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On Delta and Nabla Caputo Fractional Differences and Dual Identities

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On Delta and Nabla Caputo Fractional Differences and Dual Identities

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On Delta and Nabla Caputo Fractional Difference and Dual Identities

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Dedication

I present this as a way of gratitude to my father and my mother whom I'm truly proud of and for them I am grateful as they stood by my side every day and moment.

To my dear husband Mohammad who was the light of my inspiration throw all of this.

To my soul mate my sister Isra who always encouraged me.

To my brothers who always gave me strength

I present this to all of them.

Declaration

I certify that this thesis submitted for the degree of master, is the result of my own research, except where otherwise acknowledge, and that this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Abstract

We show that two recent definitions of discrete delta and nabla fractional sum operators are related. We investigate of dual identities for Riemann fractional sum and difference and Caputo fractional difference. The first type relates nabla and delta type fractional sums and difference, and we develop a bout the commutatively of the difference fractional sum operate with the usual difference operator are established and relates nabla and delta Caputo fractional difference. The second type represented by the Q-operator relates left and right fractional sums and difference.

Caputo fractional differences are introduced; one of them (dual one) is defined so that it obeys the investigated dual identities. The relation between Riemann and Caputo fractional differences is investigated.

Introduction

During the last two decades, due to its widespread applications in different fields of science and engineering, fractional calculus has attracted the attention of many researchers [1, 2, 4, 5, 14, 19].

Starting from the idea of discretizing the Cauchy integral formula, Miller and Ross [15] and Gray and Zhang [13] obtained discrete versions of left type fractional integrals and derivatives, called fractional sums and differences. After that, several authors started to deal with discrete fractional calculus [6, 7, 8, 9, 10, 11, 12, 18, 21, 22, 23, 24], benefiting from time scales calculus originated in 1988 (see [16]).

In [22], the concept of Caputo fractional difference was introduced and investigated. In [23] we proceed deeply to investigate Caputo fractional differences under two kinds of dual identities. The first kind relates nabla and delta type Caputo fractional differences and the second one, represented by the Q -operator, and relations of left and right ones. Arbitrary order Riemann and Caputo fractional differences are related as well. By the help of the previously obtained results in [23, 24] an integration by parts formula for Caputo fractional differences is originated.

The outline of the thesis is as follows:

Chapter one: Is a general introduction a bout delta and nabla difference operator and study properties of the delta and nabla difference operators , defined rising and filling function.

Chapter two: Contains the definition in the frame of delta and nabla fractional sums and difference in the Riemann sense, some dual identities relating nabla and delta type fractional sums and difference in Riemann sense as previously investigated in [21], and develop about the commutatively of the different fractional sum operators with the usual difference operators are established by dual relation.

Chapter three: Caputo fractional difference are given and related to the Riemann ones, slightly different modified (dual) Caputo fractional differences are introduced and investigated under some dual identities, and devoted to the integration by parts for delta and nabla Caputo fractional differences and Q-operator is used to relate left and right Caputo fractional differences in the nabla and delta case.

Chapter 1

Introduction and Preliminaries

In this chapter we define the delta and nabla calculus which will be useful for our later results, we assume $a, b \in \mathbb{R}$ such that $b - a$ is positive integer, and we defined the set

$$\mathbb{N}_a = \{a, a+1, a+2, \cdots\}, \qquad \mathbb{N}_a^b = \{a, a+1, \cdots, b\} \text{ and}$$

$$_a\mathbb{N} = \{a, a-1, a-2, \cdots\}.$$

1.1 Discrete Difference

In this section, we define the difference and jumping operation as in reference [25].

Definition 1.1.1.

i. (Delta difference operator) For any function $f: \mathbb{N}_a^b \longrightarrow \mathbb{R}$, then if $b > a$ we defined the forward difference operator by $\Delta f(t) = f(t+1) - f(t)$.

ii. (Nabla difference operator) For any function $f: \mathbb{N}_a \longrightarrow \mathbb{R}$ the backward difference operator by $\nabla f(t) = f(t) - f(t-1)$.

iii. Let $m \in \mathbb{N}, m \geq 2$, then

$$\Delta^m f(t) = \Delta(\Delta^{m-1} f(t))$$

$$\nabla^m f(t) = \nabla\big(\nabla^{m-1} f(t)\big).$$

When $m = 0$, Δ^0 and ∇^0 denote the identity operators, i.e, $\Delta^0 f(t) = f(t)$ and $\nabla^0 f(t) = f(t)$.

To understand the delta and nabla difference operators, let take the following example.

Example 1.1.2. Let $f(t) = t^2$, $t \in \mathbb{N}$, then

i. $\Delta f(t) = f(t+1) - f(t) = (t+1)^2 - t^2 = 2t + 1$,

ii. $\nabla f(t) = f(t) - f(t-1) = t^2 - (t-1)^2 = 2t - 1$,

iii. $\Delta^2 f(t) = \Delta(\Delta f(t)) = \Delta(2t+1) = (2(t+1)+1) - (2t+1) = 2$,

iv. $\nabla^2 f(t) = \nabla(\nabla f(t)) = (2t-1) - (2(t-1)-1) = 2$,

v. $\Delta^3 f(t) = \Delta(\Delta^2 f(t)) = 2 - 2 = 0$,

vi. $\nabla^3 f(t) = \nabla(\nabla^2 f(t)) = 2 - 2 = 0$.

Example 1.1.3. Let $f(t) = e^t$, $t \in \mathbb{N}$, then

i. $\Delta f(t) = f(t+1) - f(t) = e^{t+1} - e^t = e^t(e-1)$,

ii. $\nabla f(t) = f(t) - f(t-1) = e^t - e^{t-1} = e^t\left(1 - \frac{1}{e}\right)$,

iii. $\Delta^2 f(t) = \Delta(\Delta f(t)) = \Delta(e^t(e-1)) = e^t(e-1)^2$

iv. $\nabla^2 f(t) = \nabla(\nabla f(t)) = \nabla\left(e^t\left(1 - \frac{1}{e}\right)\right) = e^t\left(1 - \frac{1}{e}\right)^2$.

We introduce the following properties of the delta difference operators [20].

Theorem 1.1.4. Let$f\colon \mathbb{N}_a^b \longrightarrow \mathbb{R}$ be a function defined on $\mathbb{R}$ and $\alpha \in \mathbb{R}$, $t \in \mathbb{N}_a^{b-1}$, then

i. $\Delta\alpha = 0$,

ii. $\Delta\alpha f(t) = \alpha\Delta f(t)$,

iii. $\Delta(f(t) + g(t)) = \Delta f(t) + \Delta g(t)$.

Proof: Using definition 1.1.1

i. Let $f(t) = \alpha$, then $\Delta\alpha = \Delta f(t) = f(t+1) - f(t) = \alpha - \alpha = 0$

ii. Let $g(t) = \alpha f(t)$, then $\Delta g(t) = g(t+1) - g(t)$

$$= \alpha f(t+1) - \alpha f(t) = \alpha\Delta f(t)$$

iii. $\Delta\big(f(t) + g(t)\big) = f(t+1) + g(t+1) - (f(t) + g(t))$

$$= \big(f(t+1) - f(t)\big) + \big(g(t+1) - g(t)\big) = \Delta f(t) + \Delta g(t).$$

Clearly, if $f(t) = \alpha$, then $\Delta^m f(t) = 0$, for all $m \geq 1$ and $\Delta^m\big(\alpha f(t)\big) = \alpha\Delta^m f(t)$

and $\Delta^m\big(f(t) + g(t)\big) = \Delta^m f(t) + \Delta^m g(t)$.

We introduce the following properties of the nabla difference operates [3].

Theorem 1.1.5. assume $f, g \in \mathbb{N}_a^b \longrightarrow \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ then for $t \in \mathbb{N}_a^{b-1}$.

i. $\nabla\alpha = 0,$

ii. $\nabla\alpha f(t) = \alpha\nabla f(t),$

iii. $\nabla\big(f(t) + g(t)\big) = \nabla f(t) + \nabla g(t).$

Proof: Using definition 1.1.1

i. Let $f(t) = \alpha$, then $\nabla\alpha = \nabla f(t) = f(t) - f(t-1) = \alpha - \alpha = 0.$

ii. Let $g(t) = \alpha f(t)$, then $\nabla g(t) = g(t) - g(t-1)$

$$= \alpha f(t) - \alpha f(t-1) = \alpha\nabla f(t).$$

iii. $\nabla\big(f(t) + g(t)\big) = f(t) + g(t) - (f(t-1) + g(t-1))$

$$= \big(f(t) - f(t-1)\big) + \big(g(t) - g(t-1)\big) = \nabla f(t) + \nabla g(t).$$

Clearly, if $f(t) = \alpha$, then $\nabla^m f(t) = 0$, for all $m \geq 1$ and $\nabla^m\big(\alpha f(t)\big) = \alpha\nabla^m f(t)$

and, $\nabla^m\big(f(t) + g(t)\big) = \nabla^m f(t) + \nabla^m g(t)$.

Definition 1.1.6. For all $t \in \mathbb{N}$, the forward jumping is defined be $\sigma(t) = t + 1$ and the backward jumping operator is defined by $\rho(t) = t - 1$.

Now, more properties of delta operators can be proved as the following.

Theorem 1.1.7. Let f and g be functions defined on $\mathbb{R}$ and $t \in \mathbb{N}, \alpha \in \mathbb{R}$, then

i. $\Delta\alpha^t = (\alpha - 1)\alpha^t,$

ii. $\Delta\big(f(t)g(t)\big) = f\big(\sigma(t)\big)\Delta g(t) + \big(\Delta f(t)\big)g(t),$

iii. $\Delta\left(\dfrac{1}{f(t)}\right) = -\dfrac{\Delta f(t)}{f(t)f\big(\sigma(t)\big)},$

iv. $\Delta\dfrac{f(t)}{g(t)} = \dfrac{g(t)\Delta f(t) - f(t)\Delta g(t)}{g(t)g(\sigma(t))}.$

Proof: Using definition 1.1.1

i. $\Delta\alpha^t = \alpha^{t+1} - \alpha^t = \alpha^t(\alpha - 1).$

ii. $\Delta\big(f(t)g(t)\big) = f(t+1)g(t+1) - f(t)g(t)$

$$= f(t+1)g(t+1) + f(t+1)g(t) - f(t+1)g(t) - f(t)g(t)$$

$$= f(t+1)\big(g(t+1) - g(t)\big) + g(t)(f(t+1) - f(t))$$

$$= f\big(\sigma(t)\big)\Delta g(t) + g(t)\Delta f(t).$$

iii. Using definition 1.1.1

$$\Delta\left(\frac{1}{f(t)}\right) = \frac{1}{f(t+1)} - \frac{1}{f(t)} = \frac{f(t) - f(t+1)}{f(t)f(t+1)} = -\frac{f(t+1) - f(t)}{f(t)f(t+1)}$$

$$= -\frac{\Delta f(t)}{f(t)f\big(\sigma(t)\big)}.$$

iv. Applying property ii and iii of this theorem, we have

$$\Delta\frac{f(t)}{g(t)}=\Delta\left(f(t)\times\frac{1}{g(t)}\right)=f\big(\sigma(t)\big)\Delta\left(\frac{1}{g(t)}\right)+\Delta f(t)\frac{1}{g(t)}$$

$$=f\big(\sigma(t)\big)\times\frac{-\Delta g(t)}{g(t)g\big(\sigma(t)\big)}+\Delta f(t)\frac{g\big(\sigma(t)\big)}{g(t)g\big(\sigma(t)\big)}$$

$$=\frac{\Delta f(t)g\big(\sigma(t)\big)-\Delta g(t)f(\sigma(t))}{g(t)g(\sigma(t))}$$

$$=\frac{\big(f(t+1)-f(t)\big)g\big(\sigma(t)\big)-\big(g(t+1)-g(t)\big)f\big(\sigma(t)\big)}{g(t)g\big(\sigma(t)\big)}$$

$$=\frac{f(t+1)g(t+1)-f(t)g(t+1)-g(t+1)f(t+1)+g(t)f(t+1)}{g(t)g\big(\sigma(t)\big)}$$

$$=\frac{-f(t)g(t+1)+g(t)f(t+1)+g(t)f(t)-f(t)g(t)}{g(t)g\big(\sigma(t)\big)}$$

$$=\frac{g(t)\big(f(t+1)-f(t)\big)-f(t)(g(t+1)-g(t))}{g(t)g(\sigma(t))}=\frac{g(t)\Delta f(t)-f(t)\Delta g(t)}{g(t)g(\sigma(t))}.$$

Now, more properties of nabla operators can be proved as the following.

Theorem 1.1.8. Let f and g be functions defined on $\mathbb{R}$ and $\alpha\in\mathbb{R}$, $t\in\mathbb{N}$, then

i. $\nabla\alpha^t=\dfrac{(\alpha-1)\alpha^t}{\alpha}$

ii. $\nabla\big(f(t)g(t)\big)=f\big(\rho(t)\big)\nabla g(t)+\big(\nabla f(t)\big)g(t),$

iii. $\nabla\left(\dfrac{1}{f(t)}\right)=-\dfrac{\nabla f(t)}{f(t)f\big(\rho(t)\big)},$

iv. $\nabla\dfrac{f(t)}{g(t)}=\dfrac{g(t)\nabla f(t)-f(t)\nabla g(t)}{g(t)g(\rho(t))}.$

Proof: Using definition 1.1.1

i. $\nabla\alpha^t = \alpha^t - \alpha^{t-1} = \alpha^t\left(1 - \frac{1}{\alpha}\right) = \alpha^t\left(\frac{\alpha - 1}{\alpha}\right).$

ii. $\nabla\big(f(t)g(t)\big) = f(t)g(t) - f(t-1)g(t-1)$

$$= f(t)g(t) - f(t-1)g(t-1) + f(t-1)g(t) - f(t-1)g(t)$$

$$= f(t-1)\big(g(t) - g(t-1)\big) + g(t)(f(t) - f(t-1))$$

$$= f\big(\rho(t)\big)\nabla g(t) + g(t)\nabla f(t).$$

iii. $\nabla\left(\frac{1}{f(t)}\right) = \frac{1}{f(t)} - \frac{1}{f(t-1)} = \frac{f(t-1) - f(t)}{f(t)f(t-1)}$

$$= -\frac{f(t) - f(t-1)}{f(t)f(t-1)} = \frac{-\nabla f(t)}{f(t)f\big(\rho(t)\big)}.$$

iv. $\nabla\frac{f(t)}{g(t)} = \nabla\left(f(t) \times \frac{1}{g(t)}\right)$

Applying property ii and iii of this theorem, we have

$$\nabla\left(f(t) \times \frac{1}{g(t)}\right) = f\big(\rho(t)\big)\nabla\left(\frac{1}{g(t)}\right) + \nabla f(t)\frac{1}{g(t)}$$

$$= f\big(\rho(t)\big) \times \frac{-\nabla g(t)}{g(t)g\big(\rho(t)\big)} + \nabla f(t)\frac{g\big(\rho(t)\big)}{g(t)g\big(\rho(t)\big)}$$

$$= \frac{\nabla f(t)g\big(\rho(t)\big) - \nabla g(t)f(t-1)}{g(t)g(\rho(t))}$$

$$= \frac{\big(f(t) - f(t-1)\big)g\big(\rho(t)\big) - \big(g(t) - g(t-1)\big)f(t-1)}{g(t)g\big(\rho(t)\big)}$$

$$= \frac{f(t)g(t-1) - f(t-1)g(t-1) - g(t)f(t-1) + g(t-1)f(t-1)}{g(t)g\big(\rho(t)\big)}$$

$$= \frac{f(t)g(t-1) - g(t)f(t-1) + g(t)f(t) - f(t)g(t)}{g(t)g(\rho(t))}$$

$$= \frac{g(t)\big(f(t) - f(t-1)\big) - f(t)\big(g(t) - g(t-1)\big)}{g(t)g\big(\rho(t)\big)}$$

$$= \frac{g(t)\nabla f(t) - f(t)\nabla g(t)}{g(t)g(\rho(t))}.$$

1.2 Gamma function

The gamma function was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non-integer values. Later, because of its great importance, it was studied by other eminent mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), ... as well as many others.

The gamma function belongs to the category of the special transcendental functions and we will see that some famous mathematical constants are occurring in its study.

We make use of gamma function defined as the following:

Definition 1.2.1. The gamma function is defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$

The gamma function has following properties:

i. $\Gamma(x) > 0$.

ii. $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t}|_0^\infty + x\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)$.

iii. $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$.

We have, $\Gamma(1) = \int_0^\infty e^{-t}\, t^{1-1} dt = 1$, and this is why the gamma function can be interpreted as an extension of the factorial function to nonzero positive real number .

Example 1.2.2

i. $\Gamma(2) = \Gamma(1+1) = \; 1! = 1$

ii. $\Gamma(5) = \Gamma(4+1) = 4! = 4(3)(2)(1) = 24$

iii. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: by definition 1.2.1 then $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{\frac{1}{2}-1}\, dt = \int_0^\infty e^{-t} t^{\frac{-1}{2}}\, dt$

Let $t = u^2$ then $dt = 2u\, du$ then $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} u^{\frac{-2}{2}}\, 2u\, du = 2\int_0^\infty e^{-u^2}$

$$= 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

iv. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}\,.$

v. $\Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}\,.$

1.3 Rising and Filling Factorials

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$, then from the property of gamma function

$\Gamma(t+1) = t\Gamma(t)$, we have get

$$\begin{aligned}\Gamma(t+1) &= t\Gamma(t)\\ &= t(t-1)\Gamma(t-1)\\ &= t(t-1)(t-2)\Gamma(t-2)\\ &= \cdots = t(t-1)(t-2)\ldots\big(t-(n-1)\big)\Gamma\big(t-(n-1)\big).\end{aligned}$$

so $t(t-1)(\mathrm{t}-2)\ldots\big(t-(n-1)\big)=\dfrac{\Gamma(t+1)}{\Gamma\big(t-(n-1)\big)}$.

Also, $\Gamma(t+n)=(t+n-1)\Gamma(t+n-1)$

$$=(t+n-1)(t+n-2)\Gamma(t+n-2)$$

$$=\cdots=(t+n-1)(t+n-2)\ldots t\Gamma(t).$$

so $t(t+1)(t+2)\cdots(t+n-1)=\dfrac{\Gamma(t+n)}{\Gamma(t)}$.

Now, we can generalize these formulas for any real number α as the following definition.

Definition 1.3.1

i. for any real number α and $t\in\mathbb{R}$, filling function defined by

$$t^{\underline{\alpha}}=\frac{\Gamma(t+1)}{\Gamma\big(t-(\alpha-1)\big)}$$

ii. for any real number β and $t\in\mathbb{R}-\{\ldots,-2,-1,0\}$ rising function defined by

$$t^{\overline{\beta}}=\frac{\Gamma(t+\beta)}{\Gamma(t)}.$$

Remark 1.3.2. (see [3, 20])

i. If $t-\alpha+1$ is nonpositive integer and $t+1$ is not a nonpositive integer then $t^{\underline{\alpha}}=0$.

ii. If t is nonpositive integer and $t+\beta$ is not a nonpositive integer then $t^{\overline{\beta}}=0$.

Example 1.3.3 For $t\epsilon\mathbb{R}$, then

1. $t^{\underline{3}}=\dfrac{\Gamma(t+1)}{\Gamma(t-2)}=\dfrac{t(t-1)(t-2)\Gamma(t-2)}{\Gamma(t-2)}=t(t-1)(t-2)$.

2. $t^{\overline{3}} = \dfrac{\Gamma(t+3)}{\Gamma(t)} = \dfrac{(t+2)(t+1)t\Gamma(t)}{\Gamma(t)} = (t+2)(t+1)t.$

3. $2^{\underline{3}} = \dfrac{\Gamma(3+1)}{\Gamma(2-3+1)}$, by remark 1.3 then $2^{\underline{3}} = 0$

4. $2^{\overline{3}} = \dfrac{\Gamma(2+3)}{\Gamma(2)} = \dfrac{\Gamma(5)}{\Gamma(2)} = \dfrac{4 \times 3\Gamma(2)}{\Gamma(2)} = 24.$

5. $3^{\underline{1/2}} = \dfrac{\Gamma(3+1)}{\Gamma(3-(1/2)+1)} = \dfrac{\Gamma(4)}{\Gamma(7/2)} = \dfrac{4 \times 3 \times 2 \times 1}{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2})} = \dfrac{24}{\frac{15}{6}\sqrt{\pi}} = \dfrac{64}{5\sqrt{\pi}}$

6. $3^{\overline{1/2}} = \dfrac{\Gamma\left(3+(1/2)\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{15}{8}.$

Next we state and prove some properties of the factorial function.

