_0(\alpha\, c)+(\alpha\, c)J_0'(\alpha\, c)=0 \tag{4.3.20}$$

and since $T'(t) + \lambda kT(t) = 0$, the solution of (4.3.16) is

$$u_j=R_j(\rho)T_j(t)=J_0(\alpha_j\rho)\; e^{-\alpha_j^2kt}, \quad (j=1,2,\ldots.)$$

the formal solution of the boundary value problem is

$$u(\rho,t)=\sum_{j=1}^{\infty} A_j J_0(\alpha_j\rho)\; e^{-\alpha_j^2kt} \tag{4.3.21}$$

according to the initial condition(4.3.18)

$$A_j=\frac{2\alpha_j^2}{(\alpha_j^2c^2+h^2)[J_0(\alpha_jc)]^2}\int_0^c \rho\, f(\rho)J_0(\alpha_j\rho)d\rho \;, \quad (j=1,2,\ldots)\,.(4.3.22)$$

If $h = 0$, the boundary conditions in our problem becomes $R'(c) = 0$, with

α_j $(j=1,2,...)$ are the positve roots of the equation $J_0'(\alpha\, c)=0$, or

$$J_1(\alpha\, c)=0\ . \tag{4.3.23}$$

Noting that $J_0(\alpha_1\rho)=1$, then we have

$$u(\rho\ ,t)=A_1+\sum_{j=2}^{\infty} A_j J_0(\alpha_j\rho\ )\ e^{-\alpha_j^2 kt} \tag{4.3.24}$$

where

$$A_1=\frac{2}{c^2}\int_0^c \rho\ f(\rho)\ d\rho\ , \tag{4.3.25}$$

$$A_j=\frac{2}{c^2[J_0(\alpha_j c)]^2}\int_0^c \rho\ f(\rho\ )J_0(\alpha_j\rho\ )\ d\rho\quad ,(j=2,...)\ . \tag{4.3.26}$$

3.The third example is the vibrating of a circular membrane .

A membrane stretched over a fixed circular frame $\rho=c$ in the plane $z=0$, is given an initial displacement $z=f\ (\ \rho,\phi\ )$ and released at rest from that position .

Let $z(\rho,\phi,t)$ be described by the continuous function that satisfies this boundary value problem **[15]**, **[22]**

$$z_{tt}=a^2\left(z_{\rho\rho}+\frac{1}{\rho}z_\rho+\frac{1}{\rho^2}z_{\phi\phi}\right) \tag{4.3.27}$$

$$z(c,\phi,t)=0\qquad(-\pi\le\phi\ \le\pi,\ t\ge 0) \tag{4.3.28}$$

$$z(\rho,\phi,0)=f(\rho,\phi),\quad z_t(\rho,\phi,0)=0\quad(0\le\rho\ \le c,\ -\pi\le\phi\ \le\pi) \tag{4.3.29}$$

where the function $z(\rho,\phi,t)$ is periodic with period 2π, in the variable ϕ . By using separation of variable method $z=R(\rho)\Phi(\phi)T(t)$ satisfies the equation (4.3.27) if

$$\frac{T''}{a^2T}=\frac{1}{R}\left(R''+\frac{1}{\rho}R'\right)+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}=-\lambda\ ,\quad -\lambda\ \text{is any constant} \tag{4.3.30}$$

let $\Phi''/\Phi=-M$ then we have

$$\rho^2R''(\rho)+\rho R'(\rho)+(\lambda\rho^2-M)R(\rho)=0\ ,\quad R(c)=0 \tag{4.3.31}$$

$$\Phi''(\phi)+M\Phi(\phi)=0\ ,\quad \Phi(-\pi)=\Phi(\pi)\quad ,\ \Phi'(-\pi)=\Phi'(\pi) \tag{4.3.32}$$

if $M=n^2$, let $\lambda_{nj}=\alpha_{nj}^2$ $(j=1,2,3,...)$ where α_{nj} are the positive roots of the equation $$J_n(\alpha\ c)=0 \qquad (n=0,1,2,\ldots) \tag{4.3.33}$$

where
$R(\rho)=J_n(\alpha_{nj}\rho)$ and $T(t)=\cos(\alpha_{nj}at)$.

