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Analysis of radiant heat exchange

and

interaction with conduction

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Al-Quds University
2009

Declaration

I certify that thesis, submitted for the degree of Master, is the result of my own research except where otherwise acknowledged, and that thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ……………

Imad Ali Hamdan Al-Atrash

Date: 9/ 4/ 2009

Dedication

To my parents, my wife and my sons Nour, Marah, Rand , Mariam and Mohammad whom help and support, created a great touch in my whole life, and to all

whom I love.

Acknowledgement

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Thanks is given first to God.

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Abstract

This work deals with the three fundamental concepts of heat transfer modes (radiation, conduction and convection) with great emphasis on heat radiation and interaction with conduction.

Determining radiation interchange between surface areas is needed in heat transfer, illumination engineering and applied optics. In fact, since 1960 the study of radiant interchange has been given impetus by technological advances that provides systems in which thermal radiation is very important.

The geometric configuration factors derived here are an important component for analyzing radiation exchange. The computation of configuration factors involves integration, either analytically or numerically, over the solid angles by which surfaces can view each other.

Some examples are given to demonstrate analytical integrations arise in radiant heat analysis. Moreover, the question of coupling heat radiation with conduction has been dealt with. To analyze this problem we consider a conductive - radiative heat transfer model containing two conducting and opaque materials which are in contact by radiation through a transparent medium bounded by diffuse - grey surfaces. Some properties of the radiative integral operator are presented. The question of existence and uniqueness of weak solutions for this problem is investigated. The existence of weak solution is proved by showing that our problem is pseudo-monotone and coercive. The uniqueness of solution is proved using some ideas from the analysis of nonlinear heat conduction.

الملخص

هذة الرسالة تناولت المفاهيم الأساسية لانتقال الحرارة (الاشعاع والتوصيل والحمل) مع التركيز على الأنتقال الحراري بالأشعاع ، وتوضيح حالات حدوثها ، وتقديم القوانين الرياضية القرينة لكل واحدة منها. الأنتقال الحراري بالأشعاع يحدث بين السطوح بدرجات حرارة متباينة وهذا ما وضحه قانون " The Stefan _ Boltzmann". أما الأنتقال الحراري بالتوصيل يحدث عبر الأجسام الصلبة ، والسوائل موضحا بقانون "Fourier's law " بينما الأنتقال الحراري بالحمل يحدث بين السوائل المتحركة المحاطة بسطوح ذات درجات حرارة متباينة . منذ عام 1960 أصبحت تطبيقات ألأشعاع الحراري مهمة في التقنية المتقدمة، وبناءا عليه انتجت اجهزة يستخدم فيها ألأشعاع الحراري .

تعريف " configuration factor" واشتقاقه بين السطوح السوداء ، وذكر خصائصها جزء مهم لتحليل التبادل الأشعاعي. ان حساب " configuration factor" يتضمن التكامل سواء تحليليا أم رقميا . وكذلك للسطوح غير السوداء (الرمادية) المخالفة للسطوح السوداء بوجود اشعة منعكسة ،والتي ستنضم للأشعة المنبعثة . وقد أوردت بعض الامثلة لبيان التكامل التحليلي الذي يظهر في تحليل الأشعاع الحراري. اضافة لذلك تم طرح ومناقشة تزاوج الاشعاع الحراري مع التوصيل بفرض ان السطوح المعمول بها موصلة و مشعة و قاتمة ويحويها وسط شفاف رمادي باعث للاشعة. وقد عرضت بعض خصائص " Radiative integral operator" . وقد طرح السؤال عن وجود حل وحيد لهذة المسألة ، ويمكن اثبات وجود الحل باثبات ان المسألة المطروحة لدينا تتمتع بالخواص "Pseudomonotone and coercive". وأن وحدانية الحل سيثبت بالاستعانة ببعض الأفكار من "The analysis of nonlinear heat conduction".

Contents

Introduction

Thermal radiation is very important in some applications because of the manner in which radiant emission depends on temperature. For conduction and convection, energy transfer between two locations depends on their temperature difference to approximately the first power. For free convection, or when variable property effects are included, the power of the temperature difference may become larger than one, but usually in conduction and convection it is less than two.

Thermal radiation energy transfer between two bodies, however, depends on the difference between their absolute temperatures, each raised to about the fourth power. From this basic difference between radiation and conduction or convection, it is clear that the importance of radiation is intensified at high absolute temperatures.

Consequently, radiation contributes substantially to energy transfer in furnaces, combustion chambers, fires, and to the energy emission from a nuclear explosion. Radiative behavior governs the temperature distribution within the sun, and the solar emission. The nature of radiation from the sun is important in the technology for solar-energy utilization. Some devices are designed to operate at high temperature levels to achieve good thermal efficiency. Hence radiation must be considered in calculating thermal effects in rocket nozzles, power plants, engines and high temperature heat exchangers. Another distinguishing feature is that an intervening medium is not required between two locations or radiant exchange to occur. Radiation energy passes perfectly through a vacuum. This is in contrast to convection and conduction, where the physical medium must be present to carry energy with the convective flow or transport it by conduction. When no medium is present, radiation is significant mode of heat transfer, such as for the heat leakage through the evacuated space in the world of the thermos bottle.

Radiation is important in some instances because its action from a distance provides local heat sources that modify temperature distributions, thereby influencing conduction, free convection, or forest convection. Radiation can penetrate into fiberglass insulation to add to heat flow by conduction. Radiation can heat the walls of an enclosure, producing free convection where it would not ordinarily occur.

An important application of thermal radiation is in the practical utilization of the sun's radiation as an energy source. Solar energy transferred through the vacuum of space and the earth's atmosphere is received by a solar collector that converts the solar radiation into internal energy.

In fact, we note that the thermal radiation considered here is in the wave length region that gives humans the benefit of heat light, and photosynthesis. This is strong motivation for studying thermal radiation. Our existence depends on the solar radiant energy absorbed by the earth and it's atmosphere.

Due to the importance of heat transfer modes, different methods and techniques have been introduced and developed over the years for the computations of energy transfer problems (radiation, conduction and convection) as well as the coupling of these modes (see for example [1,2,4,9,10,11-17,20,21,22]).

This thesis is organized as follows:

In chapter one we outline the fundamental concepts of heat transfer modes. These modes are radiation, conduction and convection.

In chapters two and three we analyze configuration factors for radiant black and grey surfaces respectively. The computation of configuration factors involves integration, either analytically or numerically, over the solid angels by which surfaces can view each other. Some examples are given to demonstrate analytical integrations.

In chapter four we investigate the existence and the uniqueness of the solution of the coupled conduction- radiation energy transfer on diffuse- grey surfaces.

Index of Special Notation

σ	Stefan–Boltzmann constant which has the value $5.669\times10^{-8}\,W/(m^2.K^4)$
ε	Emissivity
ρ	Reflexivity
α	Absorptivity
γ	Scattering
λ	Wavelength
ν	Frequency
c	Light speed in any medium other than a vacuum.
c_o	Light speed in a vacuum.
n	Unit normal vector
N	Number of the surfaces of the enclosure.
q	Radiative heat flux.
Q	The total heat transfer rate.
i, o	Incoming and Outgoing radiation respectively.
θ	Azimuth angle
β	The angle in Y-Z plane
ϕ	Polar angle
T	The absolute temperature
T_A	The temperature of area.
T_b	The temperature of black body.
K	The coefficient of heat conductivity.
t	The time
R^n	Euclidean n–dimensional space
R	The set of all real numbers
A_s	The surface area.
dA	Differential area
h	Convection heat transfer.
$\overline{h}$	Average convection coefficient.
h_r	The radiation heat transfer coefficient.
F_{1-2}	The Configuration Factor between two surfaces.
S	The distance between two areas in three dimensions.
L	The distance between two areas in two dimensions.
$\frac{\partial}{\partial n}$	Differentiation along the outward normal
$\langle .,. \rangle$	Inner product
$\lVert f \rVert$	Norm of bounded linear functions.

$N(T)$	The null space of an operator T.
$L^2(\Omega)$	The set of all integral functions such that $\int_{\Omega} \|f\|^2 d\mu < \infty$.
X^*	The algebraic dual space of X .

$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}$, $1 \le P < \infty$.

Fundamental concepts of heat transfer

The heat is the form of energy that can be transferred from one system to another. The heat transfer is the energy that results from a temperature difference. Whenever there exists a temperature difference in a medium or between media, heat transfer must occur. We refer to different types of heat transfer processes as modes. These modes are heat radiation, heat conduction and heat convection. All surfaces of finite temperature emit energy in the form of electromagnetic waves. See for example $[4]$.

1.1 Radiation

Thermal radiation is the energy emitted by matter that is at a finite temperature. Although radiation occurs from solid surfaces, it occurs also from liquids and gases. Thermal radiation is transmitted through a vacuum since the sun's energy emits millions of kilometers of space before reaching the earth.

Assume a solid that is initially at a higher temperature T_s than that of its surroundings temperature T_2, where T_2 is the temperature of the surroundings, but around which there exists vacuum. However, it's obvious that the solid's temperature T_s will cool and finally achieve thermal equilibrium with its surroundings. This cooling is associated with a reduction in the internal energy stored by the solid and is a direct consequence of the emission of thermal radiation from the surface. However, if $T_s > T_2$ the net heat transfer rate by radiation Q is from the surface to the surroundings, and the surface will cool until T_s reaches T_2.

Thermal radiation is emitted by all surfaces that surround us: by the walls of the room, the furniture if we are inside, or by the sun, buildings, cars and the ground if we are outside. The mechanism of emission is related to energy released as a result of many electrons that constitute matter. These oscillations are, in turn, obtained by the internal energy, and therefore, the temperature, of the matter. Hence, we associate the emission of thermal radiation with thermally excited conditions within the matter. All forms of matter emit thermal radiation. For gases and for semitransparent solids, such as glass and salt crystals at elevated temperatures, hence emission is a volumetric phenomenon. That is radiation emitting from a finite volume of matter is the integrated effect of local emission throughout the volume. However, in this thesis, we will concentrate on situations for which radiation is surface phenomenon. Accordingly, radiation that is emitted from a solid or a liquid originates from molecules that are within a distance of approximately 1 μm from the exposed surface, where $1\mu\,m=10^{-6}$ m. Because of this reason, mission from a solid or a liquid into a gas or a vacuum is called a surface phenomenon.

Since radiation transport does not require the presence of any matter, one theory viewed radiation as propagation of electromagnetic waves like radio waves and X- rays. The other theory in the twentieth century states that radiation is composed of particles called photons. In any case the two properties of waves, frequency υ and wavelength λ are related by

$$c = \upsilon \lambda \tag{1.1.1}$$

Where c is the speed of light in the medium. The speed of light in a vacuum, $c_0=2.998\text{x}10^8$ m/s.

The unit of wavelength is commonly the micrometer (μm). The short wavelength gamma (γ) rays, X- rays and ultraviolet (UV) radiation are primarily of interest to the high energy physicist, medicine and the nuclear engineer, while the long wavelength microwaves and radio waves are of concern to the electrical engineer. Thermal radiation emitted by a surface includes a range of wavelengths.

There are many applications for radiation in biological field, when radiation passes through living cells; it can damage their structure which causes death of them and consequently death of the organism. The most rapidly growing cells are immature cells; often cancer cells are rapidly growing which are highly affected to radiation. In one study it was found that children whose mothers received X- rays during pregnancy had a 30% to 40% increase in the incident of cancer. In medical researches, amino acids, sugars, DNA and penicillin are a few of hundreds of medical compounds containing ^{14}C, ^{3}H, ^{35}S, ^{32}P. The radioactivity of these elements makes it possible to follow their pathways and metabolism conveniently.

The flux at which radiation may be emitted from a black surface is given by the Stefan _ Boltzmann Law:

$$q = \sigma T_s^4 \qquad (1.1.2a)$$

Where T_s the absolute temperature of the surface, σ is the Stefan _Boltzmann constant which has the value σ =5.67 x 10^{-8} (w/m^2.k^4). Such a surface is called ideal radiator. The heat flux emitted by a real surface is less than that and is given by

$$q = \varepsilon \sigma T_s^4 \qquad (1.1.2b)$$

Where ε is the emissivity of the surface, with $0 \leq \varepsilon \leq 1$, for black bodies , $\varepsilon = 1$, the total emissivity ε

$$\varepsilon = \frac{e(T)}{e_b(T)} \qquad (1.1.3)$$

Where $e(T)$ is the emittance of any surface which is always less than the emittance of black surface $e_b(T)$ which proves that $0 \leq \varepsilon \leq 1$. Conversely, if radiation is incident upon a surface, a portion will be absorbed. On the other hand, the rate at which energy is absorbed per unit surface area termed the absorptivity α. See for example[10].

$$q_2 = \alpha\, q_{inc} \qquad (1.1.4)$$

Where q_2 is the absorbed radiation, q_{inc} is the incident radiation, and $0 \leq \alpha \leq 1$, whereas the emission reduces the thermal energy of matter, but absorption increases this energy. Equation (1.1.4) determines the rate at which radiant energy is emitted and absorbed respectively at a surface. Determination of the net rate at which radiation is exchanged between surfaces is generally good to deal with. The special case occurs frequently in practice involves the net exchange between a small surface and a much larger surface that completely surrounds the smaller surface. This is what we will introduce in chapters two and three. See for example[7].

The surface and the surroundings are separated by a gas has no effect on the radiation transfer. Assume a grey surface that has the property α = ε. The net rate of radiation exchange between the surface and its surroundings is given by

$$q = \frac{Q}{A} = \varepsilon\ \sigma (T_s^{\ 4} - T_2^{\ 4}) \qquad (1.1.5)$$

Where A is the surface area, ε is its emissivity, T_s is temperature of the surface and T_2 is temperature of the surroundings.