Theorem 1.3.4. for any $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we have

i. $\Delta(t+\alpha)^{\underline{n}} = n(t+\alpha)^{\underline{n-1}}$,

ii. $\Delta(t+\alpha)^{\overline{m}} = m(t+\alpha+1)^{\overline{m-1}}$,

iii. $\nabla(t+\alpha)^{\underline{n}} = n(t+\alpha-1)^{\underline{n-1}}$,

iv. $\nabla(t+\alpha)^{\overline{m}} = m(t+\alpha)^{\overline{m-1}}$.

Proof:

i. Using the definition 1.3.1 (i) and definition 1.1.1 then

$$\Delta(t+\alpha)^{\underline{n}} = \Delta\frac{\Gamma(t+\alpha+1)}{\Gamma(t+\alpha-n+1)}$$

$$= \frac{\Gamma(t+\alpha+2)}{\Gamma(t+\alpha-n+2)} - \frac{\Gamma(t+\alpha+1)}{\Gamma(t+\alpha-n+1)}$$

$$= \frac{(t+\alpha+1)\Gamma(t+\alpha+1)}{(t+\alpha-n+1)\Gamma(t+\alpha-n+1)} - \frac{\Gamma(t+\alpha+1)}{\Gamma(t+\alpha-n+1)}$$

$$=\frac{(t+\alpha+1)\Gamma(t+\alpha+1)}{(t+\alpha-n+1)\Gamma(t-n+1)}-\frac{(t+\alpha-n+1)\Gamma(t+\alpha+1)}{(t+\alpha-n+1)\Gamma(t+\alpha-n+1)}$$

$$=\frac{\Gamma(t+\alpha+1)[(t+\alpha+1)-(t+\alpha-n+1)]}{(t+\alpha-n+1)\Gamma(t+\alpha-n+1)}$$

$$=\frac{n\Gamma(t+\alpha+1)}{\Gamma(t+\alpha-n+2)}=\frac{n\Gamma(t+\alpha+1)}{\Gamma(t+\alpha+1-(n-1))}=n(t+\alpha)^{\underline{n-1}}.$$

ii. using the definition 1.3.1 (ii) and definition 1.1.1 then

$$\Delta(t+\alpha)^{\overline{m}}=\Delta\frac{\Gamma(t+\alpha+m)}{\Gamma(t+\alpha)}=\frac{\Gamma(t+\alpha+m+1)}{\Gamma(t+\alpha+1)}-\frac{\Gamma(t+\alpha+m)}{\Gamma(t+\alpha)}$$

$$=\frac{(t+\alpha+m)\Gamma(t+\alpha+m)}{(t+\alpha)\Gamma(t+\alpha)}-\frac{\Gamma(t+\alpha+m)}{\Gamma(t+\alpha)}$$

$$=\frac{(t+\alpha+m)\Gamma(t+\alpha+m)}{(t+\alpha)\Gamma(t+\alpha)}-\frac{(t+\alpha)\Gamma(t+\alpha+m)}{(t+\alpha)\Gamma(t+\alpha)}$$

$$=\frac{\Gamma(t+\alpha+m)[t+\alpha+m-t-\alpha]}{(t+\alpha)\Gamma(t+\alpha)}$$

$$=\frac{m\Gamma(t+\alpha+m)}{(t+\alpha)\Gamma(t+\alpha)}=\frac{m\Gamma(t+\alpha+1+m-1)}{\Gamma(t+\alpha+1)}=m(t+\alpha+1)^{\overline{m-1}}.$$

iii. Using the definition 1.3.1 and definition 1.1.1 then

$$\nabla(t+\alpha)^{\underline{n}}=\nabla\frac{\Gamma(t+\alpha+1)}{\Gamma(t+\alpha-n+1)}$$

$$=\frac{\Gamma(t+\alpha+1)}{\Gamma(t+\alpha-n+1)}-\frac{\Gamma(t+\alpha)}{\Gamma(t+\alpha-n)}$$

$$=\frac{(t+\alpha)\Gamma(t+\alpha)}{(t+\alpha-n)\Gamma(t+\alpha-n)}-\frac{\Gamma(t+\alpha)}{\Gamma(t+\alpha-n)}$$

$$=\frac{(t+\alpha)\Gamma(t+\alpha)}{(t+\alpha-n)\Gamma(t+\alpha-n)}-\frac{(t+\alpha-n)\Gamma(t+\alpha)}{(t+\alpha-n)\Gamma(t+\alpha-n)}$$

$$= \frac{\Gamma(t+\alpha)(t+\alpha-t-\alpha+n)}{\Gamma(t+\alpha-n)(t+\alpha-n)}$$

$$= \frac{n\Gamma(t+\alpha)}{(t+\alpha-n)\Gamma(t+\alpha-n)} = \frac{n\Gamma(t+\alpha)}{\Gamma(t+\alpha-n+1)}$$

$$= \frac{n\Gamma(t+\alpha-1+1)}{\Gamma(t+\alpha-1-(n-1)+1)} = n(t+\alpha-1)^{\underline{n-1}}.$$

v. Using the definition 1.3.1 (ii) and definition 1.1.1 then

$$\nabla(t+\alpha)^{\overline{m}} = \nabla\frac{\Gamma(t+\alpha+m)}{\Gamma(t+\alpha)} = \frac{\Gamma(t+\alpha+m)}{\Gamma(t+\alpha)} - \frac{\Gamma(t+\alpha+m-1)}{\Gamma(t+\alpha-1)}$$

$$= \frac{(t+\alpha+m-1)\Gamma(t+\alpha+m-1)}{(t+\alpha-1)\Gamma(t+\alpha-1)} - \frac{\Gamma(t+\alpha+m-1)}{\Gamma(t+\alpha-1)}$$

$$= \frac{(t+\alpha+m-1)\Gamma(t+\alpha+m-1)}{(t+\alpha-1)\Gamma(t+\alpha-1)} - \frac{(t+\alpha-1)\Gamma(t+\alpha+m-1)}{(t+\alpha-1)\Gamma(t+\alpha-1)}$$

$$= \frac{\Gamma(t+\alpha+m-1)[(t+\alpha+m-1)-(t+\alpha-1)]}{(t+\alpha-1)\Gamma(t+\alpha-1)}$$

$$= \frac{m\Gamma(t+\alpha+m-1)}{\Gamma(t+\alpha)} = m(t+\alpha)^{\overline{m-1}}.$$

Remark 1.3.5. when $\alpha = 0$ then by theorem 1.3.4,we have .

i. $\Delta t^{\underline{n}} = nt^{\underline{n-1}}$,

ii. $\Delta t^{\overline{m}} = m(t+1)^{\overline{m-1}}$,

iii. $\nabla t^{\underline{n}} = n(t-1)^{\underline{n-1}}$,

iv. $\nabla t^{\overline{m}} = mt^{\overline{m-1}}$.

In the following theorem, we proved more properties of rising and filling factorials.

Theorem 1.3.6. (see [6]) for any $\alpha, \beta \in \mathbb{R}$, we have.

i. $(t-\alpha)t^{\underline{\alpha}} = t^{\underline{\alpha+1}}$;

ii. $\alpha^{\underline{\alpha}} = \Gamma(\alpha+1)$;

iii. If $t \leq r$ then $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ for any $\alpha > r$;

iv. If $0 < \alpha < 1$ then $t^{\underline{\alpha v}} \geq (t^{\underline{v}})^{\alpha}$,for any $v \epsilon \mathbb{R}$;

v. $t^{\underline{(\alpha+\beta)}} = (t-\beta)^{\underline{\alpha}}\, t^{\underline{\beta}}$;

vi. $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha}}$;

vii. $t^{\underline{\alpha}} = (t-\alpha+1)^{\overline{\alpha}}$.

Proof: Using the definition 1.3.1, we get

i. $(t-\alpha)t^{\underline{\alpha}} = (t-\alpha)\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} = (\mathrm{t}-\alpha)\frac{\Gamma(t+1)}{(t-\alpha)\Gamma(t-\alpha)}$

$$= \frac{\Gamma(t+1)}{\Gamma(t-\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-(\alpha+1)+1)} = t^{\underline{\alpha+1}}.$$

ii. $\alpha^{\underline{\alpha}} = \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\alpha+1)} = \dfrac{\Gamma(\alpha+1)}{\Gamma(1)} = \Gamma(\alpha+1)$

iii. If $t \le r$ then

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} = \frac{t\Gamma(t)}{(t-\alpha)\Gamma(t-\alpha)} \le \frac{r\Gamma(r)}{(r-\alpha)\Gamma(r-\alpha)} = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} = r^{\underline{\alpha}}.$$

iv. Use the log-convexity property of the gamma function then

$$t^{\underline{\alpha v}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha v+t\alpha-t\alpha+\alpha-\alpha)}$$

$$= \frac{\Gamma(t+1)}{\Gamma(\alpha(t+1-v)+(1-\alpha)(t+1))}$$

$$\ge \frac{\Gamma(t+1)}{\big(\Gamma(t+1-v)\big)^{\alpha}(\Gamma(t+1))^{1-\alpha}}$$

$$= \left(\frac{\Gamma(t+1)}{\Gamma(t+1-v)}\right)^{\alpha} = (\mathrm{t}^{\underline{\mathrm{v}}})^{\alpha}$$

Then $\mathrm{t}^{\underline{\alpha \mathrm{v}}} \ge (\mathrm{t}^{\underline{\mathrm{v}}})^{\alpha}$.

v. $t^{\underline{\alpha+\beta}} = \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha-\beta)} \times \dfrac{\Gamma(t+1-\beta)}{\Gamma(t+1-\beta)}$

$$= \frac{\Gamma(t+1-\beta)}{\Gamma(t+1-\alpha-\beta)} \times \frac{\Gamma(t+1)}{\Gamma(t+1-\beta)} = (t-\beta)^{\underline{\alpha}}\, t^{\underline{\beta}}.$$

vi. $t^{\overline{\alpha}} = \dfrac{\Gamma(t+\alpha)}{\Gamma(\mathrm{t})} = \dfrac{\Gamma(t+\alpha-1+1)}{\Gamma(t+\alpha-\alpha+1-1)} = (t+\alpha-1)^{\underline{\alpha}}$.

vii. $t^{\underline{\alpha}} = \dfrac{\Gamma(t+1)}{\Gamma(t-\alpha+1)} = \dfrac{\Gamma(t-\alpha+1+\alpha)}{\Gamma(t-\alpha+1)} = (t-\alpha+1).^{\overline{\alpha}}$

We see that (vi) and (vii) are equivalent because by direct substitution

$(t-\alpha+1)^{\overline{\alpha}} = (t-\alpha+1+\alpha-1))^{\underline{\alpha}} = t^{\underline{\alpha}}$.

Also, for our purposes we list down the following properties, the proofs of which are straightforward:

Theorem 1.3.7 : for any $\alpha, \beta \in \mathbb{R}$, we have.

i. For fixed $t \in \mathbb{R}$,

$$\nabla_s (s-t)^{\underline{\alpha-1}} = (\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}};$$

ii. For fixed $s \in \mathbb{R}$,

$$\nabla_t (\rho(s)-t)^{\underline{\alpha-1}} = -(\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}}.$$

Proof:

i. Direct by theorem 1.3.4 (iii)

ii. Using the definition 1.3.1 and definition 1.1.1 then

$$\nabla_t (\rho(s)-t)^{\underline{\alpha-1}} = \nabla_t \frac{\Gamma(\rho(s)-t+1)}{\Gamma(\rho(s)-t-(\alpha-1)+1)}$$

$$= \frac{\Gamma(\rho(s)-t+1)}{\Gamma(\rho(s)-t-(\alpha-1)+1)} - \frac{\Gamma(\rho(s)-(t-1)+1)}{\Gamma(\rho(s)-(t-1)-(\alpha-1)+1)}$$

$$= \frac{\Gamma(\rho(s)-t+1)}{\Gamma(\rho(s)-t-(\alpha-1)+1)} - \frac{\Gamma(\rho(s)-t+2)}{\Gamma(\rho(s)-t+1-\alpha+1+1)}$$

$$= \frac{\Gamma(\rho(s)-t+1)}{\Gamma(\rho(s)-t-(\alpha-2))} - \frac{(\rho(s)-t+1)\Gamma(\rho(s)-t+1)}{(\rho(s)-t-(\alpha-2))\Gamma(\rho(s)-t-(\alpha-2))}$$

$$= -(\alpha-1)\frac{\Gamma(\rho(s)-t+1)}{\Gamma(\rho(s)-t-(\alpha-2)+1)} = -(\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}}.$$

Theorem 1.3.8. The following are hold for any $\alpha, \beta \in \mathbb{R}$.

i. For fixed $s \in \mathbb{R}$, then

$$\Delta_t (s-\rho(t))^{\overline{\alpha}} = -\alpha\big(s-\rho(t)\big)^{\overline{\alpha-1}}.$$

ii. For fixed $t \in \mathbb{R}$, then

$$\Delta_s (s-t)^{\overline{\alpha}} = \alpha (s-\rho(t))^{\overline{\alpha-1}}.$$

Proof:

i. Using the definition 1.3.1 and definition 1.1.1 then

$$\Delta_t \big(s-\rho(t)\big)^{\overline{\alpha}} = \Delta_t \frac{\Gamma(s-\rho(t)+\alpha)}{\Gamma\big(s-\rho(t)\big)}$$

$$= \frac{\Gamma\big(s-\big(\rho(t+1)\big)+\alpha\big)}{\Gamma\Big(s-\big(\rho(t+1)\big)\Big)} - \frac{\Gamma(s-\rho(t)+\alpha)}{\Gamma\big(s-\rho(t)\big)}$$

$$= \frac{\Gamma(s-\rho(t)-1+\alpha)}{\Gamma(s-\rho(t)-1))} - \frac{(s-\rho(t)-1+\alpha)\Gamma(s-\rho(t)-1+\alpha)}{(s-\rho(t)-1)\Gamma(s-\rho(t)-1))}$$

$$= \frac{\Gamma(s-\rho(t)-1+\alpha)(-\alpha)}{\Gamma\big(s-\rho(t)\big)} = -\alpha\big(s-\rho(t)\big)^{\overline{\alpha-1}}.$$

ii. Direct by theorem 1.3.4 (ii)

Theorem 1.3.10. The following are hold

i. for fixed $t \in \mathbb{R}$, $\alpha \in \mathbb{R}$ then

$$\Delta_s (t-s)^{\underline{\alpha-1}} = -(\alpha-1)(t-\sigma(s))^{\underline{\alpha-2}},$$

ii. for fixed $s \in \mathbb{R}$, $\beta \in \mathbb{R}$ the

$$\Delta_t (t-\sigma(s))^{\underline{\alpha-1}} = (\alpha-1)(t-\sigma(s))^{\underline{\alpha-2}}$$

Proof:

i. Using the definition 1.3.1 and definition 1.1.1 then

$$\Delta_s (t-s)^{\underline{(\alpha-1)}} = \Delta_s \frac{\Gamma(t-s+1)}{\Gamma(t-s-(\alpha-1)+1)} = \Delta_s \frac{\Gamma(t-s+1)}{\Gamma(t-s-\alpha+2)}$$

$$= \frac{\Gamma(t-(s+1)+1)}{\Gamma(t-(s+1)-\alpha+2)} - \frac{\Gamma(t-s+1)}{\Gamma(t-s-\alpha+2)}$$

$$= \frac{\Gamma(t-s)}{\Gamma(t-s-\alpha+1)} - \frac{\Gamma(t-s+1)}{\Gamma(s-t-\alpha+2)}$$

$$= \frac{\Gamma(t-s)}{\Gamma(t-s-\alpha+1)} - \frac{(t-s)\Gamma(t-s)}{(t-s-\alpha+1)\Gamma(t-s-\alpha+1)}$$

$$= \frac{-(\alpha-1)\Gamma(s-t)}{\Gamma(t-s-\alpha+2)} = \frac{-(\alpha-1)\Gamma(t-s+1-1)}{\Gamma(t-s-(\alpha-2)+1-1)}$$

$$= \frac{-(\alpha-1)\Gamma(t-(s+1)+1)}{\Gamma(t-(s+1)+1-(\alpha-2))}$$

$$= -(\alpha-1)\frac{\Gamma(t-\sigma(s)+1)}{\Gamma\big(t-\sigma(s)+1-(\alpha-2)\big)}$$

$$= -(\alpha-1)(t-\sigma(s))^{\underline{\alpha-2}}.$$

ii. Direct by theorem 1.3.4 (i)

Theorem 1.3.11. The following are hold

i. for fixed $t \in \mathbb{R}$

$$\nabla_s (t-s)^{\overline{\alpha-1}} = -(\alpha-1)\big(t-\rho(s)\big)^{\overline{\alpha-2}},$$

ii. for fixed $s \in \mathbb{R}$

$$\nabla_t (t-\rho(s))^{\overline{\alpha-1}} = (\alpha-1)\big(t-\rho(s)\big)^{\overline{\alpha-2}}.$$

Proof:

i. Using the definition 1.3.1 and definition 1.1.1 then

$$\nabla_s (t-s)^{\overline{\alpha-1}} = \nabla_s \frac{\Gamma\big(t-s+(\alpha-1)\big)}{\Gamma(t-s)}$$

$$= \frac{\Gamma(t-s+(\alpha-1))}{\Gamma(t-s)} - \frac{\Gamma(t-(s-1)+(\alpha-1))}{\Gamma(t-(s-1))}$$

$$= \frac{\Gamma(t-s+(\alpha-1))}{\Gamma(t-s)} - \frac{\Gamma(t-s+(\alpha-1)+1)}{\Gamma(t-s+1)}$$

$$= \frac{\Gamma(t-s+(\alpha-1))}{\Gamma(t-s)} - \frac{(t-s+(\alpha-1))\Gamma(t-s+(\alpha-1))}{(t-s)\Gamma(t-s)}$$

$$= \frac{\Gamma(t-s+(\alpha-1))(t-s-(t-s+(\alpha-1))}{\Gamma(t-s+1)}$$

$$= -(\alpha-1)\frac{\Gamma\big(t-s+(\alpha-1)\big)}{\Gamma(t-s+1)}$$

$$= -(\alpha-1)\frac{\Gamma(t-(s-1)+(\alpha-2))}{\Gamma(t-(s-1))}$$

$$= -(\alpha-1)\frac{\Gamma(t-\rho(s)+(\alpha-2))}{\Gamma(t-\rho(s))} \;\; = -(\alpha-1)\big(t-\rho(s)\big)^{\overline{\alpha-2}}.$$

ii. Direct by theorem 1.3.4 (iv)

Theorem 1.3.12. The following is hold for fixed $t \in \mathbb{R}$,

$$\Delta_s(\rho(s)-\rho(t))^{\overline{\alpha-1}} = (\alpha-1)(s-\rho(t))^{\overline{\alpha-2}}.$$

Proof: Using theorem 1.3.8 then

$$\Delta_s\big(\rho(s)-\rho(t)\big)^{\overline{\alpha-1}} = \Delta_s\big((s-1)-(t-1)\big)^{\overline{\alpha-1}}$$

$$= \Delta_s(s-t)^{\overline{\alpha-1}} = (\alpha-1)(s-\rho(t))^{\overline{\alpha-2}}$$

Chapter 2

Delta and Nabla Fractional Sum and Difference

This chapter contains summary to some of the basic notations and definition of delta and nabla.

2.1 Fractional sum and difference.

In this section, we define the frame of delta and nabla fractional sum and difference in the Riemann sense.

Definition 2.1.1 . For any $a, b \in \mathbb{R}$, and any function f on $\mathbb{R}$ we can define

$$\sum_{a}^{b} f(s) = \sum_{[a]}^{[b]} f(s).$$

Where [a] is the greatest integer less than or equal to a.