The generalized Linear combination of our functions $R\Phi T$ is

$$z(\rho,\phi,t)=\frac{1}{2}\sum_{j=1}^{\infty} A_{0j}J_0(\alpha_{0j}\rho)\cos(\alpha_{0j}at)$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} J_n(\alpha_{nj}\rho)(A_{nj}\cos n\phi+B_{nj}\sin n\phi\ )\cos(\alpha_{nj}at) \tag{4.3.34}$$

from the initial conditions, we have

$$f(\rho,\phi)=\frac{1}{2}\sum_{j=1}^{\infty} A_{0j}J_0(\alpha_{0j}\rho)$$
$$+\sum_{n=1}^{\infty}\left\{\left[\sum_{j=1}^{\infty}A_{nj}J_n(\alpha_{nj}\rho)\right]\cos n\phi+\left[\sum_{j=1}^{\infty} B_{nj}J_n(\alpha_{nj}\rho)\right]\sin n\phi\right\} \tag{4.3.35}$$

when $0\le\rho\le c\ ,\quad -\pi\le\phi\le\pi$.

For each fixed value of ρ, series (4.3.35) is the *Fourier* series for $f(\rho,\phi)$ on the interval $-\pi\le\phi\le\pi$ if

$$\sum_{j=1}^{\infty} A_{nj}J_n(\alpha_{nj}\rho)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\rho,\phi)\cos n\phi\ d\phi \qquad (n=0,1,2,\ldots)$$
$$\sum_{j=1}^{\infty} B_{nj}J_n(\alpha_{nj}\rho)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\rho,\phi)\sin n\phi\ \ d\phi \qquad (n=1,2,\ldots)$$

For every fixed n and we have

$$A_{nj}=\frac{2}{\pi\ c^2[J_{n+1}(\alpha_{nj}c)]^2}\int_0^c \rho\ J_n(\alpha_{nj}\rho)\int_{-\pi}^{\pi} f(\rho,\phi)\cos n\phi\ d\phi\ d\rho$$
$$B_{nj}=\frac{2}{\pi\ c^2[J_{n+1}(\alpha_{nj}c)]^2}\int_0^c \rho\ J_n(\alpha_{nj}\rho)\int_{-\pi}^{\pi} f(\rho,\phi)\sin n\phi\ d\phi\ d\rho\ .$$

4.The fourth example related to the vibration of a disklike membrane .

The eigenvalue problem is **[24]**

$$\Delta\ u+\lambda\ K(|\,x\,|)u=0 \qquad \text{in } R^n \tag{4.3.36}$$

in the space

$$D=\{u \mid u \text{ is mesurable}, \int_{R^n} K(|x|)u^2 dx < \infty\},$$

where $n \geq 3$ and $\lambda \in R$ is a parameter. Let the weight function $K(r)$, $r=|x|$ satisfies

$$\begin{cases} K(r) > 0 \quad \text{on } (0,\infty) \\ K(r) \in C((0,\infty)) \\ rK(r) \in L^1(0,\infty) . \end{cases}$$

All the eigenfunctions can be expressed as a product of the Bessel functions and functions of the argument θ. Moreover they form a complete orthonormal basis.

Since the weight K is radially symmetric, (4.3.36) can have radial solutions $u = u(r)$ which are obtained as asolution of the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \lambda K(r)u = 0, \ r > 0 \\ u(0) = 1 . \end{cases} \tag{4.3.37}$$

We note that (4.3.37) has a unique solution for any $\lambda > 0$ under the assumption (K). Under a stronger condition than (K), Natio **[12]** shows the existence of the smallest eigenvalue $\lambda_0 > 0$ for which (4.3.37) has a positive solution satisfying $\lim_{r\to\infty} r^{n-2} < \infty$.