There are many applications for which it is convenient to express the net radiation heat exchange in the form

$$\mathrm{Q} = h_r\ \mathrm{A}\,(\mathrm{T_s} - \mathrm{T_2}) \qquad (1.1.6)$$

Where h_r is the radiation heat transfer coefficient

$$h_r = \varepsilon\,\sigma\,(T_s + \mathrm{T_2})(T_s^{\ 2} + \mathrm{T_2^2}) \qquad (1.1.7)$$

The total emissivity ε of a surface is determined only by the physical properties and temperature of that surface. The total absorptivity α on the other hand depends on the source from which the surface absorbs radiation as well as the surface's own characteristics. This happens because the surface may absorb some wavelengths better than others. Thus, the total absorptivity will depend on the way that incoming radiation is distributed in wavelength. And that distribution, in turn, depends on the temperature and physical properties of the surface or surfaces from which radiation is absorbed . The total absorptivity α thus depends on the temperature and physical properties of all bodies involved in the heat exchange process.

There is a relationship between the emissivity and the absorptivity for a surface that is in thermodynamic equilibrium with its surroundings in which Kirchhoff's law deals

$$\varepsilon_\lambda(T,\theta,\varnothing) = \alpha_\lambda(T,\theta,\varnothing) \tag{1.1.8}$$

Where θ is angle between the incident rays and the normal line with $0 < \theta < \frac{\pi}{2}$, $\varnothing$ is the angle at the base of the hemisphere with $0 < \varnothing < 2\pi$, ε_λ and α_λ are the emissivity and the absorptivity for the surface respectively and T is the surface temperature.

Kirchhoff's law states that a body in thermodynamic equilibrium emits as much energy as it absorbs in each direction $(\theta,\varnothing)$ and at each wavelength λ. If this were not so, for example, a body might absorb more energy than it emits in one direction θ_1, and might also emit more than it absorbs in another direction, θ_2. Then the body would thus pump heat out of its surroundings from the first direction , θ_1 , and into its surroundings in the second direction , θ_2 . See for example [10, 20].

For a diffuse body, the emissivity and the absorptivity do not depend on the angles, and Kirchhoff's law becomes

$$\varepsilon_\lambda(T) = \alpha_\lambda(T) \tag{1.1.9}$$

If, in addition, the body is grey, Kirchhoff's law is further simplified

$$\varepsilon (T) = \ (T) \qquad (1.1.10\ )$$

Equation (1.1.10) is the most widely used form of Kirchhoff's law. However, this form is not valid if surfaces are not grey.

1.2 Conduction

Conduction may be viewed as the transfer of energy from the higher energetic to the lower energetic particles of a substance. Conduction in the case of gases and liquids takes place as a result of the diffusion and collisions of the material molecules during its random motion. However, in solids conduction occurs as a result of vibrations of the molecules at fixed positions called a lattice vibration and as a free flow of electrons. Thermal conduction needs the presence of medium. In solids, molecules have greater energies with high temperature, but in fluids, the thermal energy incident in molecules. The movement of these molecules from high temperature places to the lower places formed a flow of heat. The rate of heat conduction depends on many factors such as the geometry of the medium, its thickness, the material of the medium and the temperature difference a cross the medium. See for example[7].

For heat conduction, the rate equation is known as Fourier's law expressed algebraically as

$$q_x = -K\frac{dT}{dx} \qquad (1.2.1)$$

Where K is the thermal conductivity of the material which is a measure of the ability of a material to conduct heat and is given in tables[4]. $\frac{dT}{dx}$ is the temperature gradient, which is a vector quantity curve on a $T - x$ diagram. In fact equation (1.2.1) can be further expressed as

$$q_x = K\,\frac{(T_1 - T_2)}{L} \qquad (1.2.2)$$

Where T_1 and T_2 are the temperatures of the surfaces of the wall, and L is the thickness of the wall. In general, metals are good conductors. Copper is the common substance with highest conductivity at ordinary temperature which value is 401W/m.^{0}C which means that the wall of copper of thick 1 m can conduct heat at a rate of 401 W/m.^{0}C a cross the wall. Note that there are good electric conductors and heat conductors such as copper, silver which have high values of thermal conductivity. On the other hand, there are poor conductors such as rubber and wood which have low conductivity values. The thermal conductivity of materials vary over a wide range as noted in the table[7]. The thermal conductivity of gases are too smaller than metals like copper. Note also metals have the highest thermal conductivity and gases the lowest thermal conductivity. Pure metals have high thermal conductivity, alloy metals should also have high conductivity, where the alloy is made up of two metals of thermal conductivities k_1 and k_2 to have a conductivity k, where k between k_1 and k_2 in some cases, usually, the thermal conductivity of an alloy is much lower than that of either metals. For example, the thermal conductivity of steel containing 1 percent of chrome is 62 W/m.^{0}C, while the thermal conductivities of iron and chromonium are 83 and 95 W/m.^{0}C respectively. The thermal conductivity is normally highest in solids phase and lowest in gases phase. There are metals which are good heat conductors but poor electrical conductors, such as silicon, such materials find wide spread use in electronics industry.

There are many examples. The exposed end of a metal spoon suddenly immersed in a cup of hot coffee will eventually be warmed due to the conduction of energy through the spoon. On winter day, there is significant energy loss from a heated room to the outside air. This loss is due to conduction heat transfer through the wall that separates the room from the outside air. For more examples see [4, 7].

1.3 Convection

The convection heat transfer mode is obtained by random molecular motion and by the bulk motion of the fluid within the boundary surface with different temperatures. The contribution due to random molecular motion occurs near the surface where the fluid velocity is low. In fact, at the interface between the surface and the fluid closed to the surface, the fluid velocity is zero and heat transferred by random molecular motion only.

Convection heat transfer may be classified according to the nature of the flow. There is forced convection when the flow is caused by external means, such as by a fan to provide forced convection air cooling of hot electrical components, and there is free (natural) convection in which the flow is induced by buoyancy forces which arise from density differences caused by temperature variations in the fluid. The convection heat transfer is given as

$$q = h\,(T_s - T_\infty) \qquad (1.3.1)$$

Where q the convection heat flux which is proportional to the difference between the surface and fluid temperatures T_s , T_∞ respectively and h is the convection heat transfer coefficient. This expression is known as Newton's of cooling, h depends on the boundary of the layer, which is influenced by surface geometry, the nature of the fluid motion, and on assortment of fluid thermodynamic and transport properties. In the solution of such problem we assume h to be known. When equation (1.3.1) is used, the convection heat flux is positive when ($T_s \geq T_\infty$) and negative when ($T_s \leq T_\infty$). There are many examples of heat convection; one of them is the movement of water through the solar panels on top of our houses.

The total heat transfer rate Q may be obtained by integrating the local flux over the entire surface, that is

$$Q = \int_{A_s} q \, dA_s \tag{1.3.2}$$

But $q = h\,(T_s - T_\infty)$ consequently, equation (1.3.2) becomes

$$Q = (T_s - T_\infty) \int_{A_s} h \, dA_s \tag{1.3.4}$$

Defining an average convection coefficient $\bar{h}$ for the entire surface, the total heat transfer rate may be expressed as

$$Q = \bar{h} A_s (T_s - T_\infty) \tag{1.3.5}$$

Hence the average and local convection coefficients are related by

$$\bar{h} = \frac{Q}{A_s(T_s - T_\infty)} \tag{1.3.6}$$

Substituting equation (1.3.4) into (1.3.6) we obtain

$$\bar{h} = \frac{1}{A_s}\int_{As} h\, dA_s \tag{1.3.7}$$

Note that for special case of flow over a flat plate with length L, h varies with the distance x from the leading edge, hence equation (1.3.7) becomes

$$\bar{h} = \frac{1}{L}\int_0^l h\, dx \tag{1.3.8}$$

The surface within the surroundings may also simultaneously transfer heat by convection to the adjoining gas. The total rate of heat transfer from the surface is then the sum of the heat rates due to the two modes. That is

$$Q = Q_{con} + \mathrm{Q_{rad}} = \mathrm{hA}(\mathrm{T_s} - T_\infty) + \varepsilon\sigma A(T_s^4 - T_2^4). \tag{1.3.9}$$

CHAPTER TWO

Configuration factor for radiant black surfaces

A blackbody is a body that can emit the maximum amount of radiation by the surface at a given temperature. Or it is defined as a perfect absorber and emitter of radiation. At a given temperature no surface can emit more energy than a black body. A blackbody absorbs all incident radiation with all wavelengths and radiation. Also black bodies emit radiation uniformly in all directions that is; a blackbody is a diffuse emitter. Any body that absorbs light completely would appear black to the eye. On the other hand, any surface that reflects it completely would appear white. Consequently, some surfaces such as snow and white paint reflect light and thus appear white. But they are black with respect to the infrared radiation since they absorb the radiations. The blackbody properties can be summarized in

1- Black surfaces are completely absorbers, which simplifies the energy exchange process, since there is no reflected energy to be considered.

2- All black surfaces emit in a perfectly diffuse fashion. For more[20]

Definition: A configuration factor is a fraction of radiation leaving one surface reaches another surface, denoted by

$$F_{1-2} = \frac{Q_{1-2}}{Q_1} \tag{2.1}$$

Where Q_{1-2} is the energy radiated from A_1 to A_2 only, Q_1 is the total energy radiated from A_1.

Now, we will illustrate the calculation of configuration factor in the following cases

2.1 Configuration factor between two differential elements

Assume there is a differential element dA_1 with temperature T_1 at a distance S from another differential element dA_2 with temperature T_2 in R^3 as shown in figure (2.1), then the configuration factor **is** derived from the definition as:

$$F_{d_1-d_2} = \frac{(\sigma T_1^{\,4} \cos\theta_1 \cos\theta_2 / \pi S^{\,2}) dA_1 \, dA_2}{\sigma T_1^{\,4} dA_1} \qquad (2.1.1)$$

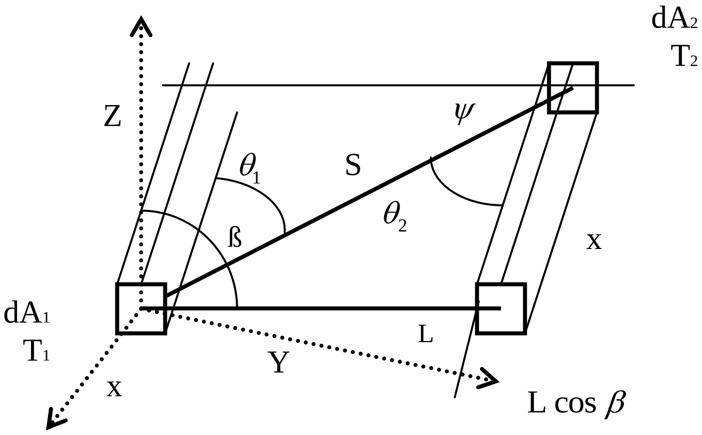


Fig. (2.1)

Where θ_1 is the angle between S and the normal to dA_1, θ_2 is the angle between S and the normal to dA_2, (2.1.1) can be simplified as

$$F_{d_1-d_2} = \frac{\cos\theta_1 \, \cos\theta_2 \; dA_2}{\pi S^{\,2}} \qquad (2.1.2)$$

Equation (2.1.2) shows that $F_{d_1-d_2}$ depends only on the size of dA_2 and its orientation with respect to dA_1.

$$d_{w_1} = \frac{|\cos\theta_2| dA_2}{S^2} \tag{2.1.3}$$

Where d_{w_1} is the solid angle subtended by dA_2 when viewed from dA_1. Hence equation (2.1.2) becomes

$$\mathrm{F}_{\mathrm{d}_1-\mathrm{d}_2} = \frac{\cos\,\theta_1\, dw_1}{\pi} \tag{2.1.4}$$

$$\cos\,\theta_1 = \frac{\mathrm{L}\cos\,\beta}{\mathrm{S}} \tag{2.1.5}$$

$$\mathrm{d}_{\mathrm{w}_1} = \frac{\text{Projected area of } \mathrm{dA}_2}{\mathrm{S}^2} \tag{2.1.6}$$

$$\mathrm{d}_{\mathrm{w}_1} = \frac{(\text{Projected width of } \mathrm{dA}_2)(\text{ Projected length of } \mathrm{dA}_2)}{\mathrm{S}^2} \tag{2.1.7}$$

$$= \frac{(L\, d\,\beta)\ (dx\ \cos\psi)}{S^2} \tag{2.1.8}$$

With $\cos\psi = \dfrac{L}{S}$, substitute equations (2.1.5) and (2.1.8) into (2.1.4) , we get

$$F_{d_1-d_2} = \frac{L^3 \cos\beta \, d\beta \, dx}{\pi(L^2+x^2)^2} \tag{2.1.9}$$

Where β is the angle in the Y – Z plane, L is the distance between dA_1 and dA_2 after the projection in Y-Z plane, and $S^2 = L^2 + x^2$ as shown in figure (2.1).

We can further illustrate the computation of the configuration factor between differential elements by the following example.