Definition 2.1.2. [25] Let f be a function on $\mathbb{R}$ and $a, b \in \mathbb{R}$.

i. The (delta) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \left(t - \sigma(s)\right)^{\underline{\alpha-1}} f(s) , \qquad t \in \mathbb{N}_{a+\alpha}.$$

ii. The (delta) right fractional sum of order $\alpha > 0$ (ending at b) is defined by

$$ {}_b\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} \left(s - \sigma(t)\right)^{\underline{\alpha-1}} f(s)\,, $$

$$ = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s) - t)^{\underline{\alpha-1}} f(s), \qquad t \in {}_{b-\alpha}\mathbb{N} \quad . $$

iii. The (nabla) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$ \nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t - \rho(s)\right)^{\overline{\alpha-1}} f(s)\,, \qquad t \in \mathbb{N}_{a+1}. $$

iv. The (nabla) left fractional sum of order $\alpha > 0$ (ending at b) is defined by

$$ {}_b\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \left(t - \rho(s)\right)^{\overline{\alpha-1}} f(s) $$

$$ = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s) - t)^{\overline{\alpha-1}} f(s)\,, \qquad t \in {}_{b-1}\mathbb{N}. $$

Example 2.1.3. Let f is any function and $\alpha = 1$, then by Definition 2.1.2 and definition 2.1.1

1. $\Delta_a^{-1} f(t) = \frac{1}{\Gamma(1)} \sum_{s=a}^{t-1} \left(t - \sigma(s)\right)^{\underline{1-1}} f(s) = \sum_{s=a}^{t-1} f(s)$

   Now, if $a = \frac{1}{2}$ and $f(s) = s$, then $\Delta_a^{-1} f(t) = \sum_{s=\frac{1}{2}}^{t-1} s = \sum_{s=0}^{[t]-1} s$,

   Where $t \in \mathbb{N}_{\frac{1}{2}+1}$, i.e $t = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

2. ${}_b\Delta^{-1} f(t) = \frac{1}{\Gamma(1)} \sum_{s=t+1}^{b} \left(s - \sigma(t)\right)^{\underline{1-1}} f(s) = \sum_{s=t+1}^{b} f(s)$

Now, if $b = \frac{1}{2}$ and $f(s) = s$, then ${}_{\frac{1}{2}}\Delta^{-1} = \sum_{s=t+1}^{\frac{1}{2}} s = \sum_{[t]+1}^{\left[\frac{1}{2}\right]} s = \sum_{[t]+1}^{0} s$, where

$t \in {}_{\frac{1}{2}-1}\mathbb{N} = {}_{-\frac{1}{2}}\mathbb{N} = \left\{\frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, \cdots\right\}$.

3. $\nabla_a^{-1} f(t) = \dfrac{1}{\Gamma(1)} \displaystyle\sum_{s=a+1}^{t} \left(t - \rho(s)\right)^{\overline{1-1}} f(s) = \sum_{a+1}^{t} f(s)$

Now, if $a = \frac{1}{2}$ and $f(s) = s$, then $\nabla_{\frac{1}{2}}^{-1} f(t) = \sum_{s=\frac{3}{2}}^{t} s = \sum_{[3/2]}^{[t]} s = \sum_{s=1}^{[t]} s$,Where

$t \in \mathbb{N}_{\frac{1}{2}+1} = \mathbb{N}_{\frac{3}{2}} = \left\{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots\right\}$.

4. ${}_b\nabla^{-1} f(t) = \dfrac{1}{\Gamma(1)} \displaystyle\sum_{s=t}^{b-1} \left(t - \rho(s)\right)^{\overline{1-1}} f(s) = \sum_{t}^{b-1} f(s)$

Now, if $b = \frac{1}{2}$, $f(s) = s$ then ${}_{\frac{1}{2}}\nabla^{-1} f(t) = \sum_{s=t}^{-\frac{1}{2}} s = \sum_{s=t}^{-\frac{1}{2}} s = \sum_{s=[t]}^{-1} s$ Where

$t \in {}_{\frac{1}{2}-1}\mathbb{N} = \left\{\frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, \cdots\right\}$.

Note 2.1.4. [25] The domains of these fractional type sums are:

i. $\Delta_a^{-\alpha}$ maps function defined on $\mathbb{N}_a$ to function defined on $\mathbb{N}_{a+\alpha}$.

ii. ${}_b\Delta^{-\alpha}$ maps function defined on ${}_b\mathbb{N}$ to function defined on ${}_{b-\alpha}\mathbb{N}$.

iii. $\nabla_a^{-\alpha}$ maps function defined on $\mathbb{N}_a$ to function defined on $\mathbb{N}_a$.

iv. ${}_b\nabla^{-\alpha}$ maps function defined on $\mathbb{N}_a$ to function defined on ${}_b\mathbb{N}$.

Example 2.1.5. let $f, g : \mathbb{N}_a \longrightarrow \mathbb{R}$, then for $t \in \mathbb{N}_a$ the initial value problem

$$\begin{cases} \Delta^n u(t) = f(t) \\ u(a + j - 1) = 0 \qquad j = 1,2, \dots, n \end{cases}$$

Hence the solution to IVP is given by

$$\Delta^n u(t) = f(t) \Longrightarrow u(t) = \Delta_a^{-n} f(t), \text{then } u(t) = \frac{1}{\Gamma(n)} \sum_{s=a}^{t-n} \left(t - \sigma(s)\right)^{\underline{n-1}} f(s),$$

and $u(a+j-1) = \dfrac{1}{\Gamma(n)} \sum_{s=a}^{t-n} \left(a+j-1-\sigma(s)\right)^{\underline{n-1}} f(s) = 0.$

Example 2.1.6. let $f, g : \mathbb{N}_a \longrightarrow \mathbb{R}$, then for $t \in \mathbb{N}_a$ the initial value problem

$$\begin{cases} \nabla^n y(t) = f(t) \\ \nabla^i y(a) = 0 \qquad j = 0,1,\dots,n-1 \end{cases}$$

Hence the solution to IVP is given by

$$\nabla^n y(t) = f(t) \Rightarrow y(t) = \nabla_a^{-n} f(t), \text{then } y(t) = \frac{1}{\Gamma(n)} \sum_{s=t+n}^{b} \left(s-\sigma(t)\right)^{\underline{n-1}} f(s),$$

and $\nabla^i\, y(a) = \frac{\nabla^i}{\Gamma(n)} \sum_{s=t+n}^{b} \left(s-\sigma(t)\right)^{\underline{n-1}} f(s) = 0.$

Definition 2.1.7. For $n \in \mathbb{N}$, $t \in \mathbb{R}$ and f is any function, we denote

i. $_{\ominus}\Delta^n f(t) \triangleq (-1)^n \Delta^n f(t)$.

ii. $\nabla_{\ominus}^n f(t) \triangleq (-1)^n \nabla^n f(t)$.

For $m = 2,3,\dots$ we define ∇^m inductively $\nabla^m = \nabla\nabla^{m-1}$ and $\Delta^m = \Delta\Delta^{m-1}$.

Example 2.1.8. Let $u(t) = {}_b\nabla^{-1} f(t)$ when $n = 1$ then

$\nabla_{\ominus}^1 u(t) = \nabla_{\ominus}^1\, {}_b\nabla^{-1} f(t) = (-1)\nabla({}_b\nabla^{-1} f(t))$

$= -\nabla({}_b\nabla^{-1} f(t))$, use example 2.1.3 then

$$-\nabla({}_b\nabla^{-1} f(t)) = -\nabla\Big(\sum_{s=t}^{b-1} f(s)\Big) = -\Big(\sum_{s=t}^{b-1} f(s) - \sum_{s=t-1}^{b-1} f(s)\Big) = -f(t).$$

Next we define the fractional difference of order α by fractional sum [see 25].

Definition 2.1.9. Let f be a function on $\mathbb{R}$ and $n \in \mathbb{N}$.

i. The (delta) left fractional difference [15] of order $\alpha > 0$ (srarting from a) is defined by $\Delta_a^{\alpha} f(t) = \Delta^n \Delta^{-(n-\alpha)} f(t)$

$$= \frac{\Delta^n}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \left(t - \sigma(s)\right)^{\underline{n-\alpha-1}} f(s), t \in \mathbb{N}_{a+(n-\alpha)}$$

ii. The (delta) right fractional difference [21] of order $\alpha > 0$ (ending at b) is defined by ${}_b\Delta^{\alpha} f(t) = \nabla_{\ominus}^n {}_b\Delta^{-(n-\alpha)} f(t)$

$$= \frac{(-1)^n \nabla^n}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} \left(s - \sigma(t)\right)^{\underline{n-\alpha-1}} f(s), t \in {}_{b-(n-\alpha)}\mathbb{N}.$$

iii. The (nabla) left fractional difference of order $\alpha > 0$ (srarting from a) is defined by $\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t)$

$$= \nabla^n \left[\tfrac{1}{\Gamma(n-\alpha)} \textstyle\sum_{s=a+1}^{t} \left(t - \rho(s)\right)^{\overline{n-\alpha-1}} f(s) \right], t \in \mathbb{N}_{a+1}.$$

iv. The (nabla) right fractional difference of order $\alpha > 0$ (ending at b) is defined by ${}_b\nabla^{\alpha} f(t) = {}_{\ominus}\Delta^n \; {}_b\Delta^{-(n-\alpha)} f(t)$

$$= \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} \left(s - \rho(t)\right)^{\overline{n-\alpha-1}} f(s), t \in {}_{b-1}\mathbb{N}.$$

Theorem 2.1.10 (Leibniz Rule) [20] Let $g: \mathbb{N}_{a+\alpha} \times \mathbb{N}_a \to \mathbb{R}$ be given then

$$\Delta\left(\sum_{s=a}^{t-\alpha} g(t,s)\right) = \sum_{s=a}^{t-\alpha} \Delta_t g(t,s) + g(t+1, t+1-\alpha) \quad \text{for } t \epsilon \mathbb{N}_{a+\alpha}.$$

Proof: By definition 1.1.1 in chapter 1

$$\Delta\left(\sum_{s=a}^{t-\alpha} g(t,s)\right) = \sum_{s=a}^{t+1-\alpha} g(t+1,s) - \sum_{s=a}^{t-\alpha} g(t,s)$$

$$= \sum_{s=a}^{t-\alpha} g(t+1,s) + g(t+1,t+1-\alpha) - \sum_{s=a}^{t-\alpha} g(t,s)$$

$$= \sum_{s=a}^{t-\alpha} g(t+1,s) - \sum_{s=a}^{t-\alpha} g(t,s) + g(t+1,t+1-\alpha)$$

$$= \sum_{s=a}^{t-\alpha} \Delta_t g(t,s) + g(t+1,t+1-\alpha).$$

Theorem 2.1.11. The following equality holds

$$\nabla\left(\sum_{s=t+\alpha}^{b} g(t,s)\right) = \sum_{s=t+\alpha}^{b} \nabla g(t,s) - g(t-1,t-1+\alpha)$$

Proof: By definition 1.1.1 in chapter 1

$$\nabla_t\left(\sum_{s=t+\alpha}^{b} g(t,s)\right) = \sum_{s=t+\alpha}^{b} g(t,s) - \sum_{s=t-1+\alpha}^{b} g(t-1,s)$$

$$= \sum_{s=t+\alpha}^{b} g(t,s) - \left[\sum_{s=t+\alpha}^{b} g(t-1,s) + g(t-1,t-1+\alpha)\right]$$

$$= \sum_{s=t+\alpha}^{b} g(t,s) - \sum_{s=t+\alpha}^{b} g(t-1,s) - g(t-1,t-1+\alpha)$$

$$= \sum_{s=t+\alpha}^{b} [g(t,s) - g(t-1,s)] - g(t-1,t-1+\alpha)$$

$$= \sum_{s=t+\alpha}^{b} \nabla_t\, g(t,s) - g(t-1,t-1+\alpha).$$

Theorem 2.1.12. For any $\alpha > 0$, Let $f: \mathbb{N}_a \to \mathbb{R}$, the following is hold

$$\Delta\left(\Delta_a^{-\alpha} f(t)\right) = \Delta_a^{-\alpha+1} f(t) = \Delta_a^{-(\alpha-1)} f(t).$$

Proof: By definition 2.1.2

$$\Delta\left(\Delta_a^{-\alpha} f(t)\right) = \Delta\left(\frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \left(t - \sigma(s)\right)^{\underline{\alpha-1}} f(s)\right)$$

Let $(t,s) = \frac{1}{\Gamma(\alpha)}\left(t - \sigma(s)\right)^{\underline{\alpha-1}} f(s)$, by Leibniz Rule

$$\Delta\left(\sum_{s=a}^{t-\alpha} g(t,s)\right) = \sum_{s=a}^{t-\alpha} \Delta_t g(t,s) + g(t+1, t+1-\alpha),$$

Use theorem 1.3.10

$$\Delta_t g(t,s) = \Delta_t \frac{1}{\Gamma(\alpha)}\left(t - \sigma(s)\right)^{\underline{\alpha-1}} f(s) = \frac{(\alpha-1)}{\Gamma(\alpha)}\left(t - \sigma(s)\right)^{\underline{\alpha-2}} f(s) \cdots (1)$$

and, $g(t+1, t+1-\alpha) = \frac{1}{\Gamma(\alpha)}(t+1-(t+1-\alpha+1))^{\underline{\alpha-1}} f(t+1-\alpha)$

$$= \frac{(\alpha-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(t+1-\alpha) = f(t+1-\alpha) \cdots (2)$$

knowing, $$\frac{(\alpha-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-1+1)}{\Gamma(\alpha)\Gamma((\alpha-1)-(\alpha-1)+1))} = \frac{\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(1)} = 1$$

Use equation 1 and equation 2 then

$$\Delta\left(\sum_{s=a}^{t-\alpha} g(t,s)\right) = \sum_{s=a}^{t-\alpha} \Delta_t g(t,s) + g(t+1, t+1-\alpha)$$

$$= \sum_{s=a}^{t-\alpha} \frac{(\alpha-1)}{\Gamma(\alpha)}\left(t - \sigma(s)\right)^{\underline{\alpha-2}} f(s) + f(t+1-\alpha)$$

$$= \sum_{s=a}^{t+1-\alpha} \frac{(\alpha-1)}{\Gamma(\alpha)} \big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s)$$

$$= \sum_{s=a}^{t+1-\alpha} \frac{1}{\Gamma(\alpha-1)} \big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s)$$

$$= \sum_{s=a}^{t+1-\alpha} \frac{1}{\Gamma(\alpha-1)} \big(t-\sigma(s)\big)^{\underline{(\alpha-1)-1}} f(s) = \Delta_a^{-(\alpha-1)} f(t) = \Delta_a^{-\alpha+1} f(t).$$

In fact, we can prove the theorem for any $n \in \mathbb{N}$, i.e (see [17])

$$\Delta^n(\Delta_a^{-\alpha} f(t)) = \Delta_a^{-\alpha+n} f(t)\,.$$

The following theorems relation between difference operator and fractional sum.

Lemma 2.1.13. [8] For any $\alpha > 0$ the following equality

$$\Delta_a^{-\alpha}\Delta f(t) = \Delta\Delta_a^{-\alpha} f(t) - \frac{(t-a)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(a)$$

Proof: First we need to prove the following

$$\Delta_s\big((t-s)^{\underline{\alpha-1}} f(s)\big) = \big(t-\sigma(s)\big)^{\underline{\alpha-1}} \Delta_s f(s) - (\alpha-1)\big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s).$$

Use theorem 1.1.6 and use theorem 1.3.10 in chapter 1 then

$$\Delta_s\big((t-s)^{\underline{\alpha-1}} f(s)\big) = \big(t-\sigma(s)\big)^{\underline{\alpha-1}} \Delta_s f(s) + \Delta_s (t-s)^{\underline{\alpha-1}} f(s)$$

$$= \big(t-\sigma(s)\big)^{\underline{\alpha-1}} \Delta_s f(s) - (\alpha-1)\big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s) \cdots (1)$$

Now, by definition 2.1.2 then

$$\Delta_a^{-\alpha}\Delta f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \big(t-\sigma(s)\big)^{\underline{\alpha-1}} \Delta_s f(s)$$

and by equation 1 then

$$\Delta_a^{-\alpha}\Delta f(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left[(\alpha-1)\left(t-\sigma(s)\right)^{\underline{\alpha-2}}f(s) + \Delta_s((t-s)^{\underline{\alpha-1}}f(s))\right]$$

$$= \frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left(t-\sigma(s)\right)^{\underline{\alpha-2}}f(s) + \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\Delta_s((t-s)^{\underline{\alpha-1}}f(s))$$

$$= \frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left(t-\sigma(s)\right)^{\underline{\alpha-2}}f(s) + \frac{(t-s)^{\underline{\alpha-1}}f(s)}{\Gamma(\alpha)}\Bigg|_a^{t+1-\alpha}$$

$$= \frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left(t-\sigma(s)\right)^{\underline{\alpha-2}}f(s) + \frac{(\alpha-1)^{\underline{\alpha-1}}f(t+1-\alpha)}{\Gamma(\alpha)} - \frac{(t-a)^{\underline{\alpha-1}}f(a)}{\Gamma(\alpha)}$$

but, $\frac{(\alpha-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)} = 1$ so

$$\Delta_a^{-\alpha}\Delta f(t) = \frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left(t-\sigma(s)\right)^{\underline{\alpha-2}}f(s) + f(t+1-\alpha) - \frac{(t-a)^{\underline{\alpha-1}}f(a)}{\Gamma(\alpha)}$$

$$= \frac{1}{\Gamma(\alpha-1)}\sum_{s=a}^{t-(\alpha-1)}\left(t-\sigma(s)\right)^{\underline{\alpha-2}}f(s) - \frac{(t-a)^{\underline{\alpha-1}}f(a)}{\Gamma(\alpha)}$$

since $\Delta\Delta_a^{-\alpha}f(t) = \Delta\left[\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left(t-\sigma(t)\right)^{\underline{\alpha-1}}f(s)\right]$

By Leibniz rule, let $g(t,s) = \frac{1}{\Gamma(\alpha)}\left(t-\sigma(t)\right)^{\underline{\alpha-1}}f(s)$ so,

$$\Delta\sum_{s=a}^{t-\alpha}g(t,s) = \sum_{s=a}^{t-\alpha}\Delta g(t,s) + g(t+1,t+1-\alpha)$$

By theorem 1.3.10

$$\Delta g(t,s) = \Delta_t\left(\frac{1}{\Gamma(\alpha)}\big(t-\sigma(s)\big)^{\underline{\alpha-1}} f(s)\right) = \frac{1}{\Gamma(\alpha)}(\alpha-1)\big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s)$$

$$= \frac{1}{\Gamma(\alpha-1)}\big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s) \cdots\cdots (2)$$

and, $g(t+1, t+1-\alpha) = \dfrac{1}{\Gamma(\alpha)}(t+1-\sigma(t+1-\alpha))^{\underline{\alpha-1}}\, f(t+1-\alpha)$

$$= \frac{1}{\Gamma(\alpha)}\big(t+1-(t+1-\alpha+1)\big)^{\underline{\alpha-1}} f(t+1-\alpha)$$

$$= \frac{1}{\Gamma(\alpha)}(\alpha-1)^{\underline{\alpha-1}} f(t+1-\alpha)$$

since $\dfrac{1}{\Gamma(\alpha)}(\alpha-1)^{\underline{\alpha-1}} = 1$ then $g(t+1, t+1-\alpha) = f(t+1-\alpha)\ \cdots (3)$

So by equation 2 and 3 then

$$\Delta\Delta_a^{-\alpha} f(t) = \sum_{s=a}^{t-\alpha} \frac{1}{\Gamma(\alpha-1)}\big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s)\ + f\big(t-(\alpha-1)\big)$$

$$= \sum_{s=a}^{t-\alpha+1} \frac{1}{\Gamma(\alpha-1)}\big(t-\sigma(s)\big)^{\underline{\alpha-2}} f(s)$$

then $\Delta_a^{-\alpha}\Delta f(t) = \Delta\Delta_a^{-\alpha} f(t) - \dfrac{(t-a)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(a)$.

