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Optimal Homotopy Asymptotic Method for Solving Multidimensional First Order Systems of Partial Differential Equations

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Dedication

I present this as a way of gratitude to my parents whom I'm truly proud of and for them, I am grateful as they stood by my side every day and moment.

To my dear Moutaz my soulmate.
To all the people who encouraged me.

I present this to all of them.

Name: Malak Tawfiq Abdelfattah Meqbil

Declaration

I certify that this thesis submitted for the degree of Master, is the result of my own research, except where otherwise acknowledged, and that this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: malak

Name : Malak Tawfiq Abdelfattah Meqbil

Date: 15/ 07/ 2024

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Abstract

In this thesis, a semi-analytic approximating method, namely Optimal Homotopy Asymptotic Method (OHAM), which is developed from the Homotopy Analysis Method (HAM), is used to find continuous approximate solutions for linear and non-linear first-order systems of partial differential equations.

Within this work, the geometrical topological homotopy concept is used to construct the algorithm for solving such systems. A homotopy equation that depends on an embedding parameter belonging to interval [0, 1] is assumed. As the parameter varies from 0 to 1 the solution of the homotopy equation (which is assumed to be a power series of the embedding parameter) varies continuously from a solution, which is easy to find, to the exact solution. The approximate continuous solution is obtained by truncating the series and using a finite number of its terms. Least Squares Method is used to determine the so-called control-convergence parameters that appear in the approximate solution.

The derived algorithm is applied to solve some examples and the obtained solutions are compared with exact solutions. The results confirm the validity of OHAM and reveal that OHAM is effective, simple, and explicit. Moreover, it is independent of the small parameters required in perturbation methods. Furthermore, the convergence domain of OHAM is easily modifiable, depending on the convergence-control parameters that appear in the approximated solutions, enhancing its versatility.

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Chapter 1

Introduction

In this introductory chapter, we provide a concise overview of fundamental concepts associated with partial differential equations, which serve as powerful tools for describing various physical phenomena across disciplines such as fluid dynamics, electricity, magnetism, mechanics, optics, and heat flow, etc. Additionally, we briefly introduce the concepts of homotopy and perturbation, highlighting their significance in subsequent chapters. Finally, we outline the structure of the thesis.

1.1 Partial Differential Equations

A partial differential equation is an equation involving an unknown function of two or more variables and at least one of its partial derivatives. More precisely if k is an integer greater than 1 and Ω is a subset of $\mathbb{R}^n$, then an expression of the form

$$F\left(D^k u(x), D^{k-1}u(x), \cdots Du(x), u(x), x\right) = 0, \qquad (*)$$

where $x \in \Omega$, $D^k = \dfrac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_n^{k_n}}$, k_i integer, $k = (k_1, k_2, \cdots, k_n)$, $|k| = k_1+k_2+\cdots+k_n$ and

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$

is given and $u : \Omega \longrightarrow \mathbb{R}$ is the unknown, is called a partial differential equation.
A solution (classical) of $(*)$ is a function u that satisfies $(*)$ and some auxiliary boundary conditions on all or part of the boundary of Ω and (or) some given initial conditions. If the differential equation $(*)$ is of the form

$$\sum_{|\beta|\leq k} b_\beta(x) D^\beta u(x) = g(x)$$

where b_β and g are given functions then it is called linear otherwise it is called nonlinear, and when $g = 0$ it is called homogeneous otherwise non-homogeneous. A collection of partial differential equations $(*)$ of several variables is called a system of partial differential equations. A solution of a system of partial differential equations is a vector function that satisfies each scalar equation within the system and any condition associated with it.

Systems can be classified in an obvious way as linear (homogeneous, non-homogeneous), and nonlinear. A problem for a partial differential equation is called Well-posed if and only if it has a solution, the solution is unique, and the solution depends continuously on the given data. Otherwise, it is called ill-posed, for details on theoretical aspects of partial differential equations see [9, 13].
Several techniques are used to solve partial differential equations analytically and numerically such as perturbation methods, integral solutions, finite element methods, etc., see [40, 38].

1.2 Homotopy

Two continuous functions from one topological space to another are called homotopic if one can be continuously deformed into the other, such a deformation being called a homotopy between the two functions. More precisely Let X, Y be topological spaces, and $f, g : X \longrightarrow Y$ continuous maps. A homotopy from g to f is a continuous function $h : X \times [0,1] \longrightarrow Y$ satisfying $h(x,0) = g(x)$ and $h(x,1) = f(x)$, for all $x \in X$. If such a homotopy exists, we say that g is homotopic to f. Any two functions between a topological space and a convex subspace topology of $\mathbb{R}^n$, that is $((1-t)x + ty) \in$ the subspace topology for all x, y in this subspace topology and $t \in [0,1]$, are homotopic through the homotopy $h(x,t) = tf(x) + (1-t)g(x)$. Further the composite of homotopic functions is again homotopic and homotopic is an equivalence relation on the set of all continuous functions betwenn two given topological spaces.
Continuation methods are constructed using the concept of homotopy. Its idea is to embed a given problem, solving $f(x) = 0$ in its most general setting, in one-parameter family of problems using a parameter t that runs over the interval $[0,1]$. The original problem is made to correspond to $t = 1$ and another problem with known solution is made to correspond to $t = 0$. For example $h(x,t) = tf(x) + (1-t)g(x) = 0$. The equation $g(x) = 0$ should have a known solution, see [11, 16].

1.3 Perturbation

Equations stemming from mathematical models often resist exact solutions, prompting us to turn to approximation and numerical techniques. Among these, perturbation methods stand out. These methods provide an approximate resolution when the model's equations feature small terms, typically arising from underlying physical processes with minor impacts. For instance, in fluid dynamics, viscosity might pale in comparison to advection, or in projectile motion, air resistance might be dwarfed by gravity. These minute effects are encapsulated in terms within the model equations that, relative to others, are negligible. When suitably scaled, their magnitude is symbolized by a diminutive coefficient, often denoted as ϵ. A perturbation solution entails an approximate resolution, typically involving the initial terms of a Taylor-like expansion based on this small parameter ϵ. Perturbation methods are applicable across a spectrum of equations encountered in applied mathematics, including ordinary and partial differential equations, algebraic equations, integral equations, and more, underlining their indispensable role. To illustrate the concept, let's consider a differential equation represented as:

$$F(t, y, y', y'', \epsilon) = 0, \qquad t \in I$$

where t is the independent variable within the interval I, and y is the dependent variable. This equation includes a small parameter ϵ, explicitly expressed. Typically, initial or boundary conditions accompany the equation. If the parameter ϵ is large, denoted as γ, we can set $\epsilon = \frac{1}{\gamma}$, making it small.

The regular perturbation method involves expressing a solution to the differential equation as a power series in ϵ such as:

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots$$

The core of this method is assuming a solution of this form, where y_0, y_1, y_2, etc., are determined by substituting into the differential equation. Typically, only a few terms are considered for an approximate solution. Success is often determined by the uniform convergence of the approximation to the exact solution as ϵ approaches zero, uniformly on I. It's important to note that ϵ is considered to be arbitrarily small.

The leading term y_0 in the perturbation series is referred to as the leading-order term, while terms like ϵy_1, $\epsilon^2 y_2$, etc., are seen as higher-order correction terms that are expected to be small. If successful, y_0 will represent the solution of the unperturbed problem where ϵ is set to zero.

In many cases, the perturbation series is a valid representation for $\epsilon < \epsilon_0$ for some yet-to-be-determined ϵ_0. When a model equation includes a small parameter, it's often viewed as a perturbed equation, where the terms involving the small parameter denote small perturbations or deviations from a basic unperturbed problem.

To demonstrate the application of perturbation, let's solve the algebraic equation $x^2 + 2\epsilon x - 3 = 0$ where $\epsilon << 1$. Assuming a solution in the form of a perturbation series $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$, and substituting in the given equation we get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 + 2\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) - 3 = 0.$$

If we expand out and collect the terms in powers of ϵ we obtain

$$x_0^2 - 3 + 2x_0(x_1 + 1)\epsilon + (x_1^2 + 2x_0 x_2 + 2x_1)\epsilon^2 + \cdots = 0.$$

Setting the coefficients equal to 0 leads to $x_0 = \pm\sqrt{3}$, $x_1 = -1$, $x_2 = \pm\frac{1}{2\sqrt{3}}, \cdots$. Therefore we get two approximate solutions

$$\begin{aligned} x &= \sqrt{3} - \epsilon + \frac{1}{2\sqrt{3}}\epsilon^2 + \cdots , \\ x &= -\sqrt{3} - \epsilon - \frac{1}{2\sqrt{3}}\epsilon^2 + \cdots . \end{aligned}$$

For more on perturbation theory see [12, 33].

The thesis is structured as follows:

Chapter 2 introduces the foundational concepts of homotopy analysis methods, including the normal homotopy analysis method and the basic optimal homotopy analysis method. These methods are then applied to solve the Blasius equation.

In Chapter 3, we delve into a comprehensive analysis of the homotopy analysis method, exploring its application to both linear and non-linear systems of partial differential equations.

Chapter 4 focuses on the optimal homotopy asymptotic method, where we extend our analysis to encompass linear and non-linear systems of partial differential equations as well.

Finally, the thesis concludes with a summary of findings and outlines potential avenues for future research.

Chapter 2

The Origin of Optimal Homotopy Analysis Method

This chapter outlines and contrasts various approaches within the homotopy analysis method (HAM). A particular focus is given to an optimized HAM, characterized by a unified control parameter and an infinite set of parameters. Notably, the approximation yielded by this optimized HAM exhibits rapid changes. Remarkably, the default optimal HAM consistently offers the most accurate approximation in a majority of cases. Consequently, in practical applications, it stands out as the most effective implementation of the HAM.

2.1 Introduction

It is well known that nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) for boundary value problems are much more difficult to solve than linear ODEs and PDEs, especially using analytic methods. Traditionally, perturbation (Van del Pol, 1926, [7], Von Dyke, 1975, [8], Murdock, 1991,[36], Kevorkian and Cole, 1995,[14], Nayfeh, 2000,[37]) and asymptotic techniques are widely applied to obtain analytical approximations of nonlinear problems in science, finance, and engineering, [29]. Unfortunately, perturbation and asymptotic techniques are too strongly dependent upon small (large) physical parameters in the governing equations or initial (boundary) conditions and thus are often valid only for weakly nonlinear problems. For example in the viscous flow past a sphere in fluid mechanics, the perturbation formulas of the drag coefficient are valid only for rather small Reynolds number $Re << 1$. Hence, there was a need to develop analytic approximation methods, that are independent of any small (large) physical parameters and valuable for strongly nonlinear problems.

The homotopy analysis method (HAM) is one such method, it was proposed by Liao, 1992,[17]. HAM is used heavily in research by many authors, see [30, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 31, 32, 42]. The HAM is independent of any small (large) physical parameters. More importantly, unlike all other analytic techniques, it provides us a convenient way to guarantee the convergence of series solutions of nonlinear problems by means of introducing an auxiliary parameter c_0, called the convergence-control parameter. As mentioned before solving nonlinear equations is more difficult than solving linear equations, especially using analytical methods. In general, (1) a reliable analytical approximation is consistently attainable, and (2) analytical calculations can be relied upon for accuracy across all physical parameters, are important requirements for good analytical methods of nonlinear equations.

Let's rigorously assess various analysis methods for nonlinear problems by benchmarking them against the two aforementioned requirements.

All Perturbation methods are based on small (or large) physical parameters, called Perturbation quantities, in the governing equations or initial (boundary) conditions. Typically, a perturbation approximation is articulated as a series of perturbation quantities. The nonlinear equations can then be transformed into an infinite series of linear sub-problems (occasionally nonlinear problems), with the specific form determined by the nature of the original governing equation.

Perturbation methods are characterized by their simplicity and ease of comprehension. Particularly when based on small physical parameters, perturbation approximations often carry clear and intuitive physical interpretations. Unfortunately, not every nonlinear problem lends itself to such straightforward perturbation quantities. Furthermore, even if a small physical parameter exists, the associated sub-problem may lack solutions or prove to be excessively intricate, resulting in only a limited number of solvable sub-problems. Consequently, obtaining perturbation approximations for a given nonlinear problem is not guaranteed. Moreover, it is widely acknowledged that most perturbation approximations apply only to small physical parameters. Therefore, the validity of a perturbation result across the entire spectrum of physical parameters cannot be assured. Hence, perturbation techniques do not meet both of the aforementioned requirements.

To address the limitations of perturbation techniques, alternative non-perturbation methods have been developed, including Lyapunov's artificial small parameter method [34], the $\delta-$expansion method [5], and the Adomian decomposition method [1, 2, 6, 3, 4, 39], among others. These methods, in essence, introduce an artificial parameter, and the approximation solutions are expanded into series involving this artificial parameter. While these non-perturbation methods represent significant progress compared to perturbation techniques, there are theoretical challenges. The placement of the artificial small parameter is crucial for obtaining accurate approximation solutions, but there is a lack of guiding theories on optimizing its positioning. For instance, the Adomian decomposition method often employs the linear operator $\frac{d^k}{dx^k}$, making it relatively straightforward to obtain solutions for corresponding sub-problems through repeated integration. However, the use of such a simple linear operator leads to power series approximations with a finite radius of convergence. Consequently, the Adomian decomposition method cannot guarantee the convergence of its approximation series. As a result, these methods fulfill only the first requirement but not the second one mentioned above.

As said, in 1992 Liao used the topological concept of homotopy to obtain numerical approximations of nonlinear differential equations, the first homotopy analysis method (HAM) was described in his Ph.D. thesis. For a non-linear differential equation

$$N[u(x)] = 0, \qquad x \in \Omega, \tag{2.1}$$

where N is a non-linear operator and $u(x)$ is an unknown function. Liao (1992) developed a family of one-parameter in the embedding parameter $\lambda \in [0, 1]$, called the zeroth-order deformation equation

$$(1 - \lambda)L\left(u(x, \lambda) - u_0(x)\right) + \lambda N\left(u(x, \lambda)\right) = 0, x \in \Omega, \lambda \in [0, 1]. \tag{2.2}$$

Here, L represents an auxiliary linear operator, and $u_0(x)$ serves as the initial approximation. For a more in-depth exploration of homotopy, refer to [11] and [41]. Theoretically, the concept of homotopy in topology allows for significantly greater flexibility in selecting both the auxiliary linear operator L and the initial approximation compared to traditional non-perturbation methods. At the boundary values of the parameter, $\lambda = 0$ and $\lambda = 1$, we have $u(x, 0) = u_0(x)$ and $u(x, 1) = u(x)$, respectively. As the embedding parameter λ ranges from 0 to 1, the solution $u(x, \lambda)$ of the nonlinear deformation equation (2.2) undergoes a continuous transformation, transitioning from the initial approximation $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear differential equation (2.1). This continuous transformation is referred to as deformation in topology, providing the rationale behind labeling equation (2.2) as the zeroth-order deformation equation. Since $u(x, \lambda)$ also depends on the embedding parameter $\lambda \in [0, 1]$, $u(x, \lambda)$ can be expanded into the Maclaurin series with respect to λ:

$$u(x, \lambda) = u_0(x) + \sum_{n=1}^{\infty} u_n(x)\lambda^n. \tag{2.3}$$

It is called the homotopy-Maclaurin series. Note that we have considerable freedom in choosing the linear operator L and the initial approximation $u_0(x)$. Assuming that the auxiliary linear operator L and the initial approximation $u_0(x)$ are chosen appropriately so that the above homotopy-Maclaurin series converges at $\lambda = 1$, we can obtain the so-called homotopy series solution

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} u_n(x). \tag{2.4}$$

This series solution satisfies the basic equation $N[u(x)] = 0$ as proved by Liao, [19], [22]. The sequence of functions $u_n(x)$ is governed by the so-called higher-order deformation equation

$$L[(u_n(x) - \chi_n u_{n-1}(x))] = -D_{n-1}\left[N\left(u(x, \lambda)\right)\right], \qquad x \in \Omega, \lambda \in [0, 1]. \tag{2.5}$$

where χ_k is 1 when $k \geq 2$ and 0 otherwise, D_k is the kth order homotopy derivative operator defined as:

$$D_k = \frac{1}{k!}\frac{\partial^k}{\partial \lambda^k} \quad \text{at } \lambda = 0.$$

The fact that the right-hand side of the above deformation equation is known facilitates straightforward solutions, especially when we judiciously choose the auxiliary linear operator L. It is crucial to highlight that Marinca and Herisanu (2008), [35], based on the uniqueness of the Taylor series of $N(u(x, \lambda))$, reached the result (2.5) by substituting the homotopy series (2.3) into the deformation equation (2.2) and equating the coefficients of λ on both sides. To ensure the convergence of the homotopy series solution, control convergence parameters are incorporated into the zeroth deformation equation (2.2), as discussed in references [18], [20], [22], [23], [26], [28], and [35]. The flexibility to select the operator L and the initial approximation allows for the attainment of convergent homotopy series solutions by appropriately adjusting the convergence control parameters. This convergence is achieved across the entire range of physical parameters, distinguishing the Homotopy Analysis Method (HAM) from perturbation techniques and traditional non-perturbation methods. Therefore, HAM fulfills the two specified requirements mentioned above.
Various techniques are employed to ascertain the control convergent parameters, with one notably effective approach being the least squares method. This method determines the control convergent parameters by minimizing the expression:

$$E_m = \int_{\Omega} \left\{ N\left[\sum_{n=0}^{m} u_n(x) \right] \right\}^2 dx.$$

Despite its time-consuming nature, the presence of efficient computational packages like MATHEMATICA or MAPLE significantly mitigates the computational time involved in applying this method.

In the forthcoming sections, we will introduce fundamental concepts of various approaches to optimal Homotopy Analysis Method (HAM), offering comparisons between different optimal HAM variants and delivering a systematic overview of HAM.