Example 2.1

The configuration factor between differential element and an infinitely long strip of differential width as shown in figure (2.2)

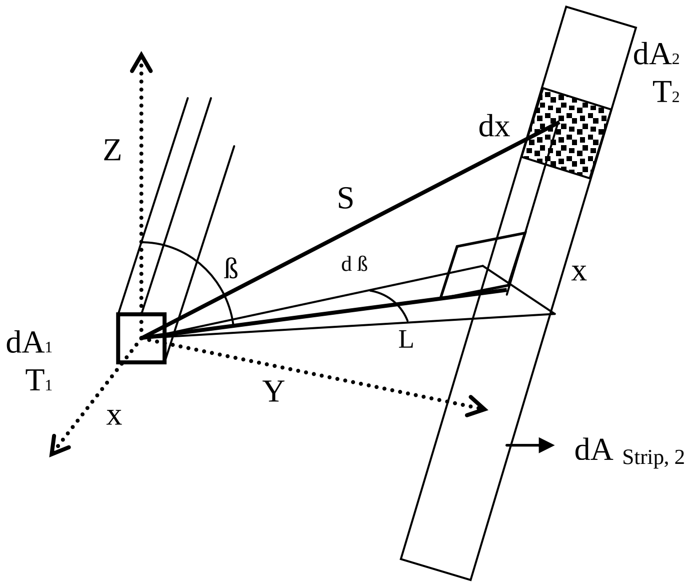


Fig. (2.2)

$$F_{d_1-stip_2} = \frac{L^3 \cos\beta \, d\beta}{\pi} \int_{-\infty}^{\infty} \frac{dx}{(x^2+L^2)^2} \tag{2.1.10}$$

Our integral is improper with symmetric integrand, equation (2.1.10) can be written as

$$F_{d_1-stip_2} = \frac{L^3 \cos\beta \; d\beta}{\pi}\left[2\int_{-\infty}^{0}\frac{dx}{(x^2+L^2)^2}\right] \tag{2.1.11}$$

$$F_{d_1-stip_2} = \frac{L^3 \cos\beta \, d\beta}{\pi}\left[2\lim_{a\to-\infty}\int_{a}^{0}\frac{dx}{(x^2+L^2)^2}\right] \tag{2.1.12}$$

Let x = L tan θ then dx =L sec$^2\theta$ dθ , consequently equation (2.1.12) becomes

$$F_{d_1-stip_2} = \frac{L^3 \cos\beta \, d\beta}{\pi}\left[2\lim_{a\to-\infty}\int_{a}^{0}\frac{L \; \sec^2\theta \, d\theta}{(L^2 \tan^2\theta \; + L^2)^2}\right] \tag{2.1.13}$$

Simplifying by using some mathematical relations we obtain

$$F_{d_1-stip_2} = \frac{\cos\beta \, d\beta}{\pi}\lim_{a\to-\infty}\left[(\tan^{-1}(\frac{x}{L}) + \frac{xL}{(x^2+L^2)}\right]_a^0 \tag{2.1.14}$$

Taking the limit of $\left[(\tan^{-1}(\frac{x}{L}) + \frac{xL}{(x^2+L^2)}\right]_a^0$ as a $\to$ - ∞ , we get

$$F_{d_1-stip_2} = \frac{\cos\beta \; d\beta}{\pi}(\frac{\pi}{2}) = \frac{d\sin\beta}{2} \tag{2.1.15}$$

2.2 Configuration factor between a differential element and a finite area

Suppose now an isothermal black element dA_1 at T_1 exchanging energy with a surface of finite area A_2 that is isothermal at temperature T_2 as shown in figure (2.3) the angle θ_1 will be different for different positions on A_2, θ_2 and S will also vary as different differential elements on A_2.

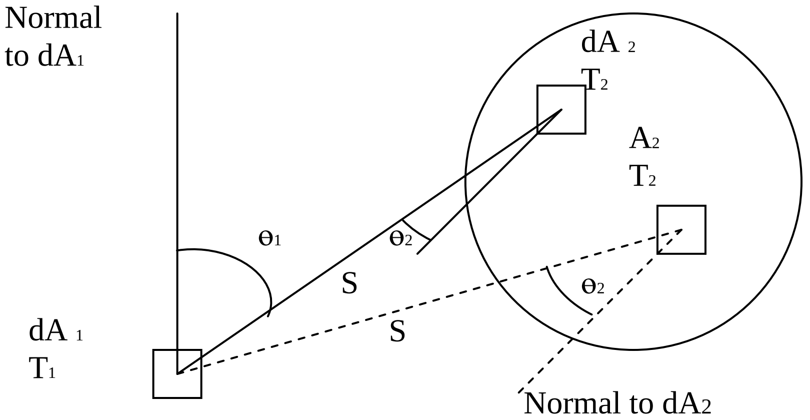


Fig. (2.3)

Here, there are two configuration factors, the configuration factor F_{d1-2} is from the differential area dA_1 to finite area A_2 and F_{2-d_1} is from A_2 to dA_1, we can derive the configuration factors using the definition as

$$F_{d_1-d_2} = \frac{(\sigma T_1^4 \cos\theta_1 \cos\theta_2 / \pi S^2)\, dA_1\, dA_2}{\sigma T_1^4\, dA_1} \tag{2.2.1}$$

Integrating over A_2 to obtain

$$F_{d_1-2} = \int_{A_2} \frac{\cos\theta_1 \cos\theta_2 \ dA_2}{\pi S^2} \tag{2.2.2a}$$

Hence,

$$F_{d_1-2} = \int_{A_2} F_{d_1-d_2} \tag{2.2.2b}$$

Equation (2.2.2b) shows the fact that the fraction of the energy reaching A_2 is the sum of fractions that reach all of the parts of A_2.

The energy reaching an element area dA_1 from a finite area A_2 is given by

$$Q_{2-d_1} = dA_1 \int_{A_2} \frac{\sigma\ T_2^4 \cos\theta_1 \cos\theta_2\ dA_2}{\pi S^2} \tag{2.2.3}$$

The total energy leaving A_2 is

$$Q_2 = \int_{A_2} \sigma T_2^4\ dA_2 \tag{2.2.4}$$

The configuration factor $F_{2\text{-}d_1}$ then is

$$F_{2-d_1} = \frac{dA_1}{A_2} \int_{A_2} \frac{\cos\theta_1 \cos\theta_2\ dA_2}{\pi S^2} \tag{2.2.5}$$

Note that there is a symmetry (reciprocity) relation for configuration factor between differential element and a finite area, that is

$$A_2 F_{2-d_1} = dA_1 F_{d1-2} \tag{2.2.6}$$

The energy radiated from dA_1 that reaches A_2 is

$$Q_{d_1-2} = \sigma T_1^4 \, dA_1 \, F_{d_1-2} \tag{2.2.7}$$

The energy radiated from A_2 that reaches dA_1 is

$$Q_{2-d_1} = \sigma T_2^4 A_2 F_{2-d_1} \tag{2.2.8}$$

Consequently, the net energy transfer from dA_1 to A_2 is

$$Q_{netd_{1,2}} = \sigma A_2 \, F_{2-d_1} \, (T_1^4 - T_2^4) \tag{2.2.9a}$$

or

$$Q_{netd_{1,2}} = \sigma \, dA_1 F_{d_1-2} (T_1^4 - T_2^4) \tag{2.2.9b}$$

Example 2.2

An elemental area dA_1 is oriented perpendicular to a circular disk of finite area A_2 with outer radius r as shown in figure (2.4)

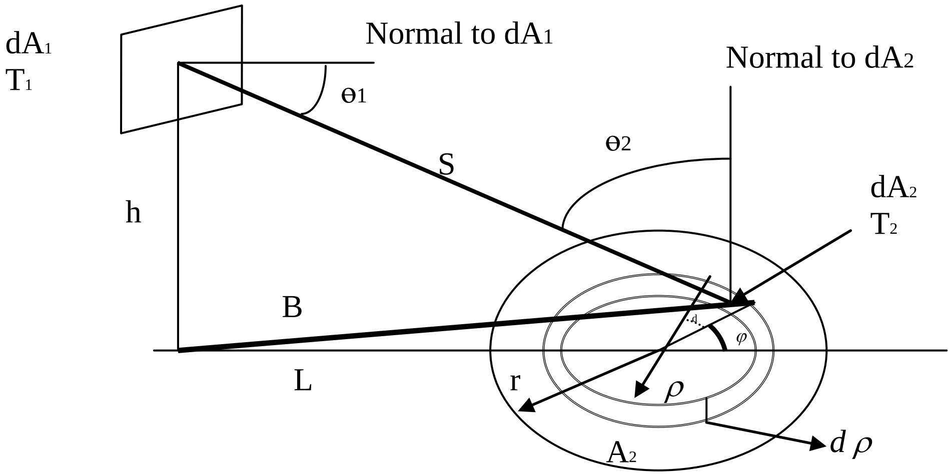


Fig. (2.4)

Using equation (2.2.2), and the following relations

$$\cos\theta_1 = \frac{L + \rho\cos\varphi}{S} \ ,$$

$$\cos\theta_2 = \frac{h}{S}$$

and

$$S^2 = h^2 + B^2$$

Where B^2 can be evaluating by using the law of cosines

$$B^2 = L^2 + \rho^2 - 2L\rho\cos(\pi - \varphi) \qquad (2.2.10a)$$

$$B^2 = L^2 + \rho^2 + 2L\rho\cos\varphi \qquad (2.2.10b)$$

And hence we obtain

$$F_{d_1-2} = \int_{A_2} \frac{h(L+\rho\cos\varphi)\rho\, d\rho\, d\varphi}{\pi S^4} \qquad (2.2.11)$$

As shown in figure (2.4) with $0 \le \rho \le r$, $0 \le \varphi \le 2\pi$ and by the symmetry of configuration factor then equation (2.2.11) becomes

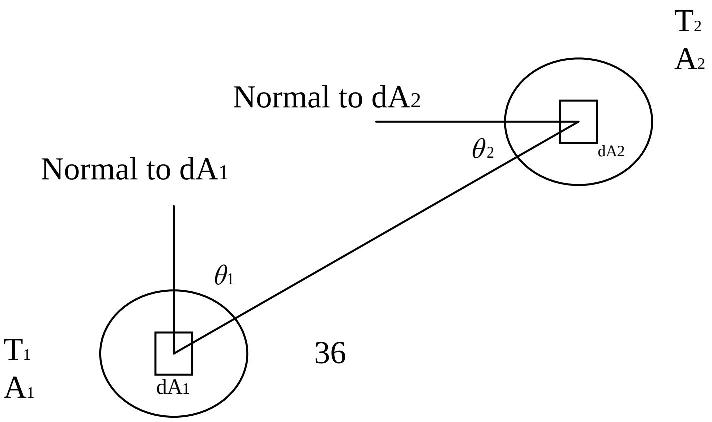
$$F_{d_1-2} = \frac{2h}{\pi} \int_{\rho=0}^{r} \int_{\varphi=0}^{\pi} \frac{\rho\,(L+\rho\,\cos\varphi)\, d\varphi\, d\rho}{(h^2+L^2+\rho^2+2\rho L\,\cos\varphi)^2} \qquad (2.2.12)$$

2.3 Configuration factor between two finite areas

Suppose there are two finite areas $A_{1.}$ and A_2 with T_1 and T_2 respectively, then we have $F_{1\text{-}2}$ and F_{2-1} configuration factors for radiation emitted from an isothermal surface A_1 as shown in figure (2.5) and reaching A_2, F_{1-2} is the fraction of energy leaving A_1 that arrives to A_2.

The total energy leaving A_1 is $\sigma T_1{}^4 A_1$ since A_1 is isothermal at T_1, the radiation leaving dA_1 reaches dA_2 is given:

$$Q_{d_1-d_2} = \sigma T_1{}^4 \frac{\cos\theta_1\, \cos\theta_2\, dA_1\, dA_2}{\pi S^2} \qquad (2.3.1)$$



If we integrate equation (2.3.1) over both A_1 and A_2 and then divide by $\sigma T_1^4 A_1$ we will obtain the fraction of energy leaving A_1 reaches A_2 as

$$F_{1-2} = \frac{1}{A_1} \int_{A_1} \int_{A_2} \frac{\cos\theta_1 \cos\theta_2 \; dA_2 \; dA_1}{\pi S^2} \qquad (2.3.2)$$

Similarly, one can show that

$$F_{2-1} = \frac{1}{A_2} \int_{A_1} \int_{A_2} \frac{\cos\theta_1 \; \cos\theta_2 \; dA_2 \, dA_1}{\pi S^2} \qquad (2.3.3)$$

As noted there is symmetry relation between configuration factors, that is

$$A_2 F_{2-1} = A_1 F_{1-2} \qquad (2.3.4)$$

Depending on the previous three cases, one can determine the configuration factor for any two surfaces, such that if A_2 is divided into $A_1 \bigcup A_2$, then

$$F_{1-2} = F_{1-3} + F_{1-4} \qquad (2.3.5)$$

This enables us to derive an expression for $F_{d_1 - ring}$

$$F_{d_1-ring} = F_{d_1-A_1} - F_{d_1-A_2} \tag{2.3.6}$$

Where A_1 and A_2 are the outer and the inner areas of the ring respectively.

Sometimes we may need to use set theory notations, if we have two areas A_1, A_2 the configuration factor from an area A_E to $A_1 \bigcup A_2$, with intersection $\neq \phi$ as shown in figure (2.6) then

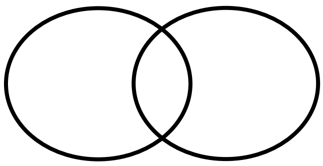


Fig. (2.6)

$$F_{E-A_1 \cup A_2} = F_{E-A_1} + F_{E-A_2} - F_{E-A_1 \cap A_2} \tag{2.3.7}$$

The relation can be cleared by knowing that the fraction of energy leaving A_E is incident upon $A_1 \bigcup A_2$ can be divided into two fractions. Once leaving A_E incident upon A_1, the other leaving A_E incident upon A_2. However, the portion $A_1 \bigcap A_2$ is covered twice, so we must subtract $F_{E-A_1 \cap A_2}$, for example the configuration factor between A_E and the L-shaped area.

