Chapter Five [Asymptotic Behavior Of (NDDE)].

Introduction

In many applications, one assumes the system under consideration is governed by a principle of causality, the future state of the system is independent of the past states and is determined solely by the present. If it is assumed that the system is governed by an equation involving the state and the rate of change of the state, then generally one is considering either ordinary or partial differential equations. However closer scrutiny it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. Also in some problems it is meaningless not to have dependence on the past. This has been known for some time, but the theory for such systems has been extensively developed only recently. In fact, until the time of Volterra most of the results obtained during the previous 150 years were concerned with special properties for very special equations. Volterra formulated some general differential equations incorporating the past states of the systems. He attempted to introduce a concept of energy function for specific physical systems. He studied the asymptotic behavior of the system in the distant future. In the late thirties and early forties Minorsky in his studies of ship stabilization and automatic steering, pointed out very clearly the importance of the consideration of the delay in the feedback mechanism. The great interest in control theory during these and later years has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state . In the late forties and early fifties a few books appeared which presented the current status of the subject and certainly greatly influenced later developments. Mishkis introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems .In their monograph at the Rand Corporation, Bellman and Danskin pointed out the diverse applications of equations containing past information to other areas such as biology and economics. They also presented a well organized theory of linear equations with constant coefficients and the beginning of stability theory. A more extensive development of these ideas is in the book of Bellman and Cooke.[1] Krasovekii presented the theory of Liapunov Functionals in his book on stability theory. The subject has undergone a rapid development in the last fifty years. New applications also continue to arise and require modifications of even the definition of the basic equation. The simplest type of past dependence in a differential equation is that in which the past dependence is through the state of variable and not the derivative of the state variable. Minorsky discussed some physical applications of the differential difference equation:

$$\frac{dx}{dt} = f(t, x(t), x(t-\tau)).$$

Lord Cherwell has encountered the differential difference equation:

$$\frac{dx}{dt} = -\alpha x(t-1)[1 + x(t)]$$

In his study of the distribution of primes, variants of this equation have also been used as models in the theory of growth of a single species. Wangersky and Cunningham used the equations:

$$\frac{dx}{dt} = \alpha x(t)[\frac{m - x(t)}{m}] - bx(t)y(t).$$

$$\frac{dy}{dt} = -\beta y(t) + cx(t-r)y(t-r)$$ for predator prey models.

In an attempt to explain the circummutation of plants (and especially the sunflower) , Isrealson and Johnsson have used the equation:

$$\frac{d\alpha}{dt} = -k\int_{1}^{\infty} f(\theta)\sin\alpha(t-\theta-t_0)d\theta$$

As a model, where α is the angle the top of the plant makes with vertical.

One of the basic applications of the **Neutral Equations** is the Flip-Flop circuit which is the basic element in a digital computer.

The usefulness of the flip-flop circuit lies in the fact that it possesses multiple equilibria and is ideally suited for its role as a memory storage device. Both lumped and distributed models have been hypothesized for these circuits but because of the flip-flop circuits high operating speed distributed models seem more reasonable. A standard model is given in Fig 1.1

In this model the section between 0 and 1 is a lossless transmission line with specific inductance L_s, specific capacitance C_s.

The voltage v across the line and current i flowing through it are functions of ξ and t and obey the following partial differential equations:

$$(1) \qquad L_s\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial \xi}, \qquad -C_s\frac{\partial v}{\partial t} = \frac{\partial i}{\partial \xi}$$

The circuits at the end of the line give rise to the boundary conditions:

$$(2) \qquad 0 = E - v_0 - R_0 i_0, \qquad -C_s\frac{dv_1}{dt} = i_1 + f(v_1).$$

Where the notations:

$v_0 = v(0,t),\ v_1 = v(1,t),\ i_0 = i(0,t),\ i_1 = i(1,t)$ are used.

The nonlinear function f gives the current through the box as shown in Fig 1.1.

The graph of the function f is indicated in Fig 1.2 and is usually associated with a tunnel diode.

