

Regular Matrix Polynomial

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Dedication

To my parents, my brothers, and to my sisters.

Ali Ahmad Mousa Abu'Aram

Declaration

I Certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Abstract

Matrix polynomials have lately received the attention of many researchers. However, the theory of matrix polynomials have been developed in the last forty years. Moreover, many applications in differential equations, boundary value problems, and numerical analysis utilize some results in the theory of matrix polynomials.

In this work we investigate regular matrix polynomials regarding there solvents, companion matrices and linearization.

Many key result about solvents of regular matrix polynomials have been studied by Prreira [10] and Brase [3].

Concerning companion matrices for regular matrix polynomials Antonio, *et al* [1]

Found out a new family of companion forms for regular matrix polynomials.

These new forms are used in solving system of differential equation and numerical analysis.

Finally, we consider some special cases [1],[2] relating linearization of self–adjoint or symmetric regular matrix polynomials and companion matrices which result in a similar type of matrices.

المخلص

مصفوفات كثيرات الحدود المنتظمة جذبت اهتمام الكثير من الباحثين مؤخرا و قد بدا تطور هذا الموضوع خلال الأربعة عقود الماضية. علاوة على ذلك هنالك تطبيقات في مجال المعادلات التفاضلية و التحليل العددي حيث يتم استخدام نتائج النظريات المتعلقة بالمصفوفات كثيرات الحدود المنتظمة .

في هذه الرسالة تركز البحث على اصفار مصفوفات كثيرات الحدود المنتظمة . هناك عدة نتائج مهمة تتعلق باصفار مصفوفات كثيرات الحدود المنتظمة وجد انتونيو [1] و آخرين عائلة جديدة من مصفوفات الرفيق لمصفوفات كثيرات الحدود المنتظمة , وهذه الصيغة الجديدة يمكن استخدامها في حل المعادلات التفاضلية و التحليل العددي .

و أخيرا اعتمدنا بعض الحالات الخاصة [1],[2] و هي حالات تتعلق بالمصفوفات كثيرات الحدود المنعكسة ذاتيا أو المتماثلة حيث في هذه الحالة تكون مصفوفات الرفيق منعكسة ذاتيا أو متماثلة أيضا.

Table of Contents

Introduction	1
Prologue to the reader	2
Chapter 1 Preliminaries	3
1.0 Introduction	3
1.1 The definition of matrix polynomial.....	3
1.2 The canonical form of matrix polynomial.....	7
1.3 Invariant polynomials.....	12
1.4 Elementary divisors.....	15
Chapter 2 Standard triples and pairs of matrix polynomials	19
2.0 Introduction.....	19
2.1 Linearization of monic matrix polynomial.....	19
2.2 Linearization of matrix polynomial.....	25
2.3 Standard triples.....	28
2.4 Standard pairs	33
2.5 Jordan chains.....	35
Chapter 3 Division and solvents of matrix polynomials	41
3.0 Introduction	41
3.1 Division of matrix polynomials.....	41
3.2 Solvent of matrix polynomials.....	47
3.3 Existence and number of solvents.....	55

Chapter 4 Companion matrices and linearization.....62

4.0 Introduction..... 62

4.1 New forms for a companion matrix of scalar polynomials..... 62

4.2 New family of companion matrices of regular matrix polynomial..... 65

4.3 Self- adjoint and symmetric linearization72

References..... 77

Symbols

\mathbb{R}	Set of real numbers
\mathbb{C}	Set of complex numbers
\mathbb{C}^n	Standard complex vector space
$\mathbb{C}^{n \times n}$	Space of $n \times n$ complex matrices
I, I_n	Identity matrix, $n \times n$ identity
$A^{[r]}(s)$	r th Derivative of $A(s)$ with respect to s
A^T	Transpose of matrix A
A^*	Conjugate transpose of matrix A
$\text{diag}\{s_1, s_2, \dots, s_n\}$	Diagonal matrix with s_1, s_2, \dots, s_n along diagonal
\in	Member symbol
$A(s)$	Matrix polynomial with $s \in \mathbb{C}$
$A(S)$	Matrix polynomial with $S \in \mathbb{C}^{n \times n}$
C_A	Companion matrix associated to $A(s)$
\cup	Union symbol
\cap	Intersection symbol
\emptyset	Empty set
$\det A$	Determinant of A
$A(s)^{-1}$	Inverse of $A(s)$, or resolvent form
$\deg A(s)$	Degree of A

$\binom{m}{n}$

Combination of m over n , that is $\frac{m!}{n!(m-n)!}$

 $A_{pq}(s)$

Cofactor of the element a_{pq} of $A(s)$

 $adjA(s)$

Adjoint of $A(s)$

 $\min(m, n)$

Minimum value

Introduction

Matrix polynomial is an important part of linear algebra, and has important applications in differential equations, boundary value problems, numerical analysis and other areas.

Problems concerning matrix polynomials initially introduced in early 1933, later the matrix polynomial appeared as a topic of research in 1976 by Gohberg, Lancaster, and Rodman.

In the thesis we consider some types of regular matrix polynomials

- 1) Monic matrix polynomial.
- 2) Self –adjoint regular matrix polynomial.
- 3) Symmetric (skew symmetric) regular matrix polynomial.

Since 1976 the matrix polynomial has received the attention of mathematicians. However, the theory of matrix polynomial has been extensively developed in the last 40 years.

Also in fact there are few specialized books appeared in the subject, such as Gohberg, Lancaster and Rodman (1982).

In 2003 M. Fiedler introduce a new family of companion matrices associated to scalar polynomials see [6]. And in 2004 E. N. Antoniou and S. Vologianidis apply the same idea used in scalar polynomials for a matrix polynomials to introduce a new family of companion matrices for regular matrix polynomials, see [1].

This research consists of four chapters:

Chapter one: contains the main concepts, definitions and preliminary basics that are essential for the rest of the research.

Chapter two: introduces the meaning of linearization, standard triples and pairs and the definition of Jordan chains.

Chapter three: contains the definition of division process of matrix polynomials, a recent results about the method of finding and the number of these solvents.

Finally,

Chapter four: contains recent results about companion matrices and gives a method for constructing different forms of companion matrices for a regular matrix polynomial.

Prologue to the reader

We refer to each one of theorems, corollaries, facts, definitions and examples by triple (A,B,C), where

A: refers to the chapter number.

B: refers to the section number.

C: refers to the number of theorem, lemma, corollary, fact, definition or example.

Each category of theorem's, lemmas, corollaries, facts, definitions, or examples has its own sequence.

Also, we refer to most of equation by triple (A, B, C). Where A, and B as defined above and C refers to the number of equation. Equations has own sequence of numbering in each section.

Each proof starts with the word "proof ", and ends with ■.

Throughout the research we mean by a constant matrix, a matrix with complex entries and by constant or scalar polynomial, a polynomial with constant coefficients.

Also, when we write $A(s)$ we mean a matrix with polynomial entries or a matrix polynomial, and A mean a matrix with constant coefficients. And when we write a matrix I we mean the identity matrix of proper order.

Chapter One

Preliminaries

1.0 Introduction

In this chapter we will give some basic definitions, notations and results related to matrix polynomials, which will be used in the rest of this manuscript.

Section 1.1, introduces the definition of matrix polynomial, gives a classifications and some properties of matrix polynomials. Section 1.2, gives the definition of the canonical form of matrix polynomials and some applications. In section 1.3 we introduce the definition of invariant polynomial to a matrix polynomial. and section 1.4, gives the definition of the elementary divisors and some theorems.

1.1 The definition of matrix polynomial

In this section we introduce the definition of the matrix polynomial and some other basic definitions.

By a matrix polynomial we mean a matrix- valued function of a complex variable of the

$$\text{form } A(s) = \sum_{i=0}^l A_i s^i, A_i \in \mathbb{C}^{p \times p} \quad (1.1.1)$$

Where A_0, A_1, \dots, A_l are $p \times p$ matrices of complex numbers, p called the order of $A(s)$,

and A_l is called the leading coefficient.

Also, we can write a matrix polynomial as a matrix with polynomial entries.

Example 1.1.1:

$$1) A(s) = \sum_{i=0}^2 A_i s^i = A_0 + A_1 s + A_2 s^2$$

Where $A_i \in \mathbb{C}^{p \times p}, i = 0, 1, 2.$

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Where A_0, A_1, \dots, A_l are $p \times p$ matrices of complex numbers, p called the order of $A(s)$, and A_l is called the leading coefficient.

Also, we can write a matrix polynomial as a matrix with polynomial entries.

Example 1.1.1:

1)
$$A(s) = \sum_{i=0}^2 A_i s^i = A_0 + A_1 s + A_2 s^2$$

Where $A_i \in \square^{p \times p}, i = 0, 1, 2.$

$$2) A(s) = \begin{bmatrix} s^2+1 & s \\ s^3+5 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 5 & 5 \end{bmatrix}.$$

Definition 1.1.1: A matrix polynomial $A(s)$ is said to be invertible if there exists a matrix polynomial $B(s)$ such that

$$A(s)B(s) = B(s)A(s) = I \quad (1.1.2)$$

And we call $B(s)$ the inverse of $A(s)$ denoted by $A^{-1}(s)$

Definition 1.1.2: The degree of a matrix polynomial $A(s)$ is defined to be the greatest degree of the polynomials appearing as entries of $A(s)$ and is denoted by $\deg A(s)$.

The determinant of a matrix polynomial is defined in usual way for matrices and is denoted by $\det A(s)$.

Definition 1.1.3: Let $A(s) = (p_{ij}(s))_{i,j=1,2,\dots,n}$ be an $n \times n$ matrix polynomial, choose k -rows, $1 \leq i_1 < \dots < i_k \leq n$, and k -columns, $1 \leq j_1 < \dots < j_k \leq n$, in $A(s)$, and consider the any $k \times k$ submatrix of $A(s)$, say $A_k(s)$, then the determinant $\det A_k(s)$ is called a minor of $A(s)$ of order k and is composed of the k -rows and k -columns of $A(s)$.

Taking another set of columns and rows, we obtain another minor of order k of $A(s)$.

Definition 1.1.4: A matrix polynomial $A(s)$ is said to be unimodular, if the determinant of $A(s)$ is a nonzero constant.

Definition 1.1.5: Let $A_{ij}(s) = (-1)^{i+j} M_{ij}$, where M_{ij} is a minor of $A(s)$ then $A_{ij}(s)$ is called the cofactor of the element a_{ij} of $A(s)$. The transposed matrix of cofactors of $A(s)$ is called the adjoint of $A(s)$ and denoted by $adjA(s)$.

Theorem 1.1.1 [9]: If $\det A \neq 0$ then

$$A^{-1} = \frac{1}{\det A} \text{adj}A \quad (1.1.3)$$

Proof: Note that the result of the product $A \text{adj}A$ is a matrix with the number $\det A$ in each position on its main diagonal and zeros elsewhere, that is

$$A \text{adj}A = I \det A, \text{ so that } A^{-1} = \frac{1}{\det A} \text{adj}A \blacksquare$$

The basic operations of addition, subtraction and multiplication of two or more matrix polynomials are defined in exactly the same way for scalar matrices.

Also, there are many theorems in usual matrices that still true in matrix polynomials but with new notations. For example in usual matrices we know the condition on a matrix to have an inverse is the nonsingularity of the matrix, and in matrix polynomials we have approximately the same condition as shown in the following theorem.

Theorem 1.1.2 [9]: An $n \times n$ matrix polynomial $A(s)$ is invertible if and only if $A(s)$ is unimodular.

Proof: Let $A(s)$ be $n \times n$ unimodular matrix polynomial then $\det A(s) = c \neq 0$, for some constant c , then by Theorem 1.1.1 $A^{-1}(s)$ exists.

Conversely, if $A(s)$ is invertible, then there exists another matrix polynomial $B(s)$ such that $A(s)B(s) = I$, so that $\det A(s)\det B(s) = \det I = 1$. Thus the product of the determinants of the matrix polynomials $A(s)$ and $B(s)$ is a nonzero constant. And this occur if they are each nonzero constants, so that $A(s)$ has a nonzero constant determinant, and hence it is unimodular matrix polynomial, and this complete the proof \blacksquare

Example 1.1.2: Consider

$$A(s) = \begin{bmatrix} (s-1)^2 & s \\ s-2 & 1 \end{bmatrix}$$

Then $\det A(s) = (s-1)^2 - s(s-2) = 1$, so that $A(s)$ is a unimodular and hence $A(s)$ is

invertible, where $A^{-1}(s) = \begin{bmatrix} 1 & -s \\ 2-s & (s-1)^2 \end{bmatrix}$.

Definition 1.1.6: A nonzero matrix polynomial $A(s)$ of dimension $n \times m$ is said to be of rank r , if r is the largest positive integer such that not all minors of $A(s)$ of order r are identically zero.

A zero matrix polynomial is said to be of rank zero.

Matrix polynomials can be classified into many types, where the classifications depend on determinant, leading coefficient and other properties.

If $A(s) = \sum_{i=0}^l A_i s^i$ is a matrix polynomial, then we can classify any matrix polynomial as follows:

a) Monic matrix polynomial:

Which is a matrix polynomial with $A_l = I_p$, where I_p is the $p \times p$ identity matrix.

Otherwise (i.e. $A_l \neq I_p$), $A(s)$ is said to be nonmonic.

b) Regular matrix polynomial:

A regular matrix polynomial is a matrix polynomial with $\det A(s) \neq 0$, except for finitely many $s \in \mathbb{C}$.

c) Self-adjoint matrix polynomial:

The adjoint of $A(s)$ denoted by $A^*(s)$ is defined by $A^*(s) = \sum_{i=0}^l A_i^* s^i$, where $A^* = \overline{A}^T$

means the operator adjoint to $A(s)$.

A matrix polynomial is said to be self-adjoint if $A(s) = A^*(s)$.

Example 1.1.3:

a) Consider $A(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Clearly $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, which is monic.

b) Let $A(s) = \begin{bmatrix} s^2 & s \\ 2 & s+5 \end{bmatrix}$, we see that $\det A(s) = s^3 + 5s^2 - 2s \neq 0$, for almost every

$s \in C$, so that $A(s)$ is regular.

c) Consider $A(s) = \begin{bmatrix} s^2 + 5s & 1 \\ 1 & s^2 + 5s \end{bmatrix}$,

we see that $A(s) = A^*(s)$, so that $A(s)$ is self-adjoint.

1.2 The canonical form of matrix polynomial

In this section we will introduce the elementary row and column operations on matrix polynomials and the definition of the canonical form of matrix polynomial.

The elementary row and column operations on matrix polynomial are given as follows:

- 1) Multiply any row (column) by a nonzero $c \in F$.
- 2) Interchange any two rows (columns).
- 3) Add to any row (column) any other row (column) multiplied by an arbitrary polynomial over F .

Clearly these operations are the elementary operations for scalar matrices except that in (3) the word "scalar" was replaced by "polynomial".

Definition 1.2.1: An elementary matrix is a matrix polynomial obtained by applying a single elementary operation to I_n , and it is unimodular.

Definition 1.2.2: Two matrix polynomials $A(s)$ and $B(s)$ are said to be equivalent if there exists unimodular matrices $P(s)$ and $Q(s)$ such that

$$B(s) = P(s)A(s)Q(s) \quad (1.2.1)$$

and we write $A(s) \square B(s)$.

Theorem 1.2.1 [9]: Any unimodular matrix polynomial is equivalent to the identity matrix.

Proof: If $A(s)$ is unimodular, then $A(s)[A(s)]^{-1} = I$, and hence $I = IA(s)[A(s)]^{-1}$, where $P(s) = I$ and $Q(s) = [A(s)]^{-1}$, see (1.2.1), which are unimodular. So that $A(s) \square I$ ■

Definition 1.2.3: Let

$$A_0(s) = \text{diag}[a_1(s), a_2(s), \dots, a_n(s)] \quad (1.2.2)$$

Where $a_j(s)$ is a zero or monic polynomial, $j = 1, \dots, n$, and $a_j(s)$ is divisible by $a_{j-1}(s)$, $j = 2, 3, \dots, n$, then $A_0(s)$ is called diagonal matrix polynomial and a matrix polynomial with these properties is called a canonical matrix polynomial.