2.2 Basic Ideas

In this section, we explain the basic idea of the optimal HAM through the so-called Blasius, non-linear differential equation in fluid mechanics, [29]. The governing equation reads

$$f'''(y) + \frac{1}{2}f(y)f''(y) = 0, f(0) = f'(0) = 0, f'(\infty) = 1. \tag{2.6}$$

Let $\mu > 0$ denote a spatial scale parameter. Using the change of variables

$$f(y) = \frac{1}{\mu}u(x), \qquad x = \mu y, \tag{2.7}$$

and the chain rule of differentiation, Equation (2.6) transformed into

$$u'''(x) + \frac{1}{2\mu^2}u(x)u''(x) = 0, u(0) = u'(0) = 0, u'(\infty) = 1. \tag{2.8}$$

Here the prime represents the derivative with respect to x.
Mathematically, due to the boundary condition $u'(\infty) = 1$, we have the asymptotic property $u \sim x$ such that $x \to \infty$. Physically, it is a common knowledge that velocity of boundary-layer flows mostly tends to mainstream flow exponentially. Thus $u(x)$ could be expressed as follows:

$$\begin{aligned} u(x) &= A_{0,0} + x + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{m,n}x^n e^{-mx}, & (2.9)\\ &= A_{0,0} + x + A_{1,0}e^{-x} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}x^n e^{-mx}. & (2.10) \end{aligned}$$

Here $A_{m,n}$ is a constant to be determined. We start by selecting our initial approximation in the following manner:

$$u_0(x) = \bar{A}_{0,0} + x + \bar{A}_{1,0}e^{-x}, \tag{2.11}$$

where $\bar{A}_{0,0}$ and $\bar{A}_{1,0}$ represent unknown constants. By ensuring that $u_0(x)$ fulfills the specified boundary conditions, we determine $\bar{A}_{0,0} = -1$ and $\bar{A}_{1,0} = 1$. Consequently, our initial approximation becomes

$$u_0(x) = x - 1 + e^{-x}. \tag{2.12}$$

Once we establish the initial guess, we select the linear operator L so that the general solution of $L(u) = 0$ encompasses the initial guess, Equation (2.12). Therefore, the linear operator that meets this criterion is defined as follows:

$$L(u) = u''' + u''. \tag{2.13}$$

Based on the governing equation (2.8), we define the nonlinear operator N as:

$$N(u) = u'''(x) + \frac{1}{2\mu^2}u(x)u''(x). \tag{2.14}$$

Let $\alpha(\lambda)$ and $\beta(\lambda)$ represent two deformation functions such that

$$\alpha(0) = \beta(0) = 0, \alpha(1) = \beta(1) = 1,$$

whose Maclaurin series

$$\alpha(\lambda) \sim \sum_{k=1}^{\infty} \alpha_k \lambda^k, \beta(\lambda) \sim \sum_{k=1}^{\infty} \beta_k \lambda^k,$$

converge at $\lambda = 1$, say,

$$\sum_{k=1}^{\infty} \alpha_k = 1, \sum_{k=1}^{\infty} \beta_k = 1.$$

Let $\lambda \in [0, 1]$ denote the embedding parameter, $c_0 \neq 0$ the convergence control parameter, and $u(x, \lambda)$ a continuous mapping, respectively. We formulate the so-called Zero-order deformation equation

$$[1 - \alpha(\lambda)]L[u(x, \lambda) - u_0(x)] = c_0\beta(\lambda)N[u(x, \lambda], \tag{2.15}$$

subject to the boundary conditions

$$u = 0, \ \frac{\partial u}{\partial x} = 0, \ \text{at}\, x = 0, \tag{2.16}$$

and

$$\frac{\partial u}{\partial x} = 1, \ \text{as}\, x \longrightarrow \infty. \tag{2.17}$$

Notice that the deformation equation (2.15) is a generalized form of Equation (2.2). Clearly, as λ varies from 0 to 1, $u(x, \lambda)$ deforms from the initial approximation $u_0(x)$ to the exact solution $u(x)$. The solution in the form of a homotopy series can be expressed as:

$$u(x) = u_0(x) + \sum_{k=1}^{\infty} u_k(x),$$

where, following Theorem (2.8), $u_m(x), m \geq 1$ is determined by the m-th order transformation equation:

$$L\left[u_m(x) - \sum_{k=1}^{m-1} \alpha_{m-k} u_k(x)\right] = c_0 \sum_{k=1}^{m} \beta_k \delta_{m-k}(x), \tag{2.18}$$

together with the boundary conditions:

$$u_m(0) = u_m'(0) = 0, \quad u_m'(\infty) = 0. \tag{2.19}$$

In Equation (2.18), the definition of δ_k is given by:

$$\delta_k(x) = D_k \left\{N[u(x, \lambda)]\right\}.$$

According to Theorem (2.3),

$$D_k \left\{N[u(x, \lambda)]\right\} = D_k[u'''(x, \lambda)] + \frac{1}{2\mu^2} D_k[u''(x, \lambda)u(x, \lambda)],$$

which, based on Theorems (2.4) and (2.7), leads to:

$$\delta_k(x) = D_k\left\{N[u(x,\lambda)]\right\} = u_k'''(x) + \frac{1}{2\mu^2}\sum_{j=0}^{k} u_j''(x)u_{k-j}(x).$$

Let $u_m^*(x)$ denote a special solution of (2.18), and L^{-1} be the inverse operator of L. Then, we have:

$$u_m^*(x) = \sum_{k=1}^{m-1} \alpha_{m-k}u_k(x) + c_0\sum_{k=1}^{m-1} \beta_k S_{m-k}(x),$$

where

$$S_k(x) = L^{-1}[\delta_k(x)].$$

The general solution is given by:

$$u_m(x) = u_m^*(x) + B_0 + B_1 x + B_2 e^{-x},$$

where the integral coefficients are determined by the boundary conditions (2.19), specifically:

$$B_1 = 0, \quad B_2 = \frac{d}{dx}\left(u_m^*(x)\right) \text{ at } x = 0, \quad B_0 = -u_m^*(0) - B_2.$$

2.3 Fundamental Theorems

Theorem 2.1. *For the product of two functions $u(x)$ and $v(x)$, and for any non-negative integer n, the Leibniz's rule for finding the nth derivative of their product is:*

$$\frac{d^n}{dx^n}\left(u(x)\cdot v(x)\right) = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}(x)\cdot v^{(n-k)}(x). \tag{2.20}$$

Where $u^{(k)}(x)$ denotes the kth derivative of $u(x)$ with respect to x, and $\binom{n}{k}$ represents the binomial coefficient "n choose k".

Proof by Mathematical Induction:

Base Case (n = 1): For $n = 1$, we have the standard product rule:

$$\frac{d}{dx}(u(x)\cdot v(x)) = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

which is the base case of Leibniz's rule.

Inductive Hypothesis: Assume that Leibniz's rule holds for some positive integer k. That is, assume that:

$$\frac{d^k}{dx^k}(u(x)\cdot v(x)) = \sum_{i=0}^{k}\binom{k}{i}u^{(i)}(x)\cdot v^{(k-i)}(x)$$

Inductive Step: We want to prove that Leibniz's rule holds for $n = k + 1$. Let's differentiate both sides of the expression obtained from the inductive hypothesis with respect to x:

$$\frac{d}{dx}\left(\frac{d^k}{dx^k}(u(x)\cdot v(x))\right) = \frac{d}{dx}\left(\sum_{i=0}^{k}\binom{k}{i}u^{(i)}(x)\cdot v^{(k-i)}(x)\right)$$

Using the product rule and the linearity of differentiation, we get:

$$\frac{d^{k+1}}{dx^{k+1}}(u(x)\cdot v(x)) = \sum_{i=0}^{k}\binom{k}{i}\left(u^{(i+1)}(x)\cdot v^{(k-i)}(x) + u^{(i)}(x)\cdot v^{(k-i+1)}(x)\right)$$

Now, let's rearrange the terms in the summation:

$$\frac{d^{k+1}}{dx^{k+1}}(u(x)\cdot v(x)) = \sum_{i=1}^{k+1}\left(\binom{k}{i-1}u^{(i)}(x)\cdot v^{(k-i+1)}(x) + \binom{k}{i}u^{(i)}(x)\cdot v^{(k-i+1)}(x)\right) + u(x)\cdot v^{(k+1)}(x)$$

Notice that the terms in the summation are in the form of binomial coefficients, allowing us to combine them into a single summation:

$$\frac{d^{k+1}}{dx^{k+1}}(u(x)\cdot v(x)) = \sum_{i=1}^{k+1}\left(\binom{k}{i-1} + \binom{k}{i}\right)u^{(i)}(x)\cdot v^{(k-i+1)}(x) + u(x)\cdot v^{(k+1)}(x)$$

By Pascal's Identity ($\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$), the terms in the summation collapse to form binomial coefficients:

$$\frac{d^{k+1}}{dx^{k+1}}(u(x)\cdot v(x)) = \sum_{i=1}^{k+1}\binom{k+1}{i}u^{(i)}(x)\cdot v^{(k+1-i)}(x) + u(x)\cdot v^{(k+1)}(x)$$

This is exactly the expression for $n = k + 1$ in Leibniz's rule. Hence, by mathematical induction, Leibniz's rule for higher derivatives of the product of functions is proven.

Notice that in Theorem (2.1), $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ and ! stands for factorial, for details on Pascal identity and other identities see [10].

Definition 2.1. *Let u be a function of the homotopy-parameter λ, then*

$$D_m(u) = \frac{1}{m!}\frac{d^m u}{d\lambda^m} \quad at\ \ \lambda = 0 \tag{2.21}$$

is called the $mth-$order homotopy-derivative of u, where $m \geq 0$ is an integer, and D_m is *called the operator of the $m-$order homotopy-derivative.*

Theorem 2.2. *For two arbitrary homotopy-Maclaurin series*

$$u = \sum_{k=0}^{\infty} u_k \lambda^k, \qquad v = \sum_{k=0}^{\infty} v_k \lambda^k$$

where u and v are analytic in $\lambda \in [0, a)$, it holds

$$(a)\ \ D_m(u) = u_m \tag{2.22}$$

$$(b)\ \ D_m(\lambda^k u) = D_{m-k}(u) = \begin{cases} u_{m-k}, & 0 \le k \le m, \\ 0, & \text{otherwise}, \end{cases} \tag{2.23}$$

$$(c)\ \ D_m(uv) = \sum_{k=0}^{m} D_k(u) D_{m-k}(v) = \sum_{k=0}^{m} u_k v_{m-k} = \sum_{k=0}^{m} D_{m-k}(u) D_k(v) = \sum_{k=0}^{m} u_{m-k} v_k \tag{2.24}$$

$$(d)\ \ D_m(u^{n+1}) = \sum_{k=0}^{m} D_k(u) D_{m-k}(u^n) = \sum_{k=0}^{m} D_{m-k}(u) D_k(u^n). \tag{2.25}$$

where $m \ge 0, n \ge 1$, and $0 \le k \le m$ are integers.

Proof:

(a) As per Taylor's theorem, the unique coefficient, denoted as u_m within the Maclaurin series representation of u with respect to the homotopy-parameter λ, is determined by

$$u_m = \frac{1}{m!} \frac{\partial^m u}{\partial \lambda^m} \qquad \text{at } \lambda = 0,$$

as described by Equation (2.21) for $D_m(u)$ leading to Equation (2.22).

(b) Since we have

$$\lambda^k u = \lambda^k \sum_{j=0}^{\infty} u_j \lambda^j = \sum_{j=0}^{\infty} u_j \lambda^{j+k} = \sum_{m=k}^{\infty} u_{m-k} \lambda^m,$$

then employing Equation (2.21) gives

$$D_m(\lambda^k u) = u_{m-k} = D_{m-k}(u), \text{ for } 0 \le k \le m,$$

and

$$D_m(\lambda^k u) = 0, \text{ for } k > m,$$

which proves part b.

(c) According to Leibniz's rule for derivatives of product functions, Theorem (2.1), it holds

$$\frac{\partial^m (uv)}{\partial \lambda^m} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{\partial^k u}{\partial \lambda^k} \frac{\partial^{m-k} v}{\partial \lambda^{m-k}} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{\partial^k v}{\partial \lambda^k} \frac{\partial^{m-k} u}{\partial \lambda^{m-k}}$$

which gives according to (2.21) and (2.22) that

$$D_m(uv) = \frac{1}{m!}\frac{\partial^m(uv)}{\partial\lambda^m}\bigg|_{\lambda=0} = \sum_{k=0}^{m}\left(\frac{1}{k!}\frac{\partial^k u}{\partial\lambda^k}\bigg|_{\lambda=0}\right)\left(\frac{1}{(m-k)!}\frac{\partial^{m-k} v}{\partial\lambda^{m-k}}\bigg|_{\lambda=0}\right)$$

$$= \sum_{k=0}^{m} D_k(u)D_{m-k}(v) = \sum_{k=0}^{m} u_{m-k}v_k$$

Similarly, it holds

$$D_m(u\psi) = \sum_{k=0}^{m} D_k(\psi)D_{m-k}(u) = \sum_{k=0}^{m} u_{m-k}v_k$$

and this proves part (2.24).

(d) Write $v = u^n$. According to (2.24), it holds

$D_m(u^{n+1}) = D_m(uv) = \sum_{k=0}^{m} D_k(u)D_{m-k}(v) = \sum_{k=0}^{m} D_k(u)D_{m-k}(u^n)$.

Similarly, it holds

$$D_m(u^{n+1}) = \sum_{k=0}^{m} D_k(u^n)D_{m-k}(u).$$

This proves part (2.25).

Theorem 2.3. *If* $u = \sum_{k=0}^{\infty} u_k\lambda^k$ *And* $v = \sum_{k=0}^{\infty} v_k\lambda^k$ *are two homotopy Maclaurin series, where u and v are analytic in* $\lambda \in [0, a)$*,* f *and g are independent of the homotopy parameter* $\lambda \in [0, 1]$*, then it holds*

$$D_m(fu + gv) = fD_m(u) + gD_m(v) = fu_m + gv_m \tag{2.26}$$

Proof:
Because D_m defined by (2.21) is a linear operator, and besides f and g are independent of λ, it has the value

$$D_m(fu + hv) = D_m(fu) + D_m(gv) = fD_m(u) + gD_m(v).$$

According to (2.22), we have $D_m(u) = u_m$ and $D_m(v) = v_m$.

Theorem 2.4. *Let L denote a linear operator independent of the homotopy parameter $\lambda \in [0, 1]$. For two homotopy Maclaurin series*

$$u = \sum_{k=0}^{\infty} u_k \lambda^k, v = \sum_{k=0}^{+\infty} v_k \lambda^k$$

where u and v are analytic in $\lambda \in [0, a)$, it holds

$$D_m[L(u)] = L[D_m(u)] = L(u_m), \tag{2.27}$$

and

$$D_m[vL(u)] = \sum_{n=0}^{m} D_{m-n}(v)L[D_n(u)] = \sum_{n=0}^{m} v_{m-n}L(u_n). \tag{2.28}$$

where $m \geq 0$ is an integer.

Proof:
Since L is linear and independent of λ, using Theorem (2.3), we have

$$L(u) = L\left(\sum_{k=0}^{\infty} u_k \lambda^k\right) = \sum_{k=0}^{\infty} L(u_k)\lambda^k.$$

Using statement (2.22), we have $D_m[L(u)] = L(u_m)$. On the other hand, according to assertion (2.22), it is true that $L\ D_m[L(u)] = L(u_m)$. So

$$D_m[L(u)] = L[D_m(u)] = L(u_m).$$

Then, according to Theorem (2.2), we have

$$D_m[vL(u)] = \textstyle\sum_{n=0}^{m} D_{m-n}(v)D_n[L(u)]$$

$$= \sum_{n=0}^{m} D_{m-n}(v)L[D_m(u)] = \sum_{n=0}^{m} v_{m-n}L(u_n).$$

Theorem 2.5. *For two homotopy Maclaurin series*

$$u = \sum_{i=0}^{\infty} u_i \lambda^i, v = \sum_{j=0}^{\infty} v_j \lambda^j,$$

where u and v are analytic in $\lambda \in [0, a)$, if $u = v$ in $\lambda \in [0, a)$, then $u_m = v_m$ and $D_m(u) = D_m(v)$ for any integer $m \geq 0$ and a real number $a > 0$.

Proof:

Since $u = v$, then

$$\sum_{k=0}^{\infty}(u_k - v_k)\lambda^k = 0.$$

The above expression is true at every point $\lambda \in [0, a)$, if and only if

$$u_m = w_m, m \geq 0,$$

using (2.22)

$$D_m(u) = D_m(v)$$

Theorem 2.6. *Let $f(u), g(v)$ denote two smooth functions. For two homotopy Maclaurin series*

$$u = \sum_{i=0}^{\infty} u_i\lambda^i, v = \sum_{j=0}^{\infty} v_j\lambda^j,$$

if $f(u) = g(v)$ in a domain $\lambda \in [0, a)$, then

$$D_m(f(u)) = D_m(g(v))$$

for any integer $m \geq 0$ and a real number $a > 0$.

Proof:
Write

$$U = f(u), V = g(v).$$

Then, using Theorem (2.5), we have

$$D_m(U) = D_m(V),$$

which gives

$$D_m[f(u)] = D_m[g(\psi)].$$

Theorem 2.7. *For an arbitrary homotopy Maclaurin series $u = \sum_{k=0}^{\infty} u_k \lambda^k$, it holds*

$$(a)\ \ D_m(u^2) = \sum_{n=0}^{m} u_{m-n}u_n, \tag{2.29}$$

$$(b)\ \ D_m(u^3) = \sum_{n=0}^{m} u_{m-n}u_n \sum_{k=0}^{n} u_{n-k}u_k, \tag{2.30}$$

$$(c)\ \ D_m(u^4) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k} \sum_{j=0}^{k} u_{k-j}u_j, \tag{2.31}$$

$$(d)\ \ D_m(u^5) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k} \sum_{j=0}^{k} u_{n-k} \sum_{j=0}^{k} u_{j-i}u_i, \tag{2.32}$$

$$(e)\ \ D_m(u^\sigma) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \dots \sum_{r_{\sigma-1}=0}^{r_{\sigma-2}} u_{r_{\sigma-2}-r_{\sigma-1}} u_{r_{\sigma-1}}. \tag{2.33}$$

where $m \geq 0$ and $\sigma \geq 2$ are positive integer.

Proof:

(a) According to (2.22) and (2.24), it is satisfied

$$D_m(u^2) = \sum_{n=0}^{m} D_{m-n}(u)D_n(u) = \sum_{n=0}^{m} u_{m-n}u_n.$$

(b) According to (2.29), we have

$$D_n(u^2) = \sum_{k=0}^{n} u_{n-k}u_k.$$

Then, according to (2.23), it satisfies

$$D_m(u^3) = \sum_{n=0}^{m} D_{m-n}(u)D_n(u^2) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k}u_k$$

(c) According to (2.30), we have

$$D_n(u^3) = \sum_{n=0}^{n} u_{n-k} \sum_{j=0}^{k} u_{k-j}u_j$$

Then, according to (2.23), it holds

$$D_m(u^4) = \sum_{n=0}^{m} D_{m-n}(u)D_n(u^3) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k} \sum_{j=0}^{k} u_{k-j}u_j$$

(d) According to (2.31), we have

$$D_n(u^4) = \sum_{n=0}^{n} u_{n-k} \sum_{j=0}^{k} u_{k-j} \sum_{i=0}^{j} u_{j-i} u_i$$

Then, according to (2.23), this has the value

$$D_m(u^5) = \sum_{n=0}^{m} D_{m-n}(u) D_n(u^4) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k} \sum_{j=0}^{k} u_{k-j} \sum_{i=0}^{j} u_{j-i} u_i.$$

(e) The proof of this statement is by mathematical induction.
(i) According to (2.29), obviously (2.33) is true when $\sigma = 2$.
(ii) Suppose that statement (2.33) is true for $\sigma = \kappa$.That is

$$D_m(u^\kappa) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{\kappa-1}=0}^{r_{\kappa-2}} u_{r_{\kappa-2}-r_{\kappa-1}} u_{r_{\kappa-1}},$$

where $m \geq 0$ and $\kappa \geq 2$ are integers. By r_j replacing r'_{j+1} and m with r'_1,the expression above reads as

$$D_{r'_1}(u^\kappa) = \textstyle\sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \ldots \sum_{r'_k=0}^{r'_{\kappa-1}} u_{r'_{\kappa-1}-r'_\kappa} u_{r'_\kappa},$$

Use the expression above and by means of (2.25) and (2.22), it holds

$$D_m(u^{\kappa+1}) = \textstyle\sum_{r'_1=0}^{m} D_{m-r'_1}(u) D_{r'_1}(u^\kappa)$$

$$= \sum_{r'_1=0}^{m} u_{m-r'_1} \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_\kappa=0}^{r'_{\kappa-1}} u_{r'_{\kappa-1}-r'_\kappa} u_{r'_\kappa},$$

Therefore, (2.33) holds for $\sigma = \kappa + 1$.
(iii) According to (i) and (ii), proposition (2.33) is true for all positive integers $\sigma \geq 2$.

Lemma 2.1. *Let*

$$u = \sum_{m=0}^{\infty} u_m \lambda^m$$

denote a homotopy Maclaurin series, where $\lambda \in [0, 1]$ *is the homotopy parameter, L an auxiliary linear operator which has the property* $L(0) = 0$ *and is independent of* λ*. If* $\alpha(\lambda)$ *is a deformation-function, i.e.* $\alpha(0) = 0$,$\alpha(1) = 1$ *and its Maclaurin series*

$$\alpha(\lambda) = \sum_{k=1}^{\infty} \alpha_k \lambda^k$$

exists and absolutely converges at $\alpha = 1$*, where* α_k *is constant, then it holds*

$$D_m \{[1 - \alpha(\lambda)]L(u - u_0)\} = L\left(u_m - \sum_{n=1}^{m-1} \alpha_n u_{m-n}\right).$$

Proof:
Write

$$U = u - u_0 = \sum_{k=1}^{\infty} \alpha_k \lambda^k, \qquad V = \alpha(\lambda) = \sum_{k=1}^{\infty} \alpha_k \lambda^k.$$

According to (2.22), we have

$$D_0(U) = D_0(V) = 0, \qquad D_n(U) = u_n,\ D_n(V) = \alpha_n,\ n \geq 1.$$

Then, using Theorems 2.3 – 2.5, we have

$$\begin{aligned}
&D_m\left[(1 - \alpha(\lambda))\, L(u - u_0)\right] = D_m\left[(1 - V)\, L(U)\right] = D_m\left[L(U) - VL(U)\right] \\
&= D_m[L(U)] - \sum_{n=0}^{m} D_n(V) D_{m-n}[L(U)] = L[D_m(U)] - \sum_{n=0}^{m} D_n(V) L[D_{m-n}(U)] \\
&= L(u_m) - \sum_{n=1}^{m-1} \alpha_n L(u_{m-n})
\end{aligned}$$

which gives, since α_k is constant, that

$$D_m\left[(1 - \alpha(\lambda))\, L(u - u_0)\right] = L\left(u_m - \sum_{n=1}^{m-1} \alpha_n u_{m-n}\right).$$

Theorem 2.8. *Let* L *denote an auxiliary linear operator which has the property* $L(0) = 0$ *and is independent of the homotopy parameter* $\lambda \in [0, 1]$*,* N *a nonlinear operator,* $u_0(x)$ *an initial approximation of the original equation* $N(u) = 0$*, respectively, where* x *denotes a vector of the spatial independent variables. Let* $\alpha(\lambda)$ *denotes a deformation function, i.e.* $\alpha(0) = 0$,$\alpha(1) = 1$ *and its Maclaurin series*

$$\alpha(\lambda) \sim \sum_{k=1}^{\infty} \alpha_k \lambda^k,$$

exists and absolutely converges at $\lambda = 1$*, where* α_k *is constant. If the series*

$$\sum_{k=0}^{\infty} \beta_k(x) \lambda^k$$

converges at $\lambda = 1$ *to a non-zero function, where* $\beta_0(x)$,$\beta_1(x)$,... *are called convergence-*

control functions, and besides if

$$u = \sum_{m=0}^{\infty} u_m(x)\lambda^m$$

is the homotopy Maclaurin series of the zeroth-order deformation equation

$$[1-\alpha(\lambda)]L(u-u_0) = \lambda\left[\sum_{k=0}^{\infty}\beta_k(x)\lambda^k\right]N(u), \tag{2.34}$$

then the corresponding m-th order deformation equation reads

$$L\left[u_m(x) - \sum_{n=1}^{m-1}\alpha_n u_{m-n}(x)\right] = \sum_{k=1}^{m}\beta_{k-1}(x)D_{m-k}[N(u)], \tag{2.35}$$

where D_{m-k} is defined by (2.1).