System (1) and (2) may in general possess one or multiple equilibrium, Equilibrium states are determined as solutions of (1) and (2) with the left-hand sides set to zero. From (1) we see the equilibriam will be constants and may then be determined from (2), i.e. (i^*,v^*) is an equilibrium if:

$$0 = E - v^* - R_0 i^*,$$
$$0 = -i^* + f(v^*)$$

Figures (1.3) and (1.4) illustrate the cases of one and three equilibrium respectively Marshall Selemond gave a heuristic argument to show that the Flip-Flop circuit operates as a memory storage device.

For other applications, see Jack Hale [15], Klaus schmitt [24].

A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system . when the derivatives of the past history enter in an arbitrary manner, most of the literature is devoted to existence, uniqueness, and continuous dependence .

Jack Hale discussed a particular class of neutral equation for which a qualitative theory is available in the last few years **Neutral delay differential equations has much effort been devoted to the study of it**. For example existence of solutions, oscillatory and nonoscillatory solutions, asymptotic behavior of oscillatory and nonoscillatory solutions.

Our project deals with **first order neutral functional differential equations.**

The first chapter consists of **definitions, main theorems and some concepts necessary for our project.**

The second chapter is about **the Existence and Uniqueness Theory of first order (NDDE)**.

The third chapter **discusses oscillatory and nonoscillatory** solutions of **first order Linear Neutral Delay Differential Equations (LNDDE).**

The fourth discusses **oscillatory and nonoscillatory** solutions of **first order Nonlinear Neutral Delay Differential Equations (NLNDDE).**

The fifth is about **the asymptotic behavior of solutions of first order Neutral Delay Differential Equations.**

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce the **basic definitions and theorems of differential equations** that are needed in later-chapters in the first section, In section two the definition of the **initial value problem (IVP).** In section three the definition of **oscillatory and nonoscillatory** solutions of **neutral** delay differential equations (NDDE) is investigated. Finally in section four a quick review of **oscillation theory of first order ordinary differential equations**.

1.1 Definitions and Examples

First of all we will define differential equations with **deviating arguments** and their kinds.

Definition 1.1.1: Differential equations with deviating arguments

Differential equations with **deviating arguments** are differential equations in which the unknown function appears with various values of the argument is known as with:

(i) **Retarded argument**

A differential equation, with **retarded argument,** is a differential equation with deviating argument, in which the highest order derivative of the unknown function appears for just one value of the argument, and this argument is not less than all arguments of the unknown function, and its derivatives appearing in the equation.

(ii) Advanced argument

A differential equation with **advanced argument** is a differential equation with deviating argument, in which the highest order derivative of the unknown function appears for just one value of the argument, and this argument is not larger than the remaining arguments of the unknown function and its derivatives appearing in the equation.

(iii) Neutral type

A differential equation with deviating arguments is called of **Neutral type** if it is not retarded nor advanced.

Now we will consider some **examples about differential equations with deviating arguments.**

Examples 1.1.1

(1) $x'(t) = f(t, x(t), x(t - \tau(t)))$

(2) $x''(t) = f(t, x(t), x'(t), x(t - \tau(t)), x'(t - \tau(t)))$

(3) $x'''(t) = f(t, x(\frac{t}{3}), x'(\frac{t}{3}), x(t), x'(t))$

(4) $x''(t) = f(t, x(t), x'(t), x(t - \tau(t)), x'(t - \tau(t)), x''(t - \tau(t)))$

(1) and (2) are equations with **retarded arguments** if $\tau(t) \geq 0$ and with **advanced argument** if $\tau(t) \leq 0$,

(4) is an equation of **Neutral type**.

(3) is an equation with **retarded argument** if $t \geq 0$, and with **advanced argument** if $t \leq 0$.

Definition 1.1.2 : Neutral delay differential equations (NDDE)

Differential equations in which the highest order derivative of the unknown function appears with the arguments (present state) as well as one or more retarded arguments (past histories) are called **Neutral delay differential equations**.

Example 1.1.2

(1) $x'(t) = f(t, x(t), x(t-\tau), x'(t-\tau)), \tau > 0$ is a first order NDDE .