Recently Kabeya **[11]** obtained the following result concerning the existence and uniqueness of radial eigenfunction .

Theorem(4.4.1): [11], [24] If $n > 0$ then there exist a unique monotone increasing

Sequence $\{\lambda_j\}_{j=0}^{\infty}$ such that the solution of (4.3.37) has exactly j zeros and

satisfying $\lim_{r\to\infty} r^{n-2} < \infty$.

4.4 *Sign of the eigenvalues; and Fourier series with respect to the eigenfunctions*

Corollary (4.4.1): [3], [21] If $r > 0$, $q \leq 0$ and if the boundary conditions imply that

$$[p\phi\phi']_{x=a}^{x=b} \leq 0 \qquad (4.4.1)$$

then all the eigenvalues of the boundary value problem for the equation (4.1.17) are nonnegative.

Proof:
Let λ be an eigenvalue and let $\phi(x)$ be the corresponding eigenfunction, if we multiply (4.1.17) by $\phi(x)$ and integrating over the interval $[a, b]$ to have

$$\int_a^b (p\phi')'\phi dx + \int_a^b q\phi^2 dx = -\lambda \int_a^b r\phi^2 dx \ .$$

Integrating by parts we obtain

$$[p\phi\phi']_{x=a}^{x=b} - \int_a^b p\phi'^2 dx + \int_a^b q\phi^2 dx = -\lambda \int_a^b r\phi^2 dx \qquad (4.4.2)$$

since $q \leq 0$ thus the left-hand side of (4.4.2) is ≤ 0, therefore $\lambda \geq 0$ and moreover, $\lambda = 0$ is possible only if $q \equiv 0$, and $\phi' \equiv 0$, that is to say only if (4.1.17) has the form

$$(p\phi')' = -\lambda r\phi$$

and the function $\phi =$ constant is an eigefunction .

Definition (4.4.2): [21] The *Fourier* series of an absolute integrable function $f(x)$ on [a,b] is $f(x) \sim c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$

where

$$\phi_0(x), \phi_1(x), \ldots, \phi_n(x), \ldots, \qquad (4.4.3)$$

are the eigenfunctions for arranged increasing order eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots,$.

And to have a normalization, let

$$\int_a^b r\phi_n^{\ 2}(x) dx = 1 \quad , n = 0,1,2,\ldots. \qquad (4.4.4)$$

and

$$c_n = \int_a^b rf(x)\phi_n(x) dx \quad , n = 0,1,2,\ldots \qquad (4.4.5)$$

4.5 *Does the eigenfunction method always lead to a solution of the problem*

If the problem can be written as a Sturm_Liouville problem then it can be solved by eigenfunction method.

Theorem (4.5.1): [21] Let the function $u(x,t)$ be a continuos solution of (4.1.1) in the region $a \le x \le b,\ t \ge 0$, satisfying the boundary conditions (4.1.3) and the initial conditions (4.1.4) then

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\phi_n(x) \tag{4.5.1}$$

where $\phi_n(x)$ are the eigenfunctions associated with the boundary value problem such that the eigenfunctions are normalized. The functions $T_n(t)$ can be found from the equation

$$T_n'' + \lambda_n T_n = 0 \quad , \ n = 0,1,2,... \tag{4.5.2}$$

subject to the initial conditions

$$T_n(0) = c_n \quad , T_n'(0) = d_n \quad , n = 0,1,...$$

where the quantities c_n and d_n are the *Fourier* coefficients of $f(x)$ and $g(x)$, $g(x) = \dfrac{\partial u(x,0)}{\partial t}$ with respect to the system of eigenfunctions $\phi_n(x)$. (It is assumed that the derivatives $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ are continuous and bounded in every region of the form $a < x < b,\ 0 < t < t_0$).