2.4 Configuration factor in arbitrary convex enclosures

For an enclosure of N surfaces shown in figure (2.7), the entire energy leaving any surface inside the enclosure, say A_k, must be incident on all the surfaces making up the enclosure. Thus, all the fractions of the energy leaving any surface reaching the surfaces of the enclosure must total to unity, that is

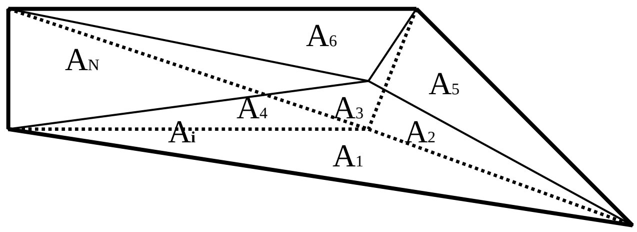


Fig. (2.7)

$$F_{k-1} + F_{k-2} + F_{k-3} + ... + F_{k-N} = 1 \qquad (2.4.1a)$$

Or

$$\sum_{j=1}^{N} F_{k-j} = 1 \qquad (2.4.1b)$$

Where k=1,2,3,4, ….. N , F_{k-k} is included if A_K is concave.

Example 2.4

We consider two black isothermal concentric spheres that exchange radiant energy, where A_1 is the surface area of the inner sphere, A_2 is the surface area of the outer sphere as shown in figure (2.8) , the configuration factor can be computed as follow:

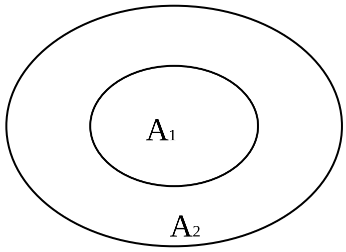


Fig. (2.8)

Since all the energy leaving A_1 is incident upon A_2 only that is $Q_{1-2} = Q_1$ then by equation (2.1), we get

$$F_{1-2} = 1 \qquad (2.4.2)$$

and by symmetry relation, we get

$$F_{2-1} = \frac{A_1}{A_2} \qquad (2.4.3)$$

also by equation (2.4.1) we obtain

$$F_{2-2} = 1 - \frac{A_1}{A_2} \qquad (2.4.4)$$

Example 2.5

An enclosure of triangular cross section is made up of three plane plate. Each of finite width and infinite length as shown in figure (2.9), we can derive an expression for the configuration factor between any two of the plates in terms of their widths L_1, L_2 and L_3 .

For plate 1,

$$F_{1-2} + F_{1-3} = 1 \qquad (2.4.5)$$

Using similar relations for each plate and multiply through by the respective plate areas:

$$A_1F_{1-2} + A_1F_{1-3} = A_1 \qquad (2.4.6a)$$

$$A_2F_{2-1} + A_2F_{2-3} = A_2 \qquad (2.4.6b)$$

$$A_3F_{3-1} + A_3F_{3-2} = A_3 \qquad (2.4.6c)$$

By applying the symmetry relations

$$A_1F_{1-2} + A_1F_{1-3} = A_1 \qquad (2.4.7a)$$

$$A_1F_{1-2} + A_2F_{2-3} = A_2 \qquad (2.4.7b)$$

$$A_1F_{1-3} + A_2F_{2-3} = A_3 \qquad (2.4.7c)$$

Subtract the third from the second adding to the first we obtain

$$2A_1F_{1-2} = A_1 + A_2 - A_3 \qquad (2.4.8)$$

$$F_{1-2} = \frac{A_1 + A_2 - A_3}{2A_1} \tag{2.4.9}$$

If $L_1 = L_2$, α is the angle between L_1 and L_2 then

$$F_{1-2} = 1 - \sin(\frac{\alpha}{2}) \tag{2.4.10}$$

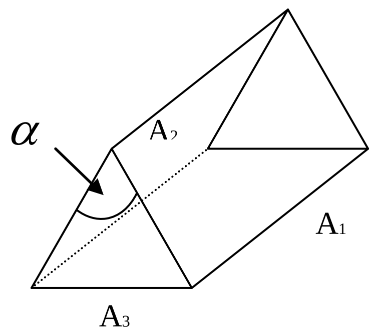


Fig (2.9)

2.5 Another approach for evaluating the configuration factor

There are many methods for evaluating the configuration factor. We will take one of them called "the unit - sphere method ". This method can determine the configuration factor by constructing a hemisphere of unit radius over the area element dA_1, and then the configuration factor from dA_1 to any other area A_2 is

$$F_{d_1-2} = \frac{1}{\pi} \int_{A_2} \frac{\cos\theta_1 \, \cos\theta_2 \, dA_2}{S^2} \tag{2.5.1}$$

Where θ_1 is the angle between the normal to dA_1 and the radiation from dA_1 to A_2 , θ_2 is the angle between the normal to dA_2 of A_2 and the radiation from dA_1 to A_2.

$$F_{d_1-2} = \frac{1}{\pi}\int_{A_2} \cos\theta_1 \, dw_1 \qquad (2.5.2)$$

Where dw_1 is the projection of dA_2 onto the surface of the unit hemisphere,

$dw_1 = \frac{|\cos\theta_2| dA_2}{S^2}$, hence equation (2.5.2) becomes

$$\mathrm{F}_{d_1-2} = \frac{1}{\pi}\int_{A_2} \cos\theta_1 \; dA_s \qquad (2.5.3)$$

Where $dw_1 = \frac{dA_s}{r^2} = dA_s$. But, $\cos\theta_1 \; dA_s$ is the projection of dA_s onto the base of unit hemisphere equals to say $\mathrm{A_b}$,(See for example [20]).

$$\mathrm{F}_{d_1-2} = \frac{A_b}{\pi} \qquad (2.5.4)$$

This relation can be further extended to any arbitrary hemisphere of radius r_e , that is

$$\mathrm{F}_{d_1-2} = \frac{A_b}{\pi r_e} \qquad (2.5.5)$$

2.6 Using the configuration factor for evaluating the radiation exchange

We can use the configuration factor for evaluating the Radiation exchange by taking the difference between the emitted from the surfaces.

2.6.1 Radiant exchange between two finite black areas

$$Q_{1-2} = \sigma T_1^4 A_1 F_{1-2} \tag{2.6.1.1}$$

$$Q_{2-1} = \sigma T_2^4 A_2 F_{2-1} \tag{2.6.1.2}$$

So the net heat transfer from A_1 to A_2 is

$$Q_{net\,1,2} = \sigma (T_1^4 - T_2^4) A_2 F_{2-1} \tag{2.6.1.3a}$$

or

$$Q_{net\,1,2} = \sigma (T_1^4 - T_2^4) A_1 F_{1-2} \tag{2.6.1.3b}$$

2.6.2 Radiation exchange in black enclosure

Consider a black enclosure with a typical surface A_k , the energy supplied to A_k by the other all surfaces of the enclosure to maintain A_k at T_k is Q_k . The emission from A_k is $\sigma T_k^4 A_k$, while the received radiant energy by A_k from the other surfaces A_j is $\sigma T_j^4 A_j F_{j-k}$, where j= 1,2, ……..,N , the heat balance is

$$Q_k = \sigma T_k^4 A_k - \sum_{j=1}^{N} \sigma T_j^4 A_j F_{j-k} \tag{2.6.2.1}$$

Where the energy arriving from A_k included if A_k is concave, applying reciprocity in equation (2.6.2.1) and we know that $\sum_{j=1}^{N} F_{k-j} = 1$, we can replace equation (2.6.2.1) by:

$$Q_k = \sigma T_k^4 A_k \sum_{j=1}^{N} F_{k-j} - \sum_{j=1}^{N} \sigma T_j^4 A_k F_{k-j} \qquad (2.6.2.2)$$

Simplifying equation (2.6.2.2) , we obtain

$$Q_k = \sigma A_k \sum_{j=1}^{N} F_{k-j} (T_k^4 - T_j^4) \qquad (2.6.2.3)$$

This relation indicates that the heat balance is the net energy transferred from the surface area A_k to each surface in the enclosure.

Example 2.6

The three sided black enclosure has it's surfaces maintained at T_1, T_2 and T_3 respectively, we can determine the amount of energy that must be supplied to each surface per unit time. Equation (2.6.6) can be written for each surface as

$$Q_1 = \sigma A_1 F_{1-2} (T_1^4 - T_2^4) + \sigma A_1 F_{1-3} (T_1^4 - T_3^4) \qquad (2.6.2.4)$$

$$Q_2 = \sigma A_2 F_{2-1} (T_2^4 - T_1^4) + \sigma A_2 F_{2-3} (T_2^4 - T_3^4) \qquad (2.6.2.5)$$

$$Q_3 = \sigma A_3 F_{3-1} (T_3^4 - T_1^4) + \sigma A_3 F_{3-2} (T_3^4 - T_2^4) \qquad (2.6.2.6)$$

All factors on the right hand side of these equations are known, and so the Q values may be computed. Using the symmetry relations on the set of Q equations, we obtain

$$\sum_{k=1}^{3} Q_k = \sigma A_1 F_{1-2}(T_1^4 - T_2^4) + \sigma A_1 F_{1-3}(T_1^4 - T_3^4)$$

$$+ \sigma A_1 F_{1-2}(T_2^4 - T_1^4) + \sigma A_2 F_{2-3}(T_2^4 - T_3^4) \qquad (2.6.2.7)$$

$$+ \sigma A_1 F_{1-3}(T_3^4 - T_1^4) + \sigma A_2 F_{2-3}(T_3^4 - T_2^4)$$

On the right hand side of equation (2.6.2.7) three terms will cancel the others and hence $\sum_{k=1}^{3} Q_k = 0$, which supports numerically the energy conservation.

CHAPTER THREE

Configuration factor for radiant grey surfaces

The analysis of radiation transfer in enclosures consisting of non black surfaces is more complicated than what we have seen in chapter two. Here a multiple reflections will occur. But radiation analysis is simplified by assumptions. It is common to assume the surfaces of an enclosure to be grey, diffuse and opaque. That is, the surfaces are diffuse emitters and diffuse reflectors and their radiation properties are independent of wavelength. In addition, each surface of the enclosure is isothermal. Moreover, the incoming and out coming radiation are uniform over each surface of the enclosure. (For example see [20])

In this chapter methods were developed for treating energy exchange within enclosures having grey surfaces. The surfaces may be of finite or infinitesimal size. Since a gray surface is not a perfect absorber, part of the incident energy on a surface is reflected. With regard to the reflected energy, there are two assumptions:

1- The reflected energy of grey surface is diffuse i.e there is a reflected energy in all directions of the boundary uniformly.
2- It is uniform over all surfaces of the enclosure.

The reflected and emitted energy can be combined into single energy quantity leaving the surface. Fore more see [20].

"The enclosure boundary is composed of areas, so that over each of these areas the following restrictions are met:

1- The temperature is uniform.
2- ε_λ, α_λ and ρ_λ are independent of wavelength and direction, so that $\varepsilon(T_A) = \alpha(T_A) = 1 - \rho(T_A)$, where ρ is the reflectivity.
3- All energy is emitted and reflected.

4- The incident and reflected energy flux is uniform over each individual area"

3.1 Radiation exchange between finite areas

A complex radiative exchange occurs inside the enclosure when radiation incident from a surface say A_K travels to the other surfaces, part of these are reflected and then re–reflected many times, as shown in the figure (3.1). Assume the k^{th} inside surface area A_K of the enclosure. The heat balance at the surface area A_K is

$$Q_k = A_k \ (q_{o,k} - q_{i,k}) \tag{3.1.1}$$

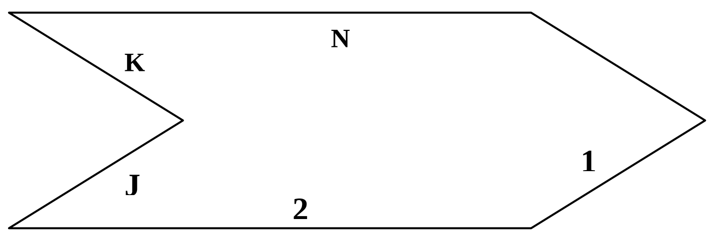


Fig. (3.1)

Where Q_k is the net radiative loss from surface k to other surfaces of the enclosure, $q_{o,k}$ is the rate of the outcoming radiant energy from A_K. $q_{i,k}$ is the rate of the incoming radiant energy to the surface area A_K. The energy flux leaving the surface is composed of directly emitted and reflected energy as

$$q_{o,k} = \varepsilon_k \ \sigma T_k^4 + \rho_k \ q_{i,k} \tag{3.1.2}$$

But $\rho_k = (1-\varepsilon_k)$ then equation (3.1.2) becomes

$$q_{o,k} = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k) q_{i,k} \qquad (3.1.3)$$

The incident flux $q_{i,k}$ is derived from the portions of the energy leaving the surfaces in the enclosure that arrive to the k^{th} surface. The incident energy is then

$$A_k q_{i,k} = \sum_{j=1}^{N} A_j q_{o,j} F_{j-k} \qquad (3.1.4)$$

Where A_j is the j^th surface and j=I,2,3,……,N, using the symmetry relation of configuration factor yields

$$A_k q_{i,k} = \sum_{j=1}^{N} A_k q_{o,j} F_{k-j} \qquad (3.1.5)$$

Simplifying, we get

$$q_{i,k} = \sum_{j=1}^{N} F_{k-j} q_{o,j} \qquad (3.1.6)$$

Substituting equations (3.1.3), (3.1.6) respectively into (3.1.1) we obtain

$$Q_k = A_k \frac{\varepsilon_k}{1-\varepsilon_k} (\sigma T_k^{\,4} - q_{o,k}) \qquad (3.1.7\text{ a})$$

$$Q_k = A_k \left(q_{o,k} - \sum_{j=1}^{N} q_{o,j} F_{k-j}\right) \qquad (3.1.7\text{ b})$$

These formulas provide 2N equations for 2N unknowns, where N belongs to the natural numbers. These 2N unknowns are $q_{o,j}$'s and Q_k's. The following examples illustrate the use of these equations.