Note that, "if there are zeros among the polynomials of $a_j(s)$ then they must be placed in the last positions of the canonical matrix, since a nonzero polynomial is not divisible by a zero polynomial, and if some of the polynomials $a_j(s)$ are nonzero constants then they must be equal to 1 and be placed in the first positions of the canonical form" [9].

Theorem 1.2.2 [9]: Any matrix polynomial over F of order n is equivalent to a canonical matrix polynomial.

Proof: The proof is given in the following four steps, also this proof gives a method for finding the canonical form to any matrix polynomial $A(s)$ of order n .

The theorem is true for $A(s) \equiv 0$. So that, without loss of generality, we assume $A(s) \not\equiv 0$.

Step1: Since we assume $A(s) \not\equiv 0$; then there is an element $a_{ij}(s) \neq 0$ of minimum degree of $A(s)$, by proper interchanges of rows and columns, one can bring this element to the (1,1) position in $A(s)$ to become the new $a_{11}(s)$. For each element of the first row and column of the resulting matrix, we find the quotient and remainder on division by $a_{11}(s)$:

$$a_{1j}(s) = a_{11}(s)q_{1j}(s) + r_{1j}(s), j = 2, 3, \dots, n$$

$$a_{i1}(s) = a_{11}(s)q_{i1}(s) + r_{i1}(s), i = 2, 3, \dots, n$$

Now, for each i and j , subtract $q_{1j}(s)$ times the first column from the j th column and subtract $q_{i1}(s)$ times the first row from the i th row. Then the elements $a_{1j}(s), a_{i1}(s)$ are replaced by $r_{1j}(s)$ and $r_{i1}(s)$, respectively ($i, j = 2, 3, \dots, n$), which must be either the zero polynomial or have degree less than that of $a_{11}(s)$. If the polynomials are not all zero, we interchange $a_{11}(s)$ with an element $r_{1j}(s)$ or $r_{i1}(s)$ of least degree. Now repeat the process of reducing the degree of the off-diagonal elements of the first row and column to be less than that of the new $a_{11}(s)$. Then by this step we obtain $A(s)$ in the form

$$\begin{bmatrix} a_{11}(s) & 0 & \dots & 0 \\ 0 & a_{22}(s) & \dots & a_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}(s) & \dots & a_{nn}(s) \end{bmatrix} \quad (1.2.3)$$

Step2: In form (1.2.3), there may now be nonzero elements $a_{ij}(s), 2 \leq i, j \leq n$, whose degree is less than that of $a_{11}(s)$. If this is the case, we repeat step1 again and arrive at another matrix of the form (1.2.3) but with the degree of $a_{11}(s)$ further reduced. Thus by repeating step1 a sufficient number of times we can find a matrix of the form (1.2.3) that is equivalent to $A(s)$ and for which $a_{11}(s)$ is a nonzero element of least degree.

Step3: If we complete step2, and there are nonzero elements that are not divisible by $a_{11}(s)$, say $a_{ij}(s)$, we add column j to column 1, and we find remainders and quotients of the new column 1 on division by $a_{11}(s)$, and go on to repeat steps 1 and 2, to obtain the new form of (1.2.3), again with $a_{11}(s)$ replaced by a polynomial of smaller degree. Then we get the form:

$$\begin{bmatrix} a_1(s) & 0 & \dots & 0 \\ 0 & b_{22}(s) & \dots & b_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2}(s) & \dots & b_m(s) \end{bmatrix} \quad (1.2.4)$$

Step4: Now in form (1.2.4), if all $b_{ij}(s)$ are zero, then this form is the canonical form of $A(s)$, and the theorem is proved. If not, the form (1.2.4) may be reduced to the form:

$$\begin{bmatrix} a_1(s) & 0 & 0 & \dots & 0 \\ 0 & a_2(s) & 0 & \dots & 0 \\ 0 & 0 & c_{33}(s) & \dots & c_{3n}(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & c_{n3}(s) & \dots & c_m(s) \end{bmatrix} \quad (1.2.5)$$

where $a_2(s)$ is divisible by $a_1(s)$ and the elements $c_{ij}(s)$, $3 \leq i, j \leq n$, are divisible by $a_2(s)$. Continuing the process, we arrive to the form:

$$\begin{bmatrix} a_1(s) & 0 & \dots & 0 \\ 0 & a_2(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n(s) \end{bmatrix} \quad (1.2.6)$$

where $a_j(s)$ is divisible by $a_{j-1}(s)$, $j = 2, 3, \dots, n$, and this complete the proof ■

Example 1.2.1: Consider

$$A(s) = \begin{bmatrix} s+3 & s+2 & s+4 \\ 2s^3+6s^2+s & 2s^3+4s^2+s & 3s^3+8s^2+s \\ s^2+4s+3 & s^2+3s+2 & 3s^2+7s+4 \end{bmatrix}$$

Using the steps described in the proof of Theorem 1.2.2 we get:

$$\begin{bmatrix} s+3 & s+2 & s+4 \\ 2s^3+6s^2+s & 2s^3+4s^2+s & 3s^3+8s^2+s \\ s^2+4s+3 & s^2+3s+2 & 3s^2+7s+4 \end{bmatrix} \xrightarrow{C_1-C_2}$$

$$\begin{bmatrix} 1 & s+2 & s+4 \\ 2s^2 & 2s^3+4s^2+s & 3s^3+8s^2+s \\ s+1 & s^2+3s+2 & 3s^2+7s+4 \end{bmatrix} \xrightarrow{R_2-2s^2(R_1) \text{ and } R_3-(s+1)(R_1)} \begin{bmatrix} 1 & s+2 & s+4 \\ 0 & s & s^3+s \\ 0 & 0 & 2s^2+2s \end{bmatrix}$$

$$\xrightarrow{C_2-(s+2)C_1 \text{ and } C_3-(s+4)C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & s^3+s \\ 0 & 0 & 2s^2+s \end{bmatrix} \xrightarrow{C_3-(s^2+1)C_2 \text{ and } \frac{1}{2}(R_3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^2+s \end{bmatrix}$$

And so

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^2+s \end{bmatrix} \text{ is the canonical form for } A(s).$$

Finally we give the following definition.

Definition 1.2.4: Let $A(s)$ be an $n \times n$ matrix polynomial, then we define the zeroes of $\det A(s)$ to be the latent roots or eigenvalues of $A(s)$, and the set of latent roots of $A(s)$ is called the spectrum of $A(s)$ which denoted by $\sigma(A)$.

And if a nonzero $v \in \mathbb{C}^n$ is such that $A(s)v = 0$, then we say that v is a right latent (or eigen) vector of $A(s)$.

1.3 Invariant polynomials

This section contains the definition of invariant polynomials and some related theorems.

Definition 1.3.1: If A is an $m \times n$ matrix, then the determinant of a $p \times p$ submatrix of A ($p \leq \min(m, n)$), obtained from A by striking out $m-p$ rows and $n-p$ columns, is called a minor of order p of A . Or, if the rows and columns retained are given by subscripts $1 \leq i_1 < i_2 < \dots < i_p \leq m, 1 \leq j_1 < j_2 < \dots < j_p \leq n$ respectively, then the corresponding $p \times p$ minor is denoted by

$$A \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} \equiv \det [a_{ik} jk]_{k=1}^p \quad (1.3.1)$$

Theorem 1.3.1 [7]: (Bient-Cauchy formula). Let A and B be $m \times n$ and $n \times m$ matrices, respectively. If $m \leq n$ and $C = AB$, then

$$\det C = \sum_{1 \leq j_1 < j_2 < \dots < j_m \leq n} A \begin{pmatrix} 1 & 2 & \dots & m \\ j_1 & j_2 & \dots & j_m \end{pmatrix} B \begin{pmatrix} j_1 & j_2 & \dots & j_m \\ 1 & 2 & \dots & m \end{pmatrix} \quad (1.3.2)$$

For the proof, see [7]

Theorem 1.3.2 [8]: Let $A(s)$ be an $m \times n$ matrix polynomial. Let $d_k(s)$ be the greatest common divisor of the minors of $A(s)$ of order k , if not all of them are zeros, and let $d_k(s) \equiv 0$, if all the minors of order k of $A(s)$ are zeros. Let $d_0(s) \equiv 1$ and

$$D(s) = \text{diag}[i_1(s), i_2(s), \dots, i_r(s), 0, 0, \dots, 0] \quad (1.3.3)$$

be the canonical form of $A(s)$, then r is the maximal integer such that $d_r(s) \neq 0$, and

$$i_l(s) = \frac{d_l(s)}{d_{l-1}(s)}, l = 1, 2, \dots, r \quad (1.3.4)$$

Proof : First we will show that if $A_1(s) \sim A_2(s)$, then the greatest common divisors $d_{k,1}(s)$ and $d_{k,2}(s)$ of the minors of order k of $A_1(s)$ and $A_2(s)$, respectively, are equal. Now,

since $A_1(s) \sim A_2(s)$, so there exists two unimodular matrices $E(s)$ and $F(s)$ such that

$$A_1(s) = E(s)A_2(s)F(s) \quad (1.3.5)$$

then by Theorem 1.3.1, we can write the minor of $A_1(s)$ of order k as a linear combination of minors of $A_2(s)$ of the same order that is

$$A_1 \begin{pmatrix} j_1 & j_2 & \dots & j_k \\ k_1 & k_2 & \dots & k_k \end{pmatrix} = \sum_{\substack{1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq m \\ 1 \leq \beta_1 < \beta_2 < \dots < \beta_k \leq n}} E \begin{pmatrix} j_1 & j_2 & \dots & j_k \\ \alpha_1 & \alpha_2 & \dots & \alpha_k \end{pmatrix} A_2 \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} F \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_k \\ k_1 & k_2 & \dots & k_k \end{pmatrix}, \quad k = 1, 2, \dots, \min(m, n) \quad (1.3.6)$$

so that $d_{k,2}(s)$ is a divisor of $d_{k,1}(s)$. Also from (1.3.5), we obtain

$$A_2(s) = E^{-1}(s)A_1(s)F^{-1}(s),$$

so again by Theorem 1.3.1, we can write the minor of $A_2(s)$ of order k as a linear combination of minors of $A_1(s)$ of the same order. Hence $d_{k,1}(s)$ is a divisor of $d_{k,2}(s)$. So that $d_{k,1}(s) = d_{k,2}(s)$. In the same way we get that the maximal

integer r_1 such that $d_{r_1,1}(s) \neq 0$, coincides with the maximal integer r_2 such that

$$d_{r_2,2}(s) \neq 0.$$

Now applying this observation for $A(s)$ and $D(s)$. From the structure of

$D(s)$ its clear that $i_1(s)i_2(s)\dots i_m(s), m = 1, 2, \dots, r$, is the greatest common divisor of the

minors of $D(s)$ of order m , so $d_m(s) = i_1(s)i_2(s)\dots i_m(s), m = 1, 2, \dots, r$, and the proof is complete ■

Definition 1.3.2: The polynomials $i_1(s), i_2(s), \dots, i_r(s)$ that appears in Theorem 1.3.2 (which

is the diagonal elements of the canonical form of $A(s)$) are called the invariant

polynomials (factors) of $A(s)$, and the canonical form (1.3.3), is known as the Smith

canonical form of $A(s)$.

Theorem 1.3.3 [9]: Let the matrix polynomials $A(s)$ and $B(s)$ of rank r be equivalent.

Then the invariant polynomials of $A(s)$ and $B(s)$ coincide.

Proof: Let $A(s) \sim B(s)$, then there exists a unimodular matrices $P(s)$ and $Q(s)$ such that

$B(s) = P(s)A(s)Q(s)$, express the minor $b(s)$ of order j of $B(s)$ in terms of minors $a_m(s)$

of $A(s)$ of the same order as follows:

$$b(s) = \sum_m p_m(s) a_m(s) q_m(s) \quad (1.3.7)$$

So that any common divisor of minors $a_m(s)$ of $A(s)$ of order $j, (1 \leq j \leq r)$ is a divisor of

$b(s)$, hence the j th invariant polynomial of $B(s)$ $k_j(s)$ is divisible by the j th invariant

polynomial of $A(s)$ $d_j(s)$. But again, the equation $A(s) = [P(s)]^{-1} B(s) [Q(s)]^{-1}$ implies that

$d_j(s)$ is divisible by $k_j(s)$. Where $d_j(s), k_j(s)$ are the j th invariant polynomials of

$A(s)$ and $B(s)$, respectively, and since both polynomials are assumed to be monic, we

obtain $d_j(s) = k_j(s)$ ■

Example 1.3.1: Consider the matrix polynomial:

$$A(s) = \begin{bmatrix} s & s^2 & 0 \\ s^3 & s^5 & 0 \\ 0 & 0 & 2s \end{bmatrix}. \text{ By row and column operations we obtain}$$

$$\begin{bmatrix} s & s^2 & 0 \\ s^3 & s^5 & 0 \\ 0 & 0 & 2s \end{bmatrix} \xrightarrow{R_2 - s^2 R_1 \text{ and } C_2 - s C_1} \begin{bmatrix} s & 0 & 0 \\ 0 & s^5 - s^4 & 0 \\ 0 & 0 & 2s \end{bmatrix} \xrightarrow{C_2 \leftrightarrow C_3 \text{ and } R_2 \leftrightarrow R_3 \text{ and } \frac{1}{2} R_2} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^5 - s^4 \end{bmatrix}$$

So the equivalent Smith canonical form is

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^5 - s^4 \end{bmatrix}, \text{ and so the invariant polynomials of } A(s) \text{ are } i_1 = s, i_2 = s, i_3 = s^5 - s^4.$$

1.4 Elementary divisors

In this section we introduce the definition of elementary divisors and some related results.

Let $i_1(s), i_2(s), \dots, i_n(s)$ be the invariant polynomials to a matrix polynomial $A(s)$, and if we decompose these invariant polynomials into irreducible factors over F such that:

$$\left. \begin{aligned} i_1(s) &= [\phi_1(s)]^{c_1} [\phi_2(s)]^{c_2} \dots [\phi_m(s)]^{c_m} \\ i_2(s) &= [\phi_1(s)]^{d_1} [\phi_2(s)]^{d_2} \dots [\phi_m(s)]^{d_m} \\ &\vdots \\ i_r(s) &= [\phi_1(s)]^{l_1} [\phi_2(s)]^{l_2} \dots [\phi_m(s)]^{l_m} \end{aligned} \right\}, c_k \geq d_k \geq l_k \geq 0, k = 1, 2, \dots, m \quad (1.4.1)$$

Where $\phi_1(s), \phi_2(s), \dots, \phi_m(s)$ are all distinct factors irreducible over F , then we have the following definition.

Definition 1.4.1: Let $A(s)$ be an n -square matrix polynomial over $F[s]$, and suppose

that the invariant polynomials are given by (1.4.1), then the factors $[\phi_1(s)]^{c_1}, \dots, [\phi_m(s)]^{l_m}$ in

(1.4.1) such that $\phi_n(s) \neq 1, 1 \leq n \leq m$, are called the elementary divisors over $F[s]$ of

$A(s)$.

Example 1.4.1: Consider

$$A(s) = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & s^2 - 2s + 1 & 0 \\ 0 & 0 & s + 1 \end{bmatrix}.$$

Then the Smith canonical form of $A(s)$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^2(s-1)^2(s+1) \end{bmatrix}$$

And hence the invariant polynomials of $A(s)$ are

$$i_1(s) = 1, i_2(s) = 1, i_3(s) = s^2(s-1)^2(s+1);$$

So the elementary divisors of $A(s)$ are

$$s^2, (s-1)^2, s+1.$$

From the preceding definition we see clearly the important relation between the elementary divisors and the invariant polynomials of any matrix polynomial. In fact, the invariant polynomials of $A(s)$ determine its rank, and its elementary divisors. And the *rank* and its elementary divisors determine its invariant polynomials.

Example 1.4.2: Suppose $A(s)$ be a matrix polynomial of order 6 with rank 5, has the elementary divisors, $s^3, s^2, s, (s-1)^2, (s-1)^2, s-1, (s+1)^2, s+1$.