Proof: If we multiply equation (4.1.1) by the function

$$r = \frac{1}{\mathrm{P}} \exp\{\int_{x_0}^{x} \frac{R}{\mathrm{P}} dx\} = \frac{P}{\mathrm{P}}$$

then according to (4.1.18) and (4.1.19) we obtain

$$p\frac{\partial^2 u}{\partial x^2} + p'\frac{\partial u}{\partial x} + qu = r\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + qu = r\frac{\partial^2 u}{\partial t^2}$$

By (4.2.2) the last equation can be written as

$$L(u) = r\frac{\partial^2 u}{\partial t^2} \tag{4.5.3}$$

and instead of (4.1.17), we can write $L(\phi) = -\lambda r\phi$ .Therefore the relation

$$L(\phi_n) = -\lambda_n r\phi_n,\ n = 0,1,2,\ldots, \tag{4.5.4}$$

holds for the eigenfunctions of the boundary value problem.

Then by the hypothesis of this theorem [and by theorem (2) in reference (21) page 261] we have for $a < x < b$ and every $t > 0$ the function $u(x,t)$can be expanded in a series of the form (4.5.1) where

$$T_n(t) = \int_a^b r u(x,t)\ \phi_n(x)dx \quad , n = 0,1,2,3,\ldots. \tag{4.5.5}$$

It follows from (4.5.4) that

$$r\phi_n = -\frac{1}{\lambda_n} L(\phi_n)$$

so that

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b u(x,t)L(\phi_n)dx$$

or by (4.2.3)

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b \phi_n(x)L(u)dx + \frac{1}{\lambda_n}\left[p(\phi_n\frac{\partial u}{\partial x} - \phi_n{}' u)\right]_{x=a}^{x=b}$$

the last term vanishes because of lemma (4.2.3) , so we obtain

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b \phi_n(x)L(u)dx \tag{4.5.6}$$

whence using (4.5.3), we get

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b r\frac{\partial^2 u}{\partial t^2}\phi_n(x)dx \tag{4.5.7}$$

on the other hand, differentiating (4.5.5) twice with respect to t gives

$$T_n''(t) = \int_a^b r\frac{\partial^2 u}{\partial t^2}\phi_n(x)dx \tag{4.5.8}$$

where the differentiation behind the integral sign is legitimate because of our hypotheses concerning $\partial u/\partial t$ and $\partial^2 u/\partial t^2$. Comparing (4.5.7) with (4.5.8), we obtain (4.5.2) furthermore, since $u(x,t)$ is continuous in the region $a \le x \le b$, $t \ge 0$ and since

$$\lim_{t\to 0} u(x,t) = f(x)$$

it follows from (4.5.5) that

$$\lim_{t\to 0} T_n(t) = \lim_{t\to 0}\int_a^b ru(x,t)\phi_n(x)dx$$
$$= \int_a^b rf(x)\phi_n(x)dx = c_n \ , \quad n = 0,1,..., \tag{4.5.9}$$

where the c_n's are the *Fourier* coefficients of the function $f(x)$. Since the function $T_n(t)$ is continuous, this is equivalent to the relations

$$T_n(0) = c_n, \ (n=0,1,2,\dots)$$

similarly, we show that

$$T_n'(0) = d_n \ , \ (n = 0,1,2,...)$$

where the d_n are the *Fourier* coefficients of the function $g(x)$.