Proof:
Writing

$$v = \sum_{k=0}^{\infty}\beta_k(x)\lambda^{k+1},$$

According to (2.22) we have

$$D_0(v) = 0, D_k(v) = \beta_{k-1}(x)$$

with $k \geq 1$. According to Theorem (2.6), we have

$$D_m[1-\alpha(\lambda)]L(u-u_0) = D_m[vN(u)].$$

According to Lemma (2.1), we have

$$D_m[1-\alpha(\lambda)]L(u-u_0) = L\left(u_m - \sum_{n=1}^{m-1}\alpha_n u_{m-n}\right).$$

Then apply (2.22) and (2.23) of Theorem (2.2), we have

$$\begin{aligned} D_m[vN(u)] &= \sum_{k=0}^{m} D_k(v)D_{m-k}[N(u)] \\ &= \sum_{k=1}^{m} D_k(v)D_{m-k}[N(u)] \\ &= \sum_{k=1}^{m}\beta_{k-1}(x)D_{m-k}[N(u)] \end{aligned}$$

Therefore, the corresponding higher-order strain equation is written as
$L\left[u_m(x) - \sum_{n=1}^{m-1} \alpha_n u_{m-n}(x)\right] = \sum_{k=0}^{m} \beta_{k-1}(x) D_{m-k}[N(u)]$.
For more details see [29].

2.4 Different Types of Optimal Methods

At the mth-order of approximation, we characterize the squared residual of the governing equation (2.8) as follows:

$$E_m = \int_0^\infty \left\{ N\left[\sum_{n=0}^{m} u_n(x)\right]\right\}^2 dx. \tag{2.36}$$

Achieving an optimal homotopy approximation involves minimizing this squared residual. Various optimal methods exist, distinguished by the number of convergence-control parameters they employ, for details on this section and the following sections of chapter one see, [29]. To reduce the CPU time espesially when m is large the discrete squared residual

$$E_m \approx \frac{1}{(n_0+1)} \sum_{j=0}^{n_0} \left\{ N\left[\sum_{i=0}^{m} u_i(x_j)\right]\right\}^2, \tag{2.37}$$

is used. For the Blasius boundary layer flow (2.8), the parameters are taken to be $x_j = j\Delta x$, $\Delta x = 0.5$ and $n_0 = 20$.

2.5 Basic Optimal HAM

If only the fundamental convergence-control parameter c_0 is unknown, we employ the basic optimal Homotopy Analysis Method (HAM). Here, the deformation functions $\alpha(\lambda)$ and $\beta(\lambda)$, the initial approximation $u_0(x)$, and the auxiliary linear operator L contain no unknown parameters. The optimal homotopy-approximation is achieved by minimizing the squared residual E_m with respect to c_0:

$$\frac{dEm(c_0)}{dc_0} = 0. \tag{2.38}$$

To simplify, we adopt the simplest deformation functions, $\alpha(\lambda) = \lambda$ and $\beta(\lambda) = \lambda$. The curves of the discrete squared residual E_m versus c_0 for different orders of approximation $m = 6$, 8 and 10 are illustrated in Fig. 2.1 Notably, the discrete squared residual decreases within the range $-1.8 \le c_0 \le -0.3$ as the order of approximation increases. This suggests convergence of the homotopy series for any value of c_0 in the interval $[-1.8, -0.3]$. Importantly, the figure suggests that the optimal value of c_0 is approximately $-\frac{3}{2}$. Using Mathematica to minimize the discrete squared residual E_m results in c_0 values that progressively approach $-\frac{3}{2}$, as shown in Table 2.1. The final column in this table demonstrates that the fundamental optimal HAM yields a series of $f''(0)$ that rapidly converges to a value

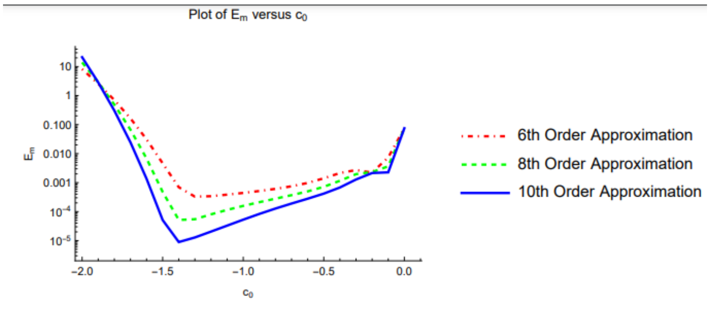

Figure 2.1: E_m versus c_0 for $m = 6,\ 8,\ 10$

closely matching Howarth's numerical findings, see [19] and the references therein.

Table 2.1: Minimum of the discrete squared residual E_m and the corresponding optimal value of c_0 given by the basic optimal HAM

m, order of approx	Optimal value of c_0	Minimum of E_m	$\lvert f''(0) - 0.332057\rvert$
2	−0.3897	3.46×10^{-3}	1.206
4	−1.0800	1.21×10^{-3}	0.0976
6	−1.2733	3.23×10^{-4}	0.0116
8	−1.3662	4.53×10^{-5}	0.00185
10	−1.4002	8.91×10^{-6}	9.71×10^{-4}
12	−1.4314	3.16×10^{-6}	2.94×10^{-4}
14	−1.4760	4.76×10^{-7}	1.89×10^{-4}
16	−1.4823	6.13×10^{-8}	8.84×10^{-5}
18	−1.4913	8.37×10^{-9}	1.02×10^{-5}
20	−1.4979	1.87×10^{-9}	8.08×10^{-6}
22	−1.5180	4.90×10^{-10}	1.06×10^{-5}

2.6 Three-Parameter Optimal HAM

There are several ways to enter additional transition control parameters in the framework of HAM. For example, Liao (2010b) proposed the following one-parameter deformation function:

$$\alpha(\lambda) = \sum_{m=1}^{+\infty} \alpha_m(c_2)\lambda^m, \beta(\lambda) = \sum_{m=1}^{+\infty} \beta_m(c_1)\lambda^m, \tag{2.39}$$

where

$$\alpha_1 = 1 - c_2, \beta_m = 1 - c_1, \alpha_m = (1 - c_2)c_2^{m-1}, \beta_m = (1 - c_1)c_1^{m-1}, m \geqslant 1 \tag{2.40}$$

and $|c_1| < 1$ and $|c_2| < 1$ are the so-called convergence control parameter. The different values of c_1 define different transformation functions $\beta(\lambda)$. In this case, the squared residual E_m contains at most three unknown convergence control parameters c_0, c_1, and c_2 in any order of approximation. In theory, the faster E_m decreases to 0, the faster the solution of the homotopy series converges. Thus, for an approximate order m, the optimal combined control parameters are determined by the smallest value of E_m corresponding to the set of three nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_0} = 0, \frac{\partial E_m}{\partial c_1} = 0, \frac{\partial E_m}{\partial c_2} = 0,$$

This provides an optimal HAM with three parameters. In the special case of $c_1 = c_2$, we have only two control convergence parameters $c_0 and c_1$ whose optimal values are given by

$$\frac{\partial E_m}{\partial c_0} = 0, \frac{\partial E_m}{\partial c_1} = 0,$$

It is like a two-parameter optimal HAM. When $c_1 = c_2 = 0$, we have $\alpha(\lambda) = \lambda$ and $\beta(\lambda) = \lambda$, so the basic transfer control parameter c_0 is unknown. This is the basic optimal HAM. For other choices of the convergence control parameters see [28]. The different choices indicate that an optimal HAM can speed up the convergence of homotopy series solutions.

2.7 Infinite-Parameter Optimal HAM

If we choose the deformation function $\alpha(\lambda) = \lambda$, and

$$\beta(\lambda) = \frac{1}{c_0} \sum_{n=1}^{\infty} c_n \lambda^n,$$

where

$$c_0 = \sum_{n=1}^{\infty} c_n \neq 0.$$

Then the zeroth-order deformation equation is:

$$(1 - \lambda) L[u(x; \lambda) - u_0(x)] = \left(\sum_{n=1}^{\infty} c_n \lambda^n \right) N[u(x; \lambda)]$$

The corresponding mth-order deformation equation is:

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \sum_{n=1}^{m} c_n \delta_{m-n}(x),$$

associated with the boundary conditions

$$u_m(0) = 0, u'_m(0) = 0, u'_m(\infty) = 0.$$

where $\chi_1 = 0$ and $\chi_m = 1$ for $m \geq 2$. Now the mth-order approximate solution

$$u(x) \sim u_0(x) + \sum_{n=1}^{m} u_n(x)$$

contains unknown m the control parameters $c_1,\ c_2,\ c_3, \ldots, c_m$ Theoretically, there is an infinite convergence control parameter $c_1,\ c_2,\ c_3, \ldots$ as $m \to \infty$. In this case, the optimal approximation of the mth- order homotopy is given by a set of m algebraic equations

$$\frac{\partial E_m}{\partial c_n} = 0, 1 \leq n \leq m.$$

This optimal HAM was proposed by Marinca and Herisanu (2008) as the "optimal homotopy asymptotic Method". Clearly, at the mth-order of approximation, the optimal HAM proposed by Marinca and Herisanu (2008) contains m convergence-control parameters. It is found that, as the order of approximation increases, the approximated squared residual of the optimal homotopy-approximation decreases faster than the basic optimal HAM that contains only one unknown convergence-control parameter c_0.

2.8 Finite-Parameter Optimal HAM

If the optimal HAM contains an infinite number of convergence control parameters, it takes a long CPU time. Thus the so-called "asymptotic homotopy optimal method" is modified to use only a small number (finite) of convergence control parameters. If we choose $\alpha(\lambda) = \lambda$ and $\beta(\lambda) = \frac{1}{c_0}\sum_{n=1}^{k} c_n\lambda^n$, where $k \geq 1$ is a positive integer and $c_0 = \sum_{n=1}^{k} c_n \neq 0$. Then the zeroth order deformation equation is:

$$(1-\lambda)L[u(x,\lambda) - u_0(x)] = \left(\sum_{n=1}^{k} c_n\lambda^n\right) N[u(x,\lambda)].$$

The corresponding mth order deformation equation is:

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \sum_{n=1}^{min(m,k)} c_n\delta_{m-n}(x),$$

associated with the boundary conditions are used

$$u_m(0) = 0, u'_m(0) = 0, u'_m(\infty) = 0.$$

Here, for $m \geq 2$, $\chi_1 = 0$ and $\chi_2 = 1$. The homotopy approximation of order m is

$$u(x) = u_0(x) + \sum_{n=1}^{m} u_n(x)$$

include at most k control parameters

$$c_1, c_2, c_3, \ldots, c_k$$

even when $m \longrightarrow \infty$. The optimal mth-order approximate solution is determined by $min(k, m)$ nonlinear equations

$$\frac{\partial E_m}{\partial c_i} = 0, \qquad 1 \leq i \leq min(m, k).$$

This method becomes exactly the so-called "optimal homotopy asymptotic method" suggested by Marinca and Herişanu (2008), if $k \longrightarrow \infty$. Besides, when $c_1 = c_0$ and $c_n = 0$ for $n > 1$, it becomes the basic optimal HAM.

Chapter 3

Homotopy Analysis Method (HAM) for Solving Systems of Partial Differential Equations

3.1 Introduction

In this chapter, we employ the Homotopy Analysis Method (HAM) to obtain analytic solutions for partial differential equations that arise in various fields such as mathematics, physics, and engineering. The subsequent section provides an in-depth analysis of the HAM method. Following that, in Sections 3 and 4, we showcase several examples to illustrate the efficacy of this approach, [15, 43].

3.2 Analysis of HAM

Consider the following operator equation

$$N[u(x,t)] = 0 \tag{3.1}$$

where N is a general differential operator, (x,t) denotes independent variables, and $u(x,t)$ is an unknown function, to be determined. For simplicity, we ignore all boundary or initial conditions, which can be treated similarly.

By means of generalizing the traditional Homotopy method, Liao constructs the so-called deformation equation, [19],

$$(1-\lambda)L\left(u(x,t,\lambda) - u_0(x,t)\right) = \lambda h N\left(u(x,t,\lambda)\right) \tag{3.2}$$

where $\lambda \in [0,1]$ is an embedding parameter, L is an auxiliary linear operator, h is a nonzero parameter, $u_0(x,t)$ is an initial guess of $u(x,t)$, and $u(x,t,\lambda)$ is an unknown function. One must have great freedom to choose objects L and h in HAM. Obviously, when $\lambda = 0$ and $\lambda = 1$, because $L[u(x,t,0) - u_0(x,t)] = 0$ then $u(x,t,0) = u_0(x,t)$, and since $hN(u(x,t,1) = 0$ then $u(x,t,1) = u(x,t)$. Hence the solution $u(x,t,\lambda)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. Expanding $u(x,t,\lambda)$ in Taylor series with respect to λ centered at

$\lambda = 0$, we have

$$u(x,t,\lambda) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)\lambda^n, \tag{3.3}$$

or we write

$$u(x,t,\lambda) - u_0(x,t) = \sum_{n=1}^{\infty} u_n(x,t)\lambda^n. \tag{3.4}$$

By substituting the above equations into Equation (3.2) we obtain

$$(1-\lambda)L\left[\sum_{n=1}^{\infty} u_n(x,t)\lambda^n\right] = \lambda h N\left(u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)\lambda^n\right) \tag{3.5}$$

Equating the corresponding coefficients of the same powers of λ on both sides of equality and using the corresponding initial conditions the coefficients $u_n(x,0) = 0$, $n = 1, 2, \ldots$, [35], we determine the coefficients $u_n(x,t)$ in the series representation given in Equation (3.3). The k^{th} order approximate solution is derived by truncating series (3.3) and retaining only the first $(k+1)^{th}$ terms.

In the next two sections, we apply the above procedure to solve several systems of Partial differential equations.

3.3 Illustrative Examples for Linear Equations

As stated above, in this section, we employ the HAM method to analyze instances of linear systems of partial differential equations in both one and two spatial dimensions, further guaranty the convergence in these examples we set $h = -1$.

Example 3.1. *Consider the following system of initial value problem:*

$u_t + v_x = u + v,$

$v_t + u_x = u + v,$

with initial conditions

$u(x,0) = sinhx,$

$v(x,0) = coshx.$

Following Equation (3.3), we assume

$$u(x,t,\lambda) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)\lambda^n, \tag{3.6}$$

and

$$v(x,t,\lambda) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t)\lambda^n. \tag{3.7}$$

Further, we define the operator N to be

$$N(u(x,t,\lambda)) = \frac{\partial}{\partial t}u(x,t,\lambda) + \frac{\partial}{\partial x}v(x,t,\lambda) - (u(x,t,\lambda) + v(x,t,\lambda)),$$

$$N(u(x,t,\lambda)) = \frac{\partial}{\partial t}u(x,t,\lambda) + \frac{\partial}{\partial x}v(x,t,\lambda) - (u(x,t,\lambda) + v(x,t,\lambda)),$$

and we take the operator L to be

$$L(u(x,t,\lambda) = \frac{\partial}{\partial t}(u(x,t,\lambda),$$

$$L(v(x,t,\lambda) = \frac{\partial}{\partial t}(v(x,t,\lambda).$$

The deformation equation, is given by

$$(1-\lambda)L\left[u(x,t,\lambda) - u_0(x,t)\right] = \lambda h N\left(u(x,t;\lambda)\right) \tag{3.8}$$

or with the help of Equation (3.3) we write

$$(1-\lambda)L\left[\sum_{n=1}^{\infty} u_n(x,t)\lambda^n\right] = \lambda h\left[\frac{\partial}{\partial t}u(x,t,\lambda) + \frac{\partial}{\partial x}v(x,t,\lambda) - (u(x,t,\lambda) + v(x,t,\lambda))\right] \tag{3.9}$$

Substitute from Equations (3.6) and (3.7) in Equation (3.9), we get

$$\begin{aligned}(1-\lambda)L\left[\sum_{n=1}^{\infty} u_n(x,t)\lambda^n\right] &= \lambda h\left[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial t}\sum_{n=1}^{\infty} u_n(x,t)\lambda^n\right. \\ &+ \frac{\partial}{\partial x}v_0(x,t) + \frac{\partial}{\partial x}\sum_{n=1}^{\infty} v_n(x,t)\lambda^n \\ &- (u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)\lambda^n \\ &+ \left.v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t)\lambda^n)\right].\end{aligned} \tag{3.10}$$

We rewrite the left and right hand sides as follows:
L.H.S. $= \sum_{n=1}^{\infty} \frac{\partial u_n(x,t)}{\partial t}\lambda^n - \sum_{n=1}^{\infty} \frac{\partial u_n(x,t)}{\partial t}\lambda^{n+1}$

$= \frac{u_1(x,t)}{\partial t}\lambda + \sum_{n=2}^{\infty} \frac{\partial u_n(x,t)}{\partial t}\lambda^n - \sum_{n=1}^{\infty} \frac{\partial u_n(x,t)}{\partial t}\lambda^{n+1}$

$= \frac{\partial u_1(x,t)}{\partial t}\lambda + \sum_{n=1}^{\infty} \frac{\partial u_{n+1}(x,t)}{\partial t}\lambda^{n+1} - \sum_{n=1}^{\infty} \frac{\partial u_n(x,t)}{\partial t}\lambda^{n+1}.$

R.H.S. $= h\left[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial x}v_0(x,t) - (u_0(x,t) + v_0(x,t))\right]\lambda$

$$+h\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial t}u_n(x,t) + \frac{\partial}{\partial x}v_n(x,t) - (u_n(x,t) + v_n(x,t))\right]\lambda^{n+1}.$$

Equating the corresponding coefficients of same powers of λ of L.H.S. and R.H.S. and using the trivial initial conditions $u_n(x,0) = 0$ for all $n = 1, 2, 3, \ldots$, we get

$\frac{\partial u_1(x,t)}{\partial t} = h[\frac{\partial u_0(x,t)}{\partial t} + \frac{\partial v_0(x,t)}{\partial x} - (u_0(x,t) + v_0(x,t)], u_1(x,0) = 0,$

$\frac{\partial u_{n+1}(x,t)}{\partial t} = (1+h)\frac{\partial u_n(x,t)}{\partial t} + h[\frac{\partial v_n(x,t)}{\partial x} - (u_n(x,t) + v_n(x,t)], u_{n+1}(x,0) = 0.$

These ordinary differential equations are called the first and the $(n+1)th$ deformation equation for the function $u(x,t)$. The zeroth order deformation equation reads

$$\frac{\partial}{\partial t}\,(u_0(x,t)) = 0,$$

its corresponding initial condition is $u(x,0) = u_0(x)$. In an analogous way, we get the zeroth, first, and $(n+1)th$ deformation equations together with the appropriate initial conditions corresponding to $v(x,t)$ as follows:

$\frac{\partial}{\partial t}\,(v_0(x,t)) = 0, v(x,0) = v_0(x),$

$\frac{\partial v_1(x,t)}{\partial t} = h[\frac{\partial v_0(x,t)}{\partial t} + \frac{\partial u_0(x,t)}{\partial x} - (u_0(x,t) + v_0(x,t)], v_1(x,0) = 0$

$\frac{\partial v_{n+1}(x,t)}{\partial t} = (1+h)\frac{\partial v_n(x,t)}{\partial t} + h[\frac{\partial u_n(x,t)}{\partial x} - (u_n(x,t) + v_n(x,t))], v_{n+1}(x,0) = 0,\ n = 1, 2, \ldots.$

We solve the previous systems of partial differential equations together with appropriate initial conditions, as follows:

$u_0(x,t) = u(x,0) = \sinh x,\ v_0(x,t) = v(x,0) = \cosh x.$

$\frac{\partial u_1(x,t)}{\partial t} = h\left[\frac{\partial u_0(x,t)}{\partial t} + \frac{\partial v_0(x,t)}{\partial x} - (u_0(x,t) + v_0(x,t))\right] = \cosh x.$

The solution reads $u_1(x,t) = t\cosh x$.