Example 3.1

Consider two uniform temperature concentric grey spheres A_1 and A_2 as in figure (3.2). We can derive an expression for the net radiation exchange between them. It is clear that $Q_{1-2} = Q_1$, by equation (2.1) $F_{1-2} = 1$

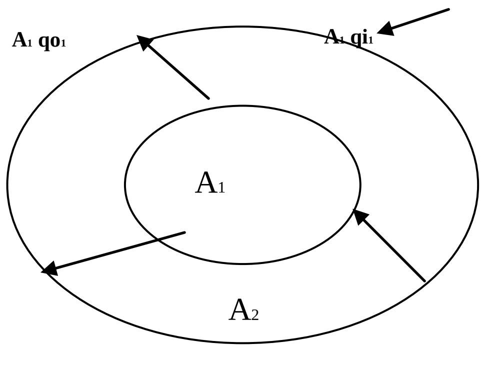


Fig. (3.2)

By symmetric relation we can compute F_{2-1} which is

$F_{2-1} = \dfrac{A_1}{A_2}$ and equation (2.4.1) enables us to obtain F_{2-2} as

$$F_{2-2} = 1 - \frac{A_1}{A_2} \quad .$$

Substitute these relations into equation (3.1.7) we obtain

For A_1

$$Q_1 = A_1 \frac{\varepsilon_1}{1-\varepsilon_1}\left[\sigma T_1^{\,4} - q_{o,1}\right] \tag{3.1.8a}$$

$$Q_1 = A_1\left[q_{o,1} - q_{o,2}\right] \tag{3.1.8b}$$

Similarly for A_2;

$$Q_2 = A_2 \frac{\varepsilon_2}{1-\varepsilon_2}\left[\sigma T_2^{\,4} - q_{o,2}\right] \tag{3.1.9a}$$

$$Q_2 = A_2\left[q_{o,2} - \frac{A_1}{A_2} q_{o,1} - (1 - \frac{A_1}{A_2}) q_{o,2}\right] \tag{3.1.9b}$$

Since we have four equations for four unknowns, so we can solve for $q_{o,1}$, $q_{o,2}$, Q_1 and Q_2, where ε is a function of T, so if T_1 and T_2 are given, ε_1 and ε_2 can be evaluated then it is easy to find the unknowns. Moreover the net heat transfer from A_1 to A_2 is

$$Q_1 = \frac{A_1 \sigma (T_1^4 - T_2^4)}{1/\varepsilon_1(T_1) + (A_1/A_2)[1/\varepsilon_2(T_2) - 1]} \qquad (3.1.10)$$

This is valid only if q , q_i and q_o are uniform over each sphere A_1 and A_2.

Example 3.2

A completely enclosed grey isothermal body with surface area A_1 and temperature T_1 is enclosed by a much larger grey isothermal enclosure with surface area A_2 with T_2. We can compute how much energy is being transferred from A_1 to A_2.

No part of A_1 can emit to any part of A_1, hence $F_{1\text{-}1}=0$, $F_{1\text{-}2}=1$, $F_{2\text{-}1} = \frac{A_1}{A_2}$ and

$F_{2\text{-}2} = 1 - \frac{A_1}{A_2}$ as shown in example 3.1. Using equation (3.1.10), and $A_1 << A_2$ the net energy transferred reduces to

$$Q_1 = A_1 \varepsilon_1(T_1) \sigma (T_1^4 - T_2^4) \qquad (3.1.11)$$

Equation (3.1.11) shows that the net energy transferred is independent of the emissivity of A_2.

Example 3.3

Consider two infinite parallel plates A_1 and A_2 with temperatures T_1 and T_2 respectively. We can compute the net radiation heat exchange between them.

Surely all the radiation leaving one plate will arrive to the other one, that is $Q_{1-2} = Q_1$ and $Q_{2-1} = Q_2$. Hence $F_{1-2} = F_{2-1} = 1$

Applying equations (3.1.7a) and (3.1.7b) for each plate:

For plate 1

$$Q_1 = A_1 \frac{\varepsilon_1}{1-\varepsilon_1} (\sigma T_1^4 - q_{0,1}) \qquad (3.1.12a)$$

$$Q_1 = A_1 (q_{0,1} - q_{0,2}) \qquad (3.1.12b)$$

For plate 2

$$Q_2 = A_2 \frac{\varepsilon_2}{1-\varepsilon_2} (\sigma T_2^4 - q_{0,2}) \qquad (3.1.13a)$$

$$Q_2 = A_2 (q_{0,2} - q_{0,1}) \qquad (3.1.13b)$$

It is clear that $q_2 = -q_1$, solving for $q_{0,1}, q_{0,2}$ from equations (3.1.12a) and (3.1.13a) respectively then substituting into (3.1.12b) we obtain

$$q_1 = \frac{\sigma (T_1^4 - T_2^4)}{1/\varepsilon_1(T_1) + 1/\varepsilon_2(T_2) - 1} \qquad (3.1.14)$$

Where ε_1 and ε_2 are functions of T_1 and T_2 respectively. So if T_1 and T_2 will be given ε_1 and ε_2 can be solved easily. Consequently q_1 and q_2 can be evaluated. Also we can find T_1 when q_1 is given and at specified value T_2 in any parallel –plates as

$$T_1 = \left\{ \frac{q_1}{\sigma} \left[\frac{1}{\varepsilon_1(T_1)} + \frac{1}{\varepsilon_2(T_2)} - 1 \right] + T_2^4 \right\}^{1/4} \tag{3.1.15}$$

Since $\varepsilon_1(T_1)$ is a function of T_1 which is required, an iterative method is used by selecting T_1, and then to choose ε_1 at this temperature. Equation (3.1.15) will find a new T_1 and for this value to choose anew ε_1. This process will continue until $\varepsilon_1(T_1)$ and T_1 no more change with more iterations.

Example 3.4

Consider a long enclosure of three surfaces. We can evaluate how much heat has to be supplied to each surface to maintain the surfaces at T_1, T_2 and T_3 respectively. Applying (3.1.7 a) and (3.1.7 b) for each surface, we get

$$Q_1 = A_1 \frac{\varepsilon_1}{1-\varepsilon_1} (\sigma T_1^4 - q_{o,1}) \tag{3.1.16a}$$

$$Q_1 = A_1 \left[q_{o,1} - F_{1-1} q_{o,1} - F_{1-2} q_{o,2} - F_{1-3} q_{o,3} \right] \tag{3.1.16 b}$$

$$Q_2 = A_2 \frac{\varepsilon_2}{1-\varepsilon_2} (\sigma T_2^4 - q_{o,2}) \tag{3.1.17 a}$$

$$Q_2 = A_2\left[q_{o,2} - F_{2-1}q_{o,1} - F_{2-2}q_{o,2} - F_{2-3}q_{o,3}\right] \tag{3.1.17 b}$$

$$Q_3 = A_3 \frac{\varepsilon_3}{1-\varepsilon_3}(\sigma T_3^{\ 4} - q_{o,3}) \tag{3.1.18 a}$$

$$Q_3 = A_3\left[q_{o,3} - F_{3-1}q_{o,1} - F_{3-2}q_{o,2} - F_{3-3}q_{o,3}\right] \tag{3.1.18 b}$$

Solving for $q_{o,1}$, $q_{o,2}$ and $q_{o,3}$ in the first equation of each pair in terms of $T_k\,,s$ and $Q_k\,,s$. Then substituting these $q_o\,'s$ into the second equation of each pair, we obtain

$$\begin{aligned}&\frac{Q_1}{A_1}\left(\frac{1}{\varepsilon_1} - F_{1-1}\frac{1-\varepsilon_1}{\varepsilon_1}\right) - \frac{Q_2}{A_2}F_{1-2}\frac{1-\varepsilon_2}{\varepsilon_2} - \frac{Q_3}{A_3}F_{1-3}\frac{1-\varepsilon_3}{\varepsilon_3}\\ &= \left(1-F_{1-1}\right)\sigma\, T_1^{\ 4} - F_{1-2}\sigma\, T_2^{\ 4} - F_{1-3}\sigma\, T_3^{\ 4}\end{aligned} \tag{3.1.19 a}$$

$$\begin{aligned}&-\frac{Q_1}{A_1}F_{2-1}\frac{1-\varepsilon_1}{\varepsilon_1} + \frac{Q_2}{A_2}\left(\frac{1}{\varepsilon_2} - F_{2-2}\frac{1-\varepsilon_2}{\varepsilon_2}\right) - \frac{Q_3}{A_3}F_{2-3}\frac{1-\varepsilon_3}{\varepsilon_3}\\ &= -\,F_{2-1}\sigma\, T_1^{\ 4} + \left(1-F_{2-2}\right)\sigma\, T_2^{\ 4} - F_{2-3}\sigma T_3^{\ 4}\end{aligned} \tag{3.1.19 b}$$

$$\begin{aligned}&-\frac{Q_1}{A_1}F_{3-1}\frac{1-\varepsilon_1}{\varepsilon_1} - \frac{Q_2}{A_2}F_{3-2}\frac{1-\varepsilon_2}{\varepsilon_2} + \frac{Q_3}{A_3}\left(\frac{1}{\varepsilon_3} - F_{3-3}\frac{1-\varepsilon_3}{\varepsilon_3}\right)\\ &= -F_{3-1}\sigma\, T_1^{\ 4} - F_{3-2}\sigma T_2^{\ 4} + \left(1-F_{3-3}\right)\sigma T_3^{\ 4}\end{aligned} \tag{3.1.19 c}$$

Since the $T_j\,'s$ are known, the $\varepsilon_j\,'s$ can be determined at these certain $T_j\,'s$. These equations solved for required Q's supplied to each surface.

3.2 Radiation exchange between infinitesimal areas

Assume as before there is an enclosure consists of N finite areas. These areas would generally be the major geometric division of the enclosure. Each of these areas is subdivided into differential area elements. A heat balance on an element d A_k located at a position r_k is

$$q_k(r_k) = q_{o,k}(r_k) - q_{i,k}(r_k) \qquad (3.2.1)$$

The outcoming flux is composed of emitted and reflected energy

$$q_{o,k}(r_k) = \varepsilon_k \sigma T_k^4(r_k) + (1-\varepsilon_k)\, q_{i,k}(r_k) \qquad (3.2.2)$$

The incoming flux is composed of the portions of the outgoing flux from the other area elements of the enclosure. Using integration to determine the total flux leaving the surfaces to $q_{i,k}(r_k)$.

$$dA_k\, q_{i,k}(r_k) = \int_{A_1} q_{o,1}(r_1)\, dF_{d_1-d_k}(r_1, r_k)\, dA_1 + \ldots\ldots +$$

$$\int_{A_N} q_{o,N}(r_N)\, dF_{d_N-d_k}(r_N, r_k)\, dA_N \qquad (3.2.3)$$

$$\mathrm{q}_{\mathrm{i,k}} = \sum_{j=1}^{N} \int_{A_j} q_{o,j}\, dF_{dk-dj}(r_j, r_k) \qquad (3.2.\ 4)$$

Substituting equations (3.2. 4) and (3.2.2) into (3.2.1) we obtain

$$q_k(r_k) = \frac{\varepsilon_k}{1-\varepsilon_k}\left[\sigma\, T_k^4(r_k) - q_{o,k}(r_k)\right] \tag{3.2.5a}$$

$$q_k(r_k) = q_{o,k}(r_k) - \sum_{j=1}^{N}\int_{A_j} q_{o,j}\, dF_{dk-dj}(r_k, r_j) \tag{3.2.5b}$$

These formulas provide 2N equations for 2N unknowns, where N belongs to the natural numbers. These 2N unknowns are $q_{o,j}(r_j)$'s and $q_j(r_j)$'s, with j=1,2,…..,N

3.3 Heat transfer in arbitrary grey enclosure bodies

We can also evaluate the heat transfer between any grey body enclosed by other grey body using equation (2.6.3b)

$$Q_{net\,1-2} = \frac{\sigma\,(T_1^4 - T_2^4)}{\left[\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}\right]} \tag{3.3.1}$$

Where $$F_{1-2} = \frac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)} \tag{3.3.2}$$

For the enclosed bodies

$$Q_{net\,1-2} = -\ Q_{net\,2-1} \tag{3.3.3}$$

Using equations (3.3.1) and (3.3.2):

$$A_1 F_{1-2}\,\sigma\,(T_1^{\,4} - T_2^{\,4}) = -A_2\,F_{2-1}\,\sigma\,(T_2^{\,4} - T_1^{\,4}) \tag{3.3.4}$$

We obtain

$$A_1 F_{1-2} = A_2 F_{2-1} \tag{3.3.5}$$

Hence, we have

$$F_{2-1} = \frac{A_1}{A_2} F_{1-2} \tag{3.3.6}$$

By equation (3.3.2), we get

$$F_{2-1} = \frac{1}{\dfrac{A_2}{\varepsilon_1 A_1} + (\dfrac{1}{\varepsilon_2} - 1)} \tag{3.3.7}$$

Also by equation (3.3.2), if $A_1 << A_2$, then

$F_{1-2} \cong \varepsilon_1$

3.4 Radiation shields

A radiation shield is a surface that has high reflectance which is placed between a high temperature and other cooler. Let us now examine what happens to the emisivity ε in the presence of the radiation shields. Assume a grey body of area A_1 surrounded by another grey body of area A_2, and assume that there is a thin sheet of reflective surface placed between them as a radiation shield. The sheet will reflect radiation arriving from A_1 back towards itself, it will radiate little energy to A_2. The radiation from A_1 either to the inside of the shield or from the outside of the shield to A_2 be two body exchange coupled by the shield temperature. Consequently, the net radiation is

$$Q_{net\,1-2} = \frac{\sigma(T_1^{\,4} - T_2^{\,4})}{\left[\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}\right] + 2(\frac{1-\varepsilon_s}{\varepsilon_s A_s}) + \frac{1}{A_s}} \tag{3.4.1}$$

Where $2(\frac{1-\varepsilon_s}{\varepsilon_s A_s}) + \frac{1}{A_s}$ is the added to the denomerator by the shield. If the radiation shield is high reflective it reduces $Q_{net\,1-2}$ more. (For more see [10]).

