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Thesis Approval

Solving an Optimal Control Problem by using Fuzzy Logic Control

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Abstract

The optimal control problem can be solved either by using the indirect methods or the direct methods. Recently, many researchers have proposed a numerical method based on the indirect methods to solve the optimal control problem via fuzzy logic control.

In addition to the parameterization and discretization techniques, and based on fuzzy logic control and fuzzy basis functions (FBF's) expansion, in this thesis we have proposed a direct method to design an optimal fuzzy controller to control a linear dynamic system under finite horizon. The method is based on parameterizing the control vector by a series of fuzzy basis functions with unknown parameters and discretizes the state vectors.

The main idea of the proposed method can be concluded as follows: Instead of minimizing a continuous performance index over the whole time interval, we minimize the performance index only over one sampling period, that is mean we minimizing the performance index over all the time interval but in a sequence.

Therefore, we convert the optimal control problem to mathematical programming problem, thus we avoid solving the two point boundary value problem (TPBVP) which may be complicated.

The proposed method was tested for the linear system with and without constraints. And it has been applied on several examples and we have found that the proposed method gives better or comparable results compared with some other methods. Several numerical examples are included to demonstrate the effectiveness of the proposed method.

الملخص

يمكن حـل مسـألة التحكم الأمثل اما بالطرق المبـاشرة او الطـرق غير المبـاشرة، في الوقت الحاضر هنالك الكثيـر من الباحثيـن اقترحوا طرق عددية تعتمـد على الطـرق غيـر المبـاشرة لحل مسـألة التحكم الخطي الأمثل و ذلك بإسـتخدام مفهوم التحكم الغامض.

في هذه الأطروحة، لقد اقترحنـا طريقة مبـاشرة لحل مسألة التحكم الخطي الأمثل في نظام معين مشـروط بشـروط متسـاوية و غيـر متسـاوية، و هـذة الطريقـة تعتمـد علـى التعبيـر عـن الضابط بإسـتخدام نظام التحكم الغامض و متغير آخـر غير معروف، علما بإن هذا المتغير هو الذي نريد أن نحصل على الحل الأفضل له، بالأضافة الى أن الطريقة المقترحة تعتمد على التعبير عن النظام بشـكل منفصل وغير متواصل وكذلك الأمر بالنسبة لمقياس الكفاءة.

بهذة الطريقة نكون قد حولنا مسألة التحكم الخطي الأمثل الى مسـألة برمجة عدديـة، مع العلم ان هذا النوع من المسـائل سهلة الحل، ليس كمسـألة التحكم الخطي الأمثل.

لقد قمنا بتجربة الطريقة المقترحة على بعض المشـاكل المحلولة بطرق أخرى، و لقد حصلنا على نتائج جيدة و مرضية و متـوافقة مع النتـائج التى حصـل عليـها باحثين سـابقين.

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Chapter 1

Optimal Control Problem

1.1 Introduction

The optimal control problem has been discussed in depth in many textbooks [1, 2, 3 and 4] and in some important survey papers [5 and 6]. The main objective of the optimal control is to determine an optimal control $u^*(t)$ which represent an open loop control, or $u^*(x,t)$ which represent an optimal feedback control that forces the system to satisfy the system physical constraints and at the same time minimizes or maximizes a specific performance index.

Optimal control is one of the oldest approaches in control engineering. It has many advantages [7] such that, the state and control constraints can be included explicitly and the cost functions to be minimized can be given in a simple intuitively appealing interpretation. But unfortunately, optimal control suffers from a major disadvantage [7]; solving optimal control problem is in general computationally difficult, except in very special cases a closed form expression of control law can be obtained.

From the point of view of the practicing control engineer [7], the main difficulty in applying optimal control is in obtaining a feedback implementation. There are many numerical approaches to solve open-loop optimal control problems; some are based on solving the Two Point Boundary Value Problems "TPBVP", others are based on dynamical programming. However, none of these numerical techniques is fast enough in general to allow a direct extension of the open loop solution to closed loop or feed-back implementation.

The basic optimal control problem consists of the following three elements [8]:
1) A mathematical representation of the system to be controlled: The dynamical system to be controlled can be described by state space representations which are a set of first order differential equations:

$$x' = f(x(t), u(t), t) \tag{1.1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control vector and f is continuously differentiable with respect to all its arguments. The control functions u are assumed to be the piecewise continuous functions from $t \in [0, t_f]$ into $\Re^m$.