(2) $x'''(t) = f(t, x(t), x'(t), x(t-\tau(t)), x''(t-\tau(t)), x'''(t-\tau(t))), \tau(t) > 0$ is a third order NDDE

Definition 1.1.3 : Solution of a Neutral functional differential equation

A function x is said to be a solution for a given (NDDE) say

$$[x(t) + c(t)x(t-\tau(t))]' = f(t, x(t, x(t-\sigma(t))), \qquad (A)$$

Where $c, \tau, \sigma \in C([t_0, \infty), R^n)$, if there are $m \in R$ and T>0, such that

$x \in C((t_0 - m, \infty), R^n)$, $t \in [t_0, t_0 + T)$ and $[x(t) + c(t)x(t-\tau)]'$ is continuously differentiable and x satisfies equation (A) on $[t_0, t_0+T)$,

$x(t) = \varphi(t)$ for $t_0 - m \le t \le t_0$.

Lebesque convergence theorem 1.1.4

Let g be integrable over a set E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g$ on E and for almost all x in E we have

$f(x) = \underset{E}{Lim} f_n(x)$, then $\int_E f = Lim \int_E f_n$. [13].

Definition 1.1.5 : Metric Space

We call a set **M a metric space** if a function $\rho(x,y)$ satisfies:

$\rho(x,y) \ge 0, \rho(y,y) = 0$ and $\rho(y,z) = 0$ implies $y = z$

$\rho(y,z) = \rho(z,y)$,

$\rho(y,z) \le \rho(y,u) + \rho(u,z)$ where $\rho(y,z)$ is called the distance between y and z in the space M, the metric space M is called complete if every fundamental sequence of points of the space (That is a sequence $y_1, y_2, y_3, \ldots, y_n, \ldots$ satisfying the condition $\rho(y_n, y_{n+m}) < \in$ where $\in > 0$, for $n \ge N(\in)$ and any number m>0) converges in the space , that is there exists a point $\bar{y} \in M$ such that $\lim_{n\to\infty} \rho(y_n, \bar{y}) = 0$.

Definition 1.1.6 Stability

Now we will consider the basic **definitions and concepts** necessary for stability of the following delay differential equation

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) \qquad (1.1.1)$$

(i) Definition of Stable solutions

A solution x(t) of the equation (1.1.1) is called stable if for every $\in > 0$ there exists a $\delta(\in) > 0$ such that the inequality $|\varphi(t) - \psi(t)| < \delta(\in)$ on the initial set implies that $\left| \underset{\varphi(t)}{x(t)} - \underset{\psi(t)}{x(t)} \right| < \in$ for $t \geq t_0$, where $\varphi(t)$ is a given continuous initial function, $\psi(t)$ is any continuous initial function.

A solution not satisfies this property is called unstable.

(ii) Definition of Asymptotic stability

A stable solution $\underset{\varphi(t)}{x(t)}$ of equation (1.1.1) is called asymptotically stable if $\lim_{t \to \infty} \left| \underset{\varphi(t)}{x(t)} - \underset{\psi(t)}{x(t)} \right| = 0$ for any continuous initial function $\psi(t)$ satisfies the condition $|\varphi(t) - \psi(t)| < \delta$ for sufficiently small δ .

(iii) Definition of Uniform asymptotic stability

A solution x(t) of equation (1.1.1) is called uniformly a symptotically stable if there exists a $\delta > 0$ such that for every $\in > 0$ there exists a $T(\in)$ such that $\left| \underset{\varphi(t)}{x(t)} - \underset{\psi(t)}{x(t)} \right| < \in$ for $t > t_1 + T(\in)$ and for any continuos initial function $\psi(t)$ satisfying the inequality $|\varphi(t) - \psi(t)| < \delta$ on the initial set E_{t_1} where δ does not depend on the choice of $t_1 \geq t_0$.