3.5 Solving equations in terms of outgoing radiation flux (q_o)

We can solve for T_k 's and q_k 's whenever T_k 's are given for $1 \le k \le m$ and q_k 's are given for $1+m \le k \le N$ with $1 \le m \le N$. N is the number of the surfaces. $q_{o,j}$ are given for $1 \le j \le N$. Substituting equation (3.2.3) into (3.2.2), we get

$$q_{o,k}(r_k) = \varepsilon_k \sigma T^{\,4}{}_k (r_k) + (1-\varepsilon_k)\sum_{j=1}^{N} \int_{A_j} q_{o,j}(r_j)\, dF_{dk-dj}(r_j, r_k) \tag{3.5.1}$$

This equation provides a relation between q_o and T along a surface. When $q_k(r_k)$ which is the heat supplied to a surface A_k is known in equation (3.2.5b), then it can be used to relate q_o and q_k. Hence, we get a complete set of N equations for q_o's in terms of either T_k's or q_k's. The obtained system of N equations consists of m equations for q_o'S are given T_k's with $1 \le m \le N$ and $1 \le k \le m$ and (N-m) equations for q_o'S are given q_k's with $m+1 \le k \le N$

$$\varepsilon_k \sigma T_k^4(r_k) = q_{o,k}(r_k) - (1-\varepsilon_k)\sum_{j=1}^{N}\int_{A_j} q_{o,j}(r_j)\, dF_{dk-dj}(r_k, r_j)\,,\ 1 \le \mathrm{k} \le \mathrm{m} \quad (3.5.2)$$

$$q_k(r_k) = \quad q_{o,k}(r_k) - \sum_{j=1}^{N}\int_{A_j} q_{o,j}(r_j)\, dF_{dk-dj}(r_k, r_j) \quad , \quad \mathrm{m}+1 \le \mathrm{k} \le \mathrm{N} \quad (3.5.3)$$

Since q_o's are given in equation (3.5.2) then we can compute T_k's with $1 \le \mathrm{k} \le \mathrm{m}$.

Consequently, by the following equation we can find the unknowns q_k's which are

$$q_k(r_k) \ = \ \frac{\varepsilon_k}{1-\varepsilon_k}\left[\sigma T_k^4(r_k) - q_{o,k}(r_k)\right] \quad , \quad 1 \le \mathrm{k} \le \mathrm{m} \qquad (3.5.4)$$

Similarly, q_o's are given in equation (3.5.3) then we can compute q_k's with $\mathrm{m} + 1 \le \mathrm{k} \le \mathrm{N}$. Then by the following equation we can find the unknowns T_k's which are

$$\sigma T_k^4(r_k) \ = q_{o,k}(r_k) + \frac{1-\varepsilon_k}{\varepsilon_k} q_k(r_k) \quad , \quad \mathrm{m}+1 \le \mathrm{k} \le \mathrm{N}. \qquad (3.5.5)$$

CHAPTER FOUR

Investigation of the existence and uniqueness solution of the coupled conduction–radiation problem

The main goal of this chapter is to prove the existence and the uniqueness of a weak solution for the following proposed problem [16]. The existence of a solution will be proved by showing that our problem is pseudo-monotone and coercive. The uniqueness of the solution will be proved using an idea borrowed from the analysis of nonlinear heat conduction. For the sake of simplicity we will use the following notations:

(i) The duality between L^p_μ and L^q_μ for a Borel measure μ is defined as

$$\langle f, g \rangle_\mu = \int f \; g \; d\mu \qquad , \quad f \in L^p_\mu \quad \text{and} \quad g \in L^q_\mu$$

with $1 \le p \le \infty$, p and q are conjugate exponents, that is, $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

(ii) An operator K is positive if $f \ge 0$ implies $K\,f \ge 0$. We denote the positive and negative parts of a function by either sub–or superscript:

$$f^{\,+} = f_{+} = \max(f\,,0) \quad \text{and} \quad f^{\,-} = f_{-} = \max(-f\,,0)\,.$$

(iii) Let Γ be a subset of $\partial\Omega$ where local heat transfer occurs and define an Operator A through

$$\int_{\Omega} a_{i\,j}\, \partial_i f \;\, \partial_j \, g \;\, dx \; + \int_{\Gamma} \xi \left|f\right|^{p-1} f \;\, g \;\, ds\,, \quad p > 1$$

The coefficients $a_{i\,j}$ and $\xi \ge 0$ are bounded. The domain of A is

$H^1(\Omega) \cap L_\gamma^{p+1}(\Gamma)$ Where the measure γ is the surface measure of Γ weighed with the coefficient ξ. The null space of A is denoted by

$$N(A) = \{ \ f \in H^1(\Omega) \cap L_\gamma^{p+1}(\Gamma) : Af = 0 \ \}.$$

(iv) $\{ a_{ij} \}$ is strictly elliptic, that is, there exists a constant $C > 0$ such that

$$\langle \, Af, f \, \rangle \ \geq \ C \int_\Omega |\nabla f|^2 \, dx \quad \text{for all} \quad f \in H^1(\Omega) \ .$$

4.1 The mathematical model

Suppose that $\Omega = \Omega_1 \cup \Omega_2 \subset R^3$ is a union of two disjoint, conductive and opaque bodies surrounded by transparent and non–conductive medium. Moreover, we suppose that the radiative surfaces Γ_1 and Γ_2 are diffuse and grey, that is, the emissivity ε of the surfaces does not depend on the wavelength of the radiation. Under the above assumptions the boundary value problem reads as

$$-\nabla . (k \nabla T) \ = \ g \quad \text{in } \Omega \tag{4.1.1}$$

$$-k \frac{\partial T}{\partial n} = \varepsilon \, \sigma \, (T^4 - T_0^4) \quad \text{on } \Gamma_1 \tag{4.1.2}$$

$$-k \frac{\partial T}{\partial n} = q = q_0 - q_i \quad \text{on } \Gamma_2 \tag{4.1.3}$$

where k is the heat conductivity, n is the outward unit normal, g is the given heat generation distribution and q is the radiative heat flux, which is defined as the difference between the outgoing radiation q_0 and the incoming radiation q_i. ε is the emissivity coefficient ($0 \leq \varepsilon < 1$), σ is the Stefan–Boltzman constant which has the value $5.669996 \times 10^{-8}\ W/(m^2 K^4)$, T is the absolute temperature and T_0 is the effective external radiation temperature. The outgoing radiation q_0 and the incoming radiation q_i are related by the relation

$$q_i = K\, q_0 \qquad \text{on} \quad \Gamma_2 . \tag{4.1.4}$$

Moreover, the outgoing radiation q_0 on Γ_2 is a combination of the emitted and reflected energy [20]. This yield

$$q_0 = \varepsilon\, \sigma\, T^4 + (1-\varepsilon) q_i = \varepsilon\, \sigma\, T^4 + (1-\varepsilon) K\, q_0 \tag{4.1.5}$$

The integral operator $K : L^{\infty}(\Gamma_2) \to L^{\infty}(\Gamma_2)$ appearing in equations (4.1.4) and (4.1.5) has the explicit form

$$K\, q_0(x) = \int_{\Gamma_2} G^*(x, y)\, \beta(x, y)\, q_0(y)\, d\Gamma_2(y), \quad x \in \Gamma_2 \tag{4.1.6}$$

Where $G^*(x, y)$ called the view factor between x and y on Γ_2 and is defined as (see, e.g., [9]).

$$G^*(x,y) = \frac{\cos\theta_x \ \cos\theta_y}{\pi|x-y|^2} \tag{4.1.7}$$

Or equivalently

$$G^*(x,y) = \frac{[\ n(y).(x-y)\ ]\ [\ n(x).(y-x)\ ]}{\pi|x-y|^4} \tag{4.1.8}$$

Where, n(x) is the inner normed to Γ at the point x and θ_x denotes the angle between n(x) and x -y, n(y) and θ_y are defined analogously. The function $\beta(x,y)$ appearing in equation (4.1.6) takes account of the shadow zones. This function, termed the visibility (shadow) function, is defined as

$$\beta(x,y) = \begin{cases} 1 & , \text{ if a point } x \text{ can be seen when} \\ & \quad \text{looking from point } y \\ & \\ 0 & , \text{ otherwise.} \end{cases} \tag{4.1.9}$$

In the following we recall some properties of the operator K defined in (4.1.6) and the corresponding kernel $G^*(x,y)$ defined in (4.1.7)–(4.1.8). These properties have already been investigated in [14, 11]. Therefore, we will state some of these results without proof unless there is a new approach for the proof. Methods for the computation of the visibility function $\beta(x,y)$ can be found in [12].

Lemma 4.1.1 [11] let Γ be a Ljapunow surface in $C^{1,\delta}$ with $\delta \in [0,1)$ then for any arbitrary point $x \in \Gamma$,

$$\int_{\Gamma_2} G^*(x,y)\, d\,\Gamma(y) = 1, \qquad (4.1.10)$$

Where $G^*(x,y)$ is given by (4.1.8) .

Proof: First we choose a local coordinate system in the point $x \in \Gamma$ so that x = (0,0,0) and the plane (ξ_1,ξ_2) is tangent to Γ in x as shown in figure (4.1). Furthermore, we choose y = $(\xi_1,\xi_2,f\,(\xi_1,\xi_2))$ in the neighborhood of $\xi_1 = \xi_2 = 0$. Using the assumption that $\Gamma \in C^{1,\delta}$ with $\delta \in [0,1)$, together with the Taylor expansion of y in the local coordinate system and some trivial estimates, we get the following inequalities:

$$\left|\frac{n(x)\,.\,(y-x)}{|y-x|^2}\right| \le c_1 |\xi_\alpha|^{\delta-1} \;,\; \left|\frac{n(y)\,.\,(x-y)}{|x-y|^2}\right| \le c_2 |\xi_\alpha|^{\delta-1} \qquad (4.1.11)$$

With $\alpha \in [1, d-1]$ and d = 2or 3. Consequently, one obtains from (4.1.11)

$$|G^*(x,y)| \le c_3 |\xi_\alpha|^{-2(1-\delta)+3-d} \qquad (4.1.12)$$

with an arbitrary constant c_3 and $d = 2$ or 3. This shows that $G^*(x,y)$ is a weakly singular kernel of type $|\text{x-y}|^{-2(1-\delta)}$ and hence it is integrable with $G\;(x,y) = G*\;(x,y)\;\beta(X,Y)$

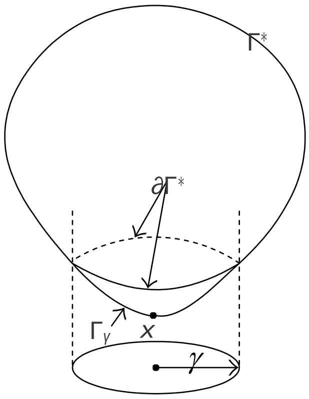


Fig.(4.1)

In order to calculate $\int_{\Gamma} G^*(x,y)\,\mathrm{d}\Gamma_{\mathrm{y}}$, we use Stoke's theorem. For the following, we consider a closed surface Γ and an arbitrary point $y = (y_{1,}\ y_2, y_3) \in \Gamma$. At this point, the normal to the area *A* is constructed. Let the functions $\mathrm{P}_1(\mathrm{y}), \mathrm{P}_2(\mathrm{y})$ and $\mathrm{P}_3(\mathrm{y})$ be any twice differentiable functions of $\mathrm{y}_1, \mathrm{y}_2$, and y_3 and *n* is the normal. Stoke's theorem in three dimensions provides the following relation:

$$\begin{aligned}&\int_{\partial \mathrm{A}} (\mathrm{p}_1 dy_1 + \mathrm{p}_2 dy_2 + \mathrm{p}_3 dy_3) = \\ &\int_A \left[\left(\frac{\partial p_3}{\partial y_2} - \frac{\partial p_2}{\partial y_3}\right) n_1(y) + \left(\frac{\partial p_1}{\partial y_3} - \frac{\partial p_3}{\partial y_1}\right) n_2(y) + \left(\frac{\partial p_2}{\partial y_1} - \frac{\partial p_1}{\partial y_2}\right) n_3(y)\right] dA\end{aligned} \tag{4.1.13}$$

Hence this relation can now be applied to express area integrals in view factor Computations in terms of boundary integrals. To this end, we consider the surface Γ as shown in figure (4.1), let $\Gamma\gamma = Z(x,\gamma) \cap \Gamma$ be a small neighborhood of the point *x*, and define Γ^* as $\Gamma^* = \Gamma \setminus \Gamma_y$.