2) A set of boundary conditions on the state variables which gives the value of the system states at the initial time:

$$x(t_0) = x_0 \tag{1.2}$$

where x_0 is a known vector of initial conditions.

3) A performance index which is to be minimized or maximized. The performance index describes some desired specifications, expressed mathematically in form of a scalar function. The optimal control problem helps the designer to select the control, by minimizing a given performance index. The performance index [2] which we are interested in can be expressed as:

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)\,dt \tag{1.3}$$

where ϕ and L are scalar functions, continuously differentiable in all arguments. The terminal cost $\phi(x(t_f), t_f)$ and the cost function $L(x(t), u(t), t)$ are selected depending on the performance objectives [3].

In designing optimal control systems, rules may be set for determining the control decisions, subject to certain constraints, such that some measure of deviations from an ideal case is minimized. That measure is usually provided by a selected performance index or cost function.

The performance index [2] is a function whose value indicates how well the actual performance of the system matches the desired performance.

Appropriate selection of the performance index is important since it determines the nature and complexity of the problem. That is, if the optimal control is linear or nonlinear, stationary or time-varying will be dependent on the selected performance index [3]. The choice of the performance index depends on system specifications, physical reliability, and the restrictions on the controls to be used. In general, the choice of an appropriate performance index involves a compromise between a meaningful evaluation of system performance and availability of feasible mathematical descriptions.

The performance index is selected by the engineer to make the system behave in a desired fashion. By definition [1], a system whose design minimizes the selected performance index is optimal. It is important to point out that a system optimal for one performance index may not be optimal under another performance index.

Before starting optimization of any system, it is necessary to formulate the system by having information on system parameters and describing equations, constraints, class of allowable control vectors, and the selected performance indexes. Then the solution can proceed by determining the optimal control vector $u^*(t)$ within the class of allowable vectors. The control vector $u(t)$ will be dependent on such factors as the nature of the performance index, constraints, initial values of states, initial outputs, and the desired state as well as the desired outputs. If analytical solutions are impossible or too complicated, then alternative computational solutions may be employed.

Linear quadratic optimal control problem is a special case from optimal control problem, the linear quadratic optimal control problem refers to problems, where the performance measure to be optimized a quadratic function of the state and control and the plant model is linear, and is given by:

$$x' = Ax(t) + Bu(t), \qquad y = Cx(t) \tag{1.4}$$

The quadratic optimal control problem is to find a control input u that minimizes the quadratic performance index J (1.5).

$$J = x^T(t_f)Sx(t_f) + \int_{t_0}^{t_f}(x^TQx + u^TRu)dt \tag{1.5}$$

where S and Q are symmetric positive semidefinite matrices, and R is a symmetric positive definite matrix.

1.2 Statement of Optimal Control Problem

The unconstrained optimal control problem can be stated as follows: Given f, x_0, t_0, t_f, ϕ and L, find an optimal open loop control or an optimal feedback control that minimizes the performance index:

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt \tag{1.6}$$

subject to the system state equations and initial conditions:

$$x' = f(x(t), u(t), t) \qquad x(t_0) = x_0 \tag{1.7}$$

In general it is not possible to solve the problem (1.6)-(1.7) analytically [8]. However, an analytical solution is possible for a special case of this problem, the linear quadratic optimal control problem, in which the performance index is quadratic and the system state equations are linear. This problem can be stated as follows: Find the optimal control that minimizes (1.8):

$$J = x^T(t_f)Sx(t_f) + \int_{t_0}^{t_f}(x^TQx + u^TRu)dt \tag{1.8}$$

subject to (1.9):

$$x' = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \tag{1.9}$$

where S, Q are positive semidefinite matrices and R is a positive definite matrix.

1.3 Solution of Optimal Control Problem

The optimal control problem can be solved by one of the following methods:

1) Bellman's dynamic programming method which is based on the principle of optimality (Hamilton-Jacobi-Bellman equation) [2, 3 and 8].

2) Variational method and Pontryagin minimum principle (Euler-Lagrange equations) [2, 8 and 9].

3) Direct methods using discretization or parameterization (nonlinear mathematical programming) [1 and 2].

1.3.1 Dynamic Programming: Hamilton-Jacobi-Bellman Equation:

The principle of optimality is usually known as dynamic programming, to derive an equation for solving optimal control problem was first proposed by Bellman [2].