Then, since $A(s)$ has an order 6, so there exists 6 invariant polynomials to $A(s)$,

also, $\text{rank}A(s) = 5$, so that $i_6(s) = 0$, and we form $i_5(s)$ from the lowest common multiple

of the elementary divisors. *i.e.* $i_5(s) = s^3(s-1)^2(s+1)^2$, and to form $i_4(s)$, we remove the

elementary divisors used in $i_5(s)$ from the original list and form the lowest common

multiple of those remaining, *i.e.* $i_4(s) = s^2(s-1)^2(s+1)$, and continue with the same

procedure as before, to get, $i_3(s) = s(s-1)$, now we note that the elementary divisors are

exhausted so that $i_2(s) = i_1(s) = 1$. And hence the Smith canonical form of $A(s)$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s(s-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2(s-1)^2(s+1) & 0 & 0 \\ 0 & 0 & 0 & 0 & s^3(s-1)^2(s+1)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following theorem gives a method for computing the elementary divisors of a block-diagonal matrix polynomial.

Theorem 1.4.1 [9]: Let $A(s), B(s)$, be matrix polynomials such that $C(s) = \text{diag}[A(s), B(s)]$ is block-diagonal matrix polynomial. Then the set of elementary divisors of $C(s)$ is the union of the elementary divisors of $A(s)$ and $B(s)$.

Proof: Let $D_1(s)$ and $D_2(s)$ be the Smith canonical forms of $A(s)$ and $B(s)$,

respectively. Then $C(s) = E(s) \begin{bmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{bmatrix} F(s)$, where $E(s)$ and $F(s)$ are a

unimodular matrix polynomials. Let $(s-s_0)^{\alpha_1}, \dots, (s-s_0)^{\alpha_p}$ and $(s-s_0)^{\beta_1}, \dots, (s-s_0)^{\beta_q}$ be the elementary divisors of $D_1(s)$ and $D_2(s)$, respectively corresponding to s_0 . Arrange the set of exponents $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$, in a nondecreasing order:

$\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q\} = \{\gamma_1, \dots, \gamma_{p+q}\}$, where $0 < \gamma_1 \leq \dots \leq \gamma_{p+q}$, so that in the Smith form

$D = \text{diag}[d_1(s), \dots, d_r(s), 0, \dots, 0]$ of $\text{diag}[D_1(s), D_2(s)]$, there is r such that the

invariant polynomial $d_r(s)$ is divisible by $(s-s_0)^{\gamma_{p+q}}$ but not by $(s-s_0)^{\gamma_{p+q}+1}$, $d_{r-1}(s)$ is

divisible by $(s-s_0)^{\gamma_{p+q-1}}$ but not by $(s-s_0)^{\gamma_{p+q-1}+1}$, and so on. It follows that the elementary

divisors of $\begin{bmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{bmatrix}$ corresponding to s_0 , are just $(s-s_0)^{\gamma_1}, \dots, (s-s_0)^{\gamma_{p+q}}$ and the

proof is complete ■

Example 1.4.3: Consider

$$A(s) = \begin{bmatrix} 1 & s-1 & s+2 \\ s & s^2 & s^2+2s \\ s-2s & s^2-3s+2 & s^2+s-3 \end{bmatrix} \quad \text{and} \quad B(s) = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & s^2-2s+1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}$$

Then the smith canonical form of $A(s)$ and $B(s)$ are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s(s+1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^2(s-1)^2(s+1) \end{bmatrix}, \text{ respectively.}$$

So that the elementary divisors of $A(s)$ are $s, s+1$ and the elementary divisors of $B(s)$ are

$$s^2, (s-1)^2, s+1.$$

Now if $C(s) = \text{diag}[A(s), B(s)]$, then the elementary divisors of $C(s)$ according to

Theorem 1.4.1 are given by:

$$s^2, s, (s-1)^2, s+1.$$

Chapter Two

Standard triples and pairs of matrix polynomials

2.0 Introduction

In this chapter, we introduce the definition, some results related to the concept of the companion matrices of a matrix polynomial and linearization, and the definition of standard triples and pairs.

In section 2.1, we give the definition of linearization of a monic matrix polynomial, and its companion matrix. In section 2.2, we will generalize the definition of companion matrix of a monic case to a matrix polynomial with invertible leading coefficient. Section 2.3 gives the definition of standard triples. Section 2.4 introduces the definition of standard pairs of a matrix polynomial and some related results. And section 2.5 introduces the definition of Jordan triples, pairs and chains of a matrix polynomial.

2.1 Linearization of monic matrix polynomial

This section introduces the definition and some basic results about linearization of monic matrix polynomial.

The word "Linearization" to a matrix polynomial, in fact, comes from the linearization of differential equations. Consider the following system of differential equation with constant coefficients

$$\frac{d^l x}{dt^l} + \sum_{j=0}^{l-1} A_j \frac{d^j x}{dt^j} = f(t), \quad -\infty < t < \infty \quad (2.1.1)$$

Where $f(t)$ is a given n-dimensional vector function and $x = x(t)$ is the unknown n-dimensional vector function. Then we can reduce this equation to a first order differential equation by using the substitution

$$x_0 = x, x_1 = \frac{dx_0}{dt}, x_2 = \frac{dx_1}{dt}, \dots, x_{l-1} = \frac{dx_{l-2}}{dt} \quad (2.1.2)$$

and rewrite (2.1.1) in the form

$$\frac{d^l x}{dt^l} + A_0 x + A_1 \frac{dx}{dt} + \dots + A_{l-1} \frac{d^{l-1} x}{dt^{l-1}} = f(t) \quad (2.1.3)$$

Then using the above substitution and rearrange the terms in (2.1.3), we get the equivalent first order differential equation

$$\frac{dx_{l-1}}{dt} + A_{l-1} x_{l-1} + \dots + A_1 x_1 + A_0 x_0 = F(t) \quad (2.1.4)$$

where $x(t) = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{l-1} \end{bmatrix}$, and each $x_i, i = 0, 1, \dots, n$ is an n -dimensional vector (*i.e* we increased

the dimension of the unknown function, which becomes ln).

This operation of reducing the l th order differential equation (2.1.1) to a first order equation (2.1.4), is called a linearization.

Definition 2.1.1: Let $A(s) = \sum_{i=0}^{l-1} A_i s^i$, be a monic matrix polynomial of order n , then we

mean by a linearization of $A(s)$, to find a linear matrix polynomial of the form $(Is - B)$,

which is equivalent to $A(s)$.

We note that we cannot find a linearization to $A(s)$ if we consider a linearization of order

equal to the order of $A(s)$ (which is n), so we will extend the order of B in the

linearization $(Is - B)$ to $n + p$, where p is some integer greater than or equal to zero.

In this case we consider instead of $A(s)$, a matrix polynomial $\begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix}$, where I is

the $p \times p$ identity matrix, now we arrive at the following definition.

Definition 2.1.2: Let $A(s)$ be an $n \times n$ matrix polynomial of degree l with nonsingular leading coefficient. A linear matrix polynomial (or some times we call it a matrix pencil), $Is - A$ of order $(n+p)(n+p)$ is called a linearization of the monic matrix polynomial $A(s)$ if

$$Is - A \square \begin{bmatrix} A(s) & 0 \\ 0 & I_p \end{bmatrix} \quad (2.1.5)$$

Also the matrix A of order $(n+p)$ in (2.1.5) is called a linearization to $A(s)$.

Note that: the equivalence in (2.1.5), means that

$$\begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix} = E(s)(Is - A)F(s),$$

where $E(s)$ and $F(s)$ are two unimodular matrices, and so $\det \begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix} = \det A(s)$,

and $\det(E(s))\det(Is - A)\det(F(s)) = m \det(Is - A)$, where m is a nonzero constant.

$$\text{So } \det A(s) = m \det(Is - A) \quad (2.1.6)$$

Comparing both sides of (2.1.6), and letting $A(s)$ be $n \times n$ matrix polynomial of degree l , we conclude that $\det(Is - A)$ must be a polynomial of degree ln , since ; $\det(A(s))$ is of degree ln , so any linearization of $A(s)$ must be of order ln , that is (2.1.5) becomes:

$$Is - A \square \begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}, \text{ where } A \text{ is an } ln \times ln \text{ matrix.}$$

For any monic matrix polynomial $A(s)$, we can find a matrix A such that A is a linearization to $A(s)$, and the following theorem shows this result.

Theorem 2.1.1 [8]: For a monic matrix polynomial of order n , $A(s) = Is^l + \sum_{i=0}^{l-1} A_i s^i$, define

the $ln \times ln$ matrix

$$C_1 = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \cdots & -A_{l-1} & \end{bmatrix} \quad (2.1.7)$$

Then $Is - C_1 \square \begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}$.

Proof: Want to show that, there exists two unimodular matrix polynomials say $E(s)$

and $M(s)$, such that $\begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} = E(s)(Is - C_1)M(s)$.

For this purpose, define the two matrix polynomials $E(s)$ and $M(s)$ by:

$$M(s) = \begin{bmatrix} I & 0 & \cdots & & 0 \\ -sI & I & & & \vdots \\ 0 & -sI & I & & \\ \vdots & \vdots & \ddots & \ddots & \\ & & & I & 0 \\ 0 & \cdots & & -sI & I \end{bmatrix}, E(s) = \left. \begin{bmatrix} B_{l-1}(s) & B_{l-2}(s) & \cdots & B_0(s) \\ -I & 0 & \cdots & 0 \\ 0 & -I & & \vdots \\ \vdots & & \ddots & \ddots \\ 0 & \cdots & & -I & 0 \end{bmatrix} \right\}$$

Where $\left. \begin{array}{l} B_0(s) = I, B_{r+1}(s) = sB_r(s) + A_{l-r-1} \\ \text{for } r = 0, 1, \dots, l-2 \text{ and } I = I_n \end{array} \right\} \quad (2.1.8)$

Now, since $M(s)$ is a block lower diagonal matrix polynomial, so $\det M(s) = \det I = 1$,

and so it has a nonzero constant determinant, so its invertible.

Also, for $E(s)$, to compute its determinant we make the following operations:

Replace each column j_{m+1} of $E(s)$ by the column j_m , for $m = 1, 2, \dots, l-1$, and replace the

first column j_1 by the last column j_l , then we get :

$$\det E(s) = (-1)^l \det \begin{bmatrix} B_0(s) & B_{l-1}(s) & B_{l-2}(s) & \cdots & B_1(s) \\ 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I \end{bmatrix}, \text{ since } B_0(s) = I,$$

so $\det E(s) = (-1)^{nl} (-1)^{nl} \det I = 1$, so that $E(s)$ also is an invertible matrix polynomial.

Now, we see that:

$$\begin{aligned} E(s)(sI - C_1) &= \begin{bmatrix} B_{l-1}(s) & B_{l-2}(s) & \cdots & B_0(s) \\ -I & 0 & \cdots & 0 \\ 0 & -I & & \vdots \\ \vdots & & \ddots & \ddots \\ 0 & \cdots & -I & 0 \end{bmatrix} \begin{bmatrix} sI & -I & 0 & \cdots & 0 \\ 0 & sI & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ & & & sI & -I \\ A_0 & A_1 & \cdots & & sI + A_{l-1} \end{bmatrix} \\ &= \begin{bmatrix} sB_{l-1}(s) + A_0 & -B_{l-1}(s) + sB_{l-2}(s) + A_1 & \cdots & & \\ sI & & I & & \cdots & 0 \\ 0 & & sI & & & \\ \vdots & & \vdots & & \ddots & \ddots & \vdots \\ 0 & & 0 & & \cdots & sI & I \end{bmatrix} \end{aligned} \quad (2.1.9)$$

And if we use (2.1.8), we get

$-B_{l-1} + sB_{l-2}(s) + A_1 = -A_1 + A_1 = 0$, and the other entries of the first row of (2.1.9) also

identically zero, so that from (2.1.9) we get

$$E(s)(sI - C_1) = \begin{bmatrix} A(s) & 0 & \cdots & & 0 \\ sI & I & & & 0 \\ 0 & sI & & & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & sI & I \end{bmatrix}, \text{ Also}$$

$$\begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix} M(s) = \begin{bmatrix} A(s) & 0 & \cdots & 0 \\ 0 & I & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & \cdots & & 0 \\ -sI & I & & & 0 \\ 0 & -sI & I & & \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & I & 0 \\ 0 & \cdots & & -sI & I \end{bmatrix}$$

$$= \begin{bmatrix} A(s) & 0 & \cdots & & 0 \\ sI & I & & & 0 \\ 0 & sI & & & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & sI & I \end{bmatrix},$$

So that, $E(s)(sI - C_1) = \begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix} M(s).$

Assume that $F(s) = M^{-1}(s),$

so $\begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix} = E(s)(Is - C_1)F(s),$

and the proof is complete ■

Definition 2.1.3: The matrix C_1 in (2.1.7) is called a first companion matrix of $A(s).$

Example 2.1.1: Consider the matrix polynomial given by

$$A(s) = \begin{bmatrix} s^2 + 2s - 2 & -3s - 1 \\ -s - 3 & s^2 + 5s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} s + \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}.$$

Then the associated companion matrix is given by

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & -2 & 3 \\ -3 & -1 & 1 & -5 \end{bmatrix},$$

and by Theorem 2.1.1 we have $sI - C_1 \square \begin{bmatrix} A(s) & 0 \\ 0 & I \end{bmatrix},$ and $sI - C_1$ is a linearization for

$A(s).$

2.2 Linearization of Matrix polynomials

In this section, we will generalize the definition of linearization of matrix polynomial and give some related theorems.

A matrix polynomial $A(s) = \sum_{i=0}^l A_i s^i$ can be transformed to a monic matrix polynomial if

and only if the leading coefficient A_l is nonsingular, since in this case we can multiply

$A(s)$ by the inverse of A_l and the resulting is a monic matrix polynomial.

Example 2.2.1: Consider:

$$A(s) = \begin{bmatrix} s^2 & s+1 \\ s-2 & 3s^2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

Then $A(s)$ is nonmonic since $A_l = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \neq I$, but A_2 is nonsingular,

and $A_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, let $B(s) = A_2^{-1} A(s)$. Then

$$B(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & 1 \end{bmatrix} = \begin{bmatrix} s^2 & s+1 \\ \frac{1}{3}s - \frac{2}{3} & s^2+1 \end{bmatrix}, \text{ which is monic matrix}$$

polynomial.

Definition 2.2.1: Let $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$ be $n \times n$ matrix polynomial then the matrix

$$C_A = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & & \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & I_n \\ -\tilde{A}_0 & -\tilde{A}_1 & \cdots & -\tilde{A}_{l-1} & \end{bmatrix} \quad (2.2.1)$$

where $\tilde{A}_j = A_l^{-1} A_j$ for $j=0,1,\dots,l-1$ (2.2.2)

is called the first companion matrix for $A(s)$.

Example 2.2.2: Let $A(s) = \begin{bmatrix} s^2+1 & 1 \\ s+3 & 2s^2+s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$

so $A_l = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow A_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, so that

$$\tilde{A}_0 = A_2^{-1} A_0 = \begin{bmatrix} 1 & 1 \\ \frac{3}{2} & 0 \end{bmatrix}$$

$$\tilde{A}_1 = A_2^{-1} A_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow C_A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ \frac{-3}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}.$$

Note that,

$$A_2^{-1} A(s) = \begin{bmatrix} s^2+1 & 1 \\ \frac{1}{2}s + \frac{3}{2} & s^2 + \frac{s}{2} \end{bmatrix}, \text{ which is monic and the companion matrix}$$

in this case is, according to Definition 2.1.3 is given by

$$C_A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ \frac{-3}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}.$$

Definition 2.2.2: The matrix

$$C_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\tilde{A}_0 \\ I & 0 & & & -\tilde{A}_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ & & & 0 & \\ 0 & 0 & \cdots & I & -\tilde{A}_{l-1} \end{bmatrix} \tag{2.2.3}$$

is called the second companion matrix of $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$ where \tilde{A}_j as defined in (2.2.2).

Theorem 2.2.1 [9]: Let $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$, then the $ln \times ln$ matrix polynomials

$$\begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} \text{ and } (sI_{ln} - C_A) \text{ are equivalent, where } C_A \text{ is defined in (2.2.1).}$$

Proof: We can use similar procedure in the proof of Theorem 2.1.1 with A_l now assumed to be nonsingular ■

Definition 2.2.3: An $n \times n$ matrix polynomial $A(s)$ is said to be similar to a matrix polynomial $B(s)$ of the same order if there exists a unimodular matrix polynomial $T(s)$ such that $A(s) = T(s)B(s)T^{-1}(s)$.