Lemma 2.1.14. The following statement is valid, for all $n \in \mathbb{N}$, and $\alpha > 0$,

1. $\Delta^n \dfrac{(t-a)^{\underline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \dfrac{(t-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}$

2. $\nabla^n \dfrac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \dfrac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}$

3. $${}_{\ominus}\Delta^{n}\frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}.$$

Proof:

i. Use identity $\Delta(t-a)^{\underline{\alpha-1}}=(\alpha-1)(t-a)^{\underline{\alpha-2}}$,then by induction

$$\Delta\frac{(t-a)^{\underline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(n-\alpha-1)(t-a)^{\underline{n-\alpha-2}}}{\Gamma(n-\alpha)}$$

$$\Delta^{2}\frac{(t-a)^{\underline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(n-\alpha-1)(n-\alpha-2)(t-a)^{\underline{n-\alpha-3}}}{\Gamma(n-\alpha)}$$

$$\vdots$$

$$\Delta^{n}\frac{(t-a)^{\underline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)(t-a)^{\underline{-\alpha-1}}}{\Gamma(n-\alpha)}$$

$$=\frac{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)(t-a)^{\underline{-\alpha-1}}}{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)\Gamma(-\alpha)}$$

$$=\frac{(t-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}.$$

ii. Use identity $\nabla(t-a)^{\overline{\alpha-1}}=(\alpha-1)(t-a)^{\overline{\alpha-2}}$ then by induction

$$\nabla\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(n-\alpha-1)(t-a)^{\overline{n-\alpha-2}}}{\Gamma(n-\alpha)}$$

$$\nabla^{2}\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(n-\alpha-1)(n-\alpha-2)(t-a)^{\overline{n-\alpha-3}}}{\Gamma(n-\alpha)}$$

$$\vdots$$

$$\nabla^{n}\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)(t-a)^{\overline{-\alpha-1}}}{\Gamma(n-\alpha)}$$

$$=\frac{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)(t-a)^{\overline{-\alpha-1}}}{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)\Gamma(-\alpha)}$$

$$=\frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}.$$

iii. Use identity ${}_{\ominus}\Delta(b-t)^{\overline{\alpha-1}} = (\alpha-1)(b-t)^{\overline{\alpha-2}}$,then

$$_{\ominus}\Delta \; \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(n-\alpha-1)(b-t)^{\overline{n-\alpha-2}}}{\Gamma(n-\alpha)}$$

$$_{\ominus}\Delta^2 \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(n-\alpha-1)(n-\alpha-2)(b-t)^{\overline{n-\alpha-3}}}{\Gamma(n-\alpha)}$$

$$\vdots$$

$$_{\ominus}\Delta^n \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)(b-t)^{\overline{-\alpha-1}}}{\Gamma(n-\alpha)}$$

$$= \frac{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)(b-t)^{\overline{-\alpha-1}}}{(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)\Gamma(-\alpha)}$$

$$= \frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}.$$

Remark 2.1.15. [8] Let $p-1<\alpha<p$, where p is positive integer. Then by the help of Lemma 2.1.13 we have

$$\Delta\Delta_a^{\alpha} f(t) = \Delta\Delta^p\left(\Delta_a^{-(p-\alpha)} f(t)\right) = \Delta^p\left(\Delta\Delta_a^{-(p-\alpha)} f(t)\right)$$

Or

$$\Delta\Delta_a^{\alpha} f(t) = \Delta^p\left[\Delta_a^{-(p-\alpha)}\Delta f(t) + \frac{(t-a)^{\underline{p-\alpha-1}}}{\Gamma(p-\alpha)} f(a)\right]$$

Lemma 2.1.16. for any $\alpha \in \mathbb{R}$, Then

$$\Delta\Delta_a^{\alpha} f(t) = \Delta_a^{\alpha}\Delta f(t) + \frac{(t-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}$$

Proof: we prove in lemma 2.1.13, for any $\alpha>0$, that

$$\Delta_a^{-\alpha}\Delta f(t) = \Delta\Delta_a^{-\alpha} f(t) - \frac{(t-a)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(a),$$

But if $\alpha < 0$, then $-\alpha > 0$, say $\beta = -\alpha > 0$, therefore

$$\Delta\Delta_a^{\beta} f(t) = \Delta^p \left[\Delta_a^{-(p-\beta)} \Delta f(t) + \frac{(t-a)^{\underline{p-\beta-1}}}{\Gamma(p-\alpha)} f(a)\right]$$

and using identity $\Delta^n \frac{(t-a)^{\underline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(t-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}$, then

$$\Delta\Delta_a^{\beta} f(t) = \Delta_a^{\beta} \Delta f(t) + \frac{(t-a)^{\underline{-\beta-1}}}{\Gamma(-\beta)}$$

Take $\alpha = -\beta$, we done.

Lemma 2.1.17.[25] For any $\alpha > 0$, the following equality holds

$${}_b\Delta^{-\alpha} \nabla_{\ominus} f(t) = \nabla_{\ominus}\, {}_b\Delta^{-\alpha} f(t) - \frac{(b-t)^{\underline{\alpha-1}} f(b)}{\Gamma(\alpha)}$$

Proof: First we need to prove the following

$$\nabla_s[(s-t)^{\underline{\alpha-1}} f(s)] = (\rho(s)-t)^{\underline{\alpha-1}} \nabla_s f(s) + (\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}} f(s)$$

Use theorem 1.1.7 and use theorem 1.3.7 in chapter 1 then

$$\nabla_s[(s-t)^{\underline{\alpha-1}} f(s)] = (\rho(s)-t)^{\underline{\alpha-1}} \nabla_s f(s) + \nabla_s (s-t)^{\underline{\alpha-2}} f(s)$$

$$= (\rho(s)-t)^{\underline{\alpha-1}} \nabla_s f(s) + (\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}} f(s) \cdots (1)$$

Now, by definition 2.1.2 and by equation 1 then

$${}_b\Delta^{-\alpha} \nabla_{\ominus} f(t) = -\frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s)-t)^{\underline{\alpha-1}}\, \nabla_s f(s)$$

$$= -\frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\nabla_s[(s-t)^{\underline{\alpha-1}} f(s)] - (\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}} f(s))$$

$$=\frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}}f(s)-\frac{1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b}\nabla_s[(s-t)^{\underline{\alpha-1}}f(s)]$$

$$=\frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}}f(s)-\frac{1}{\Gamma(\alpha)}[(s-t)^{\underline{\alpha-1}}f(s)]_{t-1+\alpha}^{b}$$

$$=\frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}}f(s)$$

$$-\frac{1}{\Gamma(\alpha)}[(b-t)^{\underline{\alpha-1}}f(b)-(t-1+\alpha-t)^{\underline{\alpha-1}}f(t-1+\alpha)]$$

$$=\frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}}f(s)$$

$$-\left[\frac{1}{\Gamma(\alpha)}(b-t)^{\underline{\alpha-1}}f(b)-\frac{1}{\Gamma(\alpha)}(\alpha-1)^{\underline{\alpha-1}}f(t-1+\alpha)\right]$$

Since $\frac{1}{\Gamma(\alpha)}(\alpha-1)^{\underline{\alpha-1}}=1$ so,

$${}_{b}\Delta^{-\alpha}\nabla_{\ominus}f(t)=\frac{\alpha-1}{\Gamma(\alpha)}\sum_{t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}}f(s)+f(t-1+\alpha)-\frac{1}{\Gamma(\alpha)}(b-t)^{\underline{\alpha-1}}f(b)$$

$$=\frac{1}{\Gamma(\alpha-1)}\sum_{s=t-1+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}}f(s)-\frac{1}{\Gamma(\alpha)}(b-t)^{\underline{\alpha-1}}f(b),$$

On other hand, $\nabla_{\ominus}\,{}_{b}\Delta^{-\alpha}f(t)=-\nabla\,{}_{b}\Delta^{-\alpha}f(t)=-\nabla\left(\frac{1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-1}}f(s)\right)$

By theorem 2.1.9, let $g(t,s)=\frac{(\rho(s)-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)}f(s)$

So, $\nabla\left(\sum_{s=t+\alpha}^{b}g(t,s)\right)=\sum_{s=t+\alpha}^{b}\nabla_t g(t,s)-g(t-1,t-1+\alpha)$

By theorem 1.3.11

$$\nabla_t g(t,s) = \nabla_t \frac{(\rho(s)-t)^{\underline{\alpha-1}} f(s)}{\Gamma(\alpha)} = -\frac{(\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}} f(s)}{\Gamma(\alpha)}$$

$$= -\frac{1}{\Gamma(\alpha-1)}(\rho(s)-t)^{\underline{\alpha-2}} f(s) \cdots (2)$$

and, $\quad g(t-1, t-1+\alpha) = \frac{1}{\Gamma(\alpha)}\big(\rho(t-1+\alpha)-(t-1)\big)^{\underline{\alpha-1}} f(t-1+\alpha)$

Since $\frac{1}{\Gamma(\alpha)}(\alpha-1)^{\underline{\alpha-1}} = 1$ then $g(t-1, t-1+\alpha) = f(t-1+\alpha) \cdots (3)$

So by equation 2 and 3 then

$$\nabla_{\ominus}\, {}_b\Delta^{-\alpha} f(t) = -\nabla\, {}_b\Delta^{-\alpha} f(t)$$

$$= -\left[-\frac{1}{\Gamma(\alpha-1)}\sum_{t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}} f(s) - f(t-1+\alpha)\right]$$

$$= \frac{1}{\Gamma(\alpha-1)}\sum_{t+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}} f(s) + f(t-1+\alpha)$$

$$= \frac{1}{\Gamma(\alpha-1)}\sum_{t-1+\alpha}^{b}(\rho(s)-t)^{\underline{\alpha-2}} f(s)$$

Then $\; {}_b\Delta^{-\alpha}\nabla_{\ominus} f(t) = \nabla_{\ominus}\, {}_b\Delta^{-\alpha} f(t) - \frac{(b-t)^{\underline{\alpha-1}} f(b)}{\Gamma(\alpha)}$.

Lemma 2.1.18. [24] For any $\alpha > 0$, the following equality holds

$$\nabla_a^{-\alpha}\nabla \mathrm{f(t)} = \nabla\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}} f(a)}{\Gamma(\alpha)}.$$

Proof: First we need to prove the following

$$\nabla_s\left[(t-s)^{\overline{\alpha-1}}f(s)\right] = \big(t-\rho(s)\big)^{\overline{\alpha-1}}\nabla_s f(s) - (\alpha-1)\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s)$$

Use theorem 1.1.7 and use theorem 1.3.11 in chapter 1 then

$$\nabla_s\left[(t-s)^{\overline{\alpha-1}}f(s)\right] = \big(t-\rho(s)\big)^{\overline{\alpha-1}}\nabla_s f(s) + \nabla_s(t-s)^{\overline{\alpha-1}}f(s)$$

$$= \big(t-\rho(s)\big)^{\overline{\alpha-1}}\nabla_s f(s) - (\alpha-1)\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s) \cdots (1)$$

Now, by definition 2.1.2 and by equation 1 then

$$\nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla \mathrm{f(t)} = \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\nabla_s f(s)$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\left[\nabla_s\Big((t-s)^{\overline{\alpha-1}}f(s)\Big) + (\alpha-1)\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s)\right]$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\nabla_s\Big((t-s)^{\overline{\alpha-1}}f(s)\Big) + \frac{(\alpha-1)}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s)$$

$$= \frac{1}{\Gamma(\alpha)}(t-s)^{\overline{\alpha-1}}f(s)\Big|_a^t + \frac{(\alpha-1)}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s)$$

$$= \frac{1}{\Gamma(\alpha)}(t-t)^{\overline{\alpha-1}}f(s) - \frac{1}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}}f(s) + \frac{1}{\Gamma(\alpha-1)}\sum_{s=a+1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s)$$

since $0^{\overline{\alpha-1}} = 0$,

so $\nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla \mathrm{f(t)} = \dfrac{1}{\Gamma(\alpha-1)}\displaystyle\sum_{s=a+1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s) - \dfrac{1}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}}f(s)$

On the other hand,

$$\nabla\nabla_a^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\nabla_t(t-\rho(s)^{\overline{\alpha-1}}f(s)=\frac{1}{\Gamma(\alpha-1)}\sum_{s=a+1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-2}}f(s)$$

Then $\nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla\mathrm{f(t)}=\nabla\nabla_a^{-\alpha}f(t)-\dfrac{(t-a)^{\overline{\alpha-1}}f(a)}{\Gamma(\alpha)}$.

Remark 2.1.19.[25] Let $\alpha>0$ and $n=[\alpha]+1$. Then, by the help of lemma 2.1.18 we have

$$\nabla\nabla_a^{\alpha}f(t)=\nabla\nabla^n\Big(\nabla_a^{-(n-\alpha)}f(t)\Big)=\nabla^n\Big(\nabla\nabla_a^{-(n-\alpha)}f(t)\Big)$$

Or

$$\nabla\nabla_a^{\alpha}f(t)=\nabla^n\left[\nabla_a^{-(n-\alpha)}\nabla f(t)+\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}f(a)\right].$$

Lemma 2.1.20. For $\alpha\in\mathbb{R}$, then

$$\nabla\nabla_a^{\alpha}f(t)=\nabla_a^{\alpha}\nabla f(t)+\frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}$$

Proof: we prove in lemma 2.1.18, for any $\alpha>0$, we have

$$\nabla_a^{-\alpha}\nabla f(t)=\nabla\nabla_a^{-\alpha}f(t)-\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a),$$

But if $\alpha<0$, then $-\alpha>0$, say $\beta=-\alpha>0$, therefore

$$\nabla\nabla_a^{\beta}f(t)=\nabla^p\left[\nabla_a^{-(p-\beta)}\nabla f(t)+\frac{(t-a)^{\overline{p-\beta-1}}}{\Gamma(p-\alpha)}f(a)\right]$$

and using identity $\nabla^n\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}=\frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}$, then

$$\nabla\nabla_a^{\beta}f(t)=\nabla_a^{\beta}\nabla f(t)+\frac{(t-a)^{\overline{-\beta-1}}}{\Gamma(-\beta)}$$

Take $\alpha = -\beta$, we done.

Theorem 2.1.21. For any real number α and any positive integer p the following equality holds:

$$\nabla_a^{-\alpha}\nabla^p f(t) = \nabla^p \nabla_a^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a)$$

Where f is defined on $\mathbb{N}_a$ and some points before a.

Proof: Use equation of lemma 2.1.18 we replace f by ∇f then

$$\nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla^2 \mathrm{f(t)} = \nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla\nabla \mathrm{f(t)} = \nabla\nabla_a^{-\alpha}\nabla f(t) - \frac{(t-a)^{\overline{\alpha-1}}\nabla f(a)}{\Gamma(\alpha)},$$

Use lemma 2.1.18 again then

$$\nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla^2 \mathrm{f(t)} = \nabla\left[\nabla\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}} f(a)}{\Gamma(\alpha)}\right] - \frac{(t-a)^{\overline{\alpha-1}}\nabla f(a)}{\Gamma(\alpha)}$$

$$= \nabla^2\nabla_a^{-\alpha} f(t) - \frac{\nabla(t-a)^{\overline{\alpha-1}} f(a)}{\Gamma(\alpha)} - \frac{(t-a)^{\overline{\alpha-1}}\nabla f(a)}{\Gamma(\alpha)},$$

Use theorem 1.3.11 in chapter 1

$$\nabla_{\boldsymbol{a}}^{-\boldsymbol{\alpha}}\nabla^2 \mathrm{f(t)} = \nabla^2\nabla_a^{-\alpha} f(t) - \frac{(\alpha-1)(t-a)^{\overline{\alpha-1}} f(a)}{\Gamma(\alpha)} - \frac{(t-a)^{\overline{\alpha-1}}\nabla f(a)}{\Gamma(\alpha)},$$

$$= \nabla^2\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}} f(a)}{\Gamma(\alpha-1)} - \frac{(t-a)^{\overline{\alpha-1}}\nabla f(a)}{\Gamma(\alpha)}$$

$$\nabla_a^{-\alpha}\nabla^2 f(t) = \nabla^2\nabla_a^{-\alpha} f(t) - \sum_{k=0}^{1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a).$$

Repeated interaction gives the desired result.

Theorem 2.1.22. For any real number α and any positive integer p the following equality holds:

$$\Delta_a^{-\alpha}\Delta^p f(t) = \Delta^p \Delta_a^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \Delta^k f(a),$$

Where f is defined on $\mathbb{N}_a$ and some points before a.

Proof: Use equation of lemma 2.1.13 we replace f by Δf then

$$\Delta_a^{-\alpha}\Delta^2 \mathrm{f(t)} = \Delta_a^{-\alpha}\Delta\Delta \mathrm{f(t)} = \Delta\Delta_a^{-\alpha}\nabla f(t) - \frac{(t-a)^{\underline{\alpha-1}}\Delta f(a)}{\Gamma(\alpha)},$$

Use lemma 2.1.13 again then

$$\Delta_a^{-\alpha}\Delta^2 \mathrm{f(t)} = \Delta\left[\Delta\Delta_a^{-\alpha} f(t) - \frac{(t-a)^{\underline{\alpha-1}} f(a)}{\Gamma(\alpha)}\right] - \frac{(t-a)^{\underline{\alpha-1}}\Delta f(a)}{\Gamma(\alpha)},$$

$$= \Delta^2\Delta_a^{-\alpha} f(t) - \frac{\Delta(t-a)^{\underline{\alpha-1}} f(a)}{\Gamma(\alpha)} - \frac{(t-a)^{\underline{\alpha-1}}\Delta f(a)}{\Gamma(\alpha)},$$

Use theorem 1.3.10 in chapter 1

$$\Delta_a^{-\alpha}\Delta^2 \mathrm{f(t)} = \Delta^2\Delta_a^{-\alpha} f(t) - \frac{(\alpha-1)(t-a)^{\underline{\alpha-1}} f(a)}{\Gamma(\alpha)} - \frac{(t-a)^{\underline{\alpha-1}}\Delta f(a)}{\Gamma(\alpha)},$$

$$= \Delta^2\Delta_a^{-\alpha} f(t) - \frac{(t-a)^{\underline{\alpha-1}} f(a)}{\Gamma(\alpha-1)} - \frac{(t-a)^{\underline{\alpha-1}}\Delta f(a)}{\Gamma(\alpha)},$$

$$\Delta_a^{-\alpha}\Delta^2 f(t) = \Delta^2\Delta_a^{-\alpha} f(t) - \sum_{k=0}^{1} \frac{(t-a)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \Delta^k f(a).$$

Repeated interaction gives the desired result.

Lemma 2.1.23. [23] For any $\alpha > 0$, the following equality holds

$$ {}_b\nabla^{-\alpha}{}_{\ominus}\Delta f(t) = {}_{\ominus}\Delta\, {}_b\nabla^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b). $$

Proof: First we need to prove the following

$$ \Delta_s\left[(\rho(s)-\rho(t))^{\overline{\alpha-1}} f(s)\right] = (s-\rho(t))^{\overline{\alpha-1}} \Delta_s f(s) + (\alpha-1)\big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) $$

Use theorem 1.1.6 and use theorem 1.3.12 in chapter 1 then

$$ \Delta_s\left[(\rho(s)-\rho(t))^{\overline{\alpha-1}} f(s)\right] = \big(s-\rho(t)\big)^{\overline{\alpha-1}} \Delta_s f(s) + \Delta_s(\rho(s)-\rho(t))^{\overline{\alpha-1}} f(s) $$

$$ = (s-\rho(t))^{\overline{\alpha-1}} \Delta_s f(s) + (\alpha-1)\big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) \cdots (1) $$

Now, by definition 2.1.2 and by equation 1 then

$$ {}_b\nabla^{-\alpha}{}_{\ominus}\Delta f(t) = -\frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-1}} \Delta_s f(s) $$

$$ = -\frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \left[\Delta_s\Big((\rho(s)-\rho(t))^{\overline{\alpha-1}} f(s)\Big) - (\alpha-1)\big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s)\right] $$

$$ = -\frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \Delta_s \left(\big(\rho(s)-\rho(t)\big)^{\overline{\alpha-1}} f(s)\right) + \frac{1}{\Gamma(\alpha-1)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) $$

$$ = \frac{1}{\Gamma(\alpha-1)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) - \frac{1}{\Gamma(\alpha)} \left(\big(\rho(s)-\rho(t)\big)^{\overline{\alpha-1}} f(s)\right)\Bigg|_t^b $$

$$ = \frac{1}{\Gamma(\alpha-1)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) - \frac{1}{\Gamma(\alpha)} \left[(b-t)^{\overline{\alpha-1}} f(b) - (0)^{\overline{\alpha-1}}\right] $$

Since $0^{\overline{\alpha-1}} = 0$ so,

$$ {}_b\nabla^{-\alpha}{}_{\ominus}\Delta f(t) = \frac{1}{\Gamma(\alpha-1)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b) $$

On the other hand,

$$ {}_{\ominus}\Delta\, {}_{b}\nabla^{-\alpha} = -\frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \Delta_t \big(s-\rho(t)\big)^{\overline{\alpha-1}} f(s) $$

By theorem 1.3.8 in chapter 1, we have

$$ {}_{\ominus}\Delta\, {}_{b}\nabla^{-\alpha} = -\frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} -(\alpha-1)\big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) $$

$$ = \frac{1}{\Gamma(\alpha-1)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-2}} f(s) $$

Then

$$ {}_{b}\nabla_{\ominus}^{-\alpha} \Delta f(t) = {}_{\ominus}\Delta\, {}_{b}\nabla^{-\alpha} - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b). $$

Remark 2.1.24. [23] Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, by the help of lemma 2.1.23 we have

$$ {}_{\ominus}\Delta\, {}_{b}\nabla^{\alpha} f(t) = {}_{\ominus}\Delta_{\ominus}\Delta^{n} \left({}_{b}\nabla^{-(n-\alpha)} f(t) \right) = {}_{\ominus}\Delta^{n} \left({}_{\ominus}\Delta\, {}_{b}\nabla^{-(n-\alpha)} f(t) \right) $$

Or,

$$ {}_{\ominus}\Delta\, {}_{b}\nabla^{\alpha} f(t) = {}_{\ominus}\Delta^{n} \left[{}_{b}\nabla^{-(n-\alpha)} {}_{\ominus}\Delta f(t) + \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(b) \right] $$

Lemma 2.1.25. For $\alpha \in \mathbb{R}$, then

$$ {}_{\ominus}\Delta^{n} \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} $$

Proof: We prove in lemma 2.1.23, for any $\alpha > 0$, we have

$$ {}_b\nabla_{\ominus}^{-\alpha}\Delta f(t) = {}_{\ominus}\Delta\ {}_b\nabla^{-\alpha} - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b). $$

But if $\alpha < 0$, then $-\alpha > 0$, say $\beta = -\alpha > 0$, therefore,

$$ {}_{\ominus}\Delta\ {}_b\nabla^{\beta} f(t) = {}_{\ominus}\Delta^{n}\left[{}_b\nabla^{-(n-\beta)}\ {}_{\ominus}\Delta f(t) + \frac{(b-t)^{\overline{n-\beta-1}}}{\Gamma(n-\beta)} f(b)\right] $$

and using the identity ${}_{\ominus}\Delta^{n}\frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}$, then

$$ {}_{\ominus}\Delta\ {}_b\nabla^{\alpha} f(t) = \left[{}_b\nabla^{-\beta}\ {}_{\ominus}\Delta f(t) + \frac{(b-t)^{\overline{-\beta-1}}}{\Gamma(-\alpha)} f(b)\right] $$

take $\alpha = -\beta$, we done .