(vi) **Definition of Asymptotic stability in the large**

A solution x(t) of equation (1.1.1) is called asymptotically stable in the large if it is stable , and $\lim_{t\to\infty}\{\underset{\varphi(t)}{x(t)}-\underset{\psi(t)}{x(t)}\}=0$ for all continuous initial function $\psi(t)$.

i.e For the equation of the Neutral type

$$x'(t)=f(t,x(t),x(t-\tau_1(t)),\dots,x(t-\tau_m(t)),x'(t-\tau_1(t)),\dots,x'(t-\tau_m(t))) \qquad (1.1.2)$$

All the above definitions are preserved except replacing

$$|\varphi(t)-\psi(t)|<\delta \qquad \text{by} \qquad |\varphi'(t)-\psi'(t)|\le\delta .$$

Weierstrass M-Test 1.1.7

(i) Let $\{M_n\}$ be a sequence of positive constants such that the series of constants $\sum_{n=1}^{\infty}M_n$ converges.

(ii) Let $\sum_{n=1}^{\infty}U_n$ be a series of real functions such that $|U_n(x)|\le M_n$ for all x such that $a\le x\le b$ and for each n=1,2,…

i.e [The series $\sum_{n=1}^{\infty}U_n$ converges uniformly on $a\le x\le b$]

Example 1.1.3

$\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ on $0\le x\le 1$, the sequence $\{\frac{1}{n^2}\}$ of positive constants , the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent , $|u_n(x)|=\left|\frac{\sin nx}{n^2}\right|\le\frac{1}{n^2}=M_n$ for all x such that $0\le x\le 1$. Implies that $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ converges uniformly.

1.2 Initial value problems

Statement of the basic initial- value problem for the following differential equation with retarded argument

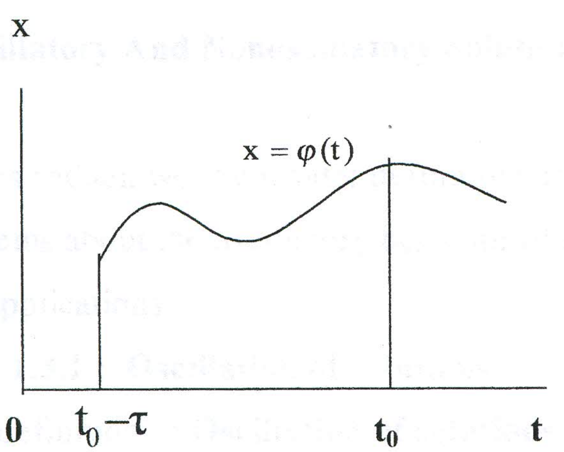
$$x'(t)=f(t,x(t),x(t-\tau)),\tau\ge 0 \qquad (1.2.1)$$

The basic initial- value problem consists of determining a continuous solution x(t) of equation 1.2.1 for $t > t_0$ such that :

$x(t) = \varphi(t)$ for $t_0 - \tau \le t \le t_0$, where $\varphi(t)$ is a given continuous function called the initial function (Fig 1.2.1).

The segment $t_0 - \tau \le t \le t_0$ on which the initial function was given is called the initial set and is denoted by E_{t_0} . It is usually assumed that

$\varphi(t_0) = x(t_0 + 0)$, [i.e $\varphi(t_0) = \lim_{t \to 0^+} x(t)$]



(**Fig 1.2.1**)

In the case of a variable retardation τ (t) in the equation

$$x'(t) = f(t, x(t), x(t - \tau(t))).$$

It is like wise required to find a solution of this equation for $t > t_0$. such that on the initial set E_{t_0} consisting of the point t_0 and those values of $t - \tau(t)$ less than t_0 for $t \ge t_0$, x(t) coincides with the given initial function $\varphi(t)$. It is usually assumed that $x(t_0+0) = \varphi(t_0)$.

Example 1.2.1

$x'(t)= f[t,x(t),x(t-\cos^2 t)]$. for $t_0=0$, the initial function $\varphi(t)$ must be given on the initial set E_{t_0} consisting of the segment $-1 \le t \le 0$.