The application of this principle on continuous optimal control problem has led to the invention of the famous Hamilton-Jacobin-Bellman "HJB" equation [2]. For the optimal control problem (1.6)-(1.7), let us now use the imbedding principle to include this problem in a larger class of problems by considering the performance measure:

$$J(x(t),t,\underset{t_0\le\tau\le t_f}{u(\tau)}) = \phi(x(t_f),t_f) + \int_{t_0}^{t_f} L(x(\tau),u(\tau),\tau)d\tau \tag{1.10}$$

where $t \le t_f$, and $x(t)$ can be any admissible state value, the performance measure will depend on the numerical values for $x(t)$ and t, and on the optimal control history in the interval $[t_0,t_f]$.

Let us now attempt to determine the controls that minimize (1.10) for all admissible $x(t)$, and for all $t \le t_f$, the minimum cost function is then:-

$$J^*(x(t),t) = \min_{\substack{u(\tau)\\ t_0\le\tau\le t_f}} \{ \int_{t_0}^{t_f} L(x(\tau),u(\tau),\tau)d\tau + \phi(x(t_f),t_f)\} \tag{1.11}$$

By subdividing the interval [2], we obtain:

$$J^*(x(t),t) = \min_{\substack{u(\tau)\\ t_0\le\tau\le t_f}} \{ \int_{t_0}^{t+\Delta t} L d\tau + \int_{t+\Delta t}^{t_f} L dt + \phi(x(t_f),t_f)\} \tag{1.12}$$

The principle of optimality requires that:-

$$J^*(x(t),t) = \min_{\substack{u(\tau)\\ t_0\le\tau\le t_f}} \{ \int_{t_0}^{t+\Delta t} L d\tau + J^*(x(t+\Delta t),t+\Delta t)\} \tag{1.13}$$

where $J^*(x(t,\Delta t),t+\Delta t)$ is the minimum cost of the process for the time interval $t+\Delta t \le \tau \le t_f$ with initial state $x(t+\Delta t)$.

Assuming that the second partial derivatives of J^* exist and are bounded, we can expand $J^*(x(t,\Delta t),t+\Delta t)$ in Taylor series about the point $x(t+\Delta t)$ [2].

For small Δt J becomes:

$$J^*(x(t),t) = \min_{u(t)}\{L(x(t),u(t),t)\Delta t + J^*(x(t),t) + J_t^*(x(t),t)\Delta t + J_x^{*T}(x(t),t)[f(x(t),u(t),t]\Delta t + o(\Delta t)\} \tag{1.14}$$

where $o(\Delta t)$ denotes the terms containing $[\Delta t]^2$ and higher orders of Δt that arise from the approximation of integral and the truncation of the Taylor series expansion. Next, removing the terms involving $J^*(x(t),t)$ and $J_t^*(x(t),t)$ from the minimization, we obtain:

$$0 = J_t^*(x(t),t)\Delta t + \min_{u(t)}\{L(x(t),u(t),t)\Delta t + J_x^{*T}(x(t),t)[f(x(t),u(t),t]\Delta t + o(\Delta t)\}. \tag{1.15}$$

Dividing by Δt and taking the limit as $\Delta t \to 0$ gives:

$$0 = J_t^*(x(t),t) + \min_{u(t)}\{L(x(t),u(t),t) + J_x^{*T}(x(t),t)[f(x(t),u(t),t)]\} \tag{1.16}$$

to find the boundary value for this partial differential equation, set $t = t_f$; from (1.11) it is apparent that:

$$J^*(x(t_f),t_f) = \phi(x(t_f),t_f) \tag{1.17}$$

Now, define the Hamiltonian H as follows:

$$H(x(t),u(t),J_x^*,t) = L(x(t),u(t),t) + J_x^{*T}(x(t),t)[f(x(t),u(t),t)] \tag{1.18}$$

and

$$H(x(t),u^*(x(t),J_x^*,t),J_x^*,t) = \min_{u(t)}\{H(x(t),u(t),J_x^*,t)\} \tag{1.19}$$

since the minimizing control will depend on x, J_x^* and t. Using these definitions, we obtained the Hamilton-Jacobi-Bellman "HJB" equation:

$$0 = J_t^*(x(t),t) + H(x(t),u^*(x(t),J_x^*,t),J_x^*,t) \tag{1.20}$$

In the case of the linear quadratic regulator optimal control problem (1.4)-(1.5), the HJB equation can be reduced to the Riccati differential equation, which is given by the following:

$$\begin{aligned} P'(x,u,t) &= P(x,u,t)A(x,u,t) + A^T(x,u,t)P(x,u,t) \\ &\quad - P(x,u,t)B(x,u,t)R^{-1}B^T(x,u,t)P(x,u,t) + Q \\ P(t_f) &= S \end{aligned} \tag{1.21}$$

1.3.2 Variational Method and Pontryagin Minimum Principle:

Variational method: Is the second approach for solving optimal control problem which is based on the calculus of variation method. This method based on the fact that at the stationary point, the variation in the cost function should vanish for arbitrary variation in the control [1].