Theorem 2.1.26. For any real number α and any positive integer p the following equality holds:

$$ {}_b\nabla^{-\alpha}\ {}_{\ominus}\Delta^{p} f(t) = {}_{\ominus}\Delta^{p}\ {}_b\nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1}\frac{(b-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\ {}_{\ominus}\Delta^{k} f(b). $$

Where f is defined on ${}_b\mathbb{N}$ and some points before b.

Proof: Use equation of lemma 2.1.23 we replace f by ${}_{\ominus}\Delta f$ then

$$ {}_b\nabla^{-\alpha}\ {}_{\ominus}\Delta^{2}\mathrm{f(t)} = {}_{\ominus}\Delta\ {}_b\nabla^{-\alpha}\ {}_{\ominus}\Delta f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\ {}_{\ominus}\Delta f(b) $$

Use lemma 2.1.23 again then

$$ {}_{\ominus}\nabla^{-\alpha}\ {}_{\ominus}\Delta^{2}\mathrm{f(t)} = {}_{\ominus}\Delta\left[{}_{\ominus}\Delta\ {}_b\nabla^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b)\right] - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\ {}_{\ominus}\Delta f(b) $$

$$= {}_{\ominus}\Delta^2 \; {}_b\nabla^{-\alpha} f(t) - {}_{\ominus}\Delta \frac{(b-t)^{\overline{\alpha-1}} f(b)}{\Gamma(\alpha)} - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \, {}_{\ominus}\Delta f(b)$$

Use theorem 1.3.8 in chapter 1

$${}_{\ominus}\nabla^{-\boldsymbol{\alpha}} \, {}_{\ominus}\Delta^2 \mathrm{f(t)} = {}_{\ominus}\Delta^2 \; {}_b\nabla^{-\alpha} f(t) - \frac{(\alpha-1)(b-t)^{\overline{\alpha-2}} f(b)}{\Gamma(\alpha)} - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \, {}_{\ominus}\Delta f(b)$$

$$= {}_{\ominus}\Delta^2 \; {}_b\nabla^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-2}} f(b)}{\Gamma(\alpha-1)} - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \, {}_{\ominus}\Delta f(b)$$

Then,

$${}_{\ominus}\nabla^{-\boldsymbol{\alpha}} \, {}_{\ominus}\Delta^2 \mathrm{f(t)} = {}_{\ominus}\Delta^p \; {}_b\nabla^{-\alpha} f(t) - \sum_{k=0}^{1} \frac{(b-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \, {}_{\ominus}\Delta^k f(b).$$

Repeated interaction gives the desired result.

Theorem 2.1.27. For any real number α and any positive integer p the following equality holds:

$${}_b\Delta^{-\alpha} \nabla_{\ominus}^p f(t) = {}_{\ominus}\nabla^p \; {}_b\Delta^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-t)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla_{\ominus}^k f(b).$$

Proof: Use equation of lemma 2.1.16 we replace f by $\nabla_{\ominus} f(t)$ then

$${}_b\Delta^{-\boldsymbol{\alpha}} \nabla_{\ominus}^2 \mathrm{f(t)} = \nabla_{\ominus} \; {}_b\nabla^{-\alpha} \nabla_{\ominus} f(t) - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \nabla_{\ominus} f(b)$$

Use lemma 2.1.16 again then

$${}_b\Delta^{-\boldsymbol{\alpha}} \nabla_{\ominus}{}^2 \mathrm{f(t)} = \nabla_{\ominus} \left[\nabla_{\ominus} \; {}_b\nabla^{-\alpha} \nabla_{\ominus} f(t) - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(b) \right] - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \nabla_{\ominus} f(b)$$

$$= \nabla_{\ominus}^2 \; {}_b\nabla^{-\alpha} \nabla_{\ominus} f(t) - \frac{\nabla_{\ominus}(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(b) - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \nabla_{\ominus} f(b)$$

Use theorem 1.3.7 in chapter 1

$$ {}_b\Delta^{-\alpha}\nabla_{\ominus}{}^{2}\mathrm{f(t)} = \nabla_{\ominus}^{2}\ {}_b\nabla^{-\alpha}\nabla_{\ominus}f(t) - \frac{(\alpha-1)(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)}f(b) - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\nabla_{\ominus}f(b) $$

$$ = \nabla_{\ominus}^{2}\ {}_b\nabla^{-\alpha}\nabla_{\ominus}f(t) - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha-1)}f(b) - \frac{(b-t)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\nabla_{\ominus}f(b) $$

Then,

$$ {}_b\Delta^{-\alpha}\nabla_{\ominus}^{p}f(t) = {}_{\ominus}\nabla^{2}\ {}_b\Delta^{-\alpha}f(t) - \sum_{k=0}^{1}\frac{(b-t)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\nabla_{\ominus}^{k}f(b). $$

Repeated interaction gives the desired result.

2.2 Dual identities for fractional sum and Riemann fractional difference

In this section, we defined relation left fractional sum and difference were investigated in [7].

The following lemmas are dual relation between the delta left fractional sum (difference) and nabla left fractional sums (difference).

Theorem 2.2.1. For any positive number m, then the following is hold

i. $\Delta^{m}y(t-m) = \nabla^{m}y(t)$.

ii. $\Delta^{m}y(t) = \nabla^{m}y(t+m)$.

Proof:

i. Use induction for $m = 1$

$\Delta y(t-1) = y(t-1+1) - y(t-1) = y(t) - y(t-1) = \nabla y(t)$.

So for $m = 1$ is true, and let $m - 1$ is true then

$$\Delta^m y(t-m) = \Delta^{m-1}(\Delta y(t-m))$$

$$= \Delta^{m-1}(y(t-m+1) - y(t-m))$$

$$= \Delta^{m-1} y(t-m+1) - \Delta^{m-1} y(t-m)$$

$$= \Delta^{m-1} y(t-(m-1)) - \Delta^{m-1} y(t-(m-1)-1)$$

when $m-1$ is true then

$$\Delta^m y(t-m) = \nabla^{m-1} y(t) - \nabla^{m-1} y(t-1)$$

$$= \nabla^{m-1}(y(t) - y(t-1)) = \nabla^{m-1}\nabla y(t) \quad = \nabla^m y(t).$$

Then the statement for m is true for all positive integer m.

ii. Use induction for $m = 1$

$$\nabla y(t+1) = y(t+1) - y(t) = \Delta y(t).$$

So for $m = 1$ is true, and let $m-1$ is true then

$$\nabla^m y(t+m) = \nabla^{m-1}(\nabla y(t-m))$$

$$= \nabla^{m-1}\big(y(t+m) - y(t+m-1)\big)$$

$$= \nabla^{m-1} y(t+m) - \nabla^{m-1} y(t+m-1)$$

$$= \nabla^{m-1} y(t+m-1+1) - \nabla^{m-1} y(t+m-1)$$

when $m-1$ is true then

$$\nabla^m y(t-m) = \Delta^{m-1} y(t+1) - \Delta^{m-1} y(t)$$

$$= \Delta^{m-1}(y(t+1) - y(t)) = \Delta^{m-1}\nabla y(t) \quad = \Delta^m y(t).$$

Then the statement for m is true for all positive integer m.

Lemma 2.2.2. [7] Let $0 \le n-1 \le \alpha \le n$ and let $y(t)$ be defined on $\mathbb{N}_a$. Then the following statement is valid

i. $\Delta_a^\alpha y(t-\alpha) = \nabla_{a-1}^\alpha y(t)$ for $t \in \mathbb{N}_{n+a}$,

ii. $\Delta_a^{-\alpha} y(t+\alpha) = \nabla_{a-1}^{-\alpha} y(t)$ for $t \in \mathbb{N}_a$.

Proof:

i. Use definition 2.1.8 and definition 2.1.2, then

$$\Delta_a^\alpha y(t-\alpha) = \Delta^m \Delta_a^{-(m-\alpha)} y(t-\alpha)$$

$$= \Delta^m \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-\alpha-(m-\alpha)} \left(t-\alpha-\sigma(s)\right)^{\underline{m-\alpha-1}} y(s)$$

$$= \Delta^m \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-m} \left(t-\alpha-\sigma(s)\right)^{\underline{m-\alpha-1}} y(s)$$

By theorem 2.2.1 then,

$$\Delta_a^\alpha y(t-\alpha) = \nabla^m \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t} (t+m-\alpha-s-1)^{\underline{m-\alpha-1}} y(s)$$

$$= \nabla^m \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t} (t+(m-\alpha-1)-1-s+1)^{\underline{m-\alpha-1}} y(s)$$

$$= \nabla^m \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t} (t+(m-\alpha-1)-1-\rho(s))^{\underline{m-\alpha-1}} y(s)$$

using identity $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha}}$, then

$$\Delta_a^\alpha y(t-\alpha) = \nabla^m \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{m-\alpha-1}} y(s) = \nabla^m \nabla_{a-1}^{-(m-\alpha)} y(s) = \nabla_{a-1}^\alpha y(t).$$

ii. Use definition 2.1.2 , then

$$\Delta_a^{-\alpha} y(t+\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t+\alpha-\alpha} \left(t+\alpha-\sigma(s)\right)^{\underline{\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-s-1)^{\underline{\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+(\alpha-1)-1-s+1)^{\underline{\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+(\alpha-1)-1-\rho(s))^{\underline{\alpha-1}} y(s)$$

using identity $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha}}$, then

$$\Delta_a^{-\alpha} y(t+\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) = \nabla_{a-1}^{-\alpha} y(t).$$

Lemma 2.2.3. [7] Let $0 \le n-1 \le \alpha \le n$ and let $y(t)$ be defined on $\mathbb{N}_{a-n}$. Then the following statement is valid

i. $\Delta_{\alpha-n}^{\alpha} y(t) = \nabla_{a-n}^{\alpha} y(t+\alpha)$ for $t \in \mathbb{N}_{\alpha-n}$,

ii. $\Delta_{\alpha-n}^{-(n-\alpha)} y(t) = \nabla_a^{-(n-\alpha)} y(t-n+\alpha)$ for $t \in \mathbb{N}_a$.

Proof:

i. Starting on the right side and use definition 2.1.9, then

$$\nabla_{\alpha-n}^{\alpha} y(t+\alpha) = \nabla^n \nabla_{\alpha-n}^{-(n-\alpha)} y(t+\alpha),$$

$$= \nabla^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t+\alpha} \left(t+\alpha-\rho(s)\right)^{\overline{n-\alpha-1}} y(s)$$

By theorem 2.2.1 then,

$$\Delta_{\alpha-n}^{\alpha} y(t+\alpha) = \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-n+\alpha-\rho(s))^{\overline{n-\alpha-1}}\, y(s)$$

$$= \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-n+\alpha-(s-1)+1-1))^{\overline{n-\alpha-1}}\, y(s)$$

$$= \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-n+\alpha+1-\rho(s)+1))^{\overline{n-\alpha-1}}\, y(s)$$

$$= \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-(n-\alpha-1)+1-\rho(s))^{\overline{n-\alpha-1}}\, y(s)$$

using identity $t^{\underline{\alpha}} = (t-\alpha+1)^{\overline{\alpha}}$, then

$$\Delta_{\alpha-n}^{\alpha} y(t+\alpha) = \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} \big((t-\rho(s))\big)^{\overline{n-\alpha-1}}\, y(s)$$

$$= \Delta^n \Delta_{\alpha-n}^{-(n-\alpha)} y(t) = \Delta_{\alpha-n}^{\alpha} y(t).$$

ii. Starting on the right side and use definition 2.1.2, then

$$\nabla_{\alpha-n}^{-(n-\alpha)} y(t-n+\alpha) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} \big(t-n+\alpha-\rho(s)\big)^{\overline{n-\alpha-1}}\, y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-n+\alpha-(s-1))^{\overline{n-\alpha-1}}\, y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-n+\alpha-s+1-1+1))^{\overline{n-\alpha-1}}\, y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-n+\alpha+1-\rho(s)+1))^{\overline{n-\alpha-1}}\, y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-(n-\alpha-1)-\rho(s)+1))^{\overline{n-\alpha-1}}\, y(s)$$

using identity $t^{\underline{\alpha}} = (t - \alpha + 1)^{\overline{\alpha}}$, then

$$\nabla_{\alpha-n}^{-(n-\alpha)} y(t-n+\alpha) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=\alpha-n}^{t-n+\alpha} (t-\rho(s))^{\underline{n-\alpha-1}} y(s)$$

$$= \nabla_{a}^{-(n-\alpha)} y(t-n+\alpha).$$

We remind the reader those previous two dual lemmas for left fractional sums and differences were obtained when the nabla left fractional sum was defined by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t-\rho(s)\right)^{\overline{\alpha-1}} f(s),$$

Now, in analogous to lemmas 2.2.2 and 2.2.3, for right fractional summation and differences the author in [24] obtained.

Lemmas 2.2.4. Let $y(t)$ be defined on ${}_{b+1}\mathbb{N}$. Then the following statement is valid.

i. ${}_{b}\Delta^{\alpha}\, y(t+\alpha) = {}_{b+1}\nabla^{\alpha} y(t)$ for $t \in {}_{b-n}\mathbb{N}$.

ii. ${}_{b}\Delta^{-\alpha} y(t-\alpha) = {}_{b+1}\nabla^{-\alpha} y(t)$ for $t \in {}_{b}\mathbb{N}$.

Proof:

i. Use definition 2.1.9 and 2.1.7 , then

$${}_{b}\Delta^{\alpha}\, y(t+\alpha) = \nabla_{\ominus}^{n}\, {}_{b}\Delta^{-(n-\alpha)} y(t+\alpha)$$

$$= (-1)^n \nabla^n\, {}_{b}\Delta^{-(n-\alpha)} y(t+\alpha)$$

$$= (-1)^n \nabla^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+\alpha+n-\alpha}^{b} (s-\sigma(t+\alpha))^{\underline{n-\alpha-1}} y(s)$$

$$= (-1)^n \nabla^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n}^{b} (s-t-\alpha-1))^{\underline{n-\alpha-1}} y(s)$$

By theorem 2.2.1, we have

$$= (-1)^n \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t-n+n}^{b} (s-(t-n)-\alpha-1))^{\underline{n-\alpha-1}} y(s)$$

$$= (-1)^n \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b} (s-t-1+n-\alpha))^{\underline{n-\alpha-1}} y(s)$$

$$= (-1)^n \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b} (s-t+1+n-\alpha-1-1))^{\underline{n-\alpha-1}} y(s)$$

$$= (-1)^n \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b} (s-\rho(t)+(n-\alpha-1)-1)^{\underline{n-\alpha-1}} y(s)$$

Using the identity $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha}}$, we arrive at

$${}_b\Delta^{\alpha} = (-1)^n \Delta^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b} \left(s-\rho(t)\right)^{\overline{n-\alpha-1}} y(s)$$

$$= (-1)^n \Delta^n {}_{b+1}\nabla^{-(n-\alpha)} y(t) \; = {}_{\ominus}\Delta^n {}_{b+1}\nabla^{-(n-\alpha)} y(t) \quad = {}_{b+1}\nabla^{\alpha} y(t)$$

ii. Use definition 2.1.2 then

$${}_b\Delta^{-\alpha} y(t-\alpha) = \frac{1}{\Gamma(\alpha)} \sum_{s=t-\alpha+\alpha}^{b} \left(s-\sigma(t-\alpha)\right)^{\underline{\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (s-(t-\alpha+1))^{\underline{\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (s-t+1+(\alpha-1)-1)^{\underline{\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (s - \rho(t) + (\alpha - 1) - 1)^{\underline{\alpha-1}} y(s)$$

Using the identity $t^{\overline{\alpha}} = (t + \alpha - 1)^{\underline{\alpha}}$, we arrive at

$${}_{b}\Delta^{-\alpha} y(t - \alpha) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (s - \rho(t))^{\overline{\alpha-1}} y(s) = {}_{b+1}\nabla^{-\alpha} y(t).$$

Lemmas 2.2.5 Let $0 \le n - 1 \le \alpha \le n$ and let $y(t)$ be defined on ${}_{n-\alpha}\mathbb{N}$. Then the following statements are valid

i. ${}_{n-\alpha}\Delta^{\alpha} y(t) = {}_{n-\alpha+1}\nabla^{\alpha} y(t - \alpha), \qquad t \in {}_{n}\mathbb{N}$

ii. ${}_{n-\alpha}\Delta^{-(n-\alpha)} y(t) = {}_{n-\alpha+1}\nabla^{-(n-\alpha)} y(t + n - \alpha), \qquad t \in {}_{0}\mathbb{N}$

Proof:

i. Starting on the right side and use definition 2.1.9, then

$${}_{n-\alpha+1}\nabla^{\alpha} y(t - \alpha) = (-1)^n \Delta^n {}_{n-\alpha+1}\nabla^{-(n-\alpha)} y(t - \alpha)$$

$$= (-1)^n \Delta^n \frac{1}{\Gamma(n - \alpha)} \sum_{s=t-\alpha}^{n-\alpha} \big(s - \rho(t - \alpha)\big)^{\overline{n-\alpha-1}} y(s)$$

By theorem 2.2.1 then,

$${}_{n-\alpha}\Delta^{\alpha} y(t) = (-1)^n \nabla^n \frac{1}{\Gamma(n - \alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} \big(s - \rho(t + n - \alpha)\big)^{\overline{n-\alpha-1}} y(s)$$

$$= (-1)^n \nabla^n \frac{1}{\Gamma(n - \alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} \big(s - (t + n - \alpha - 1)\big)^{\overline{n-\alpha-1}} y(s)$$

$$= (-1)^n \nabla^n \frac{1}{\Gamma(n - \alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} (s - t - (n - \alpha - 1))^{\overline{n-\alpha-1}} y(s)$$

$$= (-1)^n \nabla^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} (s-t-1-(n-\alpha-1)+1)^{\overline{n-\alpha-1}} y(s)$$

$$= (-1)^n \nabla^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} (s-\sigma(t)-(n-\alpha-1)+1)^{\overline{n-\alpha-1}} y(s)$$

using identity $t^{\underline{\alpha}} = (t-\alpha+1)^{\overline{\alpha}}$, then

$$_{n-\alpha}\Delta^{\alpha}\, y(t) = (-1)^n \nabla^n \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} \big(s-\rho(t)\big)^{\underline{n-\alpha-1}} y(t) = \nabla^n_{\ominus}{}_{n-\alpha}\Delta^{-(n-\alpha)} y(s)$$

$$= {}_b\Delta^{\alpha} y(t).$$

ii. Starting on the right side and definition 2.1.2, then

$$_{n-\alpha+1}\nabla^{-(n-\alpha)} y(t+n-\alpha) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} \big(s-\rho(t+n-\alpha)\big)^{\overline{n-\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} \big(s-(t+n-\alpha)+1\big)^{\overline{n-\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} (s-t-(n-\alpha-1)-1+1)^{\overline{n-\alpha-1}} y(s)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} (s-\sigma(t)-(n-\alpha-1)+1)^{\overline{n-\alpha-1}} y(s)$$

using identity $t^{\underline{\alpha}} = (t-\alpha+1)^{\overline{\alpha}}$, then

$$_{n-\alpha+1}\nabla^{-(n-\alpha)} y(t+n-\alpha) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+n-\alpha}^{n-\alpha} \big(s-\sigma(t)\big)^{\underline{n-\alpha-1}} y(s)$$

$$= {}_{n-\alpha}\Delta^{-(n-\alpha)} y(t).$$

We can apply the definition of the delta left fractional difference.