Theorem 2.2.2 [9]: Two matrix polynomials $A(s), B(s)$ are similar if and only if the matrix polynomials $sI - A$ and $sI - B$ are equivalent.

For the proof, see [9]

Theorem 2.2.3 [9]: Any matrix A is a linearization of $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$, if and only if A is similar to the first companion matrix C_A of $A(s)$.

Proof: First, Let A be an $ln \times ln$ matrix which is similar to C_A , then there exists an invertible matrix T , such that $A = T C_A T^{-1}$, so $sI - A = T(sI - C_A)T^{-1}$, so that $sI - A$ and

$$sI - C_A \text{ are similar, and since } sI - C_A \square \begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}, \text{ so } sI - A \text{ and } \text{diag}[A(s), I_{(l-1)n}]$$

are equivalent. So A is a linearization for $A(s)$.

Conversely: suppose that A is a linearization for $A(s)$, then

$$sI - A \sim \begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}. \text{ But by Theorem 2.2.1, } sI - C_A \sim \begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}$$

And so that $sI - A \sim sI - C_A$ (by transitivity of the equivalence relation) and by Theorem 2.2.2, A and C_A are similar ■

2.3 Standard Triples

This section introduces the definition of standard triples of a matrix polynomial, which is an important concept of matrix polynomials and its applications.

Definition 2.3.1: Let $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$ be an $n \times n$ matrix polynomial, then the matrix-valued function $A(s)^{-1}$ (defined for all $s \notin \sigma(A)$), (see definition (1.2.4)), is called the resolvent of $A(s)$.

Theorem 2.3.1 [9]: For every complex $s \notin \sigma(A)$, $A(s)^{-1} = P_1(sI - C_A)^{-1} R_1$, where P_1, R_1 are the matrices of dimensions $n \times nl$ and $ln \times n$ respectively, defined by

$$P_1 = [I_n \ 0 \dots 0], \quad R_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_l^{-1} \end{bmatrix} \quad (2.3.1)$$

Proof: By Theorem 2.2.1, we have $\begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}$ and $(sI_{ln} - C_A)$, are equivalent, or

$$E(s)(sI - C_A) = \begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} M(s) \quad (2.3.2)$$

for some nonsingular matrix polynomials $E(s)$ and $M(s)$,

Then, if we multiply both sides of (2.3.2) by $\begin{bmatrix} A(s) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix}^{-1}$ from the left, and by

$\left[E(s)(sI - C_A) \right]^{-1}$ from the right we get

$$\begin{bmatrix} A(s)^{-1} & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} = M(s)(sI - C_A)^{-1} E(s)^{-1} \quad (2.3.3)$$

Then use the definition of $E(s)$ in (2.1.8), and compute $E(s)R_1$ to get

$$E(s)R_1 = \begin{bmatrix} B_{l-1}(s) & B_{l-2}(s) & \cdots & B_0(s) \\ -I & 0 & \cdots & 0 \\ 0 & -I & & \vdots \\ \vdots & & \ddots & \ddots \\ 0 & \cdots & -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ A_l^{-1} \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

since $B_0(s) = A_l$, $B_{r+1}(s) = sB_r(s) + A_{l-r-1}$, for $r = 0, 1, \dots, l-2$, $I = I_n$ and

$$B_0(s)A_l^{-1}(s) = A_l(s)A_l^{-1}(s) = I.$$

So that the first n columns of $E(s)^{-1}$ are equal to R_1 . Also from the definition of

$M(s)$ in (2.1.8) and P_1 above, we see that the first n rows of $M(s)$ are equal to P_1 , so that

if we use (2.3.3), we get $A(s)^{-1} = P_1(sI - C_A)^{-1}R_1$ ■

Definition 2.3.2: Three matrices (U, T, V) are said to be admissible for the $n \times n$ matrix

polynomial $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$ of degree l , if they are of dimensions $n \times ln$, $ln \times ln$

and $ln \times n$ respectively.

Two admissible triples for $A(s)$, say (U_1, T_1, V_1) and (U_2, T_2, V_2) are said to be similar if

there is a nonsingular matrix G such that

$$U_1 = U_2 G, T_1 = G^{-1} T_2 G, V_1 = G^{-1} V_2 \quad (2.3.4)$$

Definition 2.3.3: Any admissible triple for $A(s) = \sum_{j=0}^l A_j s^j, \det A_l \neq 0$, which is similar to

(P_1, C_A, R_1) is said to be a standard triple for $A(s)$.

Note that, the triple (P_1, C_A, R_1) is a standard triple for $A(s)$, since if we choose G in

(2.3.4) to be equal to I then $P_1 = P_1 I, C_A = I^{-1} C_A I$ and $R_1 = I^{-1} R_1$.

Also from Definition 2.3.3, we see that the second member of any standard triple, say T , for $A(s)$ is always a linearization for $A(s)$, since T is similar to C_A because

$T = G C_A G^{-1}$, and by Theorem 2.2.3 T is a linearization for $A(s)$.

Theorem 2.3.2 [9]: If (U, T, V) is a standard triple for a matrix polynomial

$$A(s) = \sum_{j=0}^l A_j s^j, \det A_l \neq 0, \text{ and } s \notin \sigma(A), \text{ then } A(s)^{-1} = U(sI - T)^{-1} V.$$

Proof: Let (U, T, V) be a standard triple for $A(s)$ then it is similar to (P_1, C_A, R_1) , and

there is a nonsingular matrix G such that

$$P_1 = UG, C_A = G^{-1} T G, R_1 = G^{-1} V$$

And so,

$$\begin{aligned} (sI - C_A)^{-1} &= (sI - G^{-1} T G)^{-1} = (sG^{-1} G - G^{-1} T G)^{-1} \\ &= (G^{-1} sG - G^{-1} T G)^{-1} = (G^{-1} (sG - T G))^{-1} = (G^{-1} (sI - T) G)^{-1} \\ &= ((sI - T) G)^{-1} G = G^{-1} (sI - T)^{-1} G. \end{aligned}$$

That is, $(sI - C_A)^{-1} = G^{-1} (sI - T)^{-1} G$.

So by Theorem 2.3.1, we obtain

$$A(s)^{-1} = U G G^{-1} (sI - T)^{-1} G G^{-1} V = U (sI - T)^{-1} V,$$

and this completes the proof ■

Lemma 2.3.1 [9]: Let (U, T, V) be a standard triple for $A(s)$ and define

$$Q = \begin{bmatrix} U \\ UT \\ \vdots \\ UT^{l-1} \end{bmatrix} \quad (2.3.5)$$

then Q is nonsingular and the first companion matrix C_A of $A(s)$ is given by

$$C_A = QTQ^{-1}.$$

Proof: From the definition of P_1 and C_A in (2.3.1) and (2.2.1), respectively, we have

$$\begin{bmatrix} P_1 \\ P_1 C_A \\ \vdots \\ P_1 C_A^{l-1} \end{bmatrix} = I_n \quad (2.3.6)$$

Now, since (U, T, V) is a standard triple for $A(s)$, so the relation (2.3.4), is satisfied and

we can substitute $P_1 = UG$, $C_A = G^{-1}TG$, $R_1 = G^{-1}V$ in (2.3.6), to get

$$\begin{bmatrix} P_1 \\ P_1 C_A \\ \vdots \\ P_1 C_A^{l-1} \end{bmatrix} = \begin{bmatrix} UG \\ UGG^{-1}TG \\ \vdots \\ UG(G^{-1}TG)^{l-1} \end{bmatrix} = \begin{bmatrix} UG \\ UTG \\ \vdots \\ UT^{l-1}G \end{bmatrix} = QG = I_n.$$

So that Q is nonsingular and $Q^{-1} = G$, hence $C_A = G^{-1}TG = QTQ^{-1}$ ■

Clearly, from the definition of standard triple, every standard triple for $A(s)$ is admissible,

but the converse is not necessarily true and the following example explain this fact.

Example 2.3.1: Consider

$A(s) = \begin{bmatrix} s^2 + 3 & s - 2 \\ 5 & s^2 - 5 \end{bmatrix}$, then if we choose the triple: (U, T, V) such that

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}, V = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \\ -3 & 2 \end{bmatrix}.$$

Then this triple cannot be a standard for $A(s)$ since there is no invertible matrix G satisfies the equality $P_1 = UG$. But if (U_1, T_1, V_1) is a standard triple clearly it must satisfy the definition of the admissible triple.

Example 2.3.2: Let

$$A(s) = \begin{bmatrix} s^2 & -s \\ 0 & s^2 \end{bmatrix}, \text{ then } C_A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the triple (P_1, C_A, R_1) is a standard triple for $A(s)$

$$\text{Now, } Q = \begin{bmatrix} P_1 \\ P_1 C_A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{\ln} \Rightarrow G = I \text{ and } C_A = Q C_A Q^{-1}.$$

The following theorem gives necessary and sufficient conditions when the admissible triple forms a standard triple for $A(s)$.

Theorem 2.3.3 [9]: An admissible triple (U, T, V) for a matrix polynomial $A(s)$ is standard if and only if the following conditions are satisfied:

- 1) The matrix Q is nonsingular.

$$2) A_l UT^l + A_{l-1} UT^{l-1} + \dots + A_1 UT + A_0 U = 0$$

$$3) V = Q^{-1} R_1$$

Where R_1 and Q are given by (2.3.1) and (2.3.5), respectively.

For the proof, see [9].

2.4 Standard pairs

In the present section, we give the definition of standard pairs. From Theorem 2.3.3, we see that a standard triple is completely determined by its first two members, and this leads to the following definition.

Definition 2.4.1: A standard pair for a matrix polynomial $A(s)$ is the first two members of any standard triple for $A(s)$.

By Theorem 2.3.3, we conclude that (U, T) is a standard pair of $A(s)$ if and only if conditions (1) and (2) of the Theorem are satisfied.

Theorem 2.4.1 [9]: If (U, T) is a standard pair for $A(s)$, then (U, T) is also a standard pair for any matrix polynomial of the form $BA(s)$, where $\det B \neq 0$.

Proof: Let (U, T) be a standard pair for $A(s)$, then there exists a matrix V such that $V = Q^{-1} R_1$, where R_1 and Q are given by (2.3.1) and (2.3.5) respectively, such that (U, T, V) is a standard triple for $A(s)$.

To show that the triple (U, T, V) is a standard triple for the matrix polynomial $BA(s)$, we will check the conditions (1) and (2) of Theorem 2.3.3.

For condition (1): Let

$$A(s) = \sum_{j=0}^l s^j A_j \Rightarrow BA(s) = B \sum_{j=0}^l s^j A_j \tag{2.4.1}$$

and Q_1 and Q_2 be given by (2.3.5) for $A(s)$ and $BA(s)$, respectively, then from the equality (2.4.1), we get:

$Q_2 = BQ_1$, but (U, T, V) is assumed to be a standard triple for $A(s)$, so by Theorem 2.3.3, Q_1 is nonsingular, also by the assumption B is nonsingular, so that Q_2 is nonsingular and condition (1) of Theorem 2.3.3 is satisfied.

Now, consider the equation in condition 2 of Theorem 2.3.3 that is:

$$A_l UT^l + A_{l-1} UT^{l-1} + \dots + A_1 UT + A_0 U = 0 \quad (2.4.2)$$

is satisfied for $A(s) = \sum_{j=0}^l s^j A_j$

So for the matrix polynomial $BA(s) = B \sum_{j=0}^l s^j BA_j$, want

$$BA_l UT^l + BA_{l-1} UT^{l-1} + \dots + BA_1 UT + BA_0 U = 0$$

But

$$\begin{aligned} BA_l UT^l + BA_{l-1} UT^{l-1} + \dots + BA_1 UT + BA_0 U &= \\ B(A_l UT^l + A_{l-1} UT^{l-1} + \dots + A_1 UT + A_0 U) &= 0 \end{aligned} \quad (2.4.3)$$

By Theorem 2.3.3 condition (2), since (U, T, V) is a standard triple for $A(s)$.

Hence, condition (2) of Theorem 2.3.3 is satisfied and so the pair (U, T) is a standard pair for the matrix polynomial $BA(s)$ ■

Theorem 2.4.2 [9]: Let (U, T, V) be an admissible triple for $A(s)$ and assume that

$$A(s)^{-1} = U(sI - T)^{-1}V \quad (2.4.4)$$

Then (U, T, V) is a standard triple for $A(s)$.

For the proof, see [9].

Corollary 2.4.1 [9]: If (U, T, V) is a standard triple for $A(s)$ then (V^T, T^T, U^T)

is a standard triple for $A^T(s) \square \sum_{j=0}^l s^j A_j^T$, and (V^*, T^*, U^*) is a standard triple for

$$A^*(s) \square \sum_{j=0}^l s^j A_j^*.$$

Proof: Let (U, T, V) be a standard triple for $A(s)$, then by Theorem 2.3.2,

$$A(s)^{-1} = U(sI - T)^{-1}V \quad (2.4.5)$$

Take the transpose for both sides of (2.4.5) $\Rightarrow (A(s)^{-1})^T = (U(sI - T)^{-1}V)^T$

$$\Rightarrow [(A(s)^T)^{-1}]^{-1} = V^T (U(sI - T)^{-1})^T = V^T [(sI - T)^T]^{-1} U^T, \text{ and so by Theorem 2.4.2 the}$$

triple (V^T, T^T, U^T) is a standard triple for $A^T(s)$.

And if we take the conjugate transpose to both sides of (2.4.5), then (V^*, T^*, U^*) is a standard triple for $A^*(s)$ ■

2.5 Jordan chains

This section contains the definition of Jordan triples, pairs and chains of matrix polynomial and gives some basic theorems.

Before we define the Jordan triples and chains we will introduce the concept of Jordan normal form for scalar matrices.

Definition 2.5.1: A Jordan block is a square matrix whose diagonal elements are equal, whose super diagonal elements (those immediately above the main diagonal) all equal 1, and whose other elements are all zero, it has the form

$$\begin{bmatrix} s & 1 & 0 & \cdots & 0 & 0 \\ 0 & s & 1 & \cdots & 0 & 0 \\ 0 & 0 & s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s & 1 \\ 0 & 0 & 0 & \cdots & 0 & s \end{bmatrix}$$

A matrix B is in Jordan normal form if it is a diagonal matrix or if it has one of the following two partitioned forms

$$\begin{bmatrix} D & & & & 0 \\ & J_1 & & & \\ & & J_2 & & \\ & & & \ddots & \\ 0 & & & & J_k \end{bmatrix}, \begin{bmatrix} J_1 & & & & 0 \\ & J_2 & & & \\ & & J_3 & & \\ & & & \ddots & \\ 0 & & & & J_k \end{bmatrix}$$

Where D denotes a diagonal matrix (whose diagonal elements need not be equal), and $J_i, i = 1, 2, \dots, k$ represents a Jordan block.

Theorem 2.5.1 [8] : If $A \in \mathbb{C}^{n \times n}$ and $I_s - A$ has t elementary divisors

$(s - s_i)^{p_i}, i = 1, 2, \dots, t$, then A is similar to the matrix $J = \text{diag}[J_1, J_2, \dots, J_t] \in \mathbb{C}^{n \times n}$, where

J_i is the $p_i \times p_i$ Jordan block corresponding to $(s - s_i)^{p_i}, i = 1, 2, \dots, t$,

$$J_i = \begin{bmatrix} s_i & 1 & 0 & \cdots & 0 \\ 0 & s_i & & & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ & & & & 1 \\ 0 & & & & s_i \end{bmatrix} \quad (2.5.1)$$

For the proof, see [8].

Example 2.5.1: Let

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

Then, the eigenvalues of A are $1, 2, 4, 4$ so that,

$$J_1 = [1], \quad J_2 = [2], \quad J_3 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix},$$

and the Jordan normal form of A is

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

By the above theorem and since any matrix which is similar to C_A , (given by 2.2.1),

is a linearization for $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$, we conclude that there is a matrix in

Jordan form that is a linearization for $A(s)$ and this leads to the following definition.

Definition 2.5.2: Let (X, J, Y) be a standard triple for $A(s) = \sum_{j=0}^l A_j s^j$, $\det A_l \neq 0$, and let

J be the Jordan normal form of C_A , then the standard triple (X, J, Y) is said to be the

Jordan triple for $A(s)$, and the pair (X, J) , is said to be the Jordan pair for $A(s)$.