Here *Z*(*x*,*γ*) is a cylinder which is defined by the relation $x_1^2 + x_2^2 \le \gamma^2$. Since Γ^* is not independent of *x*, the integral $\int_{\Gamma} G^*(x,y)\,\mathrm{d}\Gamma_{\mathrm{y}}$ can be expressed as

$$F_\gamma(x) = \int_{\Gamma} G^*(x,y)\,\mathrm{d}\Gamma_{\mathrm{y}} = \int_{\Gamma_\gamma} G^*(x,y)\,\mathrm{d}\Gamma_{\mathrm{y}} + \int_{\Gamma^*} G^*(x,y)\,\mathrm{d}\Gamma_{\mathrm{y}} \tag{4.1.14}$$

where the first integral tends to zero for $\gamma \to 0$ because of the weakly singular kernel $G^*(x,y)$. Hence (4.1.14) is reduced to

$$F_\gamma(x) = \lim_{\gamma \to 0} \int_{\Gamma^*} G^*(x, y)\ d\Gamma_{\mathrm{y}} \quad ` \tag{4.1.15}$$

Since the view factor $G^*(x, y)$ is smooth in Γ^*, the application of Stoke's theorem leads

$$F_\gamma(x) = \lim_{\gamma \to 0} \int_{\Gamma^*} G^*(x, y)\ d\Gamma_y = \lim_{\gamma \to 0} \int_{\partial\Gamma^*} \nabla \times \bar{p}(y).n(y) dy \tag{4.1.16}$$

$$F_\gamma(x) = \lim_{\gamma \to 0} \int_{\partial\Gamma^*} (\mathrm{p}_1 dy_1 + \mathrm{p}_2 dy_2 + \mathrm{p}_3 dy_3) \tag{4.1.17}$$

where P_1(y), P_2(y), and P_3(y) are given respectively, by

$$p_{1(y)=} \frac{-n_2(x)(x_3 - y_3) + n_3(x)(x_2 - y_2)}{2\pi |x - y|^2},$$

$$p_{2(y)=} \frac{n_1(x)(x_3 - y_3) - n_3(x)(x_1 - y_1)}{2\pi |x - y|^2}, \tag{4.1.18}$$

$$p_{3(y)=} \frac{-n_1(x)(x_2 - y_2) + n_2(x)(x_1 - y_1)}{2\pi |x - y|^2}$$

The normal to the area element is perpendicular to both the x_1- and x_2-axes and parallel to the x_3-axis. Hence (4.1.17) becomes

$$F_\gamma(x) = \frac{1}{2\pi} \lim_{\gamma \to 0} \int_{\partial\Gamma^*} \frac{-y_2 dy_1 + y_1 dy_2}{y_1^2 + y_2^2 + y_3^2} \tag{4. 1. 19}$$

using the fact that the area element is located at the origin of the coordinate system.With the help of the relation y_2, we get

$$F_{\gamma}(x)=\frac{1}{2\pi}\lim_{\gamma\to 0}\int_{\partial\Gamma^{*}}\frac{-y_{2}dy_{1}+y_{1}dy_{2}}{\gamma^{2}}+\frac{1}{2\pi}\lim_{\gamma\to 0}\int_{\partial\Gamma^{*}}\frac{-y_{3}^{2}(-y_{2}dy_{1}+y_{1}dy_{2})}{(\gamma^{2}+y_{3}^{2})\gamma^{2}} \qquad (4.1.20)$$

Let the boundary of the domain Γ^{*} be described by the triple $(y_1, y_2, f(y_1, y_2))$ then the first integral will be integrated over the circle $y_1^2+y_2^2\le\gamma^2$. Using the polar coordinates $y_1 = \gamma\cos\theta$ and $y_2 = \gamma\sin\theta$, one obtains directly the first integral = 1 .For the second integral, we have $y_3 \underset{\Gamma^*}{=} f(y_1, y_2)$. Applying Taylor's expansion, it can easily be shown that it is equal to zero. Hence, we have the desired result for convex enclosure geometries(4.1.10). Next we have to show that this result holds also for the non convex case; see figure (4.2).Therefore, we consider the set $\Gamma\backslash\Gamma_y$, where $\Gamma_y = \{x\in\Gamma \,/\, \beta(x,y)=1\}$ This set consists in general of many disjoint components. For the sake of simplicity, we take one of these components and denote it by D_i, where D_i is the boundary of Γ_i. Clearly,.

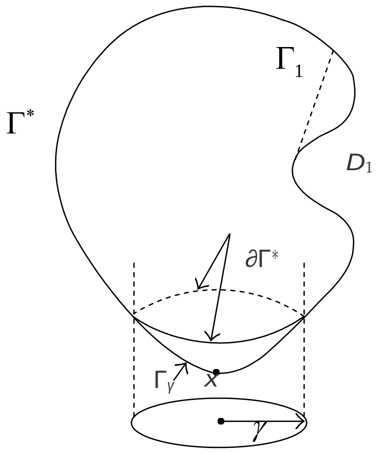


Fig.(4.2)

All Γ_i are dependent on the choice of D_i . Due to the discontinuity of the visibility Function $\beta(x,y)$, the Stoke theorem cannot be applied directly for $G(x,y)$, but we write first

$$\int_{\Gamma^*} G(x,y)\, \mathrm{d}\Gamma_y = \int_{\Gamma_\gamma} G^*(x,y)\, \mathrm{d}\Gamma_y - \sum_i \int_{D_i} \nabla \times \bar{p}(y).n(y) dy \tag{4.1.21}$$

Since the second integral vanishes over the closed surface D_i, the assertion follows directly.

Lemma 4.1.2 [11] For the integral kernel $G(x, y)$, it holds that $G(x,y) \geq 0$. The mapping $K : L^p(\Gamma_2) \to L^P(\Gamma_2)$ is compact for $1 \leq p \leq \infty$. Furthermore,

(a) K is symmetric and positive

(b) $\|K\| = 1$ in L^p for $1 \leq p \leq \infty$

(c) The eigenvalue 1 of K is simple.

(d) The spectral radius $\rho(K) = 1$.

Proof: See [14]**.**

Lemma 4.1.3 [16] For $1 \leq p \leq \infty$ and $0 \leq \varepsilon < 1$ the operator $I - (1-\varepsilon) K$ from $L^p(\Gamma_2)$ into itself is invertible and this inverse is positive.

Proof: See [11].

4.2 Variational form

In order to write (4.1.1)–(4.1.5) into variational form, we first assume that $T \in L^5(\Gamma_2)$, and solving for q_0 from equation (4.1.5), we have

$$q = (I-K)\, q_0 = (I-K)\left(I - (1-\varepsilon) K \right)^{-1} \varepsilon \sigma T^4 = E \sigma T^4 \tag{4.2.1}$$

where E is a linear operator from L^p_μ to itself for $1 \leq p \leq \infty$. Next, we define the mapping A from $\boldsymbol{H^1(\Omega) \cap L^5_\gamma(\Gamma_2)}$ to $\boldsymbol{H^1(\Omega) \cap L^5_\gamma(\Gamma_2)}$ by

$$\langle AT, \psi \rangle = \int_{\Omega} k \, \nabla T \, . \, \nabla \psi \, dx + \int_{\Gamma_1} \varepsilon \sigma |T|^3 T \psi \, ds \qquad (4.2.2)$$

Note that since the Stefan–Boltzmann law is physical only for non–negative value of temperature we can replace T^4 by $|T|^3 T$ for mathematical convenience. Finally, by setting $d\mu = \sigma \, ds$, we can write our problem in variational form as

$$\langle AT, \psi \rangle + \int_{\Gamma_2} E \, |T|^3 \, T \, \psi \, d\mu = \langle \tilde{g}, \psi \rangle , \quad \forall \psi \in X = H^1(\Omega) \cap L^5_\mu \cap L^5_\gamma \qquad (4.2.3)$$

where $\tilde{g}$ now contains also the data term on Γ_1.

Lemma 4.2.1 [16] The operator *E* is self–adjoint. As a mapping from L^2_μ into itself, *E* is positive semidefinite with respect to $\langle \, . \, , \, . \, \rangle_\mu$ inner product.

Proof: The self–adjointness of *E* is a consequence of equation (4.2.1). Let $q \in L^2_\mu$ be arbitrary and denote by q the solution of $\left(\, I - (1 - \varepsilon) \, K \, \right) q = \varepsilon q$. Then

$$\langle \, q, E \, q \rangle = \langle \, \varepsilon^{-1} \left(\, I - (1 - \varepsilon) \, K \, \right) q, (I - K) \, q \, \rangle_\mu$$

$$= \langle \, q \, , (I - K)(\varepsilon^{-1} - 1)\,(I - K) q \, \rangle_\mu \; + \; \langle \, q \, , (I - K) \, q \, \rangle_\mu \geq 0$$

as $\|K\|_2 \leq 1$ and $\varepsilon \leq 1$. □

Lemma 4.2.2 [16] The operator E can be written as $E = I - F$, where F is self–adjoint positive and $\|F\|_p \leq 1$. Moreover, every nonzero constant is an eigen function of F with eigenvalue $\lambda = 1$.

Proof: One can write

$$E = I - F = I - \left[\ (1-\varepsilon) + \varepsilon\, K \left(\ I - (1-\varepsilon) K\ \right)^{-1} \varepsilon\ \right] \tag{4.2.4}$$

where F is self–adjoint. The inverse term in F can be written as

$$\left(\ I - (1-\varepsilon) K\ \right)^{-1} = \sum_{i=0}^{\infty} \left(\ (1-\varepsilon) K\ \right)^{i}\ .$$

As K is positive, all terms in the series are also positive. This implies that F is positive. Since E is self–adjoint, then we can write

$$F = I - E = I - \varepsilon \left(\ I - K(1-\varepsilon)\ \right)^{-1} (I - K) \tag{4.2.5}$$

Next, we show that $\|F\|_1 \leq 1$ and $\|F\|_\infty \leq 1$. From Riesz–Thorin theorem [6] it follows that $\|F\|_p \leq 1$ for $1 < p < \infty$. Since F is positive we have

$$F \left(\ 1 - q / \|q\|_\infty\ \right) \geq 0 \text{, for all } \ q \in L_\mu^\infty\ , \quad q \neq 0\,.$$

Hence

$$\|F\|_\infty = \sup \frac{\|F\,q\|}{\|q\|} \leq \|F(1)\|_\infty = \|1\|_\infty = 1$$

as $F(a) = a$ for every constant a. Moreover, self–adjointness implies that

$$\|F\|_1 = \ \|F^*\|_\infty = \|F\|_\infty \leq 1\ . \qquad \square$$

4.3 Existence results

In order to prove that the original boundary value problem has a solution, it is sufficient to prove that our problem is pseudo-monotone and coercive [23, 24]. To do that we introduce next the operator $R : X \to X^*$ defined by

$$\langle R\,T\,,\psi\rangle = \langle A\,T\,,\psi\rangle + \int_{\Gamma_2} E\,|T|^3 T\ \psi\,d\,\mu = \langle\,\tilde{g}\,,\psi\,\rangle\,, \qquad \forall\psi \in X\ = H^1(\Omega)\cap L^5_\mu \cap L^5_\gamma \tag{4.3.1}$$

Note that the space X is reflexive by the arguments given in [5]. To show that R is pseudo-monotone we consider the following Lemma

Lemma 4.3.1 [16] The operator $R : X \to X^*$ is pseudo-monotone, that is, $T_i \rightharpoonup T$ weakly in X and $\overline{\lim_{i\to\infty}}\langle\,R\,T_i\,,T_i - T\,\rangle \le 0$, imply that

$$\langle R\,T, T-\psi\rangle \le \lim_{i\to\infty}\langle\,R\,T_i, T_i-\psi\rangle \qquad \forall\psi\in X \tag{4.3.2}$$

Proof: One can write $E = M - S$ where M is a multiplication operator

$(M\,T)(x) = m(x)\,T(x)$ with $0 \le m_0 \le m(x) \le 1$ and S is a compact operator in $L^{5/4}_\mu$.

Since the operator

$$\langle\tilde{A}T,\psi\rangle = \langle AT,\psi\rangle + \langle M|T|^3 T,\psi\rangle_\mu\,, \quad \forall\psi\in X \tag{4.3.3}$$

is monotone then it is sufficient to prove that the mapping $T \to S\,|T|^3 T$ is pseudo–monotone in X. Let $T_i \rightharpoonup T$ weakly in X . Then $T_i \rightharpoonup T$ weakly in L^5_μ and

$T_i \rightharpoonup T$ weakly in $H^1(\Omega)$. Thus $T_i \to T$ strongly in L^2_μ as the embedding $H^1(\Omega) \subset L^2(\Gamma_2)$ is compact [3, 6]. Consequently, $T_i \to T$ $\mu -$a.e. in Γ_2 and hence also $|T_i|^3 T_i \to |T|^3 T$ $\mu - a.e.$ Hence $|T_i|^3 T_i \rightharpoonup |T_i|^3 T$ weakly in $L^{5/4}_\mu$ as the sequence $\{ |T_i|^3 T_i \}$ is bounded in $L^{5/4}_\mu$. Finally the compactness of S implies that

$$\langle S\, |T|^3 T, T-\psi \rangle - \langle S\, |T_i|^3 T_i, T_i - \psi \rangle_\mu = \langle S\left(|T|^3 T - |T_i|^3 T_i \right), T_i - \psi \rangle_\mu$$

$$-\langle S\, |T|^3 T, T - T_i \rangle_\mu \to 0\,,\ \forall \psi \in X \quad (4.3.4)$$

The coercivity in L^5_μ can be proved through the following two Lemmas:

Lemma 4.3.2 [16] For $1 \le p \le \infty$ and $T \in L^5_\mu$, it holds $\|F\|_{L^p_\mu} \le 1$ and $\langle E|T|^3 T, T \rangle_\mu \ge 0$.

Proof: Let $T \in L^1_\mu$ be positive. Then

$$\int F T\, d\mu = \int T F^* 1\, d\mu \le \int T\, d\mu$$

Since *F* is positive, this implies that $\|F\|_{L^1_\mu} \le 1$. On the other hand $F\left(1 - \psi / \|\psi\|_{L^p_\mu}\right) \ge 0$ and thus $\|F\|_{L^\infty_\mu} \le \|F1\|_{L^\infty_\mu} \le 1$. Using Riesz interpolation theorem [24] it follows that $\|F\|_{L^p_\mu} \le 1$, $1 \le p \le \infty$. To show the second part of this Lemma we use the Holder inequality

$$\langle E\, |T|^3 T, T \rangle_\mu \ge \|T\|^5_{L^5_\mu} - \|F|T|^3 T\|_{L^{5/4}_\mu} \|T\|_{L^5_\mu} \ge \left(1 - \|F\|\right) \|T\|^5_{L^5_\mu} \ge 0 \quad \square$$

Lemma 4.3.3 [16] For $T \in L^5_\mu$, $T \notin N(E)$ implies that $\langle E|T|^3, T\rangle_\mu > 0$.