Theorem (4.5.3): **[21]** Let

$$u(x,t) \sim \sum_{n=0}^{\infty} T_n(t)\phi_n(x)$$

be either the exact or the generalized solution of equation (4.1.1) subject to the conditions (4.1.3) and (4.1.4) , also if

$$\lim_{m\to\infty} \int_a^b r[f(x) - f_m(x)]^2 dx$$
$$= \lim_{m\to\infty} \int_a^b r[g(x) - g_m(x)]^2\, dx = 0 \tag{4.5.10}$$

and if

$$u_m(x,t) = \sum_{n=0}^{\infty} T_{mn}(t)\phi_n(x)$$

is either the exact or the generalized solution of the equation (4.1.1), subject to the boundary conditions (4.1.3) and the initial condition

$$u_m(x,0) = f_m(x) \ , \frac{\partial u_m}{\partial t}(x,0) = g_m(x)$$

then $u_m(x,t)$ converges to $u(x,t)$ in the mean as $m \to \infty$.

Proof: [see [21]]

We now discuss the inhomogeneous problem

$$p\frac{\partial^2 u}{\partial x^2} + R\frac{\partial u}{\partial x} + Q\,u = \frac{\partial^2 u}{\partial t^2} + F(x,t) \tag{4.5.11}$$

subject to the same boundary conditions (4.1.3) and (4.1.5).
In vibrations problem, equation (4.5.11) corresponds to the case of forced vibrations and when multiplied by the function

$$r = \frac{1}{\mathrm{P}}\exp\left\{\int_{x_0}^{x} \frac{R}{\mathrm{P}}\,dx\right\} = \frac{p}{\mathrm{P}}$$

then (4.5.11) gives

$$L(u) = r\frac{\partial^2 u}{\partial t^2} + rF(x,t)\ . \tag{4.5.12}$$

Suppose that (4.5.11) has a solution and $F(x,t)$ has a series expansion in terms of the eigenfunctions of the boundary value problem for the equation $L(\phi) = -\lambda r\phi$ then for $t > 0$, we write $u(x,t)$ as the series

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\ \phi_n(x) \tag{4.5.13}$$

where

$$T_n(t) = \int_a^b r\,u(x,t)\phi_n(x)dx \quad , \quad n = 0,1,2,... \tag{4.5.14}$$

then by using the proof of theorem (4.5.1) we can obtain

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b \phi_n(x)L(u)dx$$

or

$$T_n(t) = -\frac{1}{\lambda_n}\int_a^b r\frac{\partial^2 u}{\partial t^2}\phi_n(x)dx - \frac{1}{\lambda_n}\int_a^b rF(x,t)\ \phi_n(x)dx \qquad (4.5.15)$$

where (4.5.12) has been used.
Assuming that $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ are bounded derivative, then by differentiating (4.5.14) twice relative to t, we get

$$T_n''(t) = \int_a^b r\frac{\partial^2 u}{\partial t^2}\phi_n(x)dx$$

and if we set

$$F(x,t) = \sum_{n=0}^{\infty} F_n(t)\phi_n(x) \qquad (4.5.16)$$

with

$$F_n(t) = \int_a^b rF(x,t)\phi_n(x)dx \quad , n = 0,1,...$$

then (4.5.15) gives

$$T_n = -\frac{1}{\lambda_n}T'' - \frac{1}{\lambda_n}F_n \quad ,$$

or

$$T_n'' + \lambda_n T_n + F_n = 0,\ n = 0,1,... \qquad (4.5.17)$$

then using the proof of theorem (4.5.1) we find

$$T_n(0) = c_n\ ,\ T_n'(0) = d_n\ ,\quad n = 0,1,... \qquad (4.5.18)$$

where c_n and d_n are the *Fourier* coefficients of the functions $f(x)$ and $g(x)$.
Remarks: All considerations for (4.1.1) are also applicable to equation (4.1.2) with the same boundary conditions and with the initial condition

$u(x,0) = f(x)$, thus by separation of variable method

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\phi_n(x)$$

where T_n's are found from the first order differential equation

$$T_n' + \lambda_n T_n = 0\ ,\ T_n(0) = c_n \quad , n = 0,1,2,...$$

and c_n's $(n = 0,1,2,\ldots)$ are the *Fourier* coefficients of $f(x)$.

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