Similarly, we have

$\frac{\partial v_1(x,t)}{\partial t} = h\left[\frac{\partial v_0(x,t)}{\partial t} + \frac{\partial u_0(x,t)}{\partial x} - (u_0(x,t) + v_0(x,t))\right] = \sinh x.$ Thus $v_1(x,t) = t\sinh x$.

Continuing this process we have

$\frac{\partial u_2(x,t)}{\partial t} = (1+h)\frac{\partial u_1(x,t)}{\partial t} + h\left[\frac{\partial v_1(x,t)}{\partial x} - (u_1(x,t) + v_1(x,t))\right] = t\sinh x,\ u_2(x,t) = \frac{t^2}{2!}\sinh x,$

$\frac{\partial v_2(x,t)}{\partial t} = (1+h)\frac{\partial v_1(x,t)}{\partial t} + h\left[\frac{\partial u_1(x,t)}{\partial x} - (u_1(x,t) + v_1(x,t))\right] = t\cosh x,\ v_2(x,t) = \frac{t^2}{2!}\cosh x,$

$\frac{\partial u_3(x,t)}{\partial t} = (1+h)\frac{\partial u_2(x,t)}{\partial t} + h\left[\frac{\partial v_2(x,t)}{\partial x} - (u_2(x,t) + v_2(x,t))\right] = \frac{t^2}{2!}\cosh x,\ u_3(x,t) = \frac{t^3}{3!}\cosh x,$

$\frac{\partial v_3(x,t)}{\partial t} = (1+h)\frac{\partial v_2(x,t)}{\partial t} + h\left[\frac{\partial u_2(x,t)}{\partial x} - (u_2(x,t) + v_2(x,t))\right] = \frac{t^2}{2!}\sinh x,\ v_3(x,t) = \frac{t^3}{3!}\sinh x,$

and so on.
Finally, with $\lambda = 1$, the series solution reads
$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_0(x,t)$
$\quad = \sinh x + t\cosh x + \frac{t^2}{2!}\sinh x + \frac{t^3}{3!}\cosh x + \ldots$
$v(x,t) = v_0(x,t) + \sum_{n=0}^{\infty} v_0(x,t)$
$\quad = \cosh x + t\sinh x + \frac{t^2}{2!}\cosh x + \frac{t^3}{3!}\sinh x + \ldots$
With the aid of the definitions of $\sinh x$ and $\cosh x$ and the Maclaurin series of e^t and e^{-t}, the obtained series solutions converge to $u(x,t) = \sinh(x+t)$ and $v(x,t) = \cosh(x+t)$.

In Figures 3.1 and 3.3 we plot the exact solutions $u(x,t) = \sinh(x+t)$ and $v(x,t) = \cosh(x+t)$ respectively. The third order approximate solutions $u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)$ and $v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t)$ are plotted in Figures 3.2 and 3.4 respectively.

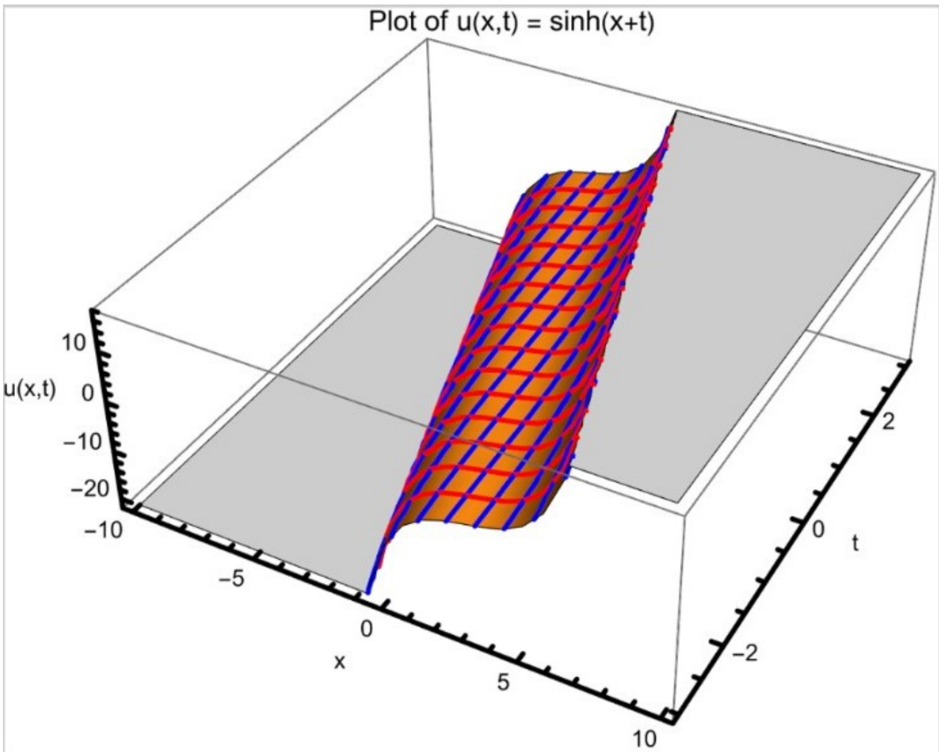

Figure 3.1: Exact solution $u(x,t) = \sinh(x+t)$

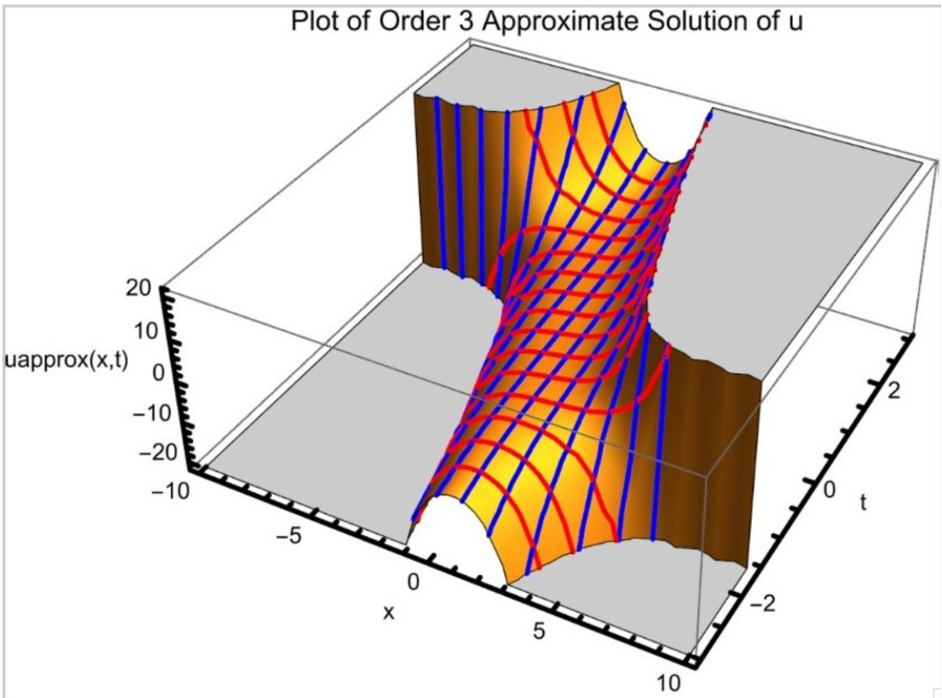

Figure 3.2: Approximate solution of $u(x,t)$ (HAM)

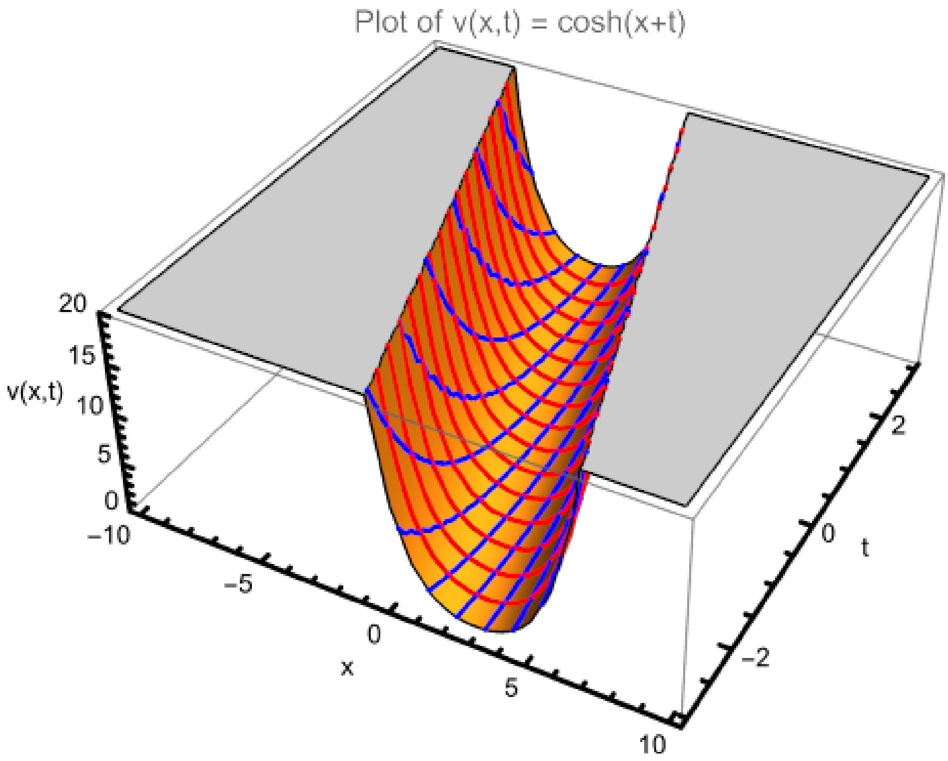

Figure 3.3: Exact solution $v(x,t) = \cosh(x+t)$

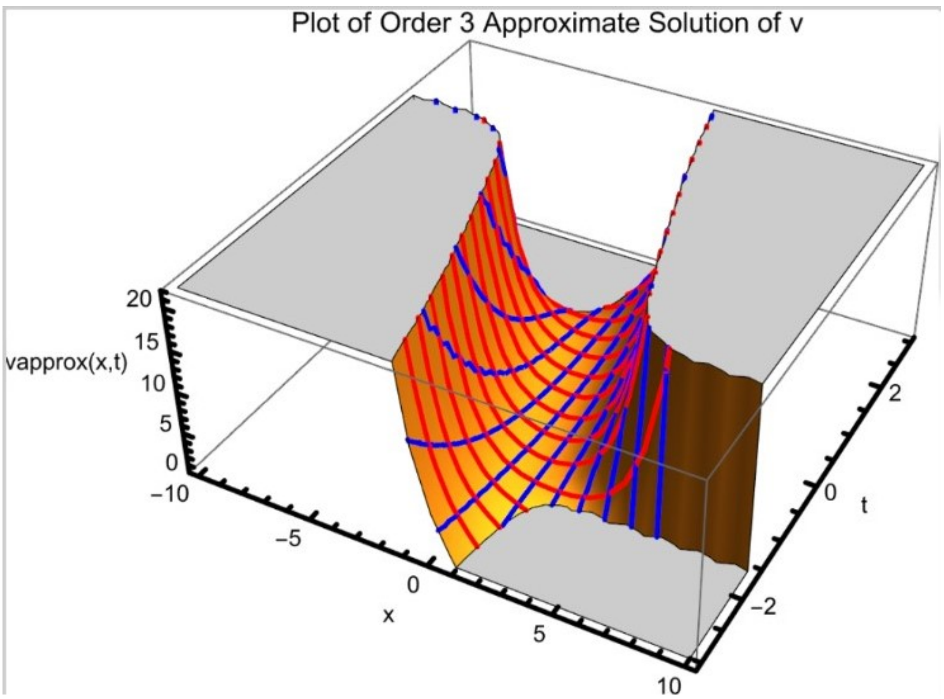

Figure 3.4: Approximate solution of $v(x,t)$ (HAM)

Example 3.2. *We consider the system of the two-dimensional spatial partial differential equations*
$u_t + v_x + w_y = w,$
$v_t + w_x + u_y = -u,$
$w_t + v_x - v_y = -v.$
together with initial conditions
$u(x, y, 0) = sin(x + y),$
$v(x, y, 0) = cos(x + y),$
$w(x, y, 0) = -sin(x + y).$

Again, following Equation (3.3) we assume that

$$u(x, y, t, \lambda) = u_0(x, y, t) + \sum_{n=1}^{\infty} u_n(x, y, t)\lambda^n, \tag{3.11}$$

$$v(x, y, t, \lambda) = v_0(x, y, t) + \sum_{n=1}^{\infty} v_n(x, y, t)\lambda^n, \tag{3.12}$$

and

$$w(x, y, t, \lambda) = w_0(x, y, t) + \sum_{n=1}^{\infty} w_n(x, y, t)\lambda^n. \tag{3.13}$$

The operators L and N are taken to be
$L(u(x, y, t, \lambda)) = \frac{\partial}{\partial t}(u(x, y, t, \lambda),$
$L(v(x, y, t, \lambda)) = \frac{\partial}{\partial t}(v(x, y, t, \lambda),$
$L(w(x, y, t, \lambda)) = \frac{\partial}{\partial t}(w(x, y, t, \lambda),$
and
$N(u(x, y, t, \lambda)) = \frac{\partial}{\partial t}u(x, y, t, \lambda) + \frac{\partial}{\partial x}v(x, y, t, \lambda) + \frac{\partial}{\partial y}w(x, y, t, \lambda) - w(x, y, t, \lambda),$
$N(v(x, y, t, \lambda)) = \frac{\partial}{\partial t}v(x, y, t, \lambda) + \frac{\partial}{\partial x}w(x, y, t, \lambda) + \frac{\partial}{\partial y}u(x, y, t, \lambda) + u(x, y, t, \lambda),$
$N(w(x, y, t, \lambda)) = \frac{\partial}{\partial t}w(x, y, t, \lambda) + \frac{\partial}{\partial x}v(x, y, t, \lambda) - \frac{\partial}{\partial y}v(x, y, t, \lambda) + v(x, y, t, \lambda).$
The deformation equation corresponding to $u(x, y, t)$ is given by

$$(1 - \lambda)L\left[u(x, y, t, \lambda) - u_0(x, y, t)\right] = \lambda h N(u(x, y, t, \lambda)). \tag{3.14}$$

or with the aid of Equation (3.11) we write

$$\begin{aligned}(1 - \lambda)L\left[\sum_{n=1}^{\infty} u_n(x, y, t)\lambda^n\right] \;=\; \lambda h \Bigg[&\frac{\partial}{\partial t}u(x, y, t, \lambda) + \frac{\partial}{\partial x}v(x, y, t, \lambda) \\ &+\frac{\partial}{\partial y}w(x, y, t, \lambda) - w(x, y, t, \lambda)\Bigg]\end{aligned} \tag{3.15}$$

Substituting from Equations (3.11-3.13) we get

$$
\begin{aligned}
(1-\lambda)L[\sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n] &= \lambda h[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial t}\sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n \\
&+ \frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial x}\sum_{n=1}^{\infty}v_n(x,y,t)\lambda^n \\
&+ \frac{\partial}{\partial y}w_0(x,y,t)-\frac{\partial}{\partial y}\sum_{n=1}^{\infty}w_n(x,y,t)\lambda^n) \\
&+ w_0(x,y,t)-\sum_{n=1}^{\infty}w_n(x,y,t)\lambda^n)]. \qquad (3.16)
\end{aligned}
$$

The left and right-hand sides are simplified to

L.H.S. $= \sum_{n=1}^{\infty}\frac{\partial u_n(x,y,t)}{\partial t}\lambda^n - \sum_{n=1}^{\infty}\frac{\partial u_n(x,y,t)}{\partial t}\lambda^{n+1}$

$= \frac{\partial u_1(x,y,t)}{\partial t}\lambda + \sum_{n=2}^{\infty}\frac{\partial u_n(x,y,t)}{\partial t}\lambda^n - \sum_{n=1}^{\infty}\frac{\partial u_n(x,y,t)}{\partial t}\lambda^{n+1}$

$= \frac{\partial u_1(x,y,t)}{\partial t}\lambda + \sum_{n=1}^{\infty}\frac{\partial u_{n+1}(x,y,t)}{\partial t}\lambda^{n+1} - \sum_{n=1}^{\infty}\frac{\partial u_n(x,y,t)}{\partial t}\lambda^{n+1}$,

R.H.S. $= h\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right]\lambda$
$+h\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial t}u_n(x,y,t)+\frac{\partial}{\partial x}v_n(x,y,t)+\frac{\partial}{\partial y}w_n(x,y,t)-w_n(x,y,t)\right]\lambda^{n+1}$.

By equating the corresponding coefficients of same powers of λ on both sides and using the trivial initial conditions $u_n(x,y,0)=0$, for all $n=1,2,3\ldots$, we obtain the zeroth, first and $(n+1)^{th}$ order following deformation equations associated with the appropriate initial conditions.

$\frac{\partial u_0(x,y,t)}{\partial t} = 0,\ u(x,y,0) = u_0(x,y)$,
$\frac{\partial u_1(x,y,t)}{\partial t} = h\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right], u_1(x,y,0)=0$,
$\frac{\partial u_{n+1}(x,y,t)}{\partial t} = (1+h)\frac{\partial}{\partial t}u_n(x,y,t)+h\left[\frac{\partial}{\partial x}v_n(x,y,t)+\frac{\partial}{\partial y}w_n(x,y,t)-w_n(x,y,t)\right]$,
$u_{n+1}(x,y,0)=0$.

Similarly, the zeroth, first, $(n+1)^th$ order deformation equations together with the appropriate initial conditions corresponding to $v(x,y,t)$ and $w(x,y,t)$ are
$\frac{\partial v_0(x,y,t)}{\partial t} = 0,\ v(x,y,0) = v_0(x,y)$,
$\frac{\partial v_1(x,y,t)}{\partial t} = h\left[\frac{\partial}{\partial t}v_0(x,y,t)+\frac{\partial}{\partial x}w_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)+u_0(x,y,t)\right], v_1(x,y,0)=0$,
$\frac{\partial v_{n+1}(x,y,t)}{\partial t} = (1+h)\frac{\partial}{\partial t}v_n(x,y,t)+h\left[\frac{\partial}{\partial x}w_n(x,y,t)+\frac{\partial}{\partial y}u_n(x,y,t)+v_n(x,y,t)\right]$,
$v_{n+1}(x,y,0)=0$.

$\frac{\partial w_0(x,y,t)}{\partial t} = 0,\ w(x,y,0) = w_0(x,y)$,
$\frac{\partial w_1(x,y,t)}{\partial t} = h\left[\frac{\partial}{\partial t}w_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)-\frac{\partial}{\partial y}v_0(x,y,t)+v_0(x,y,t)\right], w_1(x,y,0)=0$,
$\frac{\partial w_{n+1}(x,y,t)}{\partial t} = (1+h)\frac{\partial}{\partial t}w_n(x,y,t)+h\left[\frac{\partial}{\partial x}v_n(x,y,t)-\frac{\partial}{\partial y}v_n(x,y,t)+v_n(x,y,t)\right]$,
$w_{n+1}(x,y,0)=0$.

Analogous to the previous example we have
$u_0(x,y,t) = u(x,y,0) = sin(x+y)$,
$v_0(x,y,t) = v(x,y,0) = cos(x+y)$,
$w_0(x,y,t) = w(x,y,0) = -sin(x+y)$.