Example 1.2.2

$x'(t)= f[t,x(t),x(t/2)]$ for $t_0=0$,E_{t_0} consists of the single point $t_0=0$ For $t_0=1$, E_{t_1} consists of the segment $\frac{1}{2} \le t \le 1$.

1.3 Oscillatory And Nonoscillatory Solutions Of (NDDE)

In this section we'll consider definitions for oscillatory solutions, some basic theorems about the oscillatory behavior of solutions for delay equations, and some applications.

Definition 1.3.1 : Oscillation of solutions

The definition of **Oscillation of solutions** can have two different forms

(i) A nontrivial solution x(t) is said to be **oscillatory** iff it has arbitrary large zeroes, that is there exists a sequence of zeroes $\{t_n: x(t_n)=0\}$ of x(t) such that $\lim_{n\to\infty} t_n = \infty$, otherwise ,x(t) is said to be nonoscillatory.

(ii) A nontrivial solution x(t) is said to be **oscillatory** if it changes sign on (T,∞), where T is any number.

Examples 1.3.1

(1) The equation $x'(t)+ x(t-\pi/2) = 0$ has oscillatory solutions

$x(t)= \sin t$, $x(t) = \cos t$.

(2) The equation $x'(t)+x(t+\pi/2)=0$ also has oscillatory solution $x(t)=\sin t$, $x(t)=\cos t$.

(3) The equation $x'(t)-x(t)=0$ has a nonoscillatory solution $x(t)=c\,e^t$.

Now we'll consider the oscillatory behavior of solutions of the following linear inequalities and equations with retarded arguments:

$$x'(t)+p(t)x(\tau(t))\le 0 \qquad (1.3.1)$$

$$x'(t)+p(t)x(\tau(t))\ge 0 \qquad (1.3.2)$$

$$x'(t)+p(t)x(\tau(t))=0 \qquad (1.3.3),$$

Where $p,\tau,\in C[R+,R+],\tau(t)\le t, and \lim_{t\to\infty}\tau(t)=\infty.$.

Let us begin with the following result

Theorem 1.3.1

If $$\varliminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds\ge\frac{1}{e} \qquad (1.3.4)$$

Then:

(i) (1.3.1) has no eventually positive solutions.

(ii) (1.3.2) has no eventually negative solutions.

(iii) All solutions of(1.3.3) are oscillatory.

Proof

Without loss of generality , Assume that τ (t) is nondecreasing , otherwise set $\delta(t)=\max(\tau(s)),s\in[0,t])$ first we prove the validity of statement (i) Assume that x (t) is an eventually positive solution of (1.3.1) such that

$x(\tau(t))\ge 0$ for $t\ge t_1$.

$\dfrac{\lim}{t\to\infty}\int_{\tau(t)}^{t}p(s)ds\ge\dfrac{1}{e}$ so there exists a $t_2\ge t_1$ such that

$$\int_{\tau(t)}^{t}p(s)ds\ge c\ge\frac{1}{e} \quad \text{for} \quad t\ge t_2 \qquad (1.3.5)$$

Since $x'(t)<0$ for $t\ge t_1$ from (1.3.1) we get :

$$x'(t)+p(t)x(t)\le 0 \qquad (1.3.6)$$

Dividing both sides by x(t) and integrating from $\tau(t)$ to t

$$\int_{\tau(t)}^{t} \frac{x'(t)}{x(t)}dt + \int_{\tau(t)}^{t} p(s)ds \le 0,$$

We get

$$\ln\frac{x(t)}{x(\tau(t))} + \int_{\tau(t)}^{t} p(s)ds \le 0, t \ge t_2$$

and hence

$$\ln\frac{x(\tau(t))}{x(t)} \ge \int_{\tau(t)}^{t} p(s)ds \ge c, t \ge t_2 \qquad \text{since } e^x \ge ex \quad \text{for } x \ge 0,$$