Proposition 2.2.6.[23] let $\alpha > 0, \mu > 0$. Then,

$$\nabla_{b-\mu}^{-\alpha}(b-t)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-t)^{\underline{\mu+\alpha}} \cdots (1)$$

Proof: The proof can be achieved by checking that both sides of the identity (1) verify the difference equation

$$\begin{cases} (b-(\mu+\alpha)-t+1)\nabla_b g(t) = (\mu+\alpha)g(t) \\ g\big(b-(\mu+\alpha)\big) = \Gamma(\mu+1) \end{cases}$$

Let $y(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-t)^{\underline{\mu+\alpha}}$ and $h(t) = \nabla_{b-\mu}^{-\alpha}(b-t)^{\underline{\mu}}$.

First, consider $y(t)$ and use identity $\mu^{\underline{\mu}} = \Gamma(\mu+1)$ to see that

$$y\big(b-(\mu+\alpha)\big) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-(b-(\mu+\alpha))^{\underline{\mu+\alpha}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(\mu+\alpha)^{\underline{\mu+\alpha}}$$

$$= \Gamma(\mu+1).$$

also use identity $\nabla_s(\rho(s)-t)^{\underline{\alpha-1}} = -(\alpha-1)(\rho(s)-t)^{\underline{\alpha-2}}$,and similar use identity $(t-\mu)t^{\underline{\mu}} = t^{\underline{\mu+1}}$ to see that

$$(b-(\mu+\alpha)-t+1)\nabla_b y(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-(\mu+\alpha)-t+1)(\mu+\alpha)(b-t)^{\underline{\mu+\alpha-1}}$$

$$= (\mu+\alpha)y(t)$$

Second, consider $h(t)$. Noting that

$$h(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b-\mu}(\rho(s)-t)^{\underline{\alpha-1}}(b-s)^{\underline{\mu}},$$

We see that

$$h\big(b-(\mu+\alpha)\big) = \frac{1}{\Gamma(\alpha)}\sum_{s=b-\mu}^{b-\mu}(\rho(s)-b+\mu+\alpha)^{\underline{\alpha-1}}(b-s)^{\underline{\mu}}$$

$$= \frac{1}{\Gamma(\alpha)}(\alpha-1)^{\underline{\alpha-1}}\mu^{\underline{\mu}} = \Gamma(\mu+1).$$

Finally, we show that $h(t)$ satisfies the desired difference equation.

Using identity $(t-\mu)t^{\underline{\mu}} = t^{\underline{\mu+1}}$, adding and subtracting μ and adding and

Subtracting b, we see that

$$h(t)=\frac{(b-(\mu+\alpha)-t+1)}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b-\mu}(\rho(s)-t)^{\underline{\alpha-2}}(b-s)^{\underline{\mu}}$$

$$-\frac{1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b-\mu}(b-s-\mu)(\rho(s)-t)^{\underline{\alpha-2}}(b-s)^{\underline{\mu}}.$$

The rest of verification is direct.

We can apply the definition of the delta left fractional difference.

Lemmas 2.2.7.[21] Let $\alpha>0,\mu>0$.then,

$${}_{b-\mu}\Delta^{-\alpha}(b-t)^{\underline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-t)^{\underline{\mu}}.$$

The following commutative property for delta left and right fractional sums in theorem 9 in [21] and in theorem 3.1 in [17].

Theorem 2.2.8 Let $\alpha>0,\mu>0$.

i. Then, for all t such that $t\equiv b-(\mu+\alpha)(mod1)$, one has

$${}_{b}\Delta^{-\alpha}\left[{}_{b}\Delta^{-\mu}f(t)\right]={}_{b}\Delta^{-(\mu+\alpha)}f(t)={}_{b}\Delta^{-\mu}\left[{}_{b}\Delta^{-\alpha}f(t)\right],$$

Where f is defined on ${}_{b}\mathbb{N}$.

ii. Then, for all t such that $t\equiv a+(\mu+\alpha)(mod1)$, one has

$$\Delta_a^{-\alpha}\left[\Delta_a^{-\mu}f(t)\right]=\Delta_a^{-(\alpha+\mu)}f(t)=\Delta_a^{-\mu}[\Delta_a^{-\alpha}f(t)],$$

Where f is defined on $\mathbb{N}_a$.

Theorem 2.2.9.[25] Let f be a real valued function and let $\alpha,\beta>0$. Then

i. ${}_{b}\nabla^{-\alpha}\left[{}_{b}\nabla^{-\beta}f(t)\right]={}_{b}\nabla^{-(\alpha+\beta)}f(t)={}_{b}\nabla^{-\beta}\left[{}_{b}\nabla^{-\alpha}f(t)\right].$

ii. $\nabla_a^{-\alpha}\left[\nabla_a^{-\beta}f(t)\right]=\nabla_a^{-(\alpha+\beta)}f(t)=\nabla_a^{-\beta}[\nabla_a^{-\alpha}f(t)].$

Proof:

i. by applying Lemmas 2.2.4 (ii) and Theorem 2.2.8 we have,

$$ {}_{b}\nabla^{-\alpha}\left[{}_{b}\nabla^{-\beta}f(t)\right] = {}_{b}\nabla^{-\alpha}{}_{b-1}\Delta^{-\beta}f(t-\beta) $$

$$ = {}_{b-1}\Delta^{-\alpha}{}_{b-1}\Delta^{-\beta}f(t-(\alpha+\beta)) $$

$$ = {}_{b-1}\Delta^{-(\alpha+\beta)}f\big(t-(\alpha+\mu)\big) = {}_{b}\nabla^{-(\alpha+\beta)}f(t). $$

ii. by applying Lemmas 2.2.2 (ii) and Theorem 2.2.8 we have,

$$ \nabla_a^{-\alpha}\left[\nabla_a^{-\beta}f(t)\right] = \nabla_a^{-\alpha}\Delta_a^{-\beta}f(t+\alpha) $$

$$ = \Delta_a^{-\alpha}\Delta_a^{-\beta}f(t+\alpha+\beta) = \Delta_a^{-(\alpha+\beta)}f(t+\alpha+\beta) = \nabla_a^{-(\alpha+\beta)}f(t). $$

The following power rule for nabla fractional differences plays an important rule.

Proposition 2.2.10.[25] Let $\alpha > 0, \mu > -1$. Then, for $t \in {}_{b}\mathbb{N}$, one has

$$ {}_{b}\nabla^{-\alpha}(b-t)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(b-t)^{\overline{\alpha+\mu}}. $$

Proof: By the dual formula (ii) of lemma Lemmas 2.2.4, we have

$$ {}_{b}\nabla^{-\alpha}(b-t)^{\overline{\mu}} = {}_{b}\Delta^{-\alpha}(b-r)^{\overline{\mu}}\Big|_{r=t-\alpha} $$

$$ = \frac{1}{\Gamma(\alpha)}\sum_{s=t}^{b-t}(s-t+\alpha-1)^{\underline{\alpha-1}}(b-s)^{\overline{\mu}}. $$

Then by identity $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha-1}}$ and using the change of variable $r = s-\mu+1$, it follows that

$$ {}_{b}\nabla^{-\alpha}(b-t)^{\overline{\mu}} = \frac{1}{\Gamma(\alpha)}\sum_{r=t-\mu+1}^{b-\mu}\big(r-\sigma(t-\alpha-\mu+1)\big)^{\underline{\alpha-1}}(b-r)^{\overline{\mu}} $$

$$ = \left({}_{b-\mu}\Delta^{-\alpha}(b-\mu)^{\overline{\mu}}\right)\Big|_{\mu=-\alpha-\mu+1+t} $$

This by Lemmas 2.2.5 leads to

$$ {}_{b}\nabla^{-\alpha}(b-t)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(b-t+\alpha+\mu-1)^{\underline{\alpha+\mu}} $$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(b-t)^{\overline{\alpha+\mu}}.$$

Similarly, for the nabla left fractional sum we can have the following power formula and exponent law.

Proposition 2.2.11 For $\alpha > 0$, and f defined in a suitable domain $\mathbb{N}_a$, then

i. $\Delta_a^{\alpha}\Delta_a^{-\alpha}f(t) = f(t)$,

ii. $\Delta_a^{-\alpha}\Delta_a^{\alpha}f(t) = f(t)$, when $\alpha \notin \mathbb{N}$,

iii. $\Delta_a^{-\alpha}\Delta_a^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\underline{k}}}{k!}\Delta_a^{k}f(a)$, when $\alpha = n \in \mathbb{N}$

Proof: Use definition delta left fractional sum and theorem 2.2.8, then

i. $\Delta_a^{\alpha}\Delta_a^{-\alpha}f(t) = \Delta^{n}\Delta_a^{-(n-\alpha)}\Delta_a^{-\alpha}f(t) = \Delta^{n}\Delta_a^{-n}f(t) = \nabla_a^{0}f(t) = f(t)$

ii. $\Delta_a^{-\alpha}\Delta_a^{\alpha}f(t) = \Delta_a^{-\alpha}\Delta^{n}\Delta_a^{-(n-\alpha)}f(t)$, use theorem 2.1.19 then

$$\Delta_a^{-\alpha}\Delta^{n}\Delta_a^{-(n-\alpha)}f(t) = \Delta^{n}\Delta_a^{-\alpha}\Delta_a^{-(n-\alpha)}f(t)$$

$$= \Delta^{n}\Delta_a^{-\alpha}\Delta_a^{-(n-\alpha)}f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\underline{\alpha-n+k}}}{\Gamma(\alpha+k-n+1)}\Delta^{k}\Delta_a^{-(k-\alpha)}f(a)$$

Since $\Delta_a^{-(n-\alpha)}f(a) = 0$ then $\Delta_a^{-\alpha}\Delta_a^{\alpha}f(t) = \Delta^{n}\Delta_a^{-\alpha}\Delta_a^{-(n-\alpha)}f(t) = f(t)$.

iii. $\Delta_a^{-\alpha}\Delta_a^{\alpha}f(t) = \Delta_a^{-\alpha}\Delta^{n}\Delta_a^{-(n-\alpha)}f(t)$,

$$\Delta_a^{-\alpha}\Delta^{n}\Delta_a^{-(n-\alpha)}f(t) = \Delta^{n}\Delta_a^{-\alpha}\Delta_a^{-(n-\alpha)}f(t)$$

use theorem 2.1.22 since $\alpha = n$ then

$$\Delta_a^{-\alpha}\Delta^{n}\Delta_a^{-(n-\alpha)}f(t) = \Delta^{\alpha}\Delta_a^{-\alpha}f(t)$$

$$= \Delta^{\alpha}\Delta_a^{-\alpha}f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\overline{\alpha-n+k}}}{\Gamma(\alpha+k-\alpha+1)}\Delta_a^{k}f(a)$$

$$= f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{\Gamma(k+1)} \Delta_a^k f(a) \ = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta_a^k f(a).$$

Proposition 2.2.12. For $\alpha > 0$, and f defined in a suitable domain ${}_b\mathbb{N}$then

i. ${}_b\Delta^{\alpha}\ {}_b\Delta^{-\alpha} f(t) = f(t)$,

ii. ${}_b\Delta^{-\alpha}\ {}_b\Delta^{\alpha} f(t) = f(t)$, when $\alpha \notin \mathbb{N}$,

iii. ${}_b\Delta^{-\alpha}\ {}_b\Delta^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k}}}{k!}\ {}_b\Delta^{\alpha} f(a)$, when $\alpha = n \in \mathbb{N}$

The proof will be done by using the method of proof in proposition 2.2.11.

Proposition 2.2.13.[23]For $\alpha > 0$, and f defined in a suitable domain $\mathbb{N}_a$, then

i. $\nabla_a^{\alpha} \nabla_a^{-\alpha} f(t) = f(t)$,

ii. $\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = f(t)$, when $\alpha \notin \mathbb{N}$,

iii. $\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a)$, when $\alpha = n \in \mathbb{N}$

Proof: Use definition nabla left fractional sum and theorem 2.2.9

i. $\nabla_a^{\alpha} \nabla_a^{-\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} \nabla_a^{-\alpha} f(t) = \nabla^n \nabla_a^{-n} f(t) = \nabla_a^0 f(t) = f(t)$

ii. $\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = \nabla_a^{-\alpha} \nabla^n \nabla_a^{-(n-\alpha)} f(t)$, use theorem 2.1.21 then

$\nabla_a^{-\alpha} \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \nabla^n \nabla_a^{-\alpha} \nabla_a^{-(n-\alpha)} f(t)$

$$= \nabla^n \nabla_a^{-\alpha} \nabla_a^{-(n-\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{\alpha-n+k}}}{\Gamma(\alpha+k-n+1)} \nabla^k \nabla_a^{-(k-\alpha)} f(a)$$

Since $\nabla_a^{-(n-\alpha)} f(a) = 0$ then $\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-\alpha} \nabla_a^{-(n-\alpha)} f(t) = f(t)$.

iii. $\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = \nabla_a^{-\alpha} \nabla^n \nabla_a^{-(n-\alpha)} f(t)$,

$\nabla_a^{-\alpha} \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \nabla^n \nabla_a^{-\alpha} \nabla_a^{-(n-\alpha)} f(t)$

use theorem 2.1.21 , since $\alpha = n$ then

$$\nabla_a^{-\alpha}\nabla^n\nabla_a^{-(n-\alpha)}f(t) = \nabla^\alpha\nabla_a^{-\alpha}f(t)$$

$$= \nabla^\alpha\nabla_a^{-\alpha}f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\overline{\alpha-\alpha+k}}}{\Gamma(\alpha+k-\alpha+1)}\nabla^k f(a)$$

$$= f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\overline{k}}}{\Gamma(k+1)}\nabla^k f(a) = f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\overline{k}}}{k!}\nabla_a^k f(a).$$

Proposition 2.2.14.[23] For $\alpha > 0$, and f defined in a suitable domain ${}_b\mathbb{N}$then

i. ${}_b\nabla^{\alpha}\ {}_b\nabla^{-\alpha}f(t) = f(t)$,

ii. ${}_b\nabla^{-\alpha}\ {}_b\nabla^{\alpha}\ f(t) = f(t)$, when $\alpha \notin \mathbb{N}$,

iii. ${}_b\nabla^{-\alpha}\ {}_b\nabla^{\alpha}\ f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(b-t)^{\overline{k}}}{k!}\ {}_b\nabla^k\ f(a),$ when $\alpha = n \in \mathbb{N}$

The proof will be done by using the method of proof in proposition 2.2.13.

The following theorems develop relation between fractional sum and difference operator by dual identity.

Theorems 2.2.15. For any $\alpha > 0$, the following is hold:

$${}_b\Delta^{-\alpha}{}_{\ominus}\Delta f(t-1) = {}_{\ominus}\Delta\ {}_b\Delta^{-\alpha}f(t-1) - \frac{(b-t)^{\underline{\alpha-1}}f(b)}{\Gamma(\alpha)}.$$

Proof: Use by theorem 2.2.1 and lemma 2.1.17, then

$${}_b\Delta^{-\alpha}{}_{\ominus}\Delta f(t-1) = {}_b\Delta^{-\alpha}\ \ \nabla_{\ominus}f(t)$$

$$= \nabla_{\ominus}\ {}_b\Delta^{-\alpha}f(t) - \frac{(b-t)^{\overline{\alpha-1}}f(b)}{\Gamma(\alpha)}$$

$$= {}_{\ominus}\Delta\ {}_b\Delta^{-\alpha}f(t-1) - \frac{(b-t)^{\underline{\alpha-1}}f(b)}{\Gamma(\alpha)}.$$

Theorems 2.2.16. For any real number α and any positive integer p, the following equality is hold

$$ {}_{b}\Delta^{-\alpha}{}_{\ominus}\Delta^{p} f(t-p) = \Delta_{\ominus}^{p}\ {}_{b}\Delta^{-\alpha} f(t-p) - \sum_{k=0}^{p-1} \frac{(b-t)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla_{\ominus}^{k} f(b). $$

Proof: By theorem 2.2.1 and lemma 2.1.27, then

$$ {}_{b}\Delta^{-\alpha}{}_{\ominus}\Delta^{p} f(t-p) = {}_{b}\Delta^{-\alpha}\ \ \nabla_{\ominus}{}^{p} f(t) $$

$$ = {}_{\ominus}\nabla^{p}\ {}_{b}\Delta^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-t)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla_{\ominus}^{k} f(b) $$

$$ = \Delta_{\ominus}^{p}\ {}_{b}\Delta^{-\alpha} f(t-p) - \sum_{k=0}^{p-1} \frac{(b-t)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla_{\ominus}^{k} f(b). $$

Theorems 2.2.17. For any $\alpha > 0$, the following is hold:

$$ \Delta_{a}^{-\alpha} \nabla f(t+1) = \nabla \Delta_{a}^{-\alpha} f(t+1) - \frac{(t-a)^{\underline{\alpha-1}} f(a)}{\Gamma(a)}. $$

Proof: By theorem 2.2.1 and lemma 2.1.13, then

$$ \Delta_{a}^{-\alpha} \nabla f(t+1) = \Delta_{a}^{-\alpha} \Delta f(t) = \Delta \Delta_{a}^{-\alpha} f(t) - \frac{(t-a)^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(a) $$

$$ = \nabla \Delta_{a}^{-\alpha} f(t+1) - \frac{(t-a)^{\underline{a-1}} f(a)}{\Gamma(a)}. $$

Theorems 2.2.19. For any real number α and any positive integer p, the following equality is hold

$$ \Delta_{a}^{-\alpha} \nabla^{p} f(t+p) = \nabla^{p} \Delta_{a}^{-\alpha} f(t+p) - \sum_{k=0}^{p-1} \frac{(t-a)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \Delta^{k} f(a). $$

Proof: By theorem 2.2.1 and lemma 2.1.22, then

$$\Delta_a^{-\alpha}\nabla^p f(t+p) = \Delta_a^{-\alpha}\Delta^p f(t)$$

$$= \Delta^p \Delta_a^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \Delta^k f(a)$$

$$= \nabla^p \Delta_a^{-\alpha} f(t+p) - \sum_{k=0}^{p-1} \frac{(t-a)^{\underline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \Delta^k f(a).$$

Theorems 2.2.19. For any $\alpha > 0$, the following is hold:

$$\nabla_a^{-\alpha}\Delta f(t-1) = \Delta\nabla_a^{-\alpha} f(t-1) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).$$

Proof: By r theorem 2.2.1 and lemma 2.1.18, then

$$\nabla_a^{-\alpha}\Delta f(t-1) = \nabla_a^{-\alpha}\nabla f(t) = \nabla\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)$$

$$= \Delta\nabla_a^{-\alpha} f(t-1) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).$$

Theorems 2.2.20. For any real number α and any positive integer p, the following equality is hold

$$\nabla_a^{-\alpha}\Delta^p f(t-p) = \Delta^p\nabla_a^{-\alpha} f(t-p) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a).$$

Proof: By theorem 2.2.1 and theorem 2.1.21, then

$$\nabla_a^{-\alpha}\Delta^p f(t-p) = \nabla_a^{-\alpha}\nabla^p f(t)$$

$$= \nabla^p \nabla_a^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a)$$

$$= \Delta^p \nabla_a^{-\alpha} f(t-p) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a).$$

Theorems 2.2.21. For any $\alpha > 0$, the following is hold:

$${}_b\nabla^{-\alpha} \nabla_{\ominus} f(t+1) = \nabla_{\ominus}\, {}_b\nabla^{-\alpha} f(t+1) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b)$$

Proof: By theorem 2.2.1 and lemma 2.1.23, then

$${}_b\nabla^{-\alpha} \nabla_{\ominus} f(t+1) = {}_b\nabla^{-\alpha}\, {}_{\ominus}\Delta \; f(t) = {}_{\ominus}\Delta\, {}_b\nabla^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b)$$

$$= \nabla_{\ominus}\, {}_b\nabla^{-\alpha} f(t+1) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b).$$

Theorems 2.2.22. For any real number α and any positive integer p, the following equality is hold

$${}_b\nabla^{-\alpha} \nabla_{\ominus}^p f(t+p) = \nabla_{\ominus}^p\, {}_b\nabla^{-\alpha} f(t+p) - \sum_{k=0}^{p-1} \frac{(b-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\, {}_{\ominus}\Delta^k f(b)$$

Proof: By theorem 2.2.1 and lemma 2.1.27, then

$${}_b\nabla^{-\alpha} \nabla_{\ominus}^p f(t+p) = {}_b\nabla^{-\alpha}\, {}_{\ominus}\Delta^p f(t)$$

$$= {}_{\ominus}\Delta^p \; {}_b\nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\, {}_{\ominus}\Delta^k f(b)$$

$$= \nabla_{\ominus}^p\, {}_b\nabla^{-\alpha} f(t+p) - \sum_{k=0}^{p-1} \frac{(b-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\, {}_{\ominus}\Delta^k f(b).$$

Chapter 3

Caputo Fractional Difference

In this chapter, we defined left and right Caputo fractional difference, study some of their properties and then relate them to Riemann-Linville operator is used to relate the left and right Caputo fractional differences.

3.1 Delta and Nabla Caputo Fractional Difference

In this section, we defined left and right Caputo fractional difference.