We solve the previous governing equations as follows (to get convergence we set h=-1):

$$\begin{aligned}\frac{\partial u_1(x,y,t)}{\partial t} &= h\left[\frac{\partial}{\partial t}u_0(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t) + \frac{\partial}{\partial y}w_0(x,y,t) - w_0(x,y,t)\right]\\ &= \cos(x+y).\end{aligned}$$

Hence $u_1(x,y,t) = t\cos(x+y)$.

$$\begin{aligned}\frac{\partial v_1(x,y,t)}{\partial t} &= h\left[\frac{\partial}{\partial t}v_0(x,y,t) + \frac{\partial}{\partial x}w_0(x,y,t) + \frac{\partial}{\partial y}u_0(x,y,t) + u_0(x,y,t)\right]\\ &= -\sin(x+y).\end{aligned}$$

Therefore $v_1(x,y,t) = -t\sin(x+y)$.
and

$$\begin{aligned}\frac{\partial w_1(x,y,t)}{\partial t} &= h\left[\frac{\partial}{\partial t}w_0(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t) - \frac{\partial}{\partial y}v_0(x,y,t) + v_0(x,y,t)\right]\\ &= -\cos(x+y),\end{aligned}$$

which leads to $w_1(x,y,t) = -t\cos(x+y)$.
Continuing in this process we get

$$\frac{\partial}{\partial t}u_2(x,y,t) = -t\sin(x+y),\ u_2(x,y,t) = -\frac{t^2}{2!}\sin(x+y)$$

$$\frac{\partial}{\partial t}v_2(x,y,t) = -t\cos(x+y),\ v_2(x,y,t) = -\frac{t^2}{2!}\cos(x+y)$$

$$\frac{\partial}{\partial t}w_2(x,y,t) = t\sin(x+y),\ w_2(x,y,t) = \frac{t^2}{2!}\sin(x+y)$$

$$\frac{\partial}{\partial t}u_3(x,y,t) = -\frac{t^2}{2!}\cos(x+y),\ u_3(x,y,t) = -\frac{t^3}{3!}\cos(x+y)$$

$$\frac{\partial}{\partial t}v_3(x,y,t) = \frac{t^2}{2!}\sin(x+y),\ v_3(x,y,t) = \frac{t^3}{3!}\sin(x+y)$$

$$\frac{\partial}{\partial t}w_3(x,y,t) = \frac{t^2}{2!}\cos(x+y),\ w_3(x,y,t) = \frac{t^3}{3!}\cos(x+y)$$

and so on. With $\lambda = 1$, the series solutions are

$$\begin{aligned}u(x,y,t) &= u_0(x,y,t) + \sum_{n=1}^{\infty} u_n(x,y,t)\\ &= \sin(x+y) + t\cos(x+y) - \frac{t^2}{2!}\sin(x+y) - \frac{t^3}{3!}\cos(x+y)\ldots\end{aligned}$$

$$\begin{aligned}v(x,y,t) &= v_0(x,y,t) + \sum_{n=1}^{\infty} v_n(x,y,t)\\ &= \cos(x+y) - t\sin(x+y) - \frac{t^2}{2!}\cos(x+y) + \frac{t^3}{3!}\sin(x+y)\ldots\end{aligned}$$

$w(x,y,t) = w_0(x,y,t) + \sum_{n=1}^{\infty} w_n(x,y,t)$

$$= -\sin(x+y) - t\cos(x+y) + \frac{t^2}{2!}\sin(x+y) + \frac{t^3}{3!}\cos(x+y)\ldots$$

Notice that the exact solution for this system of partial differential equations reads

$u(x,y,t) = \sin(x+y+t)$,

$v(x,y,t) = \cos(x+y+t)$,

$w(x,y,t) = -\sin(x+y+t)$.

Note also that using the complex definition of $\sin(x+y+t)$ and $\cos(x+y+t)$ we can show that the series solution converges to the exact solution. In Figures 3.5, 3.7, 3.9 we plot the exact solution at $t = 2$, while in Figures 3.6, 3.8, 3.10 third order approximate solutions

$$\begin{aligned} u(x,y,t) &= u_0(x,y,t) + \sum_{n=1}^{3} u_n(x,y,t), \\ v(x,y,t) &= v_0(x,y,t) + \sum_{n=1}^{3} v_n(x,y,t), \\ w(x,y,t) &= w_0(x,y,t) + \sum_{n=1}^{3} w_n(x,y,t). \end{aligned}$$

are plotted at the same value of t.

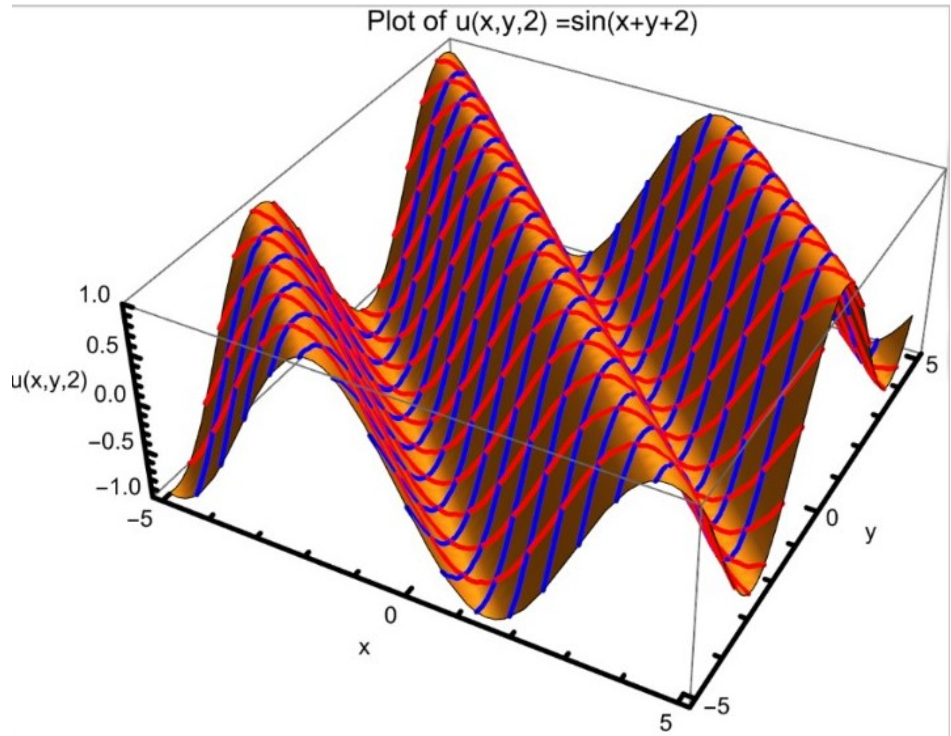

Figure 3.5: Exact solution $u(x, y, 2) = \sin(x + y + 2)$

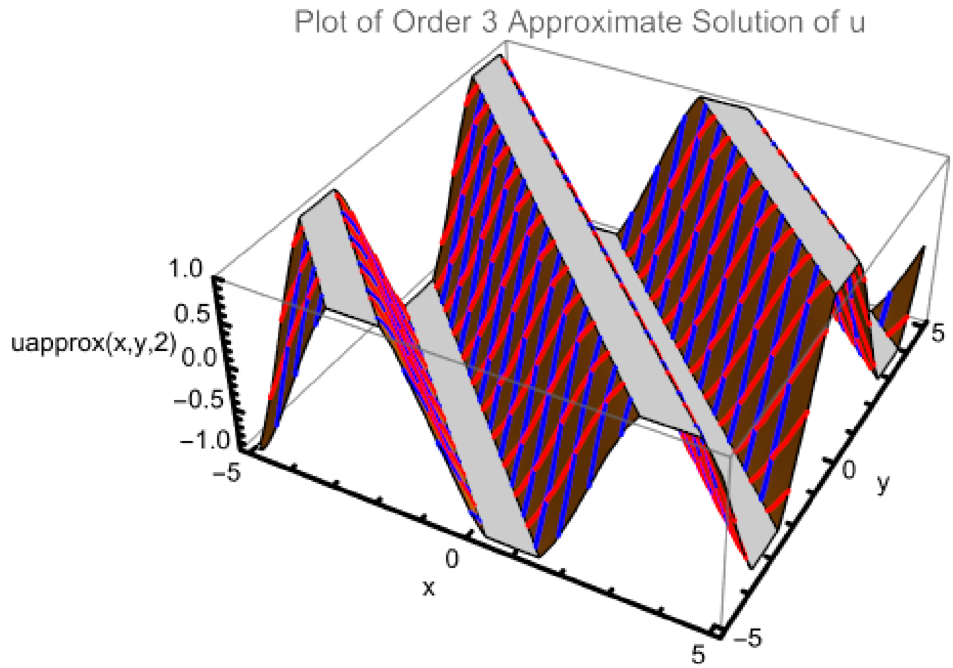

Figure 3.6: Approximate solution of $u(x, y, 2)$ (HAM)

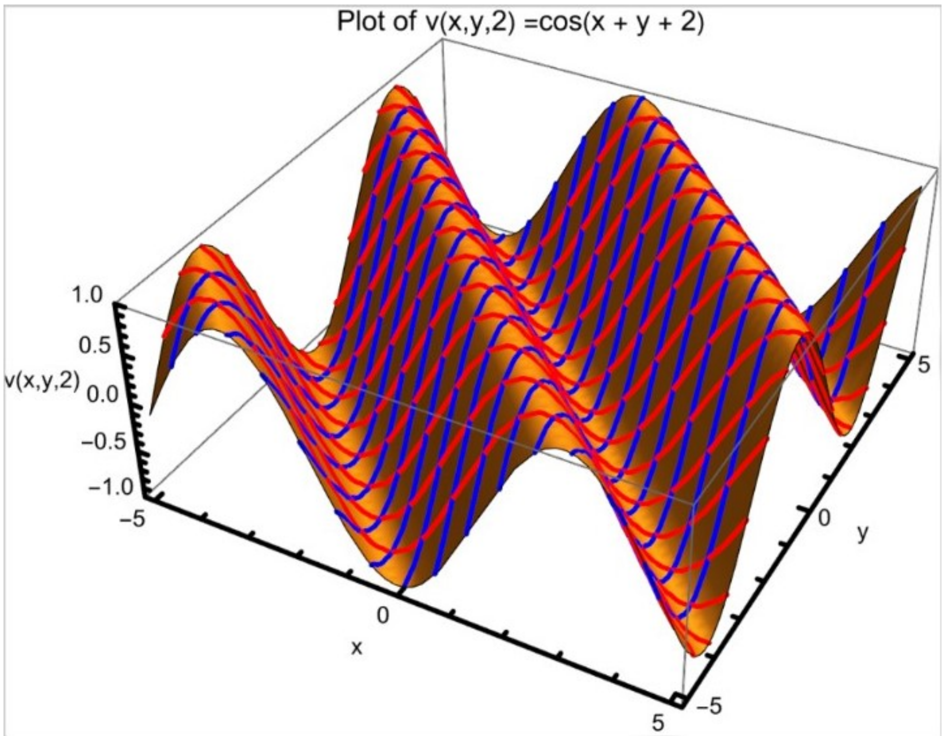

Figure 3.7: Exact solution $v(x, y, 2) = \cos(x + y + 2)$

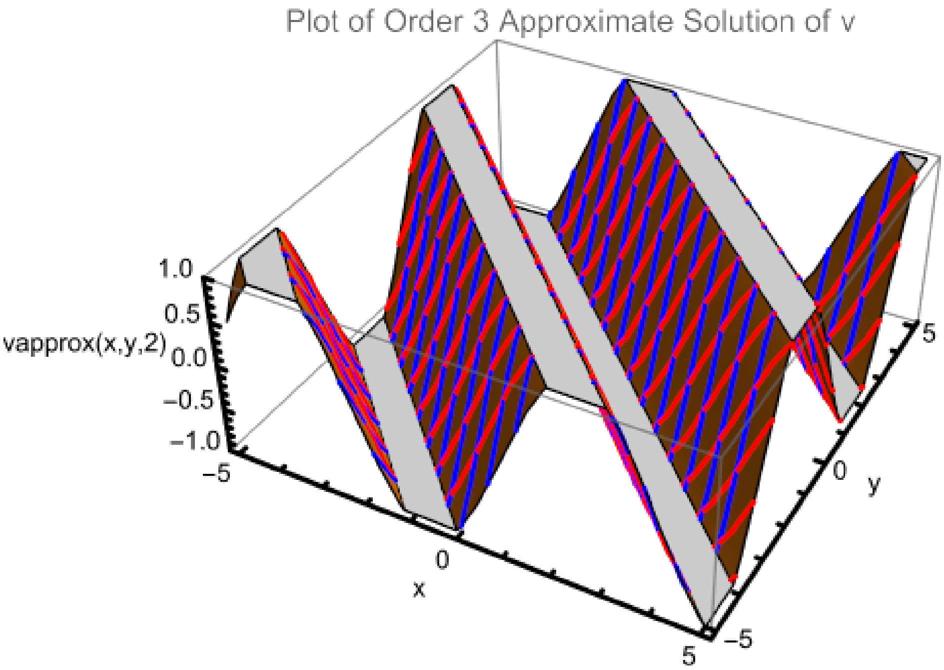

Figure 3.8: Approximate solution of $v(x, y, 2)$ (HAM)

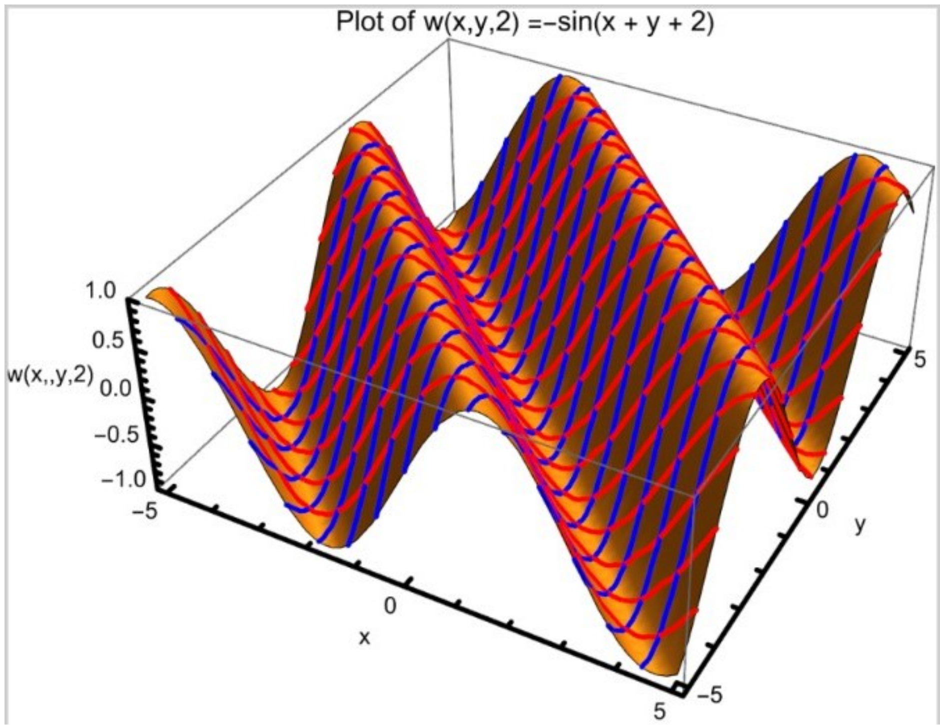

Figure 3.9: Exact solution $w(x, y, 2) = -\sin(x + y + 2)$

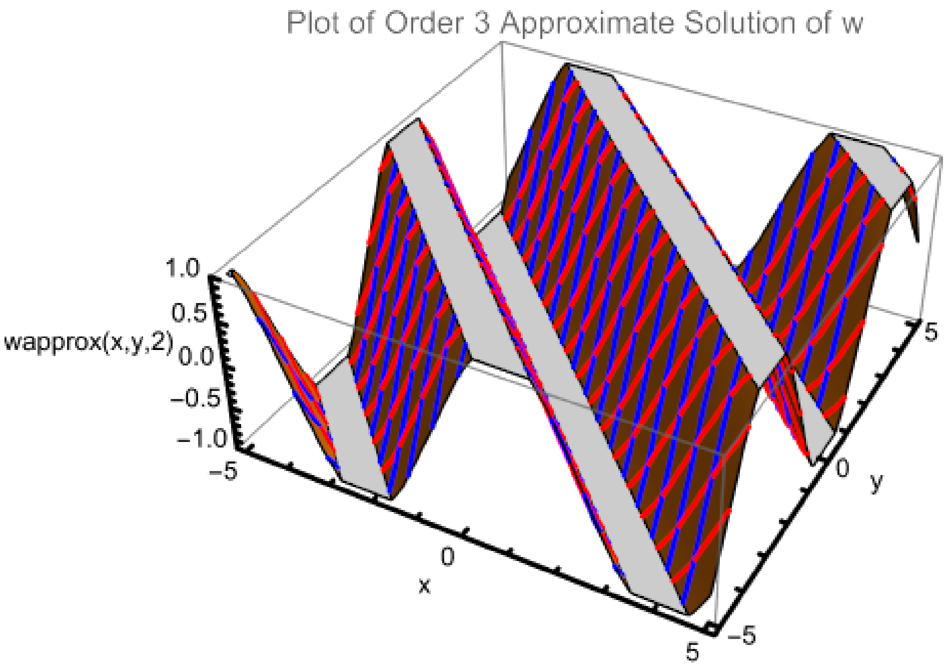

Figure 3.10: Approximate solution of $w(x, y, 2)$ (HAM)

3.4 Illustrative Example for Nonlinear Equations

In this section, we apply Homotopy Analysis Method (HAM) to get analytic solutions of nonlinear partial differential equations.