Note that any Jordan triple (pair) is standard triple (pair), the matrix

$$Q = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{l-1} \end{bmatrix} \tag{2.5.2}$$

is nonsingular and $C_A = QJQ^{-1}$, (see Lemma 2.3.1).

Definition 2.5.3: Let $A^{[r]}(s)$ be the matrix polynomial obtained by differentiating

$$A(s) = \sum_{j=0}^l A_j s^j, \det A_l \neq 0, \quad r\text{-times with respect to } s. \text{ Thus when } A(s) \text{ has degree } l,$$

$A^{[r]}(s) = 0$ for $r > l$. The set of vectors x_0, x_1, \dots, x_k , with $x_0 \neq 0$, is a Jordan chain of

length $k+1$ for $A(s)$ corresponding to the latent root s_0 if the following $k+1$ relations

hold:

$$\left. \begin{aligned} A(s_0)x_0 &= 0; \\ A(s_0)x_1 + \frac{1}{1!}A^{[1]}(s_0)x_0 &= 0; \\ \vdots & \\ A(s_0)x_k + \frac{1}{1!}A^{[1]}(s_0)x_{k-1} + \dots + \frac{1}{k!}A^{[k]}(s_0)x_0 &= 0; \end{aligned} \right\} \quad (2.5.3)$$

Lemma 2.5.1 [3]: The vectors x_0, x_1, \dots, x_k from \mathbb{C}^n , with $x_0 \neq 0$, form a Jordan chain for

$A(s)$ corresponding to the latent root s_0 if and only if

$$A_l X J^l + A_{l-1} X J^{l-1} + \dots + A_1 X J + A_0 X = 0$$

Where $X = [x_0 \ x_1 \ \dots \ x_k]$, and J is the Jordan block of order k with s_0 on the main diagonal.

For the proof, see [3].

Example 2.5.2: Consider the 2×2 matrix polynomial of degree 2

$$A(s) = \begin{bmatrix} s^2 & 0 \\ s+1 & s(s-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $\det A(s) = s^3(s-1)$, $\Rightarrow s_1 = 0, s_2 = 1$, so that $A(s)$ has two distinct latent roots.

To compute the latent vectors corresponding to $s_1 = 0$:

$$A(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ so } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0, x_2 \text{ arbitrary,}$$

so $x_0 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$ ($\alpha \neq 0$) is a latent vector corresponding to s_1 .

And for $s_2 = 1$, $A(1) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, so

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0, 2x_1 = 0 \Rightarrow x_2 \text{ arbitrary,}$$

so that the vector of the form $\begin{bmatrix} 0 \\ \alpha \end{bmatrix}$ ($\alpha \neq 0$), is a latent vector for $A(s)$ corresponding to

either of the latent roots .

To find the Jordan chains for $A(s)$. First, for $s_1 = 0$, by (2.5.3),

$$A(s_0)x_1 + A^{[1]}(s_0)x_0 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ x_1^{(1)} \end{bmatrix} + \begin{bmatrix} 0 \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ x_1^{(1)} - \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1^{(1)} = \alpha ,$$

and x_2 arbitrary.

$$\Rightarrow x_1 = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} \end{bmatrix}^T = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T, \beta \in C (\alpha \neq 0).$$

And to find x_2 we use x_1 :

$$A(s_0)x_2 + A^{[1]}(s_0)x_1 + \frac{1}{2}A^{[2]}(s_0)x_0 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2^{(1)} \\ x_2^{(2)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ x_2^{(1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha - \beta \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ x_2^{(1)} + 2\alpha - \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2^{(1)} = \beta - 2\alpha, x_2^{(2)} \text{ arbitrary, } \Rightarrow x_2 = \begin{bmatrix} \beta - 2\alpha \\ \gamma \end{bmatrix}.$$

Now choose $\alpha = 1, \beta = \gamma = 0$, we get $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, are the latent vectors

corresponding to $s_1 = 0$, so that $X = \begin{bmatrix} 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

For $s_2 = 1$, since $x_0 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$, then the equation for x_1 (see 2.5.3),

$$A(1)x_1 + A^{[1]}(1)x_0 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}x_1 + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1^{(1)} \\ 2x_1^{(1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since $\alpha \neq 0$, so there is no solution, so we have only one member, and this chain has length 1.

So the Jordan chain of $A(s)$ is (X, J) ,

$$\text{where } X = \begin{bmatrix} 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \text{ and } J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find V (the third member of Jordan triple), we use Theorem 2.3.3 that is $V = Q^{-1}R_1$,

$$\text{where } Q = \begin{bmatrix} 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \Rightarrow Q^{-1} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_1^{-1} \end{bmatrix}, A_1^{-1} = A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Rightarrow R_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ so that}$$

$$V = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & -1 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Chapter Three

Division and solvents of matrix polynomials

3.0 Introduction

This chapter contains the definition of division process for matrix polynomials, the definition of solvents and some recent results about solvents.

Section 3.1 introduces the division process for matrix polynomial. Section 3.2, gives the meaning of solvent of a matrix polynomial. And section 3.3 gives some recent results about the number of solvents of matrix polynomial.

3.1 Division of matrix polynomials

in this section we shall describe the division process of matrix polynomials and some results related. Because of the noncommutativity, we have to distinguish between right division and left division and the following definition explains this remark.

Definition 3.1.1: Suppose $B(s)$ is a matrix polynomial of degree m , and that there exist matrix polynomials $Q(s)$, $R(s)$, with $R(s) \equiv 0$ or the degree of $R(s)$ less than m , such that $A(s) = Q(s)B(s) + R(s)$, then we call $Q(s)$ is a right quotient of $A(s)$ on division by $B(s)$, and $R(s)$ is a right remainder of $A(s)$ on division by $B(s)$.

Similarly we can define a left quotient and left remainder of $A(s)$.

We call $Q(s)$ a right divisor of $A(s)$ if $R(s) \equiv 0$ is a right remainder on division by $B(s)$, and similarly for the definition of left divisor for $A(s)$.

The following Theorem gives the existence of left (right) quotients and remainders of two matrix polynomials.

Theorem 3.1.1 [9]: If $A(s) = \sum_{i=0}^l s^i A_i, B(s) = \sum_{i=0}^m s^i B_i$ are two $n \times n$ matrix polynomials of degree l and m respectively, with $\det B_m \neq 0$, then there exists a left (right) quotient and left (right) remainder of $A(s)$ on division by $B(s)$.

Proof: First, we note that if the degree of $A(s)$ is less than the degree of $B(s)$, then

$Q(s) \equiv 0$ and then $A(s) = R(s)$ and the Theorem is proved.

So, let $l \geq m$, and divide by the leading term of $B(s)$, which is $B_m s^m$ and note that the

term of highest degree of $B(s)A_l B_m^{-1} s^{l-1}$ (3.1.1)

is just $A_l s^l$. Hence

$$A(s) = B(s)A_l B_m^{-1} s^{l-m} + A^{(1)}(s) \quad (3.1.2)$$

Where $A^{(1)}(s)$ is a matrix polynomial whose degree l_1 does not exceed $l-1$, write

$A^{(1)}(s)$ in decreasing powers, that is $A^{(1)}(s) = A_{l_1}^{(1)} s^{l_1} + \dots + A_0^{(1)}$, $\det A_{l_1}^{(1)} \neq 0$.

Now, if $l_1 \geq m$, we repeat the process for $A^{(1)}(s)$ instead of $A(s)$ to obtain

$$A^{(1)}(s) = B(s)A_{l_1}^{(1)} B_m^{-1} s^{l_1-m} + A^{(2)}(s) \quad (3.1.3)$$

Where $A^{(2)}(s) = A_{l_2}^{(2)} s^{l_2} + \dots + A_0^{(2)}$, $\det A_{l_2}^{(2)} \neq 0$, $l_2 < l_1$, and continue in this process we get a

sequence of matrix polynomials, $A^{(1)}(s), A^{(2)}(s), \dots$ whose degrees are strictly decreasing,

after a finite number of terms we get $A^{(r)}(s)$ of degree $l_r < m$ with $l_{r-1} \geq m$.

Write $A(s) = A^{(0)}(s)$, then $A^{(k-1)}(s) = B(s)A_{l_{k-1}}^{(k-1)} B_m^{-1} s^{l_{k-1}-m} + A^{(k)}(s), k = 1, 2, \dots, r$.

So that

$$A(s) = B(s) \left(A_l B_m^{-1} s^{l-m} + A_l^{(1)} B_m^{-1} s^{l_1-m} + \dots + A_{l_{r-1}}^{(r-1)} B_m^{-1} s^{l_{r-1}-m} \right) + A^{(r)}(s).$$

And then

$$A_l B_m^{-1} s^{l-m} + A_l^{(1)} B_m^{-1} s^{l_1-m} + \dots + A_{l_{r-1}}^{(r-1)} B_m^{-1} s^{l_{r-1}-m}$$

is the left quotient and $A^{(r)}(s)$ is the left remainder.

Similarly we can show the existence of right quotient and right remainder ■

Example 3.1.1: Consider the matrix polynomials

$$A(s) = \begin{bmatrix} s^3 + s^2 + s & s^3 + s^2 + 2 \\ 2s^3 - s & s^2 - 3s \end{bmatrix}, \quad B(s) = \begin{bmatrix} s^2 + 3 & 2 \\ s & s^2 + s \end{bmatrix}$$

Use the procedure described in the proof of Theorem 3.1.1.

So, write $A(s)$, and $B(s)$ in the forms

$$A(s) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} s + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \text{ and}$$

$$B(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} s + \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix},$$

$$\text{so that } A(s) = B(s) \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} s + A^{(1)}(s)$$

$$\text{or } A^{(1)}(s) = A(s) - B(s) \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} s$$

$$\Rightarrow A^{(1)}(s) = \begin{bmatrix} s^3 + s^2 + s & s^3 + s^2 + 2 \\ 2s^3 - s & s^2 - 3s \end{bmatrix} - \begin{bmatrix} s^3 + 7s & s^3 + 3s \\ 2s^3 + 3s^2 & s^2 \end{bmatrix} = \begin{bmatrix} s^2 - 6s & s^2 - 3s + 2 \\ -3s^2 - s & 3s \end{bmatrix}.$$

Now, since $A^{(1)}(s)$ has degree equal to the degree of $B(s)$, we repeat the above process

for $A^{(1)}(s)$. So that

$$A^{(1)}(s) = \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -6 & -3 \\ -1 & 3 \end{bmatrix} s + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^{(1)}(s) = B(s) \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix} + A^{(2)}(s) = \begin{bmatrix} s^2+3 & 2 \\ s & s^2+s \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix} + A^{(2)}(s)$$

$$\Rightarrow A^{(2)}(s) = \begin{bmatrix} s^2-6s & s^2-3s+2 \\ -3s^2-s & 3s \end{bmatrix} - \begin{bmatrix} s^2-3 & s^2+3 \\ -3s^2-2s & s \end{bmatrix} = \begin{bmatrix} -6s+3 & -3s-1 \\ s & 2s \end{bmatrix}.$$

And hence the degree of $A^{(2)}(s)$ is less than the degree of $B(s)$, so we stop and we have

$$A(s) = B(s) \left[\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix} \right] + \begin{bmatrix} -6s+3 & -3s-1 \\ 2 & 2s \end{bmatrix}$$

$$= \begin{bmatrix} s^2+3 & 2 \\ s & s^2+s \end{bmatrix} \begin{bmatrix} s+1 & s+1 \\ 2s-3 & 0 \end{bmatrix} + \begin{bmatrix} -6s+3 & -3s-1 \\ s & 2s \end{bmatrix}$$

So that

$$Q(s) = \begin{bmatrix} s+1 & s+1 \\ 2s-3 & 0 \end{bmatrix}, \text{ is the left quotient of } A(s) \text{ on division by } B(s), \text{ also}$$

$$, R(s) = \begin{bmatrix} -6s+3 & -3s-1 \\ s & 2s \end{bmatrix}, \text{ is the left remainder.}$$

The quotient and remainder of a matrix polynomial $A(s)$ on division by $B(s)$ are unique, that is $R(s)$ and $Q(s)$ which we found in Example 3.1.1 are unique and the following theorem shows this result.

Theorem 3.1.2 [9]: With the hypotheses of Theorem 3.1.1, the right quotient, right remainder, left quotient, and left remainder are each unique.

Proof: Let $Q(s), R(s), Q_1(s)$ and $R_1(s)$ are matrix polynomials such that

$$A(s) = Q(s)B(s) + R(s) \tag{3.1.4}$$

$$\text{And } A(s) = Q_1(s)B(s) + R_1(s) \tag{3.1.5}$$

Where $\deg R(s)$ and $\deg R_1(s)$ are less than the degree of $B(s)$. If we subtract (3.1.5)

from (3.1.4), we get

$$0 \equiv Q(s)B(s) + R(s) - Q_1(s)B(s) - R_1(s)$$

$$\Rightarrow (Q(s) - Q_1(s))B(s) = R_1(s) - R(s).$$

Now, if $Q(s) \neq Q_1(s)$, then the degree of $(Q(s) - Q_1(s))B(s)$ must be at least m , but the degree of $R_1(s) - R(s)$ is less than m which is a contradiction, so that

$Q_1(s) = Q(s)$, and hence $R_1(s) = R(s)$. And similarly for left quotient and left

remainder ■

We know, in scalar polynomials, that the remainder of the polynomial $p(s)$ on division by the polynomial $q(s) = s - c$, is $P(c)$, $c \in \mathbb{F}$, and the following theorem generalize this result for matrix polynomials.

Theorem 3.1.3 [9]: The right and left remainders of a matrix polynomial $A(s)$ on division by $sI - B$ are $A(B)$ and $\tilde{A}(B)$, respectively, where

$$A(s) = A_l s^l + A_{l-1} s^{l-1} + \dots + A_0, \text{ and } \tilde{A}(s) = s^l A_l + s^{l-1} A_{l-1} + \dots + A_0.$$

Proof: We will prove the Theorem for left remainder and similarly for the right case.

Note that, the product:

$$(sI - B)(s^{j-1}I + s^{j-2}B + \dots + sB^{j-2} + B^{j-1}) = s^j I - B^j \quad (3.1.6)$$

And multiply (3.1.6) by A_j from the right, we get

$$s^j A_j - B^j A_j = (sI - B)(s^{j-1}I + s^{j-2}B + \dots + sB^{j-2} + B^{j-1})A_j \quad (3.1.7)$$

and sum the equations in (3.1.7) for $j = 1, 2, \dots, l$. Then the right hand side is of the form

$(sI - B)C(s)$, where $C(s)$ is a matrix polynomial, and the left hand side is

$$\sum_{j=1}^l s^j A_j - \sum_{j=1}^l B^j A_j = \sum_{j=0}^l s^j A_j - \sum_{j=0}^l B^j A_j = \tilde{A}(s) - \tilde{A}(B). \text{ Thus,}$$

$\tilde{A}(s) = (sI - B)C(s) + \tilde{A}(B)$ and by Theorem 3.1.2, $\tilde{A}(B)$ is the left remainder of $A(s)$ on division by $sI - B$ ■

Example 3.1.2: Let

$$A(s) = \begin{bmatrix} s^2 & s+1 \\ s-2 & s^2+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix},$$

$$\tilde{A}(s) = s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

If we divide $A(s)$ and $\tilde{A}(s)$ by $sI - B$, where

$$sI - B = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}.$$

Then

$$A(s) = \begin{bmatrix} s+1 & 3 \\ 4 & s+4 \end{bmatrix} \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix} + \begin{bmatrix} 10 & 15 \\ 14 & 26 \end{bmatrix}, \text{ and}$$

$$\tilde{A}(s) = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix} \begin{bmatrix} s+1 & 3 \\ 4 & s+4 \end{bmatrix} + \begin{bmatrix} 9 & 12 \\ 17 & 27 \end{bmatrix}$$

i.e. the right remainder of $A(s)$ on division by $Is - B$ is $\begin{bmatrix} 10 & 15 \\ 14 & 26 \end{bmatrix}$

and the left remainder is $\begin{bmatrix} 9 & 12 \\ 17 & 27 \end{bmatrix}$.