Now, based on the Lagrange multiplayer $\lambda(t) \in \Re^n$ to adjoin to the system state equation (1.9) [2], the performance index becomes as in (1.22).

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [L(x(t), u(t), t) + \lambda^T (f(x(t), u(t), t) - x')]dt \tag{1.22}$$

Define the Hamiltonian function H as in (1.23) [9].

$$H(x(t), u(t), \lambda(t)) = L(x(t), u(t)) + \lambda^T(t)[f(x(t), u(t), t)] \tag{1.23}$$

Based on the Hamiltonian function, (1.22) becomes:

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} H(x(t), u(t), \lambda(t))dt - \int_{t_0}^{t_f} \lambda^T(t) x'(t)dt \tag{1.24}$$

Then, integration $\int_0^{t_f} \lambda^T(t)x'(t)dt$ by parts, therefore (1.24) becomes as in (1.25).

$$\begin{aligned} J = {} & \phi(x(t_f), t_f) - \lambda^T(t_f)x(t_f) + \lambda^T(t_0)x(t_0) \\ & + \int_{t_0}^{t_f} [H(x(t), u(t), \lambda(t)) + \lambda'^T(t)x(t)]dt \end{aligned} \tag{1.25}$$

We can note that, the original problem (1.6) - (1.7), has been converted to the problem of minimizing (1.25). If there is a stationary, the first order effect of control variations on the cost function must be zero for $t_0 \le t \le t_f$, based on this assumption and from [2 and 8], we can summarize the necessary conditions of the optimality as follows:

$$x' = f(x(t), u(t), t), \qquad x(0) = x_0 \tag{1.26}$$

$$\lambda' = -\frac{\partial H^*}{\partial x}, \qquad \lambda(t_f) = \frac{\partial \phi}{\partial x}\Big|_{x(t_f)} \tag{1.27}$$

$$\frac{\partial H^*}{\partial u} = 0 \tag{1.28}$$

To solve the optimal control problem, we need to solve the TPBVP (1.26) and (1.27).

We have assumed that, the admissible controls are not constrained by any boundaries; however, in realistic systems such constraints do commonly occur, therefore we can not differentiate the Hamiltonian with respect to the control. Thus, the necessary conditions of optimality can be derived from the Minimum Principle which was developed by Pontryagin.

Pontryagin minimum principle [2]: By definition, the control u^* causes functional J to have a relative minimum if

$$J(u) - J(u^*) = \Delta J \geq 0 \tag{1.29}$$

for all admissible controls sufficiently close to u^*. Assume that, u^* is the optimal control with corresponding optimal trajectories x^*, let the Hamiltonian as in (1.23), In order to u^* and x^* be optimal of the problem (1.6)-(1.7), then there must exist a co-state vector $\lambda^*(t)$ such that the following conditions hold [8]:

$$\dot{\lambda}^* = -\frac{\partial H^T}{\partial x}, \; \lambda(t_f) = \left(\frac{\partial \phi}{\partial x}\right)^T \tag{1.30}$$

and

$$H(x^*, u^*, \lambda^*) \leq H(x^*, u, \lambda^*) \tag{1.31}$$

for any $t_0 \leq t \leq t_f$ and for all admissible controls u, which indicates that the optimal control must minimize the Hamiltonian.

1.3.3 Direct Methods:

The direct methods are based on discretization or parameterization; these methods offer some advantages when applied to optimal control problems. The first advantage is that the difficult dynamic optimal control problem can be converted into static parameters optimization problem which is easier than the original one. The second advantage is that there are well-developed algorithms to solve the nonlinear programming problems. The third advantage is that it is possible to treat different types of constraints easily [8].

The direct methods are based on obtaining the solution through a direct minimization of the performance index, subject to constraints, of the optimal control problem. These methods can be applied by converting the nonlinear optimal control problem into a nonlinear mathematical programming problem [8].