Example 3.3. *In this example we consider the following system of nonlinear PDEs:*

$$
\begin{aligned}
u_t + u_x v_x + u_y v_y &= -u, \\
v_t + v_x w_x - v_y w_y &= v, \\
w_t + w_x u_x + w_y u_y &= w,
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
u(x, y, 0) &= e^{x+y}, \\
v(x, y, 0) &= e^{x-y}, \\
w(x, y, 0) &= e^{-x+y}.
\end{aligned}
$$

According to HAM, we assume that the solution has the series representation

$$u(x, y, t, \lambda) = u_0(x, y, t) + \sum_{n=1}^{\infty} u_n(x, y, t)\lambda^n, \tag{3.17}$$

$$v(x, y, t, \lambda) = v_0(x, y, t) + \sum_{n=1}^{\infty} v_n(x, y, t)\lambda^n, \tag{3.18}$$

and

$$w(x, y, t, \lambda) = w_0(x, y, t) + \sum_{n=1}^{\infty} w_n(x, y, t)\lambda^n. \tag{3.19}$$

We define the operators L and N as

$$
\begin{aligned}
L(u(x, y, t, \lambda)) &= \frac{\partial}{\partial t}(u(x, y, t, \lambda), \\
L(v(x, y, t, \lambda)) &= \frac{\partial}{\partial t}(v(x, y, t, \lambda), \\
L(w(x, y, t, \lambda)) &= \frac{\partial}{\partial t}(w(x, y, t, \lambda),
\end{aligned}
$$

and

$$
\begin{aligned}
N(u(x,y,t,\lambda)) &= \frac{\partial}{\partial t}u(x,y,t,\lambda) + \frac{\partial}{\partial x}u(x,y,t,\lambda)\frac{\partial}{\partial x}v(x,y,t,\lambda) \\
&\quad + \frac{\partial}{\partial y}u(x,y,t,\lambda)\frac{\partial}{\partial y}v(x,y,t,\lambda) + u(x,y,t,\lambda), \\
N(v(x,y,t,\lambda)) &= \frac{\partial}{\partial t}v(x,y,t,\lambda) + \frac{\partial}{\partial x}v(x,y,t,\lambda)\frac{\partial}{\partial x}w(x,y,t,\lambda) \\
&\quad - \frac{\partial}{\partial y}v(x,y,t,\lambda)\frac{\partial}{\partial y}w(x,y,t,\lambda) - v(x,y,t,\lambda), \\
N(w(x,y,t,\lambda)) &= \frac{\partial}{\partial t}w(x,y,t,\lambda) + \frac{\partial}{\partial x}w(x,y,t,\lambda)\frac{\partial}{\partial x}u(x,y,t,\lambda) \\
&\quad + \frac{\partial}{\partial y}w(x,y,t,\lambda)\frac{\partial}{\partial y}u(x,y,t,\lambda) - w(x,y,t,\lambda).
\end{aligned}
$$

The associated deformation equation corresponding to $u(x,y,t)$ reads

$$
(1-\lambda)L[u(x,y,t,\lambda) - u_0(x,y,t)] = \lambda h N(u(x,y,t,\lambda)), \tag{3.20}
$$

or with the aid of Equation (3.17), we get

$$
\begin{aligned}
(1-\lambda)L\left[\sum_{n=1}^{\infty} u_n(x,y,t)\lambda^n\right] &= \lambda h\left[\frac{\partial}{\partial t}u(x,y,t,\lambda) + \frac{\partial}{\partial x}u(x,y,t,\lambda)\frac{\partial}{\partial x}v(x,y,t,\lambda)\right. \\
&+ \left.\frac{\partial}{\partial y}u(x,y,t,\lambda)\frac{\partial}{\partial y}v(x,y,t,\lambda) + u(x,y,t,\lambda)\right].
\end{aligned} \tag{3.21}
$$

Substituting from Equations (3.17) and (3.18), we get

$$
\begin{aligned}
&(1-\lambda)\frac{\partial}{\partial t}\left[\sum_{n=1}^{\infty} u_n(x,y,t)\lambda^n\right] \\
&= \lambda h\left[\frac{\partial}{\partial t}u_0(x,y,t) + \frac{\partial}{\partial t}\sum_{n=1}^{\infty} u_n(x,y,t)\lambda^n\right. \\
&\quad + \left(\frac{\partial}{\partial x}u_0(x,y,t) + \frac{\partial}{\partial x}\sum_{n=1}^{\infty} u_n(x,y,t)\lambda^n\right)\left(\frac{\partial}{\partial x}v_0(x,y,t) + \frac{\partial}{\partial x}\sum_{n=1}^{\infty} v_n(x,y,t)\lambda^n\right) \\
&\quad + \left(\frac{\partial}{\partial y}u_0(x,y,t) + \frac{\partial}{\partial y}\sum_{n=1}^{\infty} u_n(x,y,t)\lambda^n\right)\left(\frac{\partial}{\partial y}v_0(x,y,t) + \frac{\partial}{\partial y}\sum_{n=1}^{\infty} v_n(x,y,t)\lambda^n\right) \\
&\quad \left. + u_0(x,y,t) + \sum_{n=1}^{\infty} u_n(x,y,t)\lambda^n)\right].
\end{aligned} \tag{3.22}
$$

The left and right-hand sides can be rearranged as

$$\begin{aligned}
\text{L.H.S.} &= \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n - \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^{n+1} \\
&= \frac{\partial}{\partial t}u_1(x,y,t)\lambda + \sum_{n=2}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n - \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^{n+1} \\
&= \frac{\partial}{\partial t}u_1(x,y,t)\lambda + \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_{n+1}(x,y,t)\lambda^{n+1} - \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^{n+1}. \\
\text{R.H.S.} &= \lambda h\left[\frac{\partial}{\partial t}u_0(x,y,t) + \frac{\partial}{\partial t}\sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n\right. \\
&+ \left(\frac{\partial}{\partial x}u_0(x,y,t) + \frac{\partial}{\partial x}\sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n\right)\left(\frac{\partial}{\partial x}v_0(x,y,t) + \frac{\partial}{\partial x}\sum_{n=1}^{\infty}v_n(x,y,t)\lambda^n\right) \\
&+ \left(\frac{\partial}{\partial y}u_0(x,y,t) + \frac{\partial}{\partial y}\sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n\right)\left(\frac{\partial}{\partial y}v_0(x,y,t) + \frac{\partial}{\partial y}\sum_{n=1}^{\infty}v_n(x,y,t)\lambda^n\right) \\
&+ \left. u_0(x,y,t) + \sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n\right].
\end{aligned}$$

Equating the corresponding coefficients of same powers of λ on both sides and using the trivial initial conditions $u_n(x,y,0) = 0$, for all $n = 1,2,3\ldots$, we get the zeroth, first, and $(n+1))^{th}$ order deformation equations associated with the appropriate initial conditions.

$$\frac{\partial}{\partial t}u_0(x,y,t) = 0, u_0(x,y,0) = u_0(x,y).$$

$$\begin{aligned}
\frac{\partial}{\partial t}u_1(x,y,t) = h&\left[\frac{\partial}{\partial t}u_0(x,y,t) + \left(\frac{\partial}{\partial x}u_0(x,y,t)\right)\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) + \left(\frac{\partial}{\partial y}u_0(x,y,t)\right)\left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\right. \\
&+ \left. u_0(x,y,t)\right], u_1(x,y,0) = 0.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}u_{n+1}(x,y,t) = (1+h)\frac{\partial}{\partial t}u_n(x,y,t) + h&\left[\left(\frac{\partial}{\partial x}u_0(x,y,t)\right)\left(\frac{\partial}{\partial x}v_n(x,y,t)\right)\right. \\
&+ \left(\frac{\partial}{\partial x}u_n(x,y,t)\right)\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) + \left(\frac{\partial}{\partial y}u_0(x,y,t)\right)\left(\frac{\partial}{\partial y}v_n(x,y,t)\right) \\
&+ \left.\left(\frac{\partial}{\partial y}u_n(x,y,t)\right)\left(\frac{\partial}{\partial y}v_0(x,y,t)\right) + u_n(x,y,t)\right], u_{n+1}(x,y,0) = 0.
\end{aligned}$$

Similarly, the zeroth, first, and $(n+1))^{th}$ order deformation equations together with the appropriate initial conditions corresponding to $v(x,y,t)$ and $w(x,y,t)$ are

$$\frac{\partial}{\partial t}v_0(x,y,t) = 0, v_0(x,y,0) = v_0(x,y).$$

$$\begin{aligned}
\frac{\partial}{\partial t}v_1(x,y,t) = h&\left[\frac{\partial}{\partial t}v_0(x,y,t) + \left(\frac{\partial}{\partial x}v_0(x,y,t)\right)\left(\frac{\partial}{\partial x}w_0(x,y,t)\right) - \left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial}{\partial y}w_0(x,y,t)\right)\right. \\
&- \left. v_0(x,y,t)\right], v_1(x,y,0) = 0.
\end{aligned}$$

$$\frac{\partial}{\partial t}v_{n+1}(x,y,t) = (1+h)\frac{\partial}{\partial t}v_n(x,y,t) + h\Bigg[\left(\frac{\partial}{\partial x}v_0(x,y,t)\right)\left(\frac{\partial}{\partial x}w_n(x,y,t)\right)$$
$$+\left(\frac{\partial}{\partial x}v_n(x,y,t)\right)\left(\frac{\partial}{\partial x}w_0(x,y,t)\right) - \left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial}{\partial y}w_n(x,y,t)\right)$$
$$-\left(\frac{\partial}{\partial y}v_n(x,y,t)\right)\left(\frac{\partial}{\partial y}w_0(x,y,t)\right) - v_n(x,y,t)\Bigg], v_{n+1}(x,y,0) = 0.$$

$$\frac{\partial}{\partial t}w_0(x,y,t) = 0, w_0(x,y,0) = w_0(x,y).$$

$$\frac{\partial}{\partial t}w_1(x,y,t) = h\Bigg[\frac{\partial}{\partial t}w_0(x,y,t) + \left(\frac{\partial}{\partial x}w_0(x,y,t)\right)\left(\frac{\partial}{\partial x}u_0(x,y,t)\right) + \left(\frac{\partial}{\partial y}w_0(x,y,t)\right)\left(\frac{\partial}{\partial y}u_0(x,y,t)\right)$$
$$-\ w_0(x,y,t)\Bigg], w_1(x,y,0) = 0.$$

$$\frac{\partial}{\partial t}w_{n+1}(x,y,t) = (1+h)\frac{\partial}{\partial t}w_n(x,y,t) + h\Bigg[\left(\frac{\partial}{\partial x}w_0(x,y,t)\right)\left(\frac{\partial}{\partial x}u_n(x,y,t)\right)$$
$$+\left(\frac{\partial}{\partial x}w_n(x,y,t)\right)\left(\frac{\partial}{\partial x}u_0(x,y,t)\right) + \left(\frac{\partial}{\partial y}w_0(x,y,t)\right)\left(\frac{\partial}{\partial y}u_n(x,y,t)\right)$$
$$+\left(\frac{\partial}{\partial y}w_n(x,y,t)\right)\left(\frac{\partial}{\partial y}u_0(x,y,t)\right) - w_n(x,y,t)\Bigg], w_{n+1}(x,y,0) = 0.$$

Analogous to previous examples we have

$$u_0(x,y,t) = u(x,y,0) = e^{x+y},$$
$$v_0(x,y,t) = v(x,y,0) = e^{x-y},$$
$$w_0(x,y,t) = w(x,y,0) = e^{-x+y}.$$

We solve the previous governing problems corresponding to $u_n's$, $v_n's$ and $w_n's$, $n = 1,2,3,\ldots$. Further, for convergence, we set $h = -1$.

$$\frac{\partial}{\partial t}u_1(x,y,t) = h\Bigg[\frac{\partial}{\partial t}u_0(x,y,t) + \left(\frac{\partial}{\partial x}u_0(x,y,t)\right)\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) + \left(\frac{\partial}{\partial y}u_0(x,y,t)\right)\left(\frac{\partial}{\partial y}v_0(x,y,t)\right)$$
$$+\ u_0(x,y,t)\Bigg] = -e^{x+y}. \text{ Thus } u_1(x,y,t) = -te^{x+y},$$

$$\frac{\partial}{\partial t}v_1(x,y,t) = h\Bigg[\frac{\partial}{\partial t}v_0(x,y,t) + \left(\frac{\partial}{\partial x}v_0(x,y,t)\right)\left(\frac{\partial}{\partial x}w_0(x,y,t)\right) - \left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial}{\partial y}w_0(x,y,t)\right)$$
$$-v_0(x,y,t)] = e^{x-y}. \text{ Hence } v_1(x,y,t) = te^{x-y},$$

$$\frac{\partial}{\partial t}w_1(x,y,t) = h\Bigg[\frac{\partial}{\partial t}w_0(x,y,t) + \left(\frac{\partial}{\partial x}w_0(x,y,t)\right)\left(\frac{\partial}{\partial x}u_0(x,y,t)\right) + \left(\frac{\partial}{\partial y}w_0(x,y,t)\right)\left(\frac{\partial}{\partial y}u_0(x,y,t)\right)$$
$$-w_0(x,y,t)] = e^{-x+y}. \text{ Therefore } w_1(x,y,t) = te^{-x+y}.$$

Continuing in this process we get

$$\frac{\partial}{\partial t}u_2(x,y,t) = te^{x+y},\ u_2(x,y,t) = \tfrac{t^2}{2!}e^{x+y},$$
$$\frac{\partial}{\partial t}v_2(x,y,t) = te^{x-y},\ v_2(x,y,t) = \tfrac{t^2}{2!}e^{x-y},$$
$$\frac{\partial}{\partial t}w_2(x,y,t) = te^{-x+y},\ w_2(x,y,t) = \tfrac{t^2}{2!}e^{-x+y},$$
$$\frac{\partial}{\partial t}u_3(x,y,t) = \frac{t^2}{2!}e^{x+y}\ ,\ u_3(x,y,t) = -\tfrac{t^3}{3!}e^{x+y},$$
$$\frac{\partial}{\partial t}v_3(x,y,t) = \frac{t^2}{2!}e^{x-y}\ ,\ v_3(x,y,t) = \tfrac{t^3}{3!}e^{x-y},$$
$$\frac{\partial}{\partial t}w_3(x,y,t) = \frac{t^2}{2!}e^{-x+y}\ ,\ w_3(x,y,t) = \tfrac{t^3}{3!}e^{-x+y}.$$

and so on. with $\lambda = 1$, the series solutions are

$$\begin{aligned} u(x,y,t) &= u_0(x,y,t) + \sum\nolimits_{n=1}^{\infty} u_n(x,y,t) \\ &= e^{x+y} - te^{x+y} + \frac{t^2}{2!}e^{x+y} - \frac{t^3}{3!}e^{x+y} + \ldots, \\ v(x,y,t) &= v_0(x,y,t) + \sum\nolimits_{n=1}^{\infty} v_n(x,y,t) \\ &= e^{x-y} + te^{x-y} + \frac{t^2}{2!}e^{x-y} + \frac{t^3}{3!}e^{x-y} + \ldots, \\ w(x,y,t) &= w_0(x,y,t) + \sum\nolimits_{n=1}^{\infty} w_n(x,y,t) \\ &= e^{-x+y} + te^{-x+y} + \frac{t^2}{2!}e^{-x+y} + \frac{t^3}{3!}e^{-x+y} + \ldots. \end{aligned}$$

These series solutions simplified to
$u(x,y,t) = e^{(x+y-t)}$,
$v(x,y,t) = e^{(x-y+t)}$,
$w(x,y,t) = e^{(-x+y+t)}$.
In Figures 3.11, 3.13, 3.15 and Figures 3.12, 3.14, 3.16 we plot the exact solutions and the fourth order approximate solutions at $x = 1$, respectively.

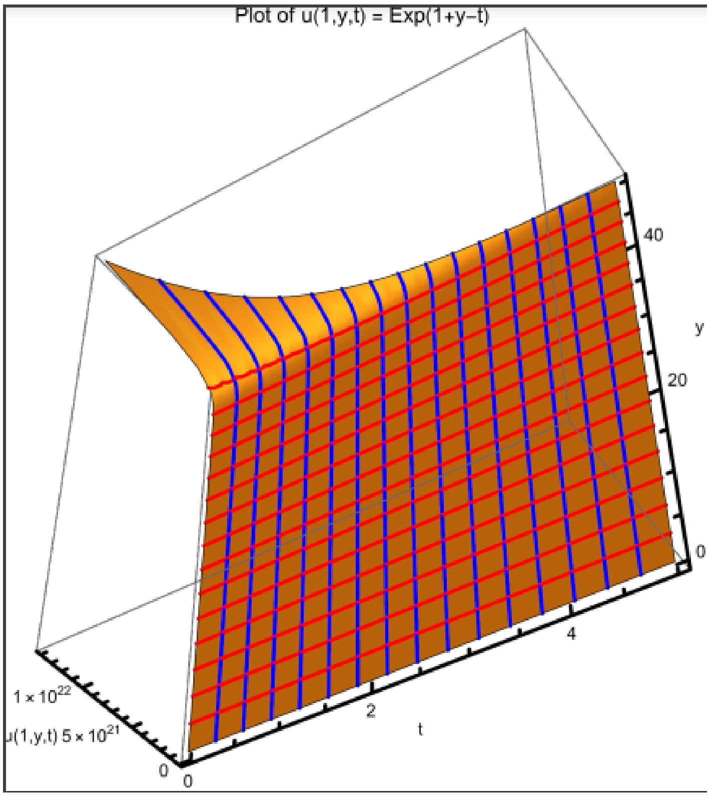

Figure 3.11: Exact solution $u(1, y, t) = e^{(1+y-t)}$

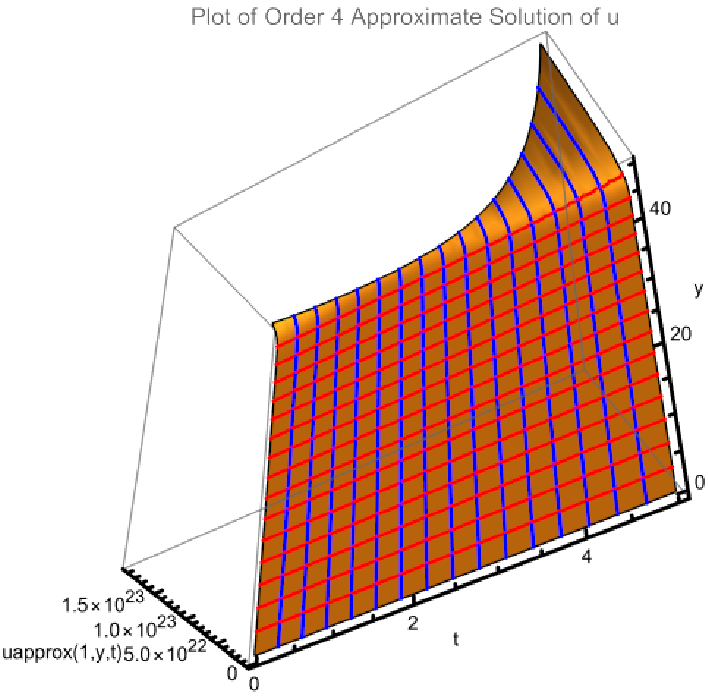

Figure 3.12: Approximate solution of $u(1, y, t)$ (HAM)

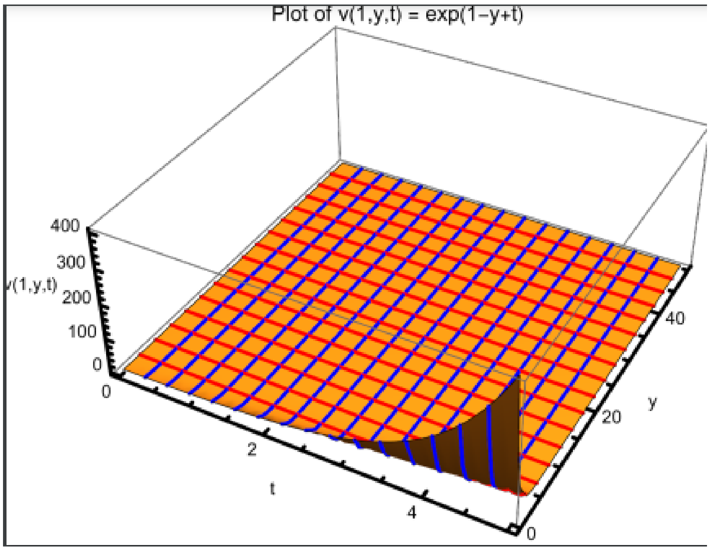

Figure 3.13: Exact solution $v(1, y, t) = e^{(1-y+t)}$

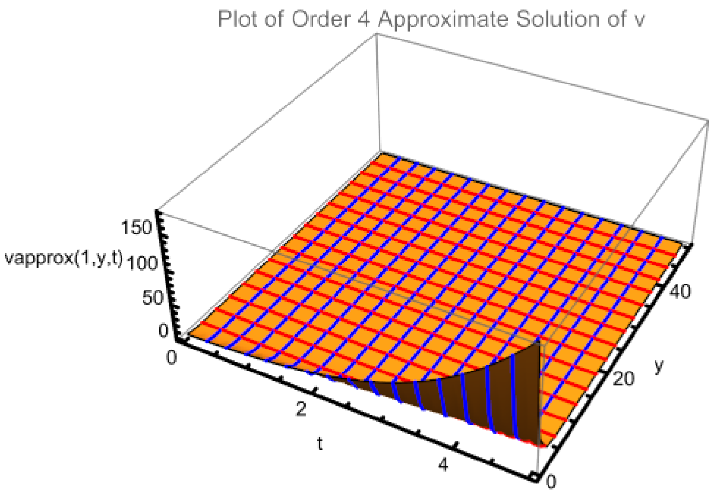

Figure 3.14: Approximate solution of $v(1, y, t)$ (HAM)

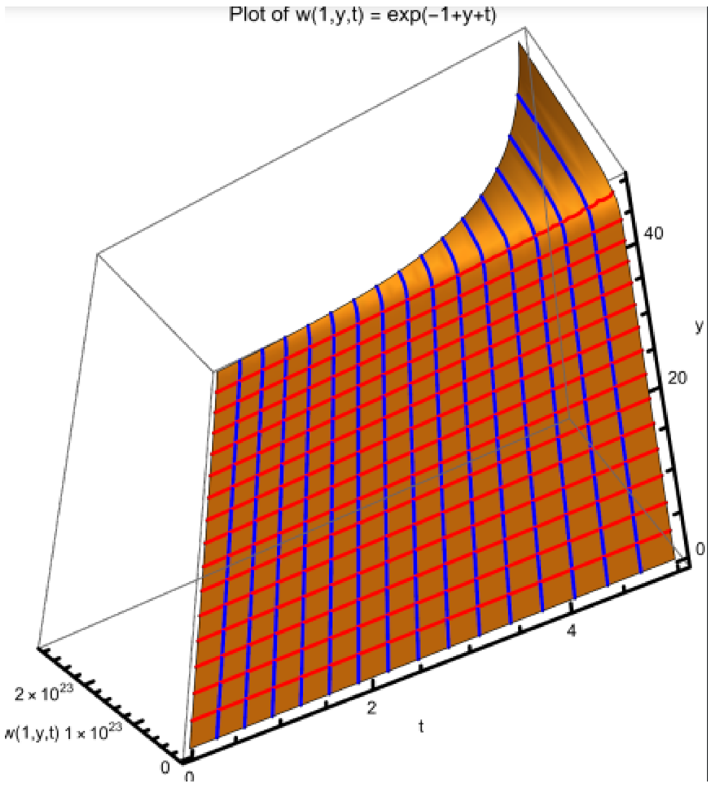

Figure 3.15: Exact solution $w(1, y, t) = e^{(-1+y+t)}$

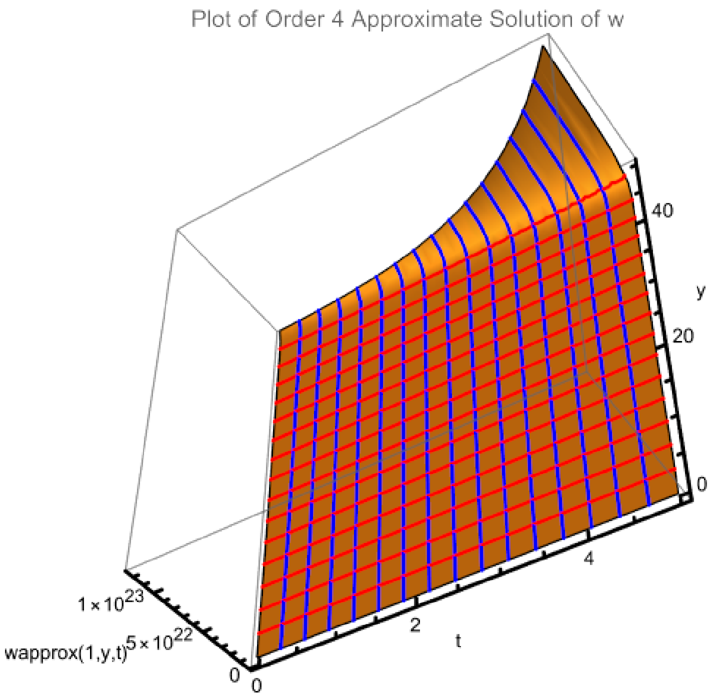

Figure 3.16: Approximate solution of $w(1, y, t)$ (HAM)

Chapter 4

Optimal Homotopy Asymptotic Method (OHAM) for Solving Systems of Partial Differential Equations

4.1 Introduction

In this chapter, we utilize the Optimal Homotopy Asymptotic Method (OHAM) to obtain an analytical solution for partial differential equations,[15, 43]. The following section provides a detailed analysis of the OHAM approach, while Sections 3 and 4 showcase various examples to illustrate its effectiveness.