Proof: Since F is positive, then we have

$$\langle E|T|^3, T\rangle_\mu \;\geq\; \langle E\,T_+^4, T_+\rangle_\mu + \langle E\,T_-^4, T_-\rangle_\mu \tag{4.3.5}$$

Under the assumption that $T \geq 0$ and $\|T\|_{L^5} = 1$, we can use the Riesz interpolation theorem [5,6] to show that

$$\langle F\,T^4, T\rangle_\mu \quad \langle T^4, T\rangle_\mu \;=\; \|T\|^5_{L^5_\mu} \text{ if } T \notin N(E) \tag{4.3.6}$$

where $N(E)$ is the null space of E defined as $N(E) = \{\, T \in L^1_\mu\ ,\ ET = 0\ \}$.

As S is compact then it can be expressed as an integral operator [6]. Moreover, one can write FT as

$$F\,T = (1-m)T + S\,T = \lim_{\varepsilon\to 0} \int f_\varepsilon(x,y)\,T(y)\,d\mu_y \quad \text{for} \quad f_\varepsilon \geq 0\,.$$

Next, we let $p = 5/4$, $p_1 = 6/5$, $p_2 = 2$ and let $q\,, q_1\,, q_2$ be the corresponding conjugate exponents. Further, let $\delta = 9/10$ so that $\dfrac{1}{p} = \dfrac{\delta}{p_1} + \dfrac{1-\delta}{p_2}$.

Hence for $T, \psi \geq 0$ we can write

$$\int T\Big(\int f_\varepsilon\,\psi\,d\mu\Big)d\mu \;=\; \int T\Big(\int f_\varepsilon^{\delta+(1-\delta)}\,\psi^{p\left(\frac{\delta}{p_1}+\frac{1-\delta}{p_2}\right)}\,d\mu\Big)\,d\mu$$

Using Holder inequality we get

$$\leq \int T \left(\int f_\varepsilon \psi^{\frac{p}{p_1}} \, d\mu \right)^{\delta} \left(\int f_\varepsilon \psi^{\frac{p}{p_2}} \, d\mu \right)^{1-\delta} d\mu$$

$$\leq \left(\int T^{\frac{q}{q_1}} \int f_\varepsilon \psi^{\frac{p}{p_1}} \, d\mu \, d\mu \right)^{\delta} \left(\int T^{\frac{q}{q_2}} \int f_\varepsilon \psi^{\frac{p}{p_2}} \, d\mu \, d\mu \right)^{1-\delta}$$

let $\varepsilon \to 0$ we obtain

$$\langle T, F\psi \rangle \leq \langle T^{\frac{q}{q_1}}, F\psi^{\frac{p}{p_1}} \rangle_\mu^{\delta} \langle T^{\frac{q}{q_2}}, F\psi^{\frac{p}{p_2}} \rangle_\mu^{1-\delta} \tag{4.3.7}$$

For $\psi = T^4$ (4.3.7) yields

$$\langle T, F T^4 \rangle_\mu \leq \langle T^{5/2}, F T^{5/2} \rangle_\mu \tag{4.3.8}$$

Finally, assume $\langle T^{5/2}, F T^{5/2} \rangle = \|T\|_{L^5_\mu}^5$. Then, letting $\psi = T^{5/2}$ we have

$$0 = \langle \psi, \psi - F\psi \rangle_\mu \geq \|\psi\|_{L^2_\mu}^2 - \|F\psi\|_{L^2_\mu} \|\psi\|_{L^2_\mu} \text{ so that } \|F\psi\|_{L^2_\mu} = \|\psi\|_{L^2_\mu}.$$

Since

$$\langle \psi, (I - F^* F\psi) \rangle_\mu = \langle \psi, \psi \rangle_\mu - \langle F\psi, F\psi \rangle_\mu = 0$$

we have

$$\|E\psi\|^2_{L^2_\mu} = \langle \psi, E^* E\psi\rangle_\mu = \langle \psi, (I-F)\psi + (I-F^*)\psi - (I-F^*F)\psi\rangle_\mu = 0$$

This implies that $T^{5/2} = \psi \in N(E)$ and hence $T \in N(E)$. Therefore, if $T \notin N(E)$ then inequalities (4.3.6) and (4.3.8) are strict . □

4.4 Uniqueness of the solution

Theorem 4.4.1. [16] Let T_1 and T_2 be solutions of (4.2.3), corresponding to the right hand sides g_1 , $g_2 \in X^*$, and suppose that

$$\langle g_1 - g_2, \psi\rangle \geq 0, \quad \forall \psi \geq 0, \quad \psi \in X .$$

Then $T_1 \geq T_2$ $L-a.e.\, in\, \Omega$, $\gamma - a.e.\, on\, \Gamma_1$ and $\mu - a.e.\, in\, \Gamma_2$. Consequently, the solution of (4.2.3) is unique.

Proof: For $\varepsilon > 0$ we denote

$$\Omega_0 = \{ x \in \overline{\Omega} : T_1(x) < T_2(x) \}$$

$$\Omega_\varepsilon = \{ x \in \Omega_0 : T_2(x) - T_1(x) > \varepsilon \}$$

$$\psi_\varepsilon = \min\{ \varepsilon, (T_2 - T_1)^+ \}.$$

We will also denote the Lebesgue measure in R^n by L . In order to prove this theorem we follow the idea from [8]. We need to show that

$\mu(\Omega_0) + L(\Omega_0) + \gamma(\Omega_0) = 0$. We argue by contradiction and assume first that

$\mu(\Omega_0) > 0$. From [22]

$$\|\psi_\varepsilon\|^2_{L^5_\mu} \le C\,\{ \int_\Omega a_{ij}\partial_i\psi_\varepsilon\partial_j\psi_\varepsilon\,dx + \Big(\int_{\Gamma_1} \xi|\psi_\varepsilon|^{p+1}\,ds\Big)^{\frac{2}{p+1}} + \Big(\int_{\Gamma_2} E\psi_\varepsilon^4\psi_\varepsilon\,d\mu\Big)^{2/5}\}$$

The next step is to estimate

$$\int_\Omega a_{ij}\partial_i\psi_\varepsilon\partial_j\psi_\varepsilon\,dx \le \varepsilon\|\psi_\varepsilon\|_{L^5_\mu}\,g - f_\varepsilon\ , \tag{4.4.1}$$

and

$$\Big(\int_{\Gamma_1}\xi|\psi_\varepsilon|^{p+1}\,ds\Big)^{\frac{2}{p+1}} + \Big(\int_{\Omega} E\psi_\varepsilon^4\psi_\varepsilon\,d\mu\Big)^{2/5} \le \varepsilon\|\psi_\varepsilon\|_{L^5_\mu}\,g_\varepsilon + h_\varepsilon \tag{4.4.2}$$

where $g_\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and* $h_\varepsilon - f_\varepsilon$ can be ignored when ε is small enough. Finally, these estimates give

$$\mu(\Omega_\varepsilon) \le \varepsilon^{-1}\Big(\int_{\Omega_\varepsilon}\varepsilon^5\,d\mu\Big)^{1/5} \le \varepsilon^{-1}\|\psi_\varepsilon\|_{L^5_\mu} \le g_\varepsilon \to 0 \tag{4.4.3}$$

This leads to a contradiction. Similarly we can prove that $L(\Omega_0) = \gamma(\Omega_0) = 0$. In the following we give a sketch for the derivation of (4.4.1)–(4.4.3). To derive the estimate (4.4.1) we can write

$$\begin{aligned}\int_\Omega a_{ij}\partial_i\psi_\varepsilon\partial_j\psi_\varepsilon\,dx &= \int_\Omega a_{ij}\partial_i(T_2 - T_1)\,\partial_j\psi_\varepsilon\,dx \\ &= \langle g_2 - g_1, \psi_\varepsilon\rangle - \int_{\Gamma_1}\xi\big(|T_2|^{p-1}T_2 - |T_1|^{p-1}T_1\big)\psi_\varepsilon\,ds \qquad (4.4.4) \\ &\quad + \int_{\Gamma_2} E\big(|T_1|^3T_1 - |T_2|^3T_2\big)\psi_\varepsilon\,d\mu\quad .\end{aligned}$$

The last term in (4.4.4) can be decomposed as

$$\int_{\Gamma_2} E\left(|T_1|^3 T_1 - |T_2|^3 T_2 \right)\psi_\varepsilon \, d\mu = \int_{\Gamma_2 \setminus \Omega_0} \left(|T_1|^3 T_1 - |T_2|^3 T_2 \right) E^* \psi_\varepsilon \, d\mu$$

$$+ \int_{\Omega_0 \setminus \Omega_\varepsilon} \left(|T_1|^3 T_1 - |T_2|^3 T_2 \right) E^* \psi_\varepsilon \, d\mu$$

$$+ \int_{\Omega_\varepsilon} \left(|T_1|^3 T_1 - |T_2|^3 T_2 \right) E^* \psi_\varepsilon \, d\mu \; .$$

In fact the first term on the right–hand side is negative as $|T_1|^3 T_1 - |T_2|^3 T_2 \geq 0$ and $E^* \psi_\varepsilon = 0 - F^* \psi_\varepsilon \leq 0 \quad in \ \Gamma_2 \setminus \Omega_0$. To investigate the second term we observe

$$|T_2|^3 T_2 - |T_1|^3 T_1 \leq \left(T_2 - T_1 \right) Q\left(|T_2|, |T_1| \right)$$

where

$Q(x, y) = x^3 + x^2 y + x y^2 + y^3$. Then

$$\int_{\Omega_0 \setminus \Omega_\varepsilon} \left(|T_1|^3 T_1 - |T_2|^3 T_2 \right) E^* \psi_\varepsilon \, d\mu \leq \int_{\Omega_0 \setminus \Omega_\varepsilon} \left(|T_2|^3 T_2 - |T_1|^3 T_1 \right) F^* \psi_\varepsilon \, d\mu$$

$$\leq \int_{\Omega_0 \setminus \Omega_\varepsilon} \left(T_2 - T_1 \right) Q\left(|T_2|, |T_1| \right) F^* \psi_\varepsilon \, d\mu$$

$$\leq \varepsilon \, \|\psi_\varepsilon\|_{L^5_\mu} \, g_\varepsilon \quad ,$$

where

$$g_\varepsilon = \left\| F\left(Q\left(|T_2|, |T_1| \right) \right\|_{L^{5/4}_\mu} \to 0 \quad as \ \ \varepsilon \to 0$$

Thus

$$\int_{\Omega} a_{ij}\partial_i \psi_\varepsilon \partial_j \psi_\varepsilon \, dx \ \le\ \varepsilon \|\psi_\varepsilon\|_{L_\mu^{5/4}} \ g_\varepsilon - f_\varepsilon$$

where

$$g_\varepsilon = \int_{\Gamma_1} \xi\big(\ |T_2|^{p-1} T_2 - |T_1|^{p-1} T_1 \ \big)\psi_\varepsilon \ ds + \int_{\Omega_\varepsilon} \big(\ |T_2|^3 T_2 - |T_1|^3 T_1 \ \big) E^* \psi_\varepsilon \ d\mu \ .$$

To derive (4.4.2) we observe that $E^* \psi_\varepsilon = \varepsilon - F^* \psi_\varepsilon \ge \varepsilon - F^* \varepsilon \ge 0 \quad in\ \Omega_\varepsilon$.

Moreover, we can show that

$$\begin{aligned} \int_{\Gamma_2} E \psi_\varepsilon^4 \psi_\varepsilon \ d\mu &\le \int_{\Omega_0 \setminus \Omega_\varepsilon} \psi_\varepsilon^4 \left| E^* \psi_\varepsilon \right| \ d\mu \ + \int_{\Omega_\varepsilon} \psi_\varepsilon^4 \ E^* \psi_\varepsilon \ d\mu \\ &\le \ \varepsilon^{5/2} \ \|\psi_\varepsilon\|_{L_\mu^5}^{5/2} \ g_\varepsilon + \varepsilon^4 \int_{\Omega_\varepsilon} E^* \ \psi_\varepsilon \ d\mu \end{aligned}$$

where $g_\varepsilon \to 0 \quad as \quad \varepsilon \to 0$. Thus we conclude that

$$\Big(\int_{\Gamma_1} \xi |\psi_\varepsilon|^{p+1} \ ds \ \Big)^{\frac{2}{p+1}} + \Big(\int_{\Gamma_2} (E\psi_\varepsilon^4)\psi_\varepsilon \ d\mu \ \Big)^{2/5} \le \varepsilon \|\psi_\varepsilon\|_{L_\mu^5} \ g_\varepsilon + h_\varepsilon$$

where

$$h_\varepsilon = \Big(\int_{\Gamma_1} \xi |\psi_\varepsilon|^{p+1} \ ds \ \Big)^{\frac{2}{p+1}} + \Big(\ \varepsilon^4 \int_{\Omega_\varepsilon} E^* \ \psi_\varepsilon \ d\mu \ \Big)^{2/5}.$$

Finally, we show that $\mu(\Omega_0) + L(\Omega_0) + \gamma(\Omega_0) = 0$.

The steps above imply that

$$\|\psi_\varepsilon\|_{L^5_\mu} \leq \varepsilon\, g_\varepsilon$$

when ε is small enough. Hence

$$\mu(\Omega_\varepsilon) = \varepsilon^{-1}\Big(\int_{\Omega_\varepsilon} \varepsilon^5 \, d\mu \Big)^{1/5} \leq \varepsilon^{-1} \|\psi_\varepsilon\|_{L^5_\mu} \leq g_\varepsilon \to 0 .$$

This is a contradiction, since also $\mu(\Omega_\varepsilon) \to \mu(\Omega_0) > 0$. Therefore $\mu(\Omega_0) = 0$. From this fact it is straight forward to conclude $L(\Omega_0) = \gamma(\Omega_0) = 0$. □

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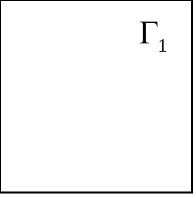

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