Definition 3.1.1. [25] Let $\alpha > 0, \alpha \notin \mathbb{N}$. Then

i. The delta α-order Caputo left fractional difference [22] of a function f defined on $\mathbb{N}_a$ is defined by

$$^{C}\Delta_a^{\alpha} f(t) \triangleq \Delta_a^{-(n-\alpha)} \Delta^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} \big(t-\sigma(s)\big)^{\underline{n-\alpha-1}} \Delta_s^n f(s).$$

ii. The delta α-order Caputo right fractional difference [22] of a function f defined on ${}_b\mathbb{N}$ is defined by

$${}_b^C\Delta^{\alpha} f(t) \triangleq {}_b\Delta^{-(n-\alpha)} \nabla_{\ominus}^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} \left(s-\sigma(t)\right)^{\underline{n-\alpha-1}} \nabla_{\ominus}^{n} f(s).$$

where $n = [\alpha] + 1$.

If $\alpha = n \in \mathbb{N}$, then

$${}^{C}\Delta_a^{\alpha} f(t) \triangleq \Delta^n f(t), \qquad {}_b^{C}\Delta^{\alpha} f(t) \triangleq \nabla_{\ominus}^{n} f(t).$$

iii. The nabla α-order Caputo left fractional difference of a function f defined on $\mathbb{N}_a$ and some point before a is defined by

$${}^{C}\nabla_a^{\alpha} f(t) \triangleq \nabla_a^{-(n-\alpha)} \nabla^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t-(n-\alpha)} \left(t-\rho(s)\right)^{\overline{n-\alpha-1}} \nabla^{n} f(s).$$

iv. The nabla α-order Caputo right fractional difference of a function f defined on ${}_b\mathbb{N}$ and some point after b is defined by

$${}_b^{C}\nabla^{\alpha} f(t) \triangleq {}_b\Delta^{-(n-\alpha)} {}_{\ominus}\Delta^{n} f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} \left(s-\rho(t)\right)^{\overline{n-\alpha-1}} {}_{\ominus}\Delta^{n} f(s).$$

If $\alpha = n \in \mathbb{N}$, then ${}^{C}\nabla_a^{\alpha} f(t) \triangleq \nabla^n f(t)$, ${}_b^{C}\nabla^{\alpha} f(t) \triangleq {}_{\ominus}\Delta^{n} f(t)$.

Note 3.1.2. It is clear that ${}^{C}\Delta_a^{\alpha}$ maps function defined on $\mathbb{N}_a$ to function defined on $\mathbb{N}_{a+(n-\alpha)}$ and that ${}_b^{C}\Delta^{\alpha}$ maps function defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-(n-\alpha)}\mathbb{N}$. Also, it is clear that the nabla left fractional difference ∇_a^{α} maps function defined on $\mathbb{N}_a$ to function on $\mathbb{N}_{a+1-n}$, and the nabla right fractional difference ${}_b\Delta^{\alpha}$ maps function defined on ${}_b\mathbb{N}$ to function defined on ${}_{b-1+n}\mathbb{N}$.

Riemann and Caputo delta fractional difference are related by following theorem.

Theorem 3.1.3. [22] For any $\alpha > 0$, one has

i. $^{C}\Delta_{a}^{\alpha} f(t) = \Delta_{a}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k-\alpha}}}{\Gamma(k-\alpha+1)} \Delta^{k} f(a),$

ii. ${}_{b}^{C}\Delta^{\alpha} f(t) = {}_{b}\Delta^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla_{\ominus}^{k} f(b).$

In particular, when $0 < \alpha < 1$, one has

$$^{C}\Delta_{a}^{\alpha} f(t) = \Delta_{a}^{\alpha} f(t) - \frac{(t-a)^{\underline{-\alpha}}}{\Gamma(1-\alpha)} f(a),$$

$${}_{b}^{C}\Delta^{\alpha} f(t) = {}_{b}\Delta^{\alpha} f(t) - \frac{(b-t)^{\underline{-\alpha}}}{\Gamma(1-\alpha)} f(b).$$

Proof:

i. Use definition 3.1.1 and theorem 2.1.19 then

$$\begin{aligned} ^{C}\Delta_{a}^{\alpha} f(t) &= \Delta_{a}^{-(n-\alpha)} \Delta^{n} f(t) \\ &= \Delta^{n} \Delta_{a}^{-(n-\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{n-\alpha-n+k}}}{\Gamma(n-\alpha+k-n+1)} \Delta^{k} f(a) \\ &= \Delta_{a}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k-\alpha}}}{\Gamma(k-\alpha+1)} \Delta^{k} f(a). \end{aligned}$$

ii. Use definition 3.1.1 and theorem 2.1.23 then

$$\begin{aligned} {}_{b}^{C}\Delta^{\alpha} f(t) &= {}_{b}\Delta^{-(n-\alpha)} \nabla_{\ominus}^{n} f(t) \\ &= \nabla_{\ominus}^{n}\, {}_{b}\Delta^{-(n-\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{n-\alpha-n+k}}}{\Gamma(n-\alpha+k-n+1)} \nabla_{\ominus}^{k} f(a) \\ &= {}_{b}\Delta^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla_{\ominus}^{k} f(a). \end{aligned}$$

One can note that the Riemann and Caputo fractional differences, for $0 < \alpha < 1$, coincide when f vanishes at the end point.

The following identity is useful to transform delta type Caputo fractional difference equation into fractional summations.

Proposition 3.1.4. [22] Assume that $\alpha > 0$ and f is defined on suitable domains $\mathbb{N}_a$ and ${}_b\mathbb{N}$. Then,

i. $\Delta_{a+(n-\alpha)}^{-\alpha} \; {}^{C}\Delta_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta^k f(a),$

ii. ${}_{b-(n-\alpha)}\Delta^{-\alpha} \; {}_b^C\Delta^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k}}}{k!} \nabla_{\ominus}^k f(b).$

In particular, if $0 < \alpha \leq 1$, then

$$\Delta_{a+(n-\alpha)}^{-\alpha} \; {}^{C}\Delta_a^{\alpha} f(t) = f(t) - f(a),$$

$${}_{b-(n-\alpha)}^{\quad -\alpha}\Delta \; {}_b^C\Delta^{\alpha} f(t) = f(t) - f(b).$$

Proof:

i. By definition Caputo delta left fractional difference and by theorem 2.2.8 proposition 2.2.11 then

$$\begin{aligned}
\Delta_{a+(n-\alpha)}^{-\alpha} \; {}^{C}\Delta_a^{\alpha} f(t) &= \Delta_{a+(n-\alpha)}^{-\alpha} \Delta_a^{-(n-\alpha)} \Delta^n f(t) \\
&= \Delta_a^{-n} \Delta^n f(t) \\
&= \Delta^n \Delta_a^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta^k f(a) \\
&= f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta^k f(a).
\end{aligned}$$

ii. By definition Caputo delta right fractional difference and by theorem 2.2.8 proposition 2.2.12 then

$${}_{b-(n-\alpha)}^{\quad -\alpha}\Delta_b^C\Delta^{\alpha} f(t) = {}_{b-(n-\alpha)}^{\quad -\alpha}\Delta \; {}_b\Delta^{-(n-\alpha)} \nabla_{\ominus}^n f(t)$$

$$= {}_{b}\Delta^{-n}\nabla_{\ominus}^{n} f(t)$$

$$= \nabla_{\ominus}^{n}\ {}_{b}\Delta^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k}}}{k!} \nabla_{\ominus}^{k} f(a)$$

$$= f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k}}}{k!} \nabla_{\ominus}^{k} f(a).$$

Similar to what we have earlier, for the nabla fractional difference we obtain the following.

Theorem 3.1.5. For any $\alpha > 0$, one has

i. $\displaystyle {}^{C}\nabla_{a}^{\alpha} f(t) = \nabla_{a}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^{k} f(a),$

ii. $\displaystyle {}_{b}^{C}\nabla^{\alpha} f(t) = {}_{b}\nabla^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} {}_{\ominus}\Delta^{k} f(b).$

In particular, when $0 < \alpha < 1$, one has

$$ {}^{C}\nabla_{a}^{\alpha} f(t) = \nabla_{a}^{\alpha} f(t) - \frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a),$$

$$ {}_{b}^{C}\nabla^{\alpha} f(t) = {}_{b}\nabla^{\alpha} f(t) - \frac{(b-t)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b).$$

Proof:

i. Use definition 3.1.1 and theorem 2.1.18 then

$$ {}^{C}\nabla_{a}^{\alpha} f(t) = \nabla_{a}^{-(n-\alpha)} \nabla^{n} f(t)$$

$$= \nabla^{n} \nabla_{a}^{-(n-\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{n-\alpha-n+k}}}{\Gamma(n-\alpha+k-n+1)} \Delta^{k} f(a)$$

$$= \nabla_{a}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^{k} f(a).$$

ii. Use definition 3.1.1 and theorem 2.1.22 then

$${}_b^C\nabla^{\alpha} f(t) = {}_b\nabla^{-(n-\alpha)} {}_{\ominus}\Delta^{n} f(t)$$

$$= {}_{\ominus}\Delta^{n}\ {}_b\nabla^{-(n-\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{n-\alpha-n+k}}}{\Gamma(n-\alpha+k-n+1)} {}_{\ominus}\Delta^{n} f(b)$$

$$= {}_b\nabla^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\underline{k-\alpha}}}{\Gamma(k-\alpha+1)} {}_{\ominus}\nabla^{k} f(b).$$

One can see that the nabla Riemann and Caputo fractional difference, for $0 < \alpha < 1$, coincide when f vanishes at the end point.

Proposition 3.1.6. Assume that $\alpha > 0$ and f is defined on suitable domains $\mathbb{N}_a$ and ${}_b\mathbb{N}$. Then,

i. $\nabla_a^{-\alpha}\, {}^C\nabla_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^{k} f(a),$

iii. ${}_b\nabla^{-\alpha}\, {}_b^C\nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k}}}{k!} {}_{\ominus}\Delta^{k} f(b).$

In particular, if $0 < \alpha \le 1$, then

$$\nabla_a^{-\alpha}\, {}^C\Delta_a^{\alpha} f(t) = f(t) - f(a),$$

$${}_b^{-\alpha}\Delta\, {}_b^C\Delta^{\alpha} f(t) = f(t) - f(b).$$

Proof:

i. By definition Caputo nabla left fractional difference and by theorem 2.2.9 proposition 2.2.13 then

$$\nabla_a^{-\alpha}\, {}^C\Delta_a^{\alpha} f(t) = \nabla_a^{-\alpha} \nabla_a^{-(n-\alpha)} \Delta^{n} f(t)$$

$$= \nabla_a^{-n} \nabla^{n} f(t)$$

$$= \nabla^n \nabla_a^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k \ f(a)$$

$$= f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k \ f(a).$$

ii. By definition Caputo nabla right fractional difference and by theorem 2.2.9 proposition 2.2.14 then

$${}_b\nabla\, {}^C_b\nabla^{\alpha} f(t) = {}_b\nabla^{-\alpha}{}_b\nabla^{-(n-\alpha)}{}_{\ominus}\Delta^n f(t)$$

$$= {}_b\nabla^{-n}{}_{\ominus}\Delta^n f(t)$$

$$= {}_{\ominus}\Delta^n \ {}_b\nabla^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k}}}{k!} {}_{\ominus}\Delta^k f(a)$$

$$= f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k}}}{k!} {}_{\ominus}\Delta^k f(a).$$

We can find the delta and nabla type Caputo fractional differences for certain power functions.

Example 3.1.7. For example, for $1 \neq \beta > 0$ and $\alpha \geq 0$,we have

i. ${}^C\nabla_a^{\alpha}(t-a)^{\overline{\beta-1}} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\overline{\beta-\alpha-1}}$,

ii. ${}^C\Delta_a^{\alpha}(t-a)^{\underline{\beta-1}} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\underline{\beta-\alpha-1}}$.

Proof:

i. By definition Caputo nabla left fractional and theorem 3.1.5 and use identity $\nabla_a^{-\alpha}(t-a)^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\overline{\alpha+\beta}}$ then

$${}^C\nabla_a^{\alpha}(t-a)^{\overline{\beta-1}} = \nabla_a^{\alpha}(t-a)^{\overline{\beta-1}} - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^k \ (a-a)^{\overline{\beta-1}},$$

$$= \nabla_a^{\alpha}(t-a)^{\overline{\beta-1}} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\overline{\beta-\alpha-1}}.$$

ii. By definition Caputo delta left fractional and theorem 3.1.3 and use identity

$$\Delta_a^{-\alpha}(t-a)^{\underline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\underline{\alpha+\beta}} \text{ then}$$

$${}^{C}\Delta_a^{\alpha}(t-a)^{\underline{\beta-1}} = \Delta_a^{\alpha}(t-a)^{\underline{\beta-1}} - \sum_{k=0}^{n-1}\frac{(t-a)^{\underline{k-\alpha}}}{\Gamma(k-\alpha+1)}\Delta^k\,(a-a)^{\underline{\beta-1}},$$

$$= \nabla_a^{\alpha}(t-a)^{\underline{\beta-1}} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\underline{\beta-\alpha-1}}.$$

Example 3.1.8. For any $\alpha > 0$,

i. ${}^{C}\nabla_a^{\alpha}1 = {}_{b}^{C}\nabla^{\alpha}\,1 = 0$

ii. ${}^{C}\Delta_a^{\alpha}1 = {}_{b}^{C}\Delta^{\alpha}\,1 = 0$

Proof:

i. ${}^{C}\nabla_a^{\alpha}1 = \nabla_a^{-(n-\alpha)}\nabla^n 1 = 0$ and ${}_{b}^{C}\nabla^{\alpha}\,1 = \nabla_a^{-(n-\alpha)}\nabla^n 1 = 0.$

ii. ${}^{C}\Delta_a^{\alpha}1 = \Delta_a^{-(n-\alpha)}\nabla^n 1 = 0$ and ${}_{b}^{C}\Delta^{\alpha}\,1 = \Delta_a^{-(n-\alpha)}\nabla^n 1 = 0.$

Example 3.1.9 For any $\alpha > 0$, and $\gamma \in \mathbb{N}_a$

i. ${}^{C}\nabla_a^{\alpha}\gamma = {}_{b}^{C}\nabla^{\alpha}\,\gamma = 0$

ii. ${}^{C}\Delta_a^{\alpha}\gamma = {}_{b}^{C}\Delta^{\alpha}\,\gamma = 0$

Proof:

i. ${}^{C}\nabla_a^{\alpha}\gamma = \nabla_a^{-(n-\alpha)}\nabla^n\gamma = 0$ and ${}_{b}^{C}\nabla^{\alpha}\,\gamma = \nabla_a^{-(n-\alpha)}\nabla^n\gamma = 0.$

ii. ${}^{C}\Delta_a^{\alpha}\gamma = \Delta_a^{-(n-\alpha)}\nabla^n\gamma = 0$ and ${}_{b}^{C}\Delta^{\alpha}\,\gamma = \Delta_a^{-(n-\alpha)}\nabla^n\gamma = 0.$

3.2 A Dual Nabla Caputo Fractional Difference

In this section we define other nabla Caputo fractional difference for which it is not necessary to request any information about f before a or after b, since we will show that these Caputo fractional differences are the dual ones for the delta Caputo fractional difference, we call them dual nabla Caputo fractional difference [25].

Definition 3.2.1. Let $\alpha > 0, n = [\alpha] + 1, a(\alpha) = a + n - 1,$ and $b(\alpha) = b - n + 1$. Then the dual nabla left and right Caputo fractional differences are defined by

$$^{C}\nabla^{\alpha}_{a(\alpha)} f(t) = \nabla^{-(n-\alpha)}_{a(\alpha)} \nabla^{n} f(t), \quad t \in \mathbb{N}_{a+n}$$

$$_{b(\alpha)}^{C}\nabla^{\alpha} f(t) = {}_{b(\alpha)}\Delta^{-(n-\alpha)}{}_{\ominus}\Delta^{n} f(t), \quad t \in {}_{b-n}\mathbb{N}$$

respectively.

Notice that the Caputo and the dual Caputo difference coincide when $0 < \alpha \leq 1$ and differ for higher order. That is, for $0 < \alpha \leq 1$,

$$^{C}\nabla^{\alpha}_{a(\alpha)} f(t) = {}^{C}\nabla^{\alpha}_{a} f(t), \qquad\qquad _{b(\alpha)}^{C}\nabla^{\alpha} f(t) = {}_{b}^{C}\nabla^{\alpha} f(t).$$

The following proposition states dual relation between left delta Caputo fractional differences and left nabla (dual) Caputo fractional difference.

Proposition 3.2.2.[25] For $\alpha > 0, n = [\alpha] + 1, a(\alpha) = a + n - 1$, one has

$$\left({}^{C}\Delta^{\alpha}_{a}\right)(t - \alpha) = \left({}^{C}\nabla^{\alpha}_{a(\alpha)}\right)(t), \quad t \in \mathbb{N}_{a+n}.$$

Proof: By definition delta caputo left fractional difference and by remark 2.2.1 for we have

$$\left({}^{C}\Delta^{\alpha}_{a}\right)(t - \alpha) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-n} \left(t - \alpha - \sigma(s)\right)^{\underline{n-\alpha-1}} \Delta^{n} f(s)$$

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{s=a}^{t-n}\big(t-\alpha-\sigma(s)\big)^{\underline{n-\alpha-1}}\nabla^n f(s+n)$$

Let $r=s+n$, when $s=a$ then $r=a+n$ and when $s=t-n$ then $r=t$

so, $$\left({}^{C}\Delta_a^\alpha\right)(t-\alpha)=\frac{1}{\Gamma(n-\alpha)}\sum_{r=a+n}^{t}(t-\alpha-\sigma(r-n)+1-1)^{\underline{n-\alpha-1}}\nabla^n f(r)$$

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{r=a+n}^{t}(t-(r-1)+n-\alpha-2)^{\underline{n-\alpha-1}}\nabla^n f(r)$$

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{r=a+n}^{t}(t-\rho(r)+(n-\alpha-1)-1)^{\underline{n-\alpha-1}}\nabla^n f(r)$$

Using identity $t^{\overline{\alpha}}=(t+\alpha-1)^{\underline{\alpha}}$, then

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{r=a+n}^{t}(t-\rho(r))^{\overline{n-\alpha-1}}\nabla^n f(r)=\left({}^{C}\nabla_{a(\alpha)}^\alpha\right)(t).$$

The following proposition relates right delta Caputo fractional difference and right nabla (delta) Caputo fractional difference [see 25].

Proposition 3.2.3. For $\alpha>0, n=[\alpha]+1, b(\alpha)=b-n+1$, one has

$$({}_{b}^{C}\Delta^\alpha)(t+\alpha)=\left({}_{b(\alpha)}^{C}\nabla^\alpha\right)(t),\qquad t\in {}_{n-n}\mathbb{N}\ .$$

Proof: By definition delta caputo right fractional difference and by remark 2.2.1 for we have

$$({}_{b}^{C}\Delta^\alpha)(t+\alpha)=\frac{1}{\Gamma(n-\alpha)}\sum_{s=t+\alpha+n-\alpha}^{b}\big(s-\sigma(t+\alpha)\big)^{\underline{n-\alpha-1}}\nabla_\ominus^n\ f(s)$$

$$=\frac{1}{\Gamma(n-\alpha)}\sum_{s=t+n}^{b}\big(s-(t+\alpha+1)\big)^{\underline{n-\alpha-1}}{}_\ominus\Delta^n f(s-n)$$

let $r = s - n$ then , when $s = t + n$ then $r = t$ and when $s = b$ then $r = b - n$

$$({}_b^C\Delta^\alpha)(t+\alpha) = \frac{1}{\Gamma(n-\alpha)} \sum_{r=t}^{b-n} (r+n-t-\alpha-1)^{\underline{n-\alpha-1}} {}_\ominus\Delta^n f(r)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{r=t}^{b-n} (r-(t-1)+n-\alpha-1-1)^{\underline{n-\alpha-1}} {}_\ominus\Delta^n f(r)$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b-n} (r-\rho(t)+(n-\alpha-1)-1)^{\underline{n-\alpha-1}} {}_\ominus\Delta^n f(r)$$

Using identity $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha}}$, then

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{b-n} (r-\rho(t))^{\overline{n-\alpha-1}} {}_\ominus\Delta^n f(r) = \left({}_{b(\alpha)}^{\;C}\nabla^\alpha\right) f(t).$$

Theorem 3.2.4. For any real number α and any positive integer p, the following equality holds.

$$\nabla_{a+p-1}^{-\alpha} \nabla^p f(t) = \nabla^p \nabla_{a+p-1}^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-(a+p-1))^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a+p-1)$$

Where f is defined on only $\mathbb{N}_a$.