It follows that $\frac{x(\tau(t))}{x(t)} \ge ec, \quad t \ge t_2,$

Repeating the above procedure, there exists a sequence $\{t_k\}$ such that

$$\frac{x(\tau(t))}{x(t)} \ge (ec)^k, t \ge t_k \tag{1.3.7}$$

from (1.3.5) there exists a t* such that

$$\int_{\tau(t)}^{t^*} p(s)ds \ge \frac{c}{2} \quad \text{and} \quad \int_{t^*}^{t} p(s)ds \ge \frac{c}{2} \quad \text{for} \quad t \ge t_k$$

Integrating (1.3.1) from $\tau(t)$ to t^* yields $x(t^*) - x(\tau(t)) + \int_{\tau(t)}^{t^*} p(s)x(\tau(s))ds \le 0$,

This implies that

$$x(\tau(t)) \ge x(\tau(t^*))\frac{c}{2} \tag{1.3.8}$$

Similarly we obtain $x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(\tau(s))ds \le 0$ and consequently

$$x(t^*) \ge x(\tau(t))\frac{c}{2} \tag{1.3.9}$$

Combining (1.3.8) with (1.3.9) we get

$$x(t^*) \ge x(\tau(t))(\frac{c}{2})^2 \tag{1.3.10}$$

From (1.3.7) and (1.3.10) it follows that

$$(\frac{2}{c})^2 \ge x(\tau(t^*))/x(t^*) \ge (ec)^k, \qquad \forall t \ge t_k. \tag{1.3.11}$$

Now we choose k sufficiently large such that

$$(ec)^k > (\frac{2}{c})^2 \qquad (1.3.12)$$

Which is possible because ec>1. Therefore (1.3.11) is a contradiction. (1.3.2) can be obtained by a parallel method .

Therefore we obtain (iii).

Theorem 1.3.2

Assume that p and τ are positive numbers in the following equation

$$x'(t) + px(t - \tau) = 0 \qquad (1.3.13),$$

Further assume that $p\tau e \leq 1$.

Then all solutions of (1.3.3) are nonoscillatory.

Proof

Let us look at a solution of (1.3.3) of the form $x(t) = e^{\lambda t}$.It follows that:

$$F(\lambda) = \lambda + pe^{-\lambda\tau} = 0, \qquad \tau(t) = t - \tau,$$

$F(0) = p>0$ and

$$F(-1/\tau) = -1/\tau + pe = (pe\tau - 1)/\tau \leq 0$$

Hence there exists a negative real number $\lambda \in [-\frac{1}{\tau}, 0)$ such that $e^{\lambda t}$ is a nonoscillatory solution of (1.3.13).

Corollary 1.3.1

If p and τ are positive numbers in (1.3.13)

Then $\qquad p\tau e > 1 \qquad (1.3.14)$

Is necessary and sufficient for all solutions of (1.3.3) to oscillate.

Example 1.3.1

The equation $x'(t) + \frac{1}{e}x(t-1) = 0 \qquad (1.3.15)$

has a nonoscillatory solution $x(t) = e^{-t}$ by theorem (1.3.2) because $p\tau e = 1$.

Example 1.3.2

Consider the equation :

$$x'(t)+\frac{1}{(e\ln 2)t}x(\frac{t}{2})=0 \qquad (1.3.16)$$

$$p(t)=\frac{1}{(e\ln 2)t}$$

$$\int_{t/2}^{t} p(s)ds=\int_{t/2}^{t}\frac{1}{(e\ln 2)s}ds=\frac{1}{e\ln 2}[\ln t-\ln t/2]=\frac{1}{e}$$

Hence (1.3.16) does not satisfy condition (1.3.4). In fact equation (1.3.16) has a nonoscillatory solution $x(t)=t^{\alpha}$ where $\alpha=-1/\ln 2$.

Now consider a new result in case that $\overline{\lim}_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds>1$ does not exist.

Theorem 1.3.3

If $p,\tau\in C[R+,R+],\tau(t)<t$ (1.3.17),

and it is nondecreasing (1.3.18),

$\lim_{t\to\infty}\tau(t)=\infty$, and $\overline{\lim}_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds>1$ (1.3.19)

Then every solution of (1.3.3) is oscillatory.