And we see that

$$A(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 14 & 26 \end{bmatrix},$$

and

$$\tilde{A}(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 17 & 27 \end{bmatrix}.$$

3.2 Solvents of matrix polynomial

In this section we consider a monic matrix polynomial $A(s) = Is^m + A_1s^{m-1} + \dots + A_m$

where $A_i \in \mathbb{C}^{n \times n}, i = 1, 2, \dots, m$ are $n \times n$ complex matrices. The polynomial equation

$A(s) = 0$ is sometimes called a unilateral matrix equation.

We will use small s to mean that $A(s)$ is a matrix polynomial with $s \in \mathbb{C}$, and capital s to

mean that $A(S)$ is a matrix polynomial with $S \in \mathbb{C}^{n \times n}$.

Definition 3.2.1: A matrix $S_1 \in \mathbb{C}^{n \times n}$ is called a right solvent for a matrix polynomial

$$A(S) = S^m + A_1S^{m-1} + \dots + A_m, \text{ if } A(S_1) = 0.$$

Definition 3.2.2: Given a matrix polynomial $A(s)$ and the matrix V of dimension $n \times k$,

$V = [v_1, \dots, v_k]$ where v_1, \dots, v_k are a Jordan chain of $A(s)$ corresponding to the latent root

α , we say that the pair (V, J) (where J is a Jordan block of size k , with the latent

root α in the main diagonal), is an eigenpair of $A(s)$.

Definition 3.2.3: Let $(V_1, J_1), (V_2, J_2), \dots, (V_l, J_l)$ be l eigenpairs of $A(s)$. If

$diag(J_1, \dots, J_l) = J_C$, where $diag(J_1, \dots, J_l)$ is a diagonal matrix of order mn , with

J_1, \dots, J_l in the main diagonal, and J_C is the Jordan normal form of the block companion

matrix C_A of $A(s)$, we say that $(V_1, J_1), (V_2, J_2), \dots, (V_l, J_l)$ is a complete system of

eigenpairs of $A(s)$.

Theorem 3.2.1 [10]: Let $A(s)$ be a matrix polynomial and let C_A be the associated companion matrix of $A(s)$. Let P be the similarity matrix of C_A , that is P is such that $C_A = PJ_C P^{-1}$.

$$\text{Then } P \text{ has the form } P = \begin{bmatrix} V_1 & V_2 & \cdots & V_l \\ V_1 J_1 & V_2 J_2 & \cdots & V_l J_l \\ \vdots & \vdots & \vdots & \vdots \\ V_1 J_1^{m-1} & V_2 J_2^{m-1} & \cdots & V_l J_l^{m-1} \end{bmatrix}, \quad (3.2.1)$$

where $(V_1, J_1), (V_2, J_2), \dots, (V_l, J_l)$ is a complete system of eigenpairs of $A(s)$, that is $\text{diag}(J_1, \dots, J_l) = J_C$.

For the proof, see [10]

Example 3.2.1 [4]: Consider

$$A(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} -1 & -6 \\ 2 & -9 \end{bmatrix} s + \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix}. \text{ Then}$$

$$\det A(s) = (s^2 - s)(s^2 - 9s + 14) - (2s - 2)(-6s + 12)$$

$\Rightarrow s_1 = 1, s_2 = 2, s_3 = 3, s_4 = 4$. To find the eigenpairs of $A(s)$,

$$\text{for } s_1 = 1, A(1) = \begin{bmatrix} 0 & 6 \\ 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 6 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \neq 0,$$

$$\text{for } s_2 = 2, A(2) = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \beta \neq 0,$$

$$\text{for } s_3 = 3, A(3) = \begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}, \gamma \neq 0, \text{ and}$$

$$\text{for } s_4 = 4, A(4) = \begin{bmatrix} 12 & -12 \\ 6 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 & -12 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_1^{(4)} \\ x_2^{(4)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_4 = \begin{bmatrix} \tau \\ \tau \end{bmatrix}, \tau \neq 0 \text{ so that we have}$$

the following eigenpairs for $A(s)$

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1\right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 2\right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 3\right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 4\right),$$

so that the similarity matrix for C_A is

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 2 & -2 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -2 & 6 & 2 & -3 \\ 1 & -4 & -1 & 2 \end{bmatrix}, \text{ and } J_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

We see that

$$\begin{aligned} PJ_C P^{-1} &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -2 & 6 & 2 & -3 \\ 1 & -4 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -12 & 1 & 6 \\ 2 & -14 & -2 & 9 \end{bmatrix} = C_A. \end{aligned}$$

Theorem 3.2.2 [10]: The pair of matrices (V, J) is an eigenpair or (Jordan chain) of a matrix polynomial $A(s)$ if and only if $VJ^m + A_1 VJ^{m-1} + \dots + A_m V = 0$.

For the proof, see [10]

Corollary 3.2.1 [10]: Given a pair of matrices (T, J_0) , where $T = [V_1 \dots V_l]$ is a matrix of dimension $n \times p$, and $J_0 = \text{diag}(J_1, \dots, J_l)$ is of order p , then (V_i, J_i) , $i = 1, \dots, l$ are eigenpairs of $A(s)$ if and only if

$$TJ_0^m + A_1 T J_0^{m-1} + \dots + A_m T = 0 \tag{3.2.2}$$

Theorem 3.2.3 [10]: If the matrix $T = [V_1 \dots V_p]$ is a nonsingular matrix of order n and $J_0 = \text{diag}(J_1, \dots, J_l)$ is also of order n , then (V_i, J_i) , $i = 1, 2, \dots, l$ are eigenpairs of a matrix polynomial $A(s)$ if and only if $S_1 = TJ_0 T^{-1}$, is a solvent of $A(S)$.

Proof: Let (V_i, J_i) , $i = 1, 2, \dots, l$ are eigenpairs of $A(s)$, then by Corollary 3.2.1, we have

$0 = TJ_0^m + A_1 TJ_0^{m-1} + \dots + A_{m-1} TJ_0^1 + A_m T$, multiply both sides of this equation by T^{-1} from the right we get

$$0 = TJ_0^m T^{-1} + A_1 TJ_0^{m-1} T^{-1} + \dots + A_{m-1} TJ_0 T^{-1} + A_m T T^{-1} \quad (3.2.3)$$

Note that, $(TJ_0 T^{-1})^m = TJ_0 T^{-1} TJ_0 T^{-1} TJ_0 T^{-1} \dots TJ_0 T^{-1} = TJ_0^m T^{-1}$, so that the equation (3.2.3)

becomes $0 = (TJ_0 T^{-1})^m + A_1 (TJ_0 T^{-1})^{m-1} + \dots + A_{m-1} (TJ_0 T^{-1}) + A_m = A(TJ_0 T^{-1}) = A(S_1)$.

Conversely, if $S_1 = TJ_0 T^{-1}$, is a solvent of $A(S)$, then reverse the above steps, we get the

equation (3.2.2), and so the pairs (V_i, J_i) , $i = 1, 2, \dots, l$ are eigenpairs of $A(s)$ ■

Definition 3.2.4: A set of solvents S_1, S_2, \dots, S_m of $A(S)$ is said to be a complete set of solvents of $A(S)$ if the corresponding Vandermonde matrix

$$V(S_1, S_2, \dots, S_m) = \begin{bmatrix} I & I & \dots & I \\ S_1 & S_2 & \dots & S_m \\ \vdots & \vdots & \vdots & \vdots \\ S_1^{m-1} & S_2^{m-1} & \dots & S_m^{m-1} \end{bmatrix} \quad (3.2.4)$$

is nonsingular.

Theorem 3.2.4 [10]: Let $A(S)$ be a matrix polynomial if S_1, S_2, \dots, S_m is a complete set of solvents of $A(S)$. Then

$$C_A = V(S_1, S_2, \dots, S_m) \text{diag}(S_1, S_2, \dots, S_m) V(S_1, S_2, \dots, S_m)^{-1}.$$

Where V is the Vandermonde matrix.

Proof: Consider

$$C_A = \begin{bmatrix} 0 & I_n & \dots & 0 \\ \vdots & & \ddots & I_n \\ -A_m & -A_{m-1} & \dots & -A_1 \end{bmatrix},$$

be a companion matrix for $A(s)$, and V as defined in (3.2.4) be the associated

Vandermonde matrix we see that

$$C_A V(S_1, S_2, \dots, S_m) = \begin{bmatrix} 0 & I_n & \dots & 0 \\ 0 & 0 & I_n & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & I_n \\ -A_m & -A_{m-1} & \dots & -A_1 & \end{bmatrix} \begin{bmatrix} I_n & \dots & I_n \\ S_1 & \dots & S_m \\ \vdots & \vdots & \vdots \\ S_1^{m-1} & \dots & S_m^{m-1} \end{bmatrix} = \begin{bmatrix} S_1 & S_2 & \dots & S_m \\ S_1^2 & S_2^2 & \dots & S_m^2 \\ \vdots & \vdots & \dots & \vdots \\ S_1^{m-1} & S_2^{m-1} & \dots & S_m^{m-1} \\ -A_m - A_{m-1}S_1 - \dots - A_1S_1^{m-1} & -A_m - A_{m-1}S_2 - \dots - A_1S_2^{m-1} & \dots & -A_m - A_{m-1}S_m - \dots - A_1S_m^{m-1} \end{bmatrix} \quad (3.2.5)$$

and, $V(S_1, S_2, \dots, S_m) \text{diag}(S_1, S_2, \dots, S_m)$

$$= \begin{bmatrix} I_n & \dots & I_n \\ S_1 & \dots & S_m \\ \vdots & & \\ S_1^{m-1} & \dots & S_m^{m-1} \end{bmatrix} \begin{bmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & S_m \end{bmatrix} = \begin{bmatrix} S_1 & S_2 & \dots & S_m \\ S_1^2 & S_2^2 & \dots & S_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ S_1^m & S_2^m & \dots & S_m^m \end{bmatrix} \quad (3.2.6)$$

Now, note that in (3.2.5)

$$-A_m - A_{m-1}S_1 - \dots - A_1S_1^{m-1} = -(A_1S_1^{m-1} + \dots + A_{m-1}S_1 + A_m) = -A(S_1) + S_1^m. \text{ Or}$$

$$A_1S_1^{m-1} + \dots + A_{m-1}S_1 + A_m = A(S_1) - S_1^m. \text{ But } S_1 \text{ is a solvent of } A(S) \text{ that is, } A(S_1) = 0,$$

$$\text{so that, } -A_m - A_{m-1}S_1 - \dots - A_1S_1^{m-1} = S_1^m,$$

also the other entries of the last row of (3.2.5), are $S_2^m, S_3^m, \dots, S_m^m$, respectively, and so

$$C_A V(S_1, S_2, \dots, S_m) = V(S_1, S_2, \dots, S_m) \text{diag}[S_1, S_2, \dots, S_m], \text{ that is}$$

$$C_A = V(S_1, S_2, \dots, S_m) \text{diag}(S_1, S_2, \dots, S_m) V(S_1, S_2, \dots, S_m)^{-1} \blacksquare$$

Example 3.2.2: Consider

$$A(S) = S^2 + \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} S + \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}. \text{ Then}$$

$$S_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \text{ are solvents of } A(S) \text{ and } V(S_1, S_2) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$\Rightarrow \det V(S_1, S_2) = -1 \neq 0, \Rightarrow V$ is nonsingular, so that S_1 and S_2 form a complete set of

$$\text{solvents for } A(S), \text{ and } V^{-1}(S_1, S_2) = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

$$\Rightarrow C_A = V(S_1, S_2) \text{diag}(S_1, S_2) V^{-1}(S_1, S_2)$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 2 & 3 \end{bmatrix} = C_A.$$

Example 3.2.3: Consider the matrix polynomial in Example 3.2.1 that is

$$A(S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^2 + \begin{bmatrix} -1 & -6 \\ 2 & -9 \end{bmatrix} S + \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix}, \text{ we see that the latent vectors of } A(s) \text{ are}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ corresponding to the latent roots,}$$

$$s_1 = 1, s_2 = 2, s_3 = 3, s_4 = 4 \text{ respectively.}$$

So that we have the following eigenpairs for $A(s)$

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1\right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 2\right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 3\right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 4\right).$$

To find the solvents of $A(S)$, we take the matrix $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is nonsingular and

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ so, by Theorem 3.2.3 the matrix}$$

$$S_1 = T_1 J_0 T_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

is a solvent of $A(S)$, clearly, we have

$$A(S_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -6 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{And, if we take } T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow T_2^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \text{ and } J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

$$\Rightarrow S_2 = T_2 J_0 T_2^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \text{ we see that}$$

$$A(S_2) = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} -1 & -20 \\ 2 & -23 \end{bmatrix} + \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Also if we choose } T_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow T_3^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \text{ and}$$

$$J_0 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow S_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix},$$

$$A(S_3) = \begin{bmatrix} 16 & 0 \\ 12 & 4 \end{bmatrix} + \begin{bmatrix} -16 & -12 \\ -10 & -18 \end{bmatrix} + \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now

$$V(S_2, S_3) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 4 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix} \Rightarrow \det V(S_2, S_3) = 1 \neq 0,$$

so the set $\{S_1, S_2\}$, is a complete set of right solvents for $A(S)$.

Theorem 3.2.5: Let $A(S) = S^m + A_1S^{m-1} + \dots + A_0$ be a matrix polynomial. Let S_1 be a right solvent of $A(S)$, then S_1^T is a left solvent for $A^T(S)$.

Proof: Let S_1 be a solvent for $A(s)$, then $S_1^m + A_1S_1^{m-1} + \dots + A_m = 0$ (3.2.7)

Take the transpose to both sides of (3.2.7), we get

$$(S_1^m + A_1S_1^{m-1} + \dots + A_m)^T = 0^T,$$

$$\Rightarrow (S_1^m)^T + (S_1^{m-1})^T A_1^T + \dots + A_m^T = 0$$

$$\Rightarrow (S_1^T)^m + (S_1^T)^{m-1} A_1^T + \dots + A_m^T = A^T(S_1^T) = 0$$

So that S_1 is a left solvent for $A^T(S)$ ■

Corollary 3.2.2: If S_1, S_2, \dots, S_m is a complete set of right solvents of $A(S)$, then the set

$\{S_1^T, S_2^T, \dots, S_m^T\}$ is a complete set of left solvents of $A^T(S)$.

Proof: If $S_1^T, S_2^T, \dots, S_m^T$ are right solvents of $A(s)$, then by Theorem (3.3.5),

$S_1^T, S_2^T, \dots, S_m^T$ are left solvents for $A(s)$. And since the set $\{S_1^T, S_2^T, \dots, S_m^T\}$, is a complete set

of right solvents of $A(s)$, so $V(S_1^T, S_2^T, \dots, S_m^T)$ is non singular, since

$\det V(S_1^T, S_2^T, \dots, S_m^T) \neq 0$. So that $\{S_1^T, S_2^T, \dots, S_m^T\}$ is a complete set of left solvents of

$A^T(S)$ ■

Example 3.2.4: Consider

$$A(S) = S^3 + \begin{bmatrix} -6 & 6 \\ -3 & -15 \end{bmatrix} S^2 + \begin{bmatrix} 2 & -42 \\ 21 & 65 \end{bmatrix} S + \begin{bmatrix} 18 & 66 \\ -33 & -81 \end{bmatrix}.$$

$$\text{If } S_1 = \begin{bmatrix} 4 & -2 \\ 1 & 7 \end{bmatrix}$$

$$\text{Then } A(S_1) = \begin{bmatrix} 34 & -182 \\ 91 & 307 \end{bmatrix} + \begin{bmatrix} -18 & 414 \\ -207 & -639 \end{bmatrix} + \begin{bmatrix} -34 & -298 \\ 149 & 413 \end{bmatrix} + \begin{bmatrix} 18 & 66 \\ -33 & -81 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So S_1 is a right solvent for $A(S)$. Now

$$A^T(S) = (S^3)^T + (S^2)^T \begin{bmatrix} -6 & -3 \\ 6 & -15 \end{bmatrix} + S^T \begin{bmatrix} 2 & 21 \\ -42 & 65 \end{bmatrix} + \begin{bmatrix} 18 & -33 \\ 66 & -81 \end{bmatrix}, \text{ and } S_1^T = \begin{bmatrix} 4 & 1 \\ -2 & 7 \end{bmatrix}.$$

We see that

$$A^T(S_1^T) = \begin{bmatrix} 34 & 91 \\ -182 & 307 \end{bmatrix} + \begin{bmatrix} -18 & -207 \\ 414 & -639 \end{bmatrix} + \begin{bmatrix} -34 & 149 \\ -298 & 413 \end{bmatrix} + \begin{bmatrix} 18 & -33 \\ 66 & -81 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow S_1^T$ is a left solvent for $A^T(S)$.