Proof: By modifies theorem 2.1.18 when replace b by $a+p-1$.

Similarly, in the right case we have the following.

Theorem 3.2.5. For any real number α and any positive integer p the following equality holds:

$$_{b-p+1}\nabla^{-\alpha} {}_\ominus\Delta^p f(t) = {}_\ominus\Delta^p {}_{b-p+1}\nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-p+1-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} {}_\ominus\Delta^k f(b-p+1)$$

Where f is defined on ${}_b\mathbb{N}$.

Proof: By modifies theorem 2.1.22 when replace b by $b - p + 1$.

Theorem 3.2.6. For any $\alpha > 0$, one has

i. $${}^{C}\nabla^{\alpha}_{a(\alpha)} f(t) = \nabla^{\alpha}_{a(\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(t - a(\alpha))^{\overline{k-\alpha}}}{\Gamma(k - \alpha + 1)} \nabla^{k} f(a(\alpha)),$$

ii. $${}_{b(\alpha)}^{\ \ C}\nabla^{\alpha} f(t) = {}_{b(\alpha)}\nabla^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(b(\alpha) - t)^{\overline{k-\alpha}}}{\Gamma(k - \alpha + 1)} {}_{\ominus}\Delta^{k} f(b(\alpha)).$$

In particular, when $0 < \alpha < 1$, then $a(\alpha) = a,$ and $b(\alpha) = b$ and hence one has

$${}^{C}\nabla^{\alpha}_{a} f(t) = \nabla^{\alpha}_{a} f(t) - \frac{(t - a)^{\overline{-\alpha}}}{\Gamma(1 - \alpha)} f(a),$$

$${}_{b}^{C}\nabla^{\alpha} f(t) = {}_{b}\nabla^{\alpha} f(t) - \frac{(b - t)^{\overline{-\alpha}}}{\Gamma(1 - \alpha)} f(b).$$

Proof: by definition dual nabla caputo left and right fractional difference

i. $${}^{C}\nabla^{\alpha}_{a(\alpha)} f(t) = \nabla^{-(n-\alpha)}_{a(\alpha)} \nabla^{n} f(t)$$

By theorem 3.2.4 then

$${}^{C}\nabla^{\alpha}_{a(\alpha)} f(t) = \nabla^{n} \nabla^{-(n-\alpha)}_{a(\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{\left(t - a(\alpha)\right)^{\overline{n-\alpha-n+k}}}{\Gamma(n - \alpha + k - n + 1)} \nabla^{k} f(a(\alpha))$$

$$= \nabla^{\alpha}_{a(\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{\left(t - a(\alpha)\right)^{\overline{k-\alpha}}}{\Gamma(k - \alpha + 1)} \nabla^{k} f\left(a(\alpha)\right).$$

ii. $${}_{b(\alpha)}^{\ \ C}\nabla^{\alpha} f(t) = {}_{b(\alpha)}\nabla^{-(n-\alpha)} {}_{\ominus}\Delta^{n} f(t)$$

By theorem 3.2.5 then

$${}^{C}\nabla^{\alpha}_{b(\alpha)} f(t) = {}_{\ominus}\Delta^{n} {}_{b(\alpha)}\nabla^{-(n-\alpha)} f(t) - \sum_{k=0}^{n-1} \frac{(b(\alpha) - t)^{\overline{n-\alpha-n+k}}}{\Gamma(n - \alpha + k - n + 1)} {}_{\ominus}\Delta^{n} f(b(\alpha))$$

$$= {}_{b(\alpha)}\nabla^{n} f(t) - \sum_{k=0}^{n-1} \frac{(b(\alpha) - t)^{\overline{k-\alpha}}}{\Gamma(k - \alpha + 1)} {}_{\ominus}\Delta^{n} f(t).$$

Theorem 3.2.7. Assume that $\alpha > 0$ and f is defined on suitable domains $\mathbb{N}_a$ and ${}_b\mathbb{N}$. Then,

i. $\nabla_{a(\alpha)}^{-\alpha}\ {}^{C}\nabla_{a(\alpha)}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a(\alpha))^{\overline{k}}}{k!} \nabla^k f(a(\alpha)),$

ii. ${}_{b(\alpha)}\nabla^{-\alpha}\ {}_{b(\alpha)}^{\ C}\nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b(\alpha)-t)^{\overline{k}}}{k!}\ {}_{\ominus}\Delta^k f(b(\alpha)).$

In particular, if $0 < \alpha \leq 1$, then $a(\alpha) = a$ and $b(\alpha) = b$.

$$\nabla_a^{-\alpha}\ {}^{C}\nabla_a^{\alpha} f(t) = f(t) - f(a),$$

$$ {}_b^{-\alpha}\nabla {}_b^{C}\nabla^{\alpha} f(t) = f(t) - f(b).$$

Proof: by dual nabla left and right caputo fractional difference

i. $\nabla_{a(\alpha)}^{-\alpha}\ {}^{C}\nabla_{a(\alpha)}^{\alpha} = \nabla_{a(\alpha)}^{-\alpha} \nabla_{a(\alpha)}^{-(n-\alpha)} \nabla^n f(t) = \nabla_{a(\alpha)}^{-n} \nabla^n f(t)$, by theorem 3.2.4

$$\nabla_{a(\alpha)}^{-\alpha}\ {}^{C}\nabla_{a(\alpha)}^{\alpha} = \nabla^n f(t) \nabla_{a(\alpha)}^{-n} - \sum_{k=0}^{n-1} \frac{\left(t-a(\alpha)\right)^{\overline{n-n+k}}}{\Gamma(n+k-n+1)} \nabla^k f(a(\alpha))$$

$$= f(t) - \sum_{k=0}^{n-1} \frac{\left(t-a(\alpha)\right)^{\overline{k}}}{\Gamma(k+1)} \nabla^k f\big(a(\alpha)\big)$$

$$= f(t) - \sum_{k=0}^{n-1} \frac{\left(t-a(\alpha)\right)^{\overline{k}}}{k!} \nabla^k f\big(a(\alpha)\big).$$

ii. ${}_{b(\alpha)}\nabla^{-\alpha}\ {}_{b(\alpha)}^{\ C}\nabla^{\alpha} f(t) = {}_{b(\alpha)}\nabla^{-\alpha}\ {}_{b(\alpha)}\nabla^{-(n-\alpha)}\ {}_{\ominus}\Delta^n f(t) = {}_{b(\alpha)}\nabla^{-n}\ {}_{\ominus}\Delta^n f(t)$

By theorem 3.2.5,

$${}_{b(\alpha)}\nabla^{-\alpha}\ {}_{b(\alpha)}^{\ C}\nabla^{\alpha} f(t) = {}_{\ominus}\Delta^n\ {}_{b(\alpha)}\nabla^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(b(\alpha)-t)^{\overline{n-n+k}}}{\Gamma(n+k-n+1)}\ {}_{\ominus}\Delta^k f(b(\alpha))$$

$$= {}_{\ominus}\Delta^n\ {}_{b(\alpha)}\nabla^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(b(\alpha)-t)^{\overline{k}}}{k!}\ {}_{\ominus}\Delta^k f\big(b(\alpha)\big).$$

3.3 Integration by Parts for Caputo Fractional Difference

In this section, we state the integration by parts formulas for nabla fractional sum and difference and use the dual identities to obtain delta integration by part formulas.

Proposition 3.3.1.[25] For

$\alpha > 0, a, b \in \mathbb{R}, f$ defined on $\mathbb{N}_a$and g defined on ${}_b\mathbb{N}$ one has

$$\sum_{s=a+1}^{b-1} g(s)\nabla_a^{-\alpha} f(s) = \sum_{s=a+1}^{b-1} f(s)\, {}_b\nabla^{-\alpha} g(s).$$

Proof: By the definition of the nabla fractional sum we have

$$\sum_{s=a+1}^{b-1} g(s)\nabla_a^{-\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{b-1} g(s) \sum_{r=a+1}^{s} \left(s-\rho(r)\right)^{\overline{\alpha-1}} f(r)$$

If we interchange the order of summation we reach at

$$\sum_{s=a+1}^{b-1} g(s)\nabla_a^{-\alpha} f(s) = \sum_{r=a+1}^{b-1} f(s)\, {}_b\nabla^{-\alpha} g(s).$$

Proposition 3.3.2.[23] let $\alpha > 0$ be noninterger and $a, b \in \mathbb{R}$ such that $a < b$ and $b \equiv a(mod\ 1)$. If f is defined on ${}_b\mathbb{N}$ and g is defined on $\mathbb{N}_a$.then

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha}\, g(s) = \sum_{s=a+1}^{b-1} g(s)\, {}_b\nabla^{\alpha} f(s).$$

Proof: By equation ${}_b\nabla^{-\alpha}\, {}_b\nabla^{\alpha} f(t) = f(t)$ and proposition 3.3.1 implies

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha}\, g(s) = \sum_{s=a+1}^{b-1} {}_b\nabla^{-\alpha}\, {}_b\nabla^{\alpha} f(s)\nabla_a^{\alpha}\, g(s) = \sum_{s=a+1}^{b-1} {}_b\nabla^{\alpha} f(s)\nabla_a^{-\alpha}\nabla_a^{\alpha}\, g(s)$$

$$= \sum_{s=a+1}^{b-1} {}_b\nabla^{\alpha} f(s) g(s) = \sum_{s=a+1}^{b-1} g(s)\, {}_b\nabla^{\alpha} f(s).$$

Proposition 3.3.3.[24] Let $\alpha > 0$ and $a, b \in \mathbb{R}$ such that $a < b$ and $b \equiv a(mod\ 1)$. If f is defined on $\mathbb{N}_a$ and g is defined on ${}_b\mathbb{N}$ .then

$$\sum_{s=a+1}^{b-1} g(s)(\Delta_{a+1}^{-\alpha} f)(s+\alpha) = \sum_{s=a+1}^{b-1} f(s)_{b-1}\Delta^{-\alpha} g(s-\alpha)$$

Proof: By theorem 2.2.2 and use theorem 3.3.1 and theorem 2.2.4, we have

$$\sum_{s=a+1}^{b-1} g(s)(\Delta_{a+1}^{-\alpha} f)(s+\alpha) = \sum_{s=a+1}^{b-1} g(s)\nabla_a^{-\alpha} f(s)$$

$$= \sum_{s=a+1}^{b-1} f(s)\, {}_b\nabla^{-\alpha} g(s) = \sum_{s=a+1}^{b-1} f(s)_{b-1}\Delta^{-\alpha} g(s-\alpha).$$

Proposition 3.3.4.[24] Let $\alpha > 0$ be noninterger and assume $b \equiv a(mod\ 1)$. If f is defined on ${}_b\mathbb{N}$ and g is defined on $\mathbb{N}_a$, then

$$\sum_{s=a+1}^{b-1} f(s)\Delta_{a+1}^{\alpha} g(s-\alpha) = \sum_{s=a+1}^{b-1} g(s)_{b-1}\Delta^{\alpha} f(s+\alpha)$$

Proof: By lemma 2.2.3 and use theorem 3.3.1

$$\sum_{s=a+1}^{b-1} f(s)\Delta_{a+1}^{\alpha} g(s-\alpha) = \sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} g(s)\, {}_b\nabla^{\alpha} f(s)$$

$$= \sum_{s=a+1}^{b-1} g(s)_{b-1}\Delta^{\alpha} f(s+\alpha).$$

Now, we proceed in this section to obtain nabla and delta integration by parts formulas for Caputo fractional differences [see 25].

Theorem 3.3.5. Let $0 < \alpha < 1$ and let f, g be functions defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$ where $a \equiv b(mod 1)$. Then,

$$\sum_{S=a+1}^{b-1} g(s)\, {}^C\nabla_a^\alpha f(s) = f(s)\, {}_b\nabla^{-(1-\alpha)} g(s)\Big|_a^{b-1} + \sum_{s=a}^{b-2} f(s)\, {}_b\nabla^\alpha g(s),$$

Where clearly ${}_b\nabla^{-(1-\alpha)} g(b-1) = g(b-1)$.

Proof: From the definition of caputo fractional difference and Proposition 3.3.1 we have

$$\sum_{S=a+1}^{b-1} g(s)\, {}^C\nabla_a^\alpha f(s) = \sum_{s=a+1}^{b-1} g(s) \nabla_\alpha^{-(1-\alpha)} \nabla f(s) = \sum_{s=a+1}^{b-1} \nabla f(s)\, {}_b\nabla^{-(1-\alpha)} g(s).$$

By integration by parts form difference calculus, $\nabla f(s) = \Delta f(s-1)$, and the definition of nabla right fractional difference, we reach at

$$\sum_{S=a+1}^{b-1} g(s)\, {}^C\nabla_a^\alpha f(s) = f(s)\, {}_b\nabla^{-(1-\alpha)} g(s)\Big|_a^{b-1} + \sum_{s=a+1}^{b-2} f(s-1)\, {}_b\nabla^\alpha g(s-1),$$

$$= f(s)\, {}_b\nabla^{-(1-\alpha)} g(s)\Big|_a^{b-1} + \sum_{s=a+1}^{b-2} f(s)\, {}_b\nabla^\alpha g(s).$$

Theorem 3.3.6. Let $0 < \alpha < 1$ and let f , g be function defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$ where $a \equiv b(mod 1)$. Then,

$$\sum_{s=a+1}^{b+1} g(s)\, {}^C\Delta_a^\alpha f(s-\alpha)$$

$$= f(s)_{b-1}\Delta^{-(1-\alpha)} g\big(s-(1-\alpha)\big)\Big|_a^{b-1} + \sum_a^{b-2} f(s)_{b-1}\Delta^\alpha g(s+\alpha)$$

Proof: By the dual Caputo identity proposition 3.2.2 and theorem 3.3.6 we have

$$\sum_{s=a+1}^{b+1} g(s)\, {}^{C}\Delta_a^{\alpha} f(s-\alpha) = f(s)\, {}_b\nabla^{-(1-\alpha)} g(s)\Big|_a^{b-1} + \sum_{a}^{b-2} f(s)_{b-1}\Delta^{\alpha} g(s+\alpha)$$

Use by lemmas 2.2.5, we get

$$\sum_{s=a+1}^{b+1} g(s)\, {}^{C}\Delta_a^{\alpha} f(s-\alpha) \ = f(s)_{b-1}\Delta_a^{\alpha} g(s-(1-\alpha))\Big|_a^{b-1} + \sum_{a}^{b-2} f(s)_{b-1}\Delta^{\alpha} g(s+\alpha).$$

3.4 The Q-operator

The Q-dual identities obtained in this section expose the validity f the definition of delta and nabla right Caputo fractional difference.

Definition 3.4.1 . If $f(s)$ is defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$ and $a \equiv b(mod\ 1)$ then the Q-operator is defined by $(Qf) = f(a+b-s)$.

The Q-operrator generates a dual identity by which the left type and the right type fractional sums and difference are related. Using the change of variable $u = a+b-s$.

Theorem 3.4.2. The following is hold

i. $\Delta_a^{-\alpha} Qf(t) = Q\ {}_b\Delta^{-\alpha} f(t)$.
ii. $\nabla_a^{-\alpha} Qf(t) = Q\ {}_b\nabla^{-\alpha} f(t)$.

Proof:

i. By definition 3.4.1 and definition 2.1.2 , we get

$$\Delta_a^{-\alpha} Qf(t) = \Delta_a^{-\alpha} f(a+b-t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{a+b-t-\alpha} \left(a+b-t-\sigma(s)\right)^{\underline{\alpha-1}} f(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{a+b-t-\alpha} \left(a+b-s-\sigma(t)\right)^{\underline{\alpha-1}} f(s)$$

Let $u = a + b - s$ when $s = a$ then $u = b$ and when $s = a + b - t - \alpha$ then

$u = a + b - (a + b - t - \alpha) = t + \alpha$ then

$$\Delta_a^{-\alpha} \mathrm{Q}\, f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=b}^{t+\alpha} \left(u-\sigma(t)\right)^{\underline{\alpha-1}} f(a+b-u)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} \left(u-\sigma(t)\right)^{\underline{\alpha-1}} f(a+b-u) = \mathrm{Q}\ {}_b\Delta^{-\alpha} f(t).$$

ii. Be definition 3.4.1 and definition 2.1.2

$$\nabla_a^{-\alpha} Qf(t) = \nabla_a^{-\alpha} f(a+b-t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{a+b-t} \left(a+b-t-\rho(s)\right)^{\overline{\alpha-1}} f(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{a+b-t} \left(a+b-s-\rho(t)\right)^{\overline{\alpha-1}} f(s)$$

Let $u = a + b - s$ when $s = a + 1$ then $u = b - 1$ and when $s = a + b - t$

then $u = a + b - (a + b - t) = t$, then

$$\nabla_a^{-\alpha} Qf(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=b-1}^{t} \left(u-\rho(t)\right)^{\overline{\alpha-1}} f(a+b-u)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} \left(u-\rho(t)\right)^{\overline{\alpha-1}} f(a+b-u) = \mathrm{Q}\ {}_b\nabla^{-\alpha} f(t).$$

Theorem 3.4.3. The following is hold

i. $-Q\nabla f(t) = \Delta Q f(t)$,

ii. $-Q\Delta f(t) = \nabla Q f(t)$.

Proof: by definition 3.4.1

i. $-Q\nabla f(t) = -Q[f(t) - f(t-1)] = -[Qf(t) - Qf(t-1)]$

$$= -[f(a+b-t) + f(a+b-(t-1)]$$

$$= [f(a+b-t+1) + f(a+b-t)] = \Delta Qf(t).$$

ii. $-Q\Delta f(t) = -Q[f(t+1) - f(t)] = -[Qf(t+1) - Qf(t)]$

$$= -[f(a+b-(t+1)) + f(a+b-t)]$$

$$= [f(a+b-t) + f(a+b-t-1)] = \nabla Qf(t).$$

The Q-operator is used to relate left and right Caputo fractional difference in the nabla and delta case.

Theorem 3.4.4. For any $\alpha > 0$,

i. ${}^{C}\Delta_a^{\alpha} Qf(t) = Q({}_b^C\Delta^{\alpha} f)(t)$,

ii. ${}^{C}\nabla_a^{\alpha} Qf(t) = Q({}_b^C\nabla^{\alpha} f)(t)$.

Proof: By definition of left and right Caputo fractional difference and by theorem 3.4.3, we get

i. ${}^{C}\Delta_a^{\alpha} Qf(t) = \Delta_a^{-(n-\alpha)} \Delta^n Qf(t) = \Delta_a^{-(n-\alpha)} Q\nabla_{\ominus}^n f(t)$, use theorem 3.4.2 then

$$= Q\ {}_b\Delta^{-(n-\alpha)} \nabla_{\ominus}^n f(t) = Q({}_b^C\Delta^{\alpha} f)(t).$$

ii. ${}^{C}\nabla_a^{\alpha} Qf(t) = \nabla_a^{-(n-\alpha)} \nabla^n Qf(t) = \nabla_a^{-(n-\alpha)} Q\,{}_{\ominus}\Delta^n f(t)$, use theorem 3.4.2 then

$$= Q\ {}_b\nabla^{-(n-\alpha)}\,{}_{\ominus}\Delta^n f(t) = \mathrm{Q}({}_b^C\nabla^{\alpha} f)(t).$$

Conclusion

Discrete fractional calculus is a relatively new theory compared to ordinary calculus. There are still many open questions in this newly developing theory and although the theory shows great potential for analyzing real world applications, ordinary calculus is still much more commonly used in such problems.

In this thesis, we proved several relations between fractional nabla and delta sum and difference. Also, we showed that there are relations between delta and nable with fractional nabla and sum. So, we believed that some one can prove that there are relation between nabla and delta with fractional nabla and delta difference. Moreover, we conjecture that there is a relation between them caputo nable and delta.

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الاشتقاق الجزئي لنبلا ودلتا كبتو و وحدة الثنائي

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الملخص

حساب التفاضل والتكامل الكسري منفصلة هي نظرية جديدة نسبيا بالمقارنة مع حساب التفاضل والتكامل العاديين. لا تزال هناك العديد من الأسئلة المفتوحة في هذه النظرية النامية حديثا وعلى الرغم من أن نظرية توضح إمكانات كبيرة لتحليل تطبيقات العالم الحقيقي، حساب التفاضل والتكامل العاديين لا تزال تستخدم أكثر من ذلك بكثير عادة في مثل هذه المشاكل.

في هذه الرسالة ، اثبتنا عدة علاقات بين مجموع كسور دلتا ونبلا مع رمز فرق دلتا ونبلا. لذلك نعتقد انو يمكن لبعض ان يثبت ان هناك علاقة بين فرق كسور نبلا و دلتا مع رموز الفرق دلتا ونبلا .

وعلاة على ذلك ، اظن ان هناك علاقة بين رموز الفرق وكابيتو النبلا و الدلتا .