4.2 Analysis of OHAM

Consider the following operation equation

$$L(u(x,t) + g(x,t) + N(u(x,t)) = 0, \tag{4.1}$$

where L is a linear operator, (x,t) denotes independent variables, $u(x,t)$ is an unknown function, $g(x,t)$ is a known function, and $N(u(x,t))$ is, in general, a nonlinear operator. For simplicity, we ignore all boundary or initial conditions, which can be treated similarly.

By means of the Optimal Homotopy Analysis method, we first construct a family of the so-called deformation equation:

$$(1-\lambda)[L(u(x,t,\lambda) + g(x,t)] = H(x,\lambda)[L(u(x,t,\lambda) + g(x,t) + N(u(x,t,\lambda)], \tag{4.2}$$

where $\lambda \in [0,1]$ is an embedding parameter, $H(x,\lambda)$ is a nonzero auxiliary function for $\lambda \neq 0$ and $H(x,0) = 0$, $u(x,t\lambda)$ is a unknown function. Obviously, when $\lambda = 0$ and $\lambda = 1$, we have respectively, since $L[u(x,t,0) + g(x,t)] = 0$, $u(x,t,0) = u_0(x,t)$, L is chosen such that $u_0(x,t)$ is easy to determine, and since $[L(u(x,t,1) + g(x,t) + N(u(x,t,1))] = 0$, $u(x,t,1) = u(x,t)$. Thus, as λ varies from 0 to 1, the solution $u(x,t,\lambda)$ varies from $u_0(x,t)$ to the exact solution $u(x,t)$.

Expanding $u(x,t,\lambda)$ in Taylor series with respect to λ centered at $\lambda = 0$, we have

$$u(x,t,\lambda) = u_0(x,t) + \sum\nolimits_{n=1}^{\infty} u_n(x,t,c_i)\lambda^n, \qquad i = 1, 2, \dots. \tag{4.3}$$

By choosing the auxiliary function $H(x,\lambda)$ to be $H(x,\lambda) = \sum_{j=1}^{\infty} c_j\lambda^j$ and substituting from Equation (4.3) into Equation (4.2), then equating the corresponding coefficients of same powers of λ of both sides of equality and using the corresponding initial conditions, $u_0(x,0) = u_0(x)$ (a given function), $u_n(x,0) = 0$, $n = 1, 2, \dots$, we get the governing problems, deformation equations and corresponding initial conditions, for the coefficients $u_n(x,t,c_i)$, $n = 0, 1, 2, \dots$, [35]. The k^{th} order approximate analytic solution reads

$$u(x,t) = u_0(x,t) + \sum_{n=0}^{k} u_n(x,t,c_i), \; i = 1, \dots, n. \tag{4.4}$$

The convergence control constants $c_1, c_2, \dots, c_k$ are determined by minimizing the L_2-error

$$\int_0^T \int_\Omega (L(u(x,t)) + g(x,t) + N(u(x,t)))^2 dxdy, \tag{4.5}$$

where $\Omega \subset (-\infty, \infty)$ and $\Omega \times [0,T]$ is the domain of the problem. Note that in the following sections, we will be writing $u_n(x,t)$ to mean $u_n(x,t,c_i), i = 1, 2, \dots$.

4.3 Illustrative Examples for Linear Equations

In this section, we return to the examples discussed in Section (3.3) and employ OHAM to approximate solutions for these cases.

Example 4.1. *Consider the following system of initial value problem:*

$u_t + v_x = u + v,$

$v_t + u_x = u + v,$

with initial conditions

$u(x, 0) = u_0(x) = \sinh x,$

$v(x, 0) = v_0(x) = \cosh x.$

Following Equation (4.3), we assume that

$$u(x, t, \lambda) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t)\lambda^n, \tag{4.6}$$

and

$$v(x, t, \lambda) = v_0(x, t) + \sum_{n=1}^{\infty} v_n(x, t)\lambda^n. \tag{4.7}$$

Further, we define the operator N to be

$$N(u(x, t, \lambda)) = \frac{\partial}{\partial x} v(x, t, \lambda) - (u(x, t, \lambda) + v(x, t, \lambda)),$$

$$N(v(x, t; \lambda)) = \frac{\partial}{\partial x} u(x, t, \lambda) - (u(x, t, \lambda) + v(x, t, \lambda)),$$

and we take the operator L to be

$$L(u(x, t, \lambda) = \frac{\partial}{\partial t}(u(x, t, \lambda),$$

$$L(v(x, t, \lambda) = \frac{\partial}{\partial t}(v(x, t, \lambda).$$

The deformation equation corresponding to the first partial differential equation is given by

$$(1 - \lambda)L[u(x, t, \lambda)] = H(x, \lambda)[L(u(x, t, \lambda) + N(u(x, t, \lambda)]. \tag{4.8}$$

Note that $g(x, t)$ in this example is identically zero. Applying the definitions of the operators L and N and substituting from Equations (4.6) and (4.7) and taking the auxiliary function

function $H(x,t) = \sum_{j=1}^{\infty} c_j \lambda^j$, the deformation equation (4.8) becomes

$$
\begin{aligned}
(1-\lambda)\left[\frac{\partial}{\partial t}u_0(x,t) + \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,t)\lambda^n\right] &= \left(\sum_{j=1}^{\infty} c_j\lambda^j\right)\Bigg[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial t}\sum_{n=1}^{\infty}u_n(x,t)\lambda^n \\
&+ \frac{\partial}{\partial x}v_0(x,t) + \frac{\partial}{\partial x}\sum_{n=1}^{\infty}v_n(x,t)\lambda^n \\
&- \Bigg(u_0(x,t) + \sum_{n=1}^{\infty}u_n(x,t)\lambda^n \\
&+ v_0(x,t) + \sum_{n=1}^{\infty}v_n(x,t)\lambda^n\Bigg)\Bigg]. \qquad (4.9)
\end{aligned}
$$

We rewrite the left and the right-hand sides as follows
$L.H.S.$

$$
\begin{aligned}
&= \frac{\partial}{\partial t}u_0(x,t) - \lambda\frac{\partial}{\partial t}u_0(x,t) + \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,t)\lambda^n - \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,t)\lambda^{n+1} \\
&= \frac{\partial}{\partial t}u_0(x.t) + \lambda\left(\frac{\partial}{\partial t}u_1(x,t) - \frac{\partial}{\partial t}u_0(x.t)\right) + \sum_{n=2}^{\infty}\frac{\partial}{\partial t}u_n(x,t)\lambda^n - \sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,t)\lambda^{n+1} \\
&= \frac{\partial}{\partial t}u_0(x.t) + \lambda\left(\frac{\partial}{\partial t}u_1(x,t) - \frac{\partial}{\partial t}u_0(x.t)\right) + \sum_{n=1}^{\infty}\left(\frac{\partial}{\partial t}u_{n+1}(x,t) - \frac{\partial}{\partial t}u_n(x,t)\right)\lambda^{n+1}.
\end{aligned}
$$

$R.H.S.$

$$
\begin{aligned}
&= \lambda c_1\left[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial x}v_0(x,t) - (u_0(x,t) + v_0(x,t))\right] \\
&+ c_1\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial t}u_n(x,t) + \frac{\partial}{\partial x}v_n(x,t) - (u_n(x,t) + v_n(x,t))\right]\lambda^{n+1} \\
&+ \lambda^2 c_2\left[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial x}v_0(x,t) - (u_0(x,t) + v_0(x,t))\right] \\
&+ c_2\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial t}u_n(x,t) + \frac{\partial}{\partial x}v_n(x,t) - (u_n(x,t) + v_n(x,t))\right]\lambda^{n+2} \\
&+ \lambda^3 c_3\left[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial x}v_0(x,t) - (u_0(x,t) + v_0(x,t))\right] \\
&+ c_3\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial t}u_n(x,t) + \frac{\partial}{\partial x}v_n(x,t) - (u_n(x,t) + v_n(x,t))\right]\lambda^{n+3} \\
&+ \ldots \\
&= \lambda c_1\left[\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial x}v_0(x,t) - (u_0(x,t) + v_0(x,t))\right] \\
&+ \lambda^2\Bigg[c_1\left(\frac{\partial}{\partial t}u_1(x,t) + \frac{\partial}{\partial x}v_1(x,t) - (u_1(x,t) + v_1(x,t))\right) \\
&\quad + c_2\left(\frac{\partial}{\partial t}u_0(x,t) + \frac{\partial}{\partial x}v_0(x,t) - (u_0(x,t) + v_0(x,t))\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda^3\left[c_1\left(\frac{\partial}{\partial t}u_2(x,t)+\frac{\partial}{\partial x}v_2(x,t)-(u_2(x,t)+v_2(x,t))\right)\right.\\
&\qquad+c_2\left(\frac{\partial}{\partial t}u_1(x,t)+\frac{\partial}{\partial x}v_1(x,t)-(u_1(x,t)+v_1(x,t))\right)\\
&\qquad\left.+c_3\left(\frac{\partial}{\partial t}u_0(x,t)+\frac{\partial}{\partial x}v_0(x,t)-(u_0(x,t)+v_0(x,t))\right)\right]\\
&+\ldots
\end{aligned}
$$

Equating the corresponding coefficients of same powers of λ of $L.H.S.$ and $R.H.S.$ and using the initial conditions $u_n(x,0)=0$, for all $n=1,2,3,\ldots$ and $u_0(x,0)=u(x,0)$, we get:
The zeroth order problem

$$\frac{\partial}{\partial t}u_0(x,t)=0,\;\; u_0(x,0)=u(x,0).$$

The first Order problem

$$\frac{\partial}{\partial t}u_1(x,t)=(1+c_1)\frac{\partial}{\partial t}u_0(x,t)+c_1\left(\frac{\partial}{\partial x}v_0(x,t)-(u_0(x,t)+v_0(x,t))\right),\;\; u_1(x,0)=0.$$

The second order problem

$$
\begin{aligned}
\frac{\partial}{\partial t}u_2(x,t)=&(1+c_1)\frac{\partial}{\partial t}u_1(x,t)+c_2\frac{\partial}{\partial t}u_0(x,t)\\
&+c_1\left(\frac{\partial}{\partial x}v_1(x,t)-(u_1(x,t)+v_1(x,t))\right)\\
&+c_2\left(\frac{\partial}{\partial x}v_0(x,t)-(u_0(x,t)+v_0(x,t))\right),\;\; u_2(x,0)=0.
\end{aligned}
$$

The third order problem

$$
\begin{aligned}
\frac{\partial}{\partial t}u_3(x,t)=&(1+c_1)\frac{\partial}{\partial t}u_2(x,t)+c_2\frac{\partial}{\partial t}u_1(x,t)+c_3\frac{\partial}{\partial t}u_0(x,t)\\
&+c_1\left(\frac{\partial}{\partial x}v_2(x,t)-(u_2(x,t)+v_2(x,t))\right)\\
&+c_2\left(\frac{\partial}{\partial x}v_1(x,t)-(u_1(x,t)+v_1(x,t))\right)\\
&+c_3\left(\frac{\partial}{\partial x}v_0(x,t)-(u_0(x,t)+v_0(x,t))\right),\;\; u_3(x,0)=0.
\end{aligned}
$$

In general the $(k+1)th$ order problem reads

$$
\begin{aligned}
\frac{\partial}{\partial t}u_{k+1}(x,t)=&(1+c_1)\frac{\partial}{\partial t}u_k(x,t)+c_2\frac{\partial}{\partial t}u_{k-1}(x,t)+c_3\frac{\partial}{\partial t}u_{k-2}(x,t)+\ldots+c_{k+1}\frac{\partial}{\partial t}u_0(x,t)\\
&+c_1\left(\frac{\partial}{\partial x}v_k(x,t)-(u_k(x,t)+v_k(x,t))\right)\\
&+c_2\left(\frac{\partial}{\partial x}v_{k-1}(x,t)-(u_{k-1}(x,t)+v_{k-1}(x,t))\right)\\
&+c_3\left(\frac{\partial}{\partial x}v_{k-2}(x,t)-(u_{k-2}(x,t)+v_{k-2}(x,t))\right)\\
&+\ldots\\
&+c_k\left(\frac{\partial}{\partial x}v_1(x,t)-(u_1(x,t)+v_1(x,t))\right)
\end{aligned}
$$

$$+c_{k+1}\left(\frac{\partial}{\partial x}v_0(x,t)-(u_0(x,t)+v_0(x,t))\right),\ \ u_{k+1}(x,0)=0.$$

The differential equations within any of these problems are called deformation equations associated with $u(x,t)$.
Analogously, we get the zeroth, first, second, third, and $(n+1)th$ deformation equations and problems corresponding to $v(x,t)$. We just interchange $u_j(x,t)$ and $v_j(x,t)$.

To solve these initial value problems associated with $u(x,t)$ and $v(x,t)$, note that $\frac{\partial}{\partial t}u_0(x,t)=0,\ \ u_0(x,0)=u(x,0)=\sinh x$ implies $u_0(x,t)=\sinh x$

and

$\frac{\partial}{\partial t}v_0(x,t)=0,\ \ v_0(x,0)=v(x,0)=\cosh x$ implies $v_0(x,t)=\cosh x$.
$\frac{\partial}{\partial t}u_1(x,t)=(1+c_1)\frac{\partial}{\partial t}u_0(x,t)+c_1\left(\frac{\partial}{\partial x}v_0(x,t)-(u_0(x,t)+v_0(x,t))\right)=-c_1\cosh x,$
$u_1(x,0)=0$. Hence $u_1(x,t)=-c_1t\cosh x$

and

$\frac{\partial}{\partial t}v_1(x,t)=(1+c_1)\frac{\partial}{\partial t}v_0(x,t)+c_1\left(\frac{\partial}{\partial x}u_0(x,t)-(u_0(x,t)+v_0(x,t))\right)=-c_1\sinh x,$
$v_1(x,0)=0$. Hence $v_1(x,t)=-c_1t\sinh x$.

Similarly, we find

$$\begin{aligned}
u_2(x,t)&=-c_1t\cosh x-c_1^2t\cosh x+c_1^2\frac{t^2}{2!}\sinh x-c_2t\cosh x,\\
v_2(x,t)&=-c_1t\sinh x-c_1^2t\sinh x+c_1^2\frac{t^2}{2!}\cosh x-c_2t\sinh x,\\
u_3(x,t)&=-c_1t\cosh x-2c_1^2t\cosh x+c_1^2t^2\sinh x-c_2t\cosh x-c_1^3t\cosh x\\
&\quad+c_1^3t^2\sinh x-c_1^3\frac{t^3}{3!}\cosh x-c_1c_2t\cosh x+c_1c_2t^2\sinh x-c_3t\cosh x,\\
v_3(x,t)&=-c_1t\sinh x-2c_1^2t\sin x+c_1^2t^2\cosh x-c_2t\sinh x-c_1^3t\sinh x\\
&\quad+c_1^3t^2\cosh x-c_1^3\frac{t^3}{3!}\sinh x-c_1c_2t\sinh x+c_1c_2t^2\cosh x-c_3t\sinh x,
\end{aligned}$$

and so on.
The third-order approximate solutions are:

$$\begin{aligned}
u&=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)\\
&=\sinh x-2c_1t\cosh x-c_1^2t\cosh x+c_1^2\frac{t^2}{2!}\sinh x-c_2t\cosh x-c_1t\cosh x-2c_1^2t\cosh x+\\
&\quad c_1^2t^2\sinh x-c_2t\cosh x-c_1^3t\cosh x+c_1^3t^2\sinh x-c_1^3\frac{t^3}{3!}\cosh x-c_1c_2t\cosh x+c_1c_2t^2\sinh x-\\
&\quad c_3t\cosh x,
\end{aligned}$$

and

$$v = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t)$$

$$= \cosh x - 2c_1 t \sinh x - c_1^2 t \sinh x + c_1^2 \frac{t^2}{2!} \cosh x - c_2 t \sinh x - c_1 t \sinh x - 2c_1^2 t \sinh x + c_1^2 t^2 \cosh x - c_2 t \sinh x - c_1^3 t \sinh x + c_1^3 t^2 \cosh x - c_1^3 \frac{t^3}{3!} \sinh x - c_1 c_2 t \sinh x + c_1 c_2 t^2 \cosh x - c_3 t \sinh x.$$

To determine the convergence control constants c_1, c_2 and c_3, we use MATHEMATICA to minimize the L_2-error, Equation (4.5). We find that $c_1 = 0.342760$, $c_2 = -0.806561$ and $c_3 = -0.251945$. In Figures 4.1 and 4.3 we plot the exact solutions $u(x,t) = \sinh(x+t)$ and $v(x,t) = \cosh(x+t)$, respectively. The third order approximate solutions are plotted in Figures 4.2 and 4.4, respectively. Comparing these results with the results of HAM Figures 3.2 and 3.4 we see a better improvement.