3.3 Existence and number of solvents

The concept of "a complete set" of solvents of a matrix polynomial analogous to the fact that, a scalar polynomial of degree n has n zeros.

The following lemma gives a sufficient condition for the existence of solvents.

Lemma 3.3.1 [5]: If $A(s) = Is^m + A_1s^{m-1} + \dots + A_0$ has n linearly independent right latent vectors, b_1, \dots, b_n , corresponding to latent roots s_1, s_2, \dots, s_n , then $PJ_C P^{-1}$ is a right solvent for $A(S)$, where $P = [b_1, \dots, b_n]$ and $J_C = \text{diag}[s_1, s_2, \dots, s_n]$.

For the proof, see [5].

Corollary 3.3.1 [5]: If $A(s) = s^m + A_1s^{m-1} + \dots + A_0$ has mn distinct latent roots, and the set of right latent vectors satisfy the condition that, every set of n elements of them are

linearly independent, then there are exactly $\binom{mn}{n}$ different right solvents.

For the proof, see [5].

Example 3.3.1: Consider the matrix polynomial

$$A(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}. \text{ So } m=1, n=2, \text{ then } \det A(s) = s^2 - s \Rightarrow s_1 = 0, s_2 = 1.$$

Now, for

$$s_1 = 0, A(0) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x_1^{(1)} = 0, \text{ and } x_2^{(1)} \text{ arbitrary,}$$

$$\Rightarrow x_2^{(1)} \neq 0, \text{ let } v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\text{And for } s_2 = 1, A(1) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1^{(2)} + x_2^{(2)} = 0 \Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

we see that the vectors v_1, v_2 are linearly independent since the matrix $P = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, is

nonsingular. By Lemma 3.3.1, $A(S)$ has a right solvent that is

$$S_1 = PJ_c P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

$$\text{Clearly } A(S_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

And by Corollary 3.3.1, we have exactly $\binom{2}{2}$ solvents.

$$\Rightarrow \text{The number of solvents} = 1, \text{ that is } S_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

Lemma 3.3.2 [10]: Let A be a matrix of order mn . If A is nonsingular then there exist a permutation of columns such that the new matrix has m diagonal blocks of order n , each of them being nonsingular.

Proof: Let $1 \leq k < n$, expand $\det A$ in terms of square matrices formed from the first k rows. Since $\det A \neq 0$, one of the products in the expansion is nonzero and the result follows ■

The following Theorem gives the minimum number of solvents of a matrix polynomial for special companion matrix.

Theorem 3.3.1 [10]: Let $A(s)$ be a matrix polynomial and let C_A be the associated companion matrix. If C_A is diagonalizable, then $A(S)$ has at least m solvents, where $m = \deg A(s)$.

Proof: Let J_C be the Jordan normal form of the companion matrix C_A , since C_A is diagonalizable, so there exists a matrix P such that $C_A = PJ_C P^{-1}$ with P nonsingular, this equality will remain valid if the same permutation of columns is applied to P and J_C , and by Lemma 3.3.2, we can suppose that the m diagonal blocks, p_i of order n , $i = 1, \dots, m$ of P are nonsingular. And by Theorem 3.2.1 the columns of p_i are eigenvectors of $A(s)$. Hence, each $p_i = p_i J_C p_i^{-1}$, $i = 1, \dots, m$, is a solvent of $A(S)$ (where $J_C, i = 1, \dots, m$ are diagonal blocks of order n of J_C). So we have at last m solvents for $A(S)$ ■

Corollary 3.3.2 [10]: For every eigenvalue α_i of C_A , there exists a solvent S_1 such that α_i is an eigenvalue of S_1 .

Example 3.3.2 [4]: Consider the matrix polynomial

$$A(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} -1 & -6 \\ 2 & -9 \end{bmatrix} s + \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix}.$$

Then $s_1 = 1, s_2 = 2, s_3 = 3, s_4 = 4$, are the latent roots of $A(s)$, (see example 3.2.3)

$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, is a solvent for $A(S)$, so $xI - S_1 = \begin{bmatrix} x-1 & 0 \\ 0 & x-2 \end{bmatrix}$, and

$\det(xI - S_1) = (x-1)(x-2) \Rightarrow x_1 = 1, x_2 = 2$, are the eigenvalues of S_1 and we note

that, $x_1 = s_1, x_2 = s_2$.

Also, $S_2 = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow xI - S_2 = \begin{bmatrix} x-4 & 0 \\ -2 & x-2 \end{bmatrix} \Rightarrow$

$\det(xI - S_2) = (x-4)(x-2) \Rightarrow x_3 = 4, x_4 = 2$ are the eigenvalues of S_2 , and note that

$x_3 = s_4, x_4 = s_2$.

And by Corollary 3.3.1, $A(S)$ has $\binom{2 \times 2}{2} = 6$ different solvents.

Note that, "if the matrix C_A is not diagonalizable then the associated matrix polynomial

$A(S)$ can have less than m or even no solvents at all" [10], and the following example

illustrate this fact.

Example 3.3.3 [10]: Consider the matrix polynomial

$$A(S) = S^2 + \begin{bmatrix} -98/25 & 108/25 & -112/25 \\ 4/5 & -24/5 & -4/5 \\ 22/25 & 38/25 & -182/25 \end{bmatrix} S + \begin{bmatrix} 89/25 & -294/25 & 316/25 \\ -7/5 & 42/5 & -8/5 \\ -46/25 & -59/25 & 251/25 \end{bmatrix}.$$

So, $m = 2, n = 3, s_1 = 2, s_2 = 3$, and the eigenpairs of $A(s)$ are

$$(V_1, J_1) = \left(\begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \right), (V_2, J_2) = \left(\begin{bmatrix} -2 & 2 & 5 & 3 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \right)$$

We see that, there are no nonsingular matrix of order 3 with leading columns of V_1 and V_2 ,

and so $A(S)$ has no solvents at all.

Theorem 3.3.2 [10]: Let $A(s)$ be a matrix polynomial, C_A the associated block companion matrix, and S_1 is a solvent of $A(S)$, if one eigenvalue of S_1 has the geometric multiplicity greater in C_A than in S_1 , then $A(S)$ has infinitely many solvents.

Proof: Let $(V_1, J_1), (V_2, J_2), \dots, (V_k, J_k)$ be a complete system of eigenpairs of S_1 where

$$V_i = [v_{i1} \dots v_{ij_i}], J_i = \begin{bmatrix} \alpha_i & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & & \alpha_i \end{bmatrix}, i = 1, \dots, k \quad \text{and } j_i \text{ is the order of the block } J_i.$$

$$\text{So that } S_1 = [V_1 \dots V_k] \begin{bmatrix} J_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_k \end{bmatrix} [V_1 \dots V_k]^{-1}.$$

Suppose that α_1 is the eigenvalue having geometric multiplicity greater in C_A than in S_1 and that v_{11} is an eigenvector of S_1 corresponding to α_1 and let v_{l1} be an eigenvector of $A(s)$ corresponding to α_1 that is not an eigenvector of S_1 (which exists by hypothesis) since v_{11} and v_{l1} are linearly independent, and the n vectors

$v_{11}, \dots, v_{1j_1}, v_{21}, \dots, v_{2j_2}, \dots, v_{k1}, \dots, v_{kj_k}$ are linearly independent, there exists an infinite number of vectors $w_h = v_{1j_1} + \beta v_{l1}$, for a nonzero complex β , such that the n vectors

$$v_{11}, \dots, v_{1(j_1-1)}, w_h, v_{21}, \dots, v_{2j_2}, \dots, v_{k1}, \dots, v_{kj_k}$$

, are also linearly independent. Hence there are infinitely many solvents

$$Y_n = [V_{1(h)} \dots V_k] \begin{bmatrix} J_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_k \end{bmatrix} [V_{1(h)} \dots V_k]^{-1}$$

Where the columns of $V_{1(h)} = [v_{11} \dots v_{1(j_1-1)}, w_h]$ are also Jordan chains of $A(s)$ of length j_1 , corresponding to α_1 but are not Jordan chains of S_1 ■

Example 3.3.4 [10]: Consider the matrix polynomial

$$A(S) = S^2 + \begin{bmatrix} -7 & -2 & -2 \\ \frac{3}{31} & -\frac{203}{31} & \frac{8}{31} \\ -\frac{13}{31} & -\frac{40}{31} & -\frac{23}{31} \end{bmatrix} S + \begin{bmatrix} 13 & 9 & 7 \\ -\frac{21}{31} & \frac{294}{31} & -\frac{36}{31} \\ \frac{60}{31} & \frac{183}{31} & \frac{435}{31} \end{bmatrix}, \text{ then } A(s) \text{ has the following}$$

eigenpairs

$$(V_1, J_1) = ([v_{11} \ v_{12}], J_1) = \left(\begin{bmatrix} -1 & 5 \\ 1 & -3 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \right), (V_2, J_2) = (v_{21}, J_2) = \left(\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, 3 \right),$$

$$(V_3, J_3) = ([v_{31} \ v_{32} \ v_{33}], J_3) = \left(\begin{bmatrix} -2 & -1 & 4 \\ 1 & 2 & 3 \\ -1 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \right).$$

And if we choose the matrix T such that

$$T = [v_{11} \ v_{12} \ v_{31}] = \begin{bmatrix} -1 & 5 & -2 \\ 1 & -3 & 1 \\ -2 & 1 & -1 \end{bmatrix}.$$

Then T is nonsingular, so that

$$S_1 = \begin{bmatrix} -1 & 5 & -2 \\ 1 & -3 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 5 & -2 \\ 1 & -3 & 1 \\ -2 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{20}{3} & 7 & \frac{5}{3} \\ -2 & -1 & -1 \\ \frac{7}{3} & 5 & \frac{13}{3} \end{bmatrix},$$

is a solvent for $A(S)$, and S_1 has the eigenvalue $\alpha_1 = 3$, with geometric multiplicity 1, and

in the matrix C_A the geometric multiplicity of the latent root 3 is 2. Thus by Theorem

3.3.2, $A(S)$ has an infinite number of solvents.

Now, consider the vector w such that $w_l = v_{12} + \beta v_{21}$, where β is a nonzero complex

$$\text{number. Then any matrix of the form } S_l = [v_{11} \ w_l \ v_{31}] \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} [v_{11} \ w_l \ v_{31}]^{-1}, \text{ for}$$

nonsingular $T_l = [v_{1l} \ w_l \ v_{3l}]$ is a solvent for $A(S)$, for example, with $\beta = 1$, then

$$w_l = v_{2l} + v_{3l} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \text{ and } T = \begin{bmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -2 & 0 & -1 \end{bmatrix}, \text{ is nonsingular.}$$

$$\Rightarrow S_2 = \begin{bmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -2 & 0 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 6 & 5 & 1 \\ -\frac{5}{3} & 0 & -\frac{2}{3} \\ 4 & 7 & 3 \end{bmatrix}, \text{ is a solvent for } A(S).$$

Chapter Four

Companion matrices and linearization

4.0 Introduction

In this chapter we consider a regular matrix polynomial of the form

$$A(s) = A_0 s^n + A_1 s + \dots + A_n \quad (4.0.1)$$

Firstly, section 4.1 introduces a new family of companion matrices to the scalar polynomials.

Section 4.2 gives a generalization to this result to the regular matrix polynomial. And finally, section 4.3 gives a special class of linearization for special class of matrix polynomials.

4.1 New forms for a companion matrix of scalar polynomials

This section gives a method for constructing new forms of companion matrix to a scalar polynomial.

Definition 4.1.1: Let $P(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$, (4.1.1)

be a polynomial with constant coefficients then the matrix

$$A \square \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (4.1.2)$$

is called a companion matrix of $P(s)$.

Note that, the matrix A in (4.1.2), has the property that $\det(sI - A) = P(s)$, and any other matrix satisfy this property is called a companion matrix for the polynomial $P(s)$.

The following lemma gives a factorization of the companion matrix in (4.1.2) into a product of n matrices

Lemma 4.1.1 [6]: For $k=1,2,\dots,n-1$, denote by A_k the matrix

$$A_k = \begin{bmatrix} I_{k-1} & & \\ & C_k & \\ & & I_{n-k-1} \end{bmatrix} \quad (4.1.3)$$

$$\text{where } C_k \text{ is a } 2 \times 2 \text{ matrix, } C_k = \begin{bmatrix} -a_k & 1 \\ 1 & 0 \end{bmatrix} \quad (4.1.4)$$

And by A_n the matrix

$$A_n = \text{diag} \{1, \dots, 1, -a_n\} \quad (4.1.5)$$

$$\text{Then, } A = A_1 A_2 \dots A_{n-1} A_n \quad (4.1.6)$$

Proof: By induction, for $n=2$ we have

$$A_1 = \begin{bmatrix} -a_1 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -a_2 \end{bmatrix} \Rightarrow A_1 A_2 = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} = A$$

Let it is true for $n=k$.

Now for $n=k+1$, note that

$$A_{k+1} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & -a_{k+1} & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and}$$

$$A_1 A_2 \dots A_k = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_k & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 1 \end{bmatrix}.$$

So that

$$A_1 A_2 \dots A_{k+1} A_n = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_k & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_k & 0 & 0 \\ 0 & -a_{k+1} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -a_n \end{bmatrix} =$$

$$\begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}$$

And the proof is complete ■

Example 4.1.1: For $n=3$, we have $P(s) = s^3 + a_1 s^2 + a_2 s + a_3$, so that $k=1, 2$.

To construct A_i 's defined in (4.1.6), we use the above Lemma, that is:

$$A_1 = \begin{bmatrix} I_0 & & \\ & C_1 & \\ & & I_1 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} I_1 & & \\ & C_2 & \\ & & I_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -a_2 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}$$

we see that $A_1 A_2 A_3 = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = A.$

Theorem 4.1.1 [6]: All matrices $A_{i_1} \dots A_{i_n}$ for any permutation (i_1, \dots, i_n) are companion

matrices of $P(s)$ and are similar to the matrix (4.1.2). In particular, this holds for the

matrix $\tilde{A} = BC$, where B is the matrix $A_1 A_3 \dots$, and C is the matrix $A_2 A_4 \dots$, where A_i

are the matrices from (4.1.3).

For the proof, see [6].

Example 4.1.2: Let $P(s) = s^3 + s^2 + 2s + 1$

$$\text{Then, } A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow A_2 A_1 A_3 = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = A^{(3)},$$

Clearly $\det(Is - A^{(3)}) = P(s)$.

$$\text{Also, } A_1 A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \square B, \text{ and}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \square C. \text{ Then}$$

$$\tilde{A} = BC = A_1 A_3 A_2 = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

is a companion matrix for $P(s)$.

4.2 New family of companion matrices for regular matrix polynomial

This section contains an equivalent definition for the linearization of regular matrix polynomial and introduces new forms of companion matrices. In fact, this section gives a generalization for Lemma 4.2.1 to matrix polynomials.

We consider polynomial matrices of the form

$$A(s) = A_0 s^n + A_1 s^{n-1} + \dots + A_n \tag{4.2.1}$$

with $A_i \in \mathbb{R}^{p \times p}$, and $\det A(s) \neq 0$, (*i.e. regular*).

Definition 4.2.1: Let $A(s)$ be a regular matrix polynomial then the associated matrix pencil

$$P(s) = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ 0 & I_p & & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & I_p \end{bmatrix} s - \begin{bmatrix} -A_1 & -A_2 & \cdots & -A_n \\ I_p & 0 & \cdots & 0 \\ \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_p & 0 \end{bmatrix} \quad (4.2.2)$$

is known as the first companion form of $A(s)$, and

$$\tilde{p}(s) = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ 0 & I_p & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_p \end{bmatrix} s - \begin{bmatrix} -A_1 & I_p & \cdots & 0 \\ -A_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & I_p \\ -A_n & 0 & \cdots & 0 \end{bmatrix}, \quad (4.2.3)$$

is called the second companion form for $A(s)$.