All discretization approaches divide the time interval into N segments and assuming that the sampling times are $0, h, 2h, 3h \cdots, (N-1)h$, where $h > 0$ is the sampling interval [10], where the time points are referred to as mesh points, grid points or nodes.

The sequence of the unknown values of state variables and control variables, $z = (x_0, x_1, x_2, \cdots, x_N, u_0, u_1, u_2, \cdots, u_{N-1})$ and the system state equations are replaced by a set of algebraic equations which are considered as equality constraints. Hence this problem can be solved using any of the programming techniques. One of the disadvantages of this approach is the high dimensionality of the vector z.

Another approach is to discretize the control variables only $z=(u_0,u_1,u_2,\cdots,u_{N-1})$, and then the system state equations have to be integrated to find the state variables as a function of the control variables. For more details of these approaches see [11].

For the parameterization technique [8, 12 and 13], it can be applied in one of the following forms:
1. Control Parameterization: The control parameterization is based on approximating the control variables by choosing an appropriate structure with finitely many unknown parameters. The control parameterization approach is the most widely used parameterization approach. Applying this technique requires the integration of the system state equations, which is a computationally expensive process.

2. Control-State Parameterization: The control-state parameterization approach is based on approximating both the state variables and the control variables by a sequence of known functions with unknown parameters. This technique is the most computationally expensive process.

3. State Parameterization: The idea of the state parameterization is to approximate only the system state variables by a sequence of given functions with unknown parameters and then the control variables are obtained from the state equations.

1.4 Thesis Contribution

The work in this thesis is based on using the discretization and parameterization techniques. We used these techniques to convert the optimal control problem into a mathematical programming problem.

For the discretization technique, we divide the time interval into N segments and assume that the sampling times are $0,h,2h,3h\ \cdots,(N-1)h$, where $h>0$ is the sampling interval, we will explain the discretization in more detail in chapter three.

On the other hand, the parameterization technique also is an essential part of this research. Based on Fuzzy Logic Control "FLC" we formulate the controller $u(t)$ in terms of Fuzzy Basis Functions "FBF" and unknown parameter matrix, therefore we will explain this approximation in details in the following chapter.

We will express the controller $u(t)$ as follows:

$$u(t)=\sum_{j=1}^{M} b_j(x(t))\theta_j \qquad j=1,2,\cdots,M \tag{1.32}$$

where $b_j(x(t))$ is the jth fuzzy basis function, and θ_j is the jth unknown parameter variable.

Chapter 6

Conclusion and Future Works

6.1 Conclusion

In this thesis, we proposed a direct method to solve unconstrained and constrained linear quadratic optimal control problem. The proposed algorithm is based on direct method techniques and fuzzy logic control.

We approximated the controller in terms of fuzzy basis functions and unknown parameter matrix, and discretize the states and the quadratic performance index to reformulate the problem into a mathematical programming problem, which can be solved easily.

The main idea of the proposed method can be state as follows: Instead of minimizing a continuous PI over the whole time interval, we minimize the PI only over one sampling period at a time, then obtain the controller $u(k)$ and the next state - $x(k+1)$, which mean that, we minimizing the PI over all sampling intervals but in a sequence form. To compensate the remaining cost we add a terminal $x^T(k+1)Px(k+1)$ in the augmented performance index.

By direct minimization, we can minimize the augmented performance index subject to state equations and obtain $x(k+1)$ in terms of $b(x)$ and θ, then we can obtain the optimal value of the parameter matrix θ, but the optimal value of the performance index at each sampling interval can be obtained from the sampling performance index

Our proposed method was tested for the unconstrained and constrained linear quadratic optimal control problems; the obtained results showed that, our proposed method gives comparable results with previously proposed methods in the literature review.

In addition, the proposed method was applied to linearized models of two practical control systems, namely: Air-Craft problem and Ball-and-Beam problem. The obtained results agree with the results for the Air-Craft and the Ball-and-Beam problems.

The main advantage of the proposed algorithm is that, instead of solving optimal control problem, we need to solve mathematical programming problem, therefore avoiding solving the TPBVP which may be complicated.

6.2 Future Works

The work on this thesis can be expanded in three ways:

- For the fuzzy controller design, we can use different types of a fuzzification, fuzzy inference and defuzzification methods.
- In the proposed method is based on the Mamdani fuzzy model, therefore, the proposed method can be treated to Takagi-Sugeno fuzzy model.

- On the other hand, how we can be expanded the proposed method to the nonlinear optimal control problems.