Proof

Without loss of generality, let $x(t)>o$ be a nonoscillatory solution such that, $x(\tau(t))\geq 0,\quad t\geq t_1$. Integrating (1.3.3) form τ (t) to t, we have

$$x(t)-x(\tau(t))+\int_{\tau(t)}^{t} p(s)x(\tau(s))ds=0$$

Or equivalently

$$x(t)+x(\tau(t))[\int_{\tau(t)}^{t} p(s)ds-1]\leq 0 \qquad (1.3.20)$$

From (1.3.20) $\int_{\tau(t)}^{t} p(s)ds\leq 1$,when t is sufficiently large, therefore (1.3.20) is a contradiction.

Example 1.3.3

Consider the equation :

$$x'(t)+(\frac{2}{\pi}(\sqrt{2}+1/e)+\cos t)x(t-\frac{2}{\pi})=0 \quad (1.3.21)$$

Where $p(t)=(\frac{2}{\pi}(\sqrt{2}+1/e)+\cos t)>0$ for $t\in R+$, and

$$\int_{-\pi/2}^{t} p(s)ds=\int_{-\pi/2}^{t}(\frac{2}{\pi}(\sqrt{2}+1/e)+\cos s)ds=\sqrt{2}+1/e+\sin t+\cos t.$$

Hence $\lim_{t\to\infty}\int_{-\pi/2}^{t} p(s)ds=1/e$.

Which does not satisfy condition (1.3.4) of theorem (1.3.1).

However $\lim_{t\to\infty}\int_{-\pi/2}^{t} p(s)ds=2\sqrt{2}+1/e>1$, Consequently ((1.3.21) satisfies condition (1.3.19) of theorem (1.3.3) therefore every solution of (1.3.21) is oscillatory.

1.4 Solutions Of First Order Ordinary differential Equations (ODE)

In this section we will consider **solutions of first order (ODE)** and well discuss some applications.

First we discuss solutions for some general forms of first order (ODE).

Example 1.4.1

Find the general solution for the following first order (ODE)

$y'+p(x)y=q(x)$, $p(x)$, $q(x)$ are continuous functions (1.4.1).

$I(x)=e^{\int p(x)dx}$ is called Integral cofactor , multiplying the equation (1.4.1) by the cofactor I(x) we get

$$e^{\int p(x)dx}y'+p(x)e^{\int p(x)dx}y=q(x)e^{\int p(x)dx} \quad \text{or} \quad \frac{d}{dx}(y\,e^{\int p(x)dx})=q(x)e^{\int p(x)dx}$$

Integrating both sides we get

$$y\, e^{\int p(x)dx} + c_1 = \int q(x)\, e^{\int p(x)dx}\, dx$$

Let $c_1 = c$ implies that

$$y = c\, e^{-\int p(x)dx} + e^{-\int p(x)dx} \int q(x)\, e^{\int p(x)dx}\, dx .$$

Example 1.4.2

Solve the following (ODE)

$$y' - 5y = 0$$

$$I(x) = e^{\int -5dx} = e^{-5x}$$

$$e^{-5x} y' - 5 e^{-5x} y = 0 \quad \text{or} \quad \frac{d}{dx}(y\, e^{-5x}) = 0 \text{ implies that } y = c\, e^{5x}.$$

Example *1.4.3*

Solve the following (ODE)

$$y' + 2xy = 0$$

$I(x) = e^{x^2}$, multiplying both sides by $I(x)$ we get

$$\frac{d}{dx}(ye^{x^2}) = 0 \text{ implies that } y = ce^{-x^2}$$

Example 1.4.4

Solve the following (ODE)

$$\frac{dx}{d\theta} - x\sin\theta = 0$$

$I(\theta) = e^{\cos\theta}$, multiplying both sides by $I(\theta)$ we get

$$\frac{d}{d\theta}(xe^{\cos\theta}) = 0, \text{ implies that } x = ce^{-\cos\theta}.$$

There are several methods for solving first order (ODE), some of them give exact solutions and some of them approximate it. All solutions of first order ordinary differential equations are nonoscillatory solutions, oscillatory solutions can only be obtained for ordinary differential equations with order more than one.