Definition 4.2.2: A matrix pencil $P_0s - P_1$ is called a linearization for $A(s)$ if

$$P_0s - P_1 \square \begin{bmatrix} A(s) & 0 \\ 0 & I_{p(n-1)} \end{bmatrix} \quad (4.2.4)$$

that is, there exists a unimodular matrix polynomials $E(s)$ and $F(s)$ such that

$$\begin{bmatrix} A(s) & 0 \\ 0 & I_{p(n-1)} \end{bmatrix} = E(s)(P_0s - P_1)F(s) \quad (4.2.5)$$

Note that: The first and second companion forms of a matrix polynomial is a linearization for $A(s)$.

Example 4.2.1: Consider

$$A(s) = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} s^2 + \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then:

$$P(s) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s - \begin{bmatrix} -2 & 1 & -1 & 0 \\ -4 & 0 & -2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{and } \tilde{P}(s) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s - \begin{bmatrix} -2 & 1 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 \end{bmatrix}$$

is the first and second companion forms for $A(s)$, respectively.

The companion matrices in (4.2.2) and (4.2.3), can be written as a product of matrices,

Lemma 4.1.1 gave the companion matrix of scalar polynomial as a product of matrices, the following lemma generalizes this result to a matrix polynomial.

Lemma 4.2.1 [1]: Let $A(s) = A_0 s^n + A_1 s^{n-1} + \dots + A_n$ where $A_i \in \mathbb{R}^{p \times p}, i = 0, \dots, n$, be regular matrix polynomial and define the matrices

$$\left. \begin{aligned} T_0 &= \text{diag} \{A_0, I_{p(n-1)}\} \\ T_k &= \begin{bmatrix} I_{p(k-1)} & & \\ & C_k & \\ & & I_{p(n-k-1)} \end{bmatrix} \\ \text{where } C_k &= \begin{bmatrix} -A_k & I_p \\ I_p & 0 \end{bmatrix}, k = 1, 2, \dots, n-1 \\ \text{and } T_n &= \text{diag} \{I_{p(n-1)}, -A_n\} \end{aligned} \right\} \quad (4.2.6)$$

Then the first and second companion matrices to $A(s)$ are given by

$$\left. \begin{aligned} P(s) &= sT_0 - T_1 T_2 \dots T_n \\ \tilde{P}(s) &= sT_0 - T_n T_{n-1} \dots T_1 \end{aligned} \right\} \quad (4.2.7)$$

, respectively.

Proof: Clearly the matrix product $T_1 \dots T_n$ equal to the second term of (4.2.2), and

obviously T_0 = the first term of (4.2.2), so $P(s) = sT_0 - T_1 T_2 \dots T_n$ is the first companion form of $A(s)$.

Similarly we can show $\tilde{P}(s)$ is the second companion form for $A(s)$.

Example 4.2.2: Let $A(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

then $P = n = 2, k = 1$, so that, by Lemma 4.2.1 we have:

$$T_0 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_1 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

we see that

$$sT_0 - T_1 T_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s - \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and this is the first companion form for $A(s)$.

The companion form for a regular matrix polynomial $A(s)$ is not unique, the following theorem gives a formula for constructing new forms of companion matrices, which is a linearization for $A(s)$.

Theorem 4.2.1 [1]: For every possible permutation (i_1, i_2, \dots, i_n) of the n-tuple $(1, 2, \dots, n)$, the matrix polynomial

$$P(s) = sT_0 - T_{i_1}T_{i_2} \dots T_{i_n} \quad (4.2.8)$$

is a linearization for a regular matrix polynomial $A(s)$, where $T_i, i = 1, \dots, n$ as defined in (4.2.6).

For the proof, see [1].

Example 4.2.3: Consider $A(s) = \begin{bmatrix} s^2+1 & 3s \\ 2 & s^2+3 \end{bmatrix}$

$$\text{Then, } T_0 = \begin{bmatrix} A_0 & 0 \\ 0 & I_{2(1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} -A_1 & I_2 \\ I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{and } T_2 = \begin{bmatrix} I_2 & \\ & -A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & -3 \end{bmatrix}$$

Now, by Theorem 4.2.1, we have for example

$$sT_0 - T_2T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s - \begin{bmatrix} 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 \end{bmatrix}$$

is a linearization for $A(s)$.

Corollary 4.2.1 [1]: Let $A(s) = A_0s^n + A_1s^{n-1} + \dots + A_n$ be regular matrix polynomial,

then for any four ordered sets of indices

$$M_k = (i_{k,1}, i_{k,2}, \dots, i_{k,n_k}), k = 1, 2, 3, 4$$

such that $M_i \cap M_j = \emptyset$ for $i \neq j$ and $\bigcup_{k=1}^4 M_k = \{1, 2, \dots, n-1\}$ the matrix polynomial

$$R(s) = sT_{M_1}^{-1}T_0T_{M_2}^{-1} - T_{M_3}T_nT_{M_4}$$

(4.2.9)

is a linearization for $T(s)$, where $T_{M_k} = T_{i_{k,1}}T_{i_{k,2}} \dots T_{i_{k,n_k}}$ for $M_k \neq \emptyset$

and $T_{I_k} = I$ for $M_k = \emptyset$.

Proof: First, the matrices $T_{M_k}, k = 1, 2, 3, 4$ are invertible since it's a product of the

matrices T_i with $0 < i < n$, so that we can write $R(s)$ in the form

$$R(s) = T_{M_1}^{-1}(sT_0T_{M_2}^{-1} - T_{M_1}T_{M_3}T_nT_{M_4}) \text{ or } R(s) = T_{M_1}^{-1}(sT_0 - T_{M_1}T_{M_3}T_nT_{M_4}T_{M_2})T_{M_2}^{-1}$$

By Theorem 4.2.1, $sT_0 - T_{M_1}T_{M_3}T_nT_{M_4}T_{M_2} \square B(s)$ is a companion matrix for $A(s)$. So we

can write $R(s) = T_{M_1}^{-1}B(s)T_{M_2}^{-1}$. That is $R(s)$ is equivalent to $B(s)$ and so $R(s)$ is a

companion matrix for $A(s)$ ■

Note that the inverse of $T_k, k = 1, 2, \dots, n-1$, where T_k as defined in (4.2.6) have a simple form, that is

$$T_k^{-1} = \left. \begin{array}{c} \left[\begin{array}{cc} I_{p(k-1)} & 0 \\ & C_k^{-1} \\ 0 & I_{p(n-k-1)} \end{array} \right] \\ \text{with } C_k^{-1} = \left[\begin{array}{cc} 0 & I_p \\ I_p & A_k \end{array} \right], k = 1, 2, \dots, n-1 \end{array} \right\} \quad (4.2.10)$$

Example 4.2.3: Consider $A(s) = A_0s^3 + A_1s^2 + A_2s + A_3$

So, $n = 3$, $\bigcup_{k=1}^4 M_k = \{1, 2\}$, and $M_i \cap M_j = \emptyset$ for $i \neq j$,

so that, by Corollary 4.2.1, we can choose for example M_k 's as follows

$$\text{a) } M_1 = \emptyset, M_2 = \{1\}, M_3 = \{2\}, M_4 = \emptyset,$$

for this choice, we have

$$T_{M_1} = I \Rightarrow T_{M_1}^{-1} = I, T_{M_2} = T_1, T_{M_3} = T_2 \text{ and } T_{M_4} = I$$

$$\text{Then } R(s) = sT_0T_1^{-1} - T_2T_3$$

$$\text{Where } T_0 = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

$$T_1 = \begin{bmatrix} -A_1 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & I_p \end{bmatrix}, T_2 = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -A_2 & I_p \\ 0 & I_p & 0 \end{bmatrix} \text{ and } T_3 = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & -A_3 \end{bmatrix}$$

$$\text{Also, using (4.2.10), we get } T_1^{-1} = \begin{bmatrix} 0 & I_p & 0 \\ I_p & A_1 & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

$$\text{So that } R(s) = \begin{bmatrix} 0 & A_0 & 0 \\ I_p & A_1 & 0 \\ 0 & 0 & I_p \end{bmatrix} s - \begin{bmatrix} I_p & 0 & 0 \\ 0 & -A_2 & -A_3 \\ 0 & I_p & 0 \end{bmatrix},$$

is a linearization for $A(s)$.

$$\text{b) If we choose } M_1 = \emptyset, M_2 = \{2\}, M_3 = \{1\} \text{ and } M_4 = \emptyset.$$

Then, follow the procedure in part a to get

$$R(s) = \begin{bmatrix} 0 & A_3 & 0 \\ I_p & A_2 & 0 \\ 0 & 0 & I_p \end{bmatrix} s - \begin{bmatrix} I_p & 0 & 0 \\ 0 & -A_1 & -A_0 \\ 0 & I_p & 0 \end{bmatrix}$$

is a linearization for $A(s)$.

4.3 Self- adjoint and symmetric linearization

In this section we consider the matrix polynomial with self-adjoint or symmetric coefficients and obtain a self-adjoint or symmetric linearization.

In general it's not true that any linearization preserves the special eigenstructure of the matrix polynomial, but in special cases it preserves it, and the following theorem gives one of these cases.

Theorem 4.3.1 [1]: Let $A(s)$ be a regular self-adjoint matrix polynomial of degree n , with $\det A_0 \neq 0$, then the matrix polynomial

$$R_s(s) = \begin{cases} sT_{odd}^{-1} - T_{even} & \text{for } n \text{ even} \\ sT_{even}^{-1} - T_{odd} & \text{for } n \text{ odd} \end{cases} \quad (4.3.1)$$

Where

$$T_{even} = T_0^{-1} T_2 T_4 T_6 \dots$$

$$T_{odd} = T_1 T_3 T_5 \dots$$

And $T_i, i = 1, \dots, n$ are as defined in (4.2.6), is a self-adjoint linearization for $A(s)$.

Proof: Notice that the matrices T_{2k}, T_{2k+1} in T_{even} and T_{odd} , respectively, for $k = 0, 1, \dots$

commute, so the particular order of indices in T_{even} and T_{odd} , respectively is not

significant. For n even, applying Corollary 4.2.1 for the ordered sets of indices

$M_1 = \emptyset, M_2 = (1, 3, \dots, n-1), M_3 = (2, 4, \dots, n-2), M_4 = \emptyset$, we take the companion pencil

$R(s) = sT_0 T_{odd}^{-1} - T_{M_3} T_n$. Since T_0 is invertible, we easily see that $R_s(s) = T_0^{-1} R(s)$

by noting that $T_{even} = T_0^{-1} T_{M_3} T_n$. So $R_s(s)$ is a companion form for $A(s)$. Similarly we

can show the second term of $R_s(s)$ ■

Example 4.3.1:

a) For $n = 4$ Consider $A(s) = A_0s^4 + A_1s^3 + A_2s^2 + A_3s + A_4$. Then

$$T_0 = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}, T_1 = \begin{bmatrix} -A_1 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}, T_2 = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & -A_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}$$

$$, T_3 = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & -A_3 & I_p \\ 0 & 0 & I_p & 0 \end{bmatrix} \text{ and } T_4 = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -A_4 \end{bmatrix}$$

so that

$$T_{odd}^{-1} = (T_1 T_3)^{-1} = T_3^{-1} T_1^{-1} = \begin{bmatrix} 0 & I_p & 0 & 0 \\ I_p & A_1 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & A_3 \end{bmatrix},$$

$$\text{and } T_{even} = T_0^{-1} T_2 T_4 = \begin{bmatrix} A_0^{-1} & 0 & 0 & 0 \\ 0 & -A_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & -A_4 \end{bmatrix}$$

by Theorem 4.3.1 the matrix polynomial

$$R(s) = \begin{bmatrix} 0 & I_p & 0 & 0 \\ I_p & A_1 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & A_3 \end{bmatrix} s - \begin{bmatrix} A_0^{-1} & 0 & 0 & 0 \\ 0 & -A_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & -A_4 \end{bmatrix}$$

is a self-adjoint linearization for $A(s)$.

b) For $n = 5$. Consider $A(s) = A_0s^5 + A_1s^4 + A_2s^3 + A_3s^2 + A_4s + A_5$

As in part (a), we compute T_0, \dots, T_5 , and get

$$R_s(s) = \begin{bmatrix} A_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 \\ 0 & I_p & A_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p \\ 0 & 0 & 0 & I_p & A_4 \end{bmatrix} s - \begin{bmatrix} -A_1 & I_p & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_3 & I_p & 0 \\ 0 & 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_5 \end{bmatrix}$$

is a self-adjoint linearization for $A(s)$.

Remark [2]: Let $A(s) = A_0s^n + A_1s^{n-1} + \dots + A_n$ be a regular symmetric (skew symmetric) matrix polynomial with $\det A_0 \neq 0$. Then the matrix polynomial

$$R(s) = sT_{\text{odd}}^{-1} - T_{\text{even}} \quad (4.3.2)$$

where

$$\begin{cases} T_{\text{even}} = T_n T_{n-2} \dots T_0^{-1}, & \text{for } n \text{ even} \\ T_{\text{odd}} = T_{n-1} T_{n-3} \dots T_1 \end{cases} \quad (4.3.3)$$

$$\text{and } \begin{cases} T_{\text{even}} = T_n T_{n-2} \dots T_1, & \text{for } n \text{ odd} \\ T_{\text{odd}} = T_{n-1} T_{n-3} \dots T_0^{-1} \end{cases} \quad (4.3.4)$$

and $T_i, i = 1, \dots, n$, are defined in (4.2.6),

is asymmetric (skew symmetric) companion form for $A(s)$.

Example 4.3.2: If we take $n = 4$, then

$$T_0 = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}, T_1^{-1} = \begin{bmatrix} 0 & I_p & 0 & 0 \\ I_p & A_1 & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}, T_2 = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & -A_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix},$$

$$T_3^{-1} = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & A_3 \end{bmatrix} \text{ and } T_4 = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -A_4 \end{bmatrix} \Rightarrow T_4^{-1} = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -A_4^{-1} \end{bmatrix}$$

$$\text{so that } T_{\text{even}} = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & -A_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & -A_4^{-1} \end{bmatrix}, \text{ and } T_{\text{odd}}^{-1} = \begin{bmatrix} 0 & I_p & 0 & 0 \\ I_p & A_1 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & A_3 \end{bmatrix}$$

$$\text{and so } R(s) = \begin{bmatrix} 0 & I_p & 0 & 0 \\ I_p & A_1 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & A_3 \end{bmatrix} s - \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & -A_2 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & -A_4^{-1} \end{bmatrix}$$

is a symmetric linearization for $A(s)$.

As applications of (4.3.2), consider the system of second order differential equation

$$A_0 x'' + A_1 x' + A_2 x = Bu \quad (4.3.5)$$

Where A_0, A_1, A_2 are symmetric matrices, then the associated matrix polynomial is

$$A(s) = A_0 s^2 + A_1 s + A_2 \quad (4.3.6)$$

Now, by (4.3.2), we see that

$$T_0 = \begin{bmatrix} A_0 & 0 \\ 0 & I_p \end{bmatrix}, T_1 = \begin{bmatrix} -A_1 & I_p \\ I_p & 0 \end{bmatrix}, T_2 = \begin{bmatrix} I_p & 0 \\ 0 & -A_2 \end{bmatrix}, T_1^{-1} = \begin{bmatrix} 0 & I_p \\ I_p & A_1 \end{bmatrix}, T_0^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ 0 & I_p \end{bmatrix}$$

so that

$$T_{\text{even}} = T_2 T_0^{-1} = \begin{bmatrix} I_p & 0 \\ 0 & -A_2 \end{bmatrix} \begin{bmatrix} A_0^{-1} & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} A_0^{-1} & 0 \\ 0 & -A_2 \end{bmatrix}$$

$$\text{and } T_{\text{odd}} = T_1, T_{\text{odd}}^{-1} = \begin{bmatrix} 0 & I_p \\ I_p & A_1 \end{bmatrix}$$

so

$$R(s) = \begin{bmatrix} 0 & I_p \\ I_p & A_1 \end{bmatrix} s - \begin{bmatrix} A_0^{-1} & 0 \\ 0 & -A_2 \end{bmatrix}$$

is a symmetric linearization for $A(s)$. For more applications see [1,2].

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