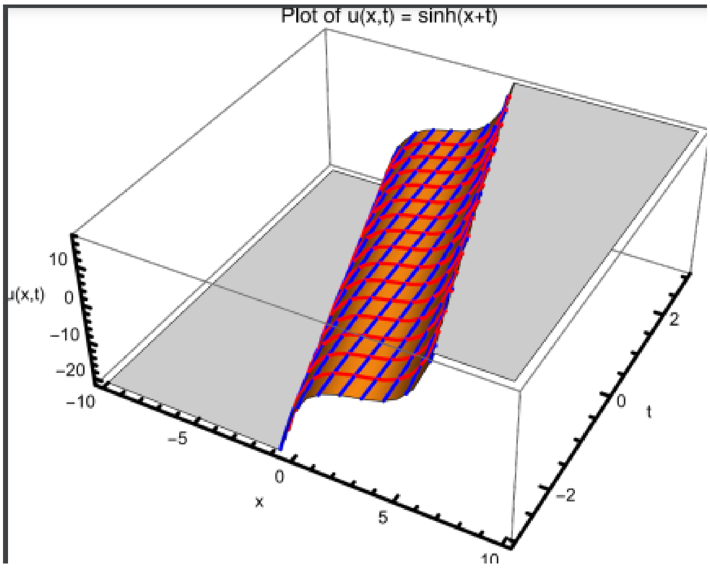

Figure 4.1: Exact solution $u(x,t) = \sinh(x+t)$

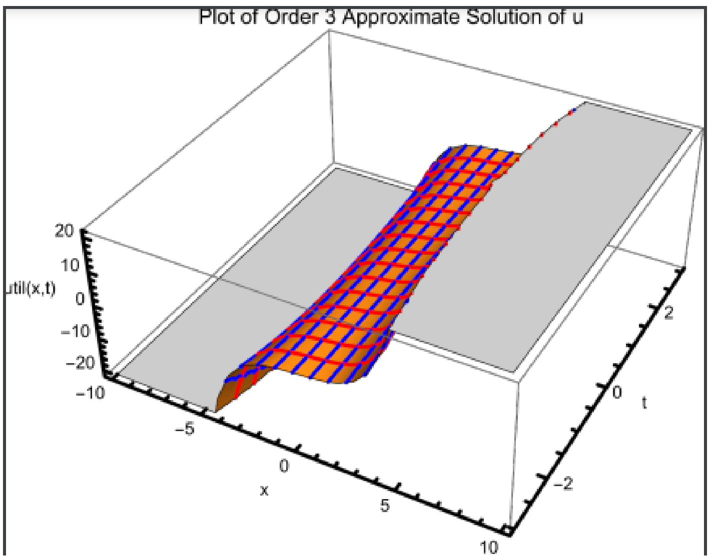

Figure 4.2: Approximate solution of $u(x,t)$ (OHAM)

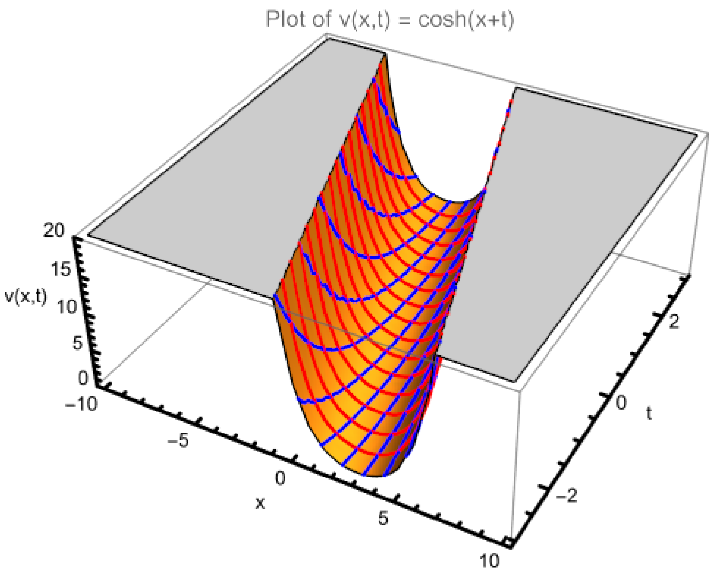

Figure 4.3: Exact solution $v(x,t) = \cosh(x+t)$

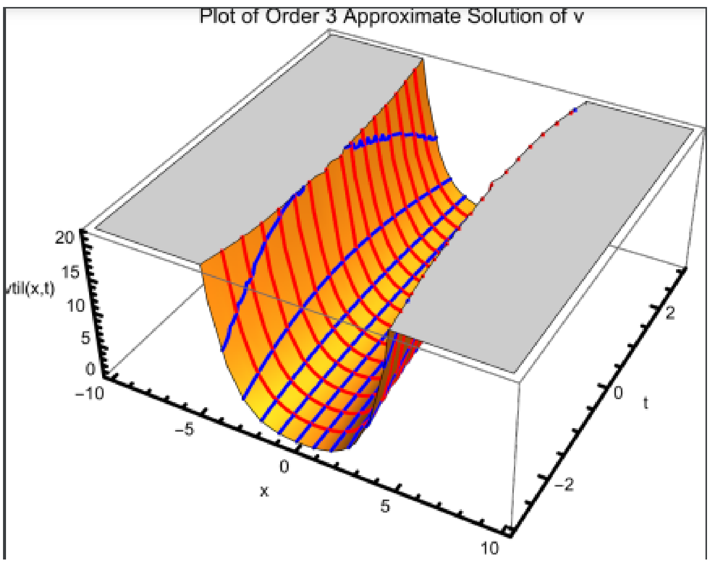

Figure 4.4: Approximate solution of $v(x,t)$ (OHAM)

Example 4.2. *In this example we Consider the two space variables system of partial differential equations:*

$$
\begin{aligned}
u_t + v_x + w_y &= \ \ w,\\
v_t + w_x + u_y &= -u,\\
w_t + v_x - v_y &= -v,
\end{aligned}
$$

together with initial conditions

$$
\begin{aligned}
u(x, y, 0) \ &= u_0(x, y) = \quad \sin(x + y),\\
v(x, y, 0) \ &= v_0(x, y) = \quad \cos(x + y),\\
w(x, y, 0) &= w_0(x, y) = -\sin(x + y).
\end{aligned}
$$

Again, following Equation(4.6), we assume that

$$u(x, y, t, \lambda) = u_0(x, y, t) + \sum_{n=1}^{\infty} u_n(x, y, t)\lambda^n, \tag{4.10}$$

and

$$v(x, y, t, \lambda) = v_0(x, y, t) + \sum_{n=1}^{\infty} v_n(x, y, t)\lambda^n, \tag{4.11}$$

$$w(x, y, t, \lambda) = w_0(x, y, t) + \sum_{n=1}^{\infty} w_n(x, y, t)\lambda^n. \tag{4.12}$$

The operators L and N are given by

$$
\begin{aligned}
L(u(x, y, t, \lambda)) &= \frac{\partial}{\partial t} u(x, y, t, \lambda),\\
L(v(x, y, t, \lambda)) &= \frac{\partial}{\partial t} v(x, y, t; \lambda),\\
L(w(x, y, t, \lambda)) &= \frac{\partial}{\partial t} w(x, y, t, \lambda),
\end{aligned}
$$

and

$$
\begin{aligned}
N(u(x, y, t, \lambda)) &= \frac{\partial}{\partial x} v(x, y, t, \lambda) + \frac{\partial}{\partial y} w(x, y, t, \lambda) - w(x, y, t, \lambda),\\
N(v(x, y, t, \lambda)) &= \frac{\partial}{\partial x} w(x, y, t, \lambda) + \frac{\partial}{\partial y} u(x, y, t, \lambda) + u(x, y, t, \lambda),\\
N(w(x, y, t; \lambda)) &= \frac{\partial}{\partial x} v(x, y, t, \lambda) - \frac{\partial}{\partial y} v(x, y, t, \lambda) + v(x, y, t, \lambda).
\end{aligned}
$$

The deformation equation corresponding to $u(x, y, t)$ is given by

$$(1 - \lambda)L[(u(x, y, t, \lambda)) + g(x, y, t)] = H(\lambda)[L(u(x, y, t, \lambda)) + N(u(x, y, t, \lambda)) + g(x, y, t)] \tag{4.13}$$

Notice, again, that $g(x, y, t)$ is identically zero. With the aid of Equations (4.10-4.12) and the definitions of the operators L and N we write Equation (4.13) as

$$\begin{aligned}
(1-\lambda)\left[\frac{\partial}{\partial t}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n\right] &= \left(\sum_{j=1}^{\infty}c_j\lambda^j\right)\left[\frac{\partial}{\partial t}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n\right.\\
&+ \frac{\partial}{\partial x}v_0(x,y,t)+\sum_{n=0}^{\infty}\frac{\partial}{\partial x}v_n(x,y,t)\lambda^n\\
&+ \frac{\partial}{\partial y}w_0(x,y,t)+\sum_{n=0}^{\infty}\frac{\partial}{\partial}w_n(x,y.t)\lambda^n\\
&- \left. w_0(x,y,t)-\sum_{n=1}^{\infty}w_n(x,y,t)\lambda^n\right] \qquad (4.14)
\end{aligned}$$

The left and right-hand sides are rearranged in the forms

$$\begin{aligned}
L.H.S. &= \frac{\partial}{\partial t}u_0(x,y,t)-\lambda\frac{\partial}{\partial t}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n-\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^{n+1}\\
&= \frac{\partial}{\partial t}u_0(x,y,t)+\left(\frac{\partial}{\partial t}u_1(x,y,t)-\frac{\partial}{\partial t}u_0(x,y,t)\right)\lambda\\
&+ \sum_{n=1}^{\infty}\left(\frac{\partial}{\partial t}u_{n+1}(x,y,t)-\frac{\partial}{\partial t}u_n(x,y,t)\right)\lambda^{n+1}
\end{aligned}$$

$$\begin{aligned}
R.H.S. &= c_1\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right]\lambda\\
&+ \left(c_1\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial y}w_1(x,y,t)-w_1(x,y,t)\right]\right.\\
&+ \left. c_2\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right]\right)\lambda^2\\
&+ \left(c_1\left[\frac{\partial}{\partial t}u_2(x,y,t)+\frac{\partial}{\partial x}v_2(x,y,t)+\frac{\partial}{\partial y}w_2(x,y,t)-w_2(x,y,t)\right]\right.\\
&+ c_2\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial y}w_1(x,y,t)-w_1(x,y,t)\right]\\
&+ \left. c_3\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right]\right)\lambda^3\\
&+ \ldots\\
&+ \sum_{n=4}^{\infty}\left(c_1\left[\frac{\partial}{\partial t}u_{n-1}(x,y,t)+\frac{\partial}{\partial x}v_{n-1}(x,y,t)+\frac{\partial}{\partial y}w_{n-1}(x,y,t)-w_{n-1}(x,y,t)\right]\right.\\
&+ c_2\left[\frac{\partial}{\partial t}u_{n-2}(x,y,t)+\frac{\partial}{\partial x}v_{n-2}(x,y,t)+\frac{\partial}{\partial y}w_{n-2}(x,y,t)-w_{n-2}(x,y,t)\right]\\
&+ \ldots\\
&+ \left. c_n\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right]\right)\lambda^n
\end{aligned}$$

Equating the corresponding coefficients of same powers of λ on both sides and using the

initial conditions $u_n(x,y,0)=0$, for all $n=1,2,3\ldots$ and $u_0(x,y,0)=u_0(x,y)$, we get the following zeroth, first, and k^{th} $(k\geq 2)$ order problems (consist of deformation equations and initial conditions) corresponding to $u(x,y,t)$:
Zeroth-order problem:

$$\frac{\partial}{\partial t}u_0(x,y,t)=0, \qquad u_0(x,y,0)=u_0(x,y).$$

First-order problem:

$$\begin{aligned}\frac{\partial}{\partial t}u_1(x,y,t) &= (1+c_1)\frac{\partial}{\partial t}u_0(x,y,t)+c_1\left(\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right),\\ u_1(x,y,0) &= 0.\end{aligned}$$

k^{th}-order problem:

$$\begin{aligned}\frac{\partial}{\partial t}u_k(x,y,t)-\frac{\partial}{\partial t}u_{k-1}(x,y,t) &= c_1\left[\frac{\partial}{\partial t}u_{k-1}(x,y,t)+\frac{\partial}{\partial x}v_{k-1}(x,y,t)+\frac{\partial}{\partial y}w_{k-1}(x,y,t)-w_{k-1}(x,y,t)\right]\\ &+ c_2\left[\frac{\partial}{\partial t}u_{k-2}(x,y,t)+\frac{\partial}{\partial x}v_{k-2}(x,y,t)+\frac{\partial}{\partial y}w_{k-2}(x,y,t)-w_{k-2}(x,y,t)\right]\\ &+ \ldots\\ &+ c_{k-1}\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial y}w_1(x,y,t)-w_1(x,y,t)\right]\\ &+ c_k\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}w_0(x,y,t)-w_0(x,y,t)\right],\\ u_k(x,y,0) &= 0, \qquad k=2,3,\ldots.\end{aligned}$$

Similar problems can be derived for $v(x,y,t)$ and $w(x,y,t)$. Respectively, they are Zeroth-order problem:

$$\frac{\partial}{\partial t}v_0(x,y,t)=0, \qquad v_0(x,y,0)=v_0(x,y).$$

First-order problem:

$$\begin{aligned}\frac{\partial}{\partial t}v_1(x,y,t) &= (1+c_1)\frac{\partial}{\partial t}v_0(x,y,t)+c_1\left(\frac{\partial}{\partial x}w_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)+u_0(x,y,t)\right),\\ v_1(x,y,0) &= 0.\end{aligned}$$

k^{th}-order problem:

$$
\begin{aligned}
\frac{\partial}{\partial t} v_k(x,y,t) - \frac{\partial}{\partial t} v_{k-1}(x,y,t) &= c_1 \left[\frac{\partial}{\partial t} v_{k-1}(x,y,t) + \frac{\partial}{\partial x} w_{k-1}(x,y,t) + \frac{\partial}{\partial y} u_{k-1}(x,y,t) + u_{k-1}(x,y,t) \right] \\
&+ c_2 \left[\frac{\partial}{\partial t} v_{k-2}(x,y,t) + \frac{\partial}{\partial x} w_{k-2}(x,y,t) + \frac{\partial}{\partial y} u_{k-2}(x,y,t) + u_{k-2}(x,y,t) \right] \\
&+ \ldots \\
&+ c_{k-1} \left[\frac{\partial}{\partial t} v_1(x,y,t) + \frac{\partial}{\partial x} w_1(x,y,t) + \frac{\partial}{\partial y} u_1(x,y,t) + u_1(x,y,t) \right] \\
&+ \quad c_k \left[\frac{\partial}{\partial t} v_0(x,y,t) + \frac{\partial}{\partial x} w_0(x,y,t) + \frac{\partial}{\partial y} u_0(x,y,t) + u_0(x,y,t) \right], \\
v_k(x,y,0) &= 0, \qquad k = 2,3,\ldots.
\end{aligned}
$$

Zeroth-order problem:

$$
\frac{\partial}{\partial t} w_0(x,y,t) = 0, \qquad w_0(x,y,0) = w_0(x,y).
$$

First-order problem:

$$
\begin{aligned}
\frac{\partial}{\partial t} w_1(x,y,t) &= (1+c_1)\frac{\partial}{\partial t} w_0(x,y,t) + c_1 \left(\frac{\partial}{\partial x} v_0(x,y,t) - \frac{\partial}{\partial y} v_0(x,y,t) + v_0(x,y,t) \right), \\
w_1(x,y,0) &= 0.
\end{aligned}
$$

k^{th}-order problem:

$$
\begin{aligned}
\frac{\partial}{\partial t} w_k(x,y,t) - \frac{\partial}{\partial t} w_{k-1}(x,y,t) &= c_1 \left[\frac{\partial}{\partial t} w_{k-1}(x,y,t) + \frac{\partial}{\partial x} v_{k-1}(x,y,t) - \frac{\partial}{\partial y} v_{k-1}(x,y,t) + v_{k-1}(x,y,t) \right] \\
&+ c_2 \left[\frac{\partial}{\partial t} w_{k-2}(x,y,t) + \frac{\partial}{\partial x} v_{k-2}(x,y,t) - \frac{\partial}{\partial y} v_{k-2}(x,y,t) + v_{k-2}(x,y,t) \right] \\
&+ \ldots \\
&+ c_{k-1} \left[\frac{\partial}{\partial t} w_1(x,y,t) + \frac{\partial}{\partial x} v_1(x,y,t) - \frac{\partial}{\partial y} v_1(x,y,t) + v_1(x,y,t) \right] \\
&+ \quad c_k \left[\frac{\partial}{\partial t} w_0(x,y,t) + \frac{\partial}{\partial x} v_0(x,y,t) - \frac{\partial}{\partial y} v_0(x,y,t) + v_0(x,y,t) \right], \\
w_k(x,y,0) &= 0, \qquad k = 2,3,\ldots.
\end{aligned}
$$

Solving these problems up to order three gives

$$
\begin{aligned}
u_0(x,y,t) &= \sin(x+y),\ v_0(x,y,t) = \cos(x+y),\ w_0(x,y,t) = -\sin(x+y),\\
u_1(x,y,t) &= -c_1 t\cos(x+y),\ v_1(x,y,t) = c_1 t\sin(x+y),\ w_1(x,y,t) = c_1 t\cos(x+y),\\
u_2(x,y,t) &= -c_1 t\cos(x+y) - c_1^2 t\cos(x+y) + c_1^2\frac{t^2}{2!}\sin(x+y) - c_2 t\cos(x+y),\\
v_2(x,y,t) &= c_1 t\sin(x+y) + c_1^2 t\sin(x+y) - c_1^2\frac{t^2}{2!}\cos(x+y) + c_2 t\sin(x+y),\\
w_2(x,y,t) &= c_1 t\cos(x+y) + c_1^2 t\cos(x+y) + c_1^2\frac{t^2}{2!}\sin(x+y) + c_2 t\cos(x+y),\\
u_3(x,y,t) &= -c_1 t\cos(x+y) - 2c_1^2 t\cos(x+y) - c_1^2 t^2\sin(x+y) - c_1^3 t\cos(x+y)\\
&\quad + c_1^3\frac{t^3}{3!}\cos(x+y) - c_1^3 t^2\sin(x+y) - c_2 t\cos(x+y) - 2c_1c_2 t\cos(x+y)\\
&\quad - c_1c_2 t^2\sin(x+y) - c_3 t\cos(x+y),\\
v_3(x,y,t) &= c_1 t\sin(x+y) + 2c_1^2 t\sin(x+y) - c_1^2 t^2\cos(x+y) + c_1^3 t\sin(x+y)\\
&\quad - c_1^3\frac{t^3}{3!}\sin(x+y) - c_1^3 t^2\cos(x+y) + c_2 t\sin(x+y) + 2c_1c_2 t\sin(x+y)\\
&\quad - c_1c_2 t^2\cos(x+y) + c_3 t\sin(x+y),\\
w_3(x,y,t) &= c_1 t\cos(x+y) + 2c_1^2 t\cos(x+y) + c_1^2 t^2\sin(x+y) + c_1^3 t\cos(x+y)\\
&\quad - c_1^3\frac{t^3}{3!}\cos(x+y) + c_1^3 t^2\sin(x+y) + c_2 t\cos(x+y) + 2c_1c_2 t\cos(x+y)\\
&\quad + c_1c_2 t^2\sin(x+y) + c_3 t\cos(x+y)
\end{aligned}
$$

and so on. The third-order approximate solutions read
$u = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t)$,
$v = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + v_3(x,y,t)$,
$w = w_0(x,y,t) + w_1(x,y,t) + w_2(x,y,t) + w_3(x,y,t)$.

To determine the convergence control constants c_1, c_2 and c_3, we utilize MATHEMATICA to minimize the L_2−error, Equation(4.5). As a result, we obtain $c_1 = -0.000453$, $c_2 = -367.916735$ and $c_3 = 735.168163$. In Figures 4.5, 4.7 and 4.9 we will illustrate the exact solutions $u(x,y,t) = \sin(x+y+t)$, $v(x,y,t) = \cos(x+y+t)$ and $w(x,y,t) = -\sin(x+y+t)$ at $t = 2$. Meanwhile, in Figures 4.6, 4.8 and 4.10, we depict the third order approximate solutions u, v and w on same domain. Upon comparison with the results obtained from the HAM method shown in Figures 3.6, 3.8, and 3.10, we observe some improvement.

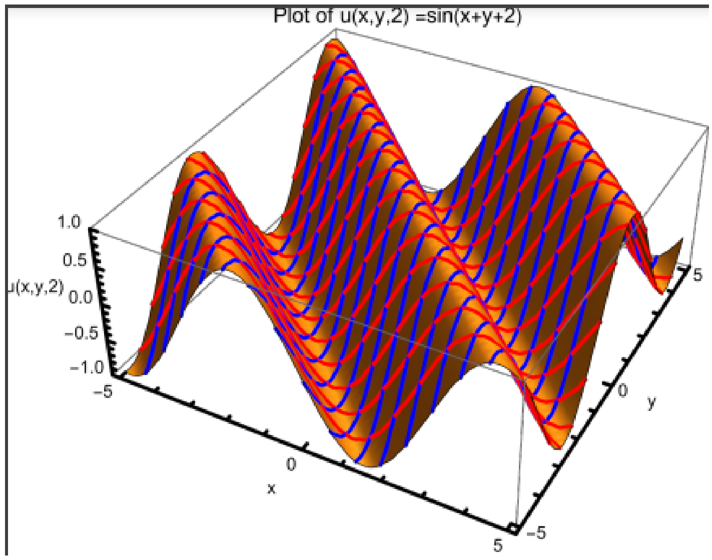

Figure 4.5: Exact solution $u(x, y, 2) = \sin(x + y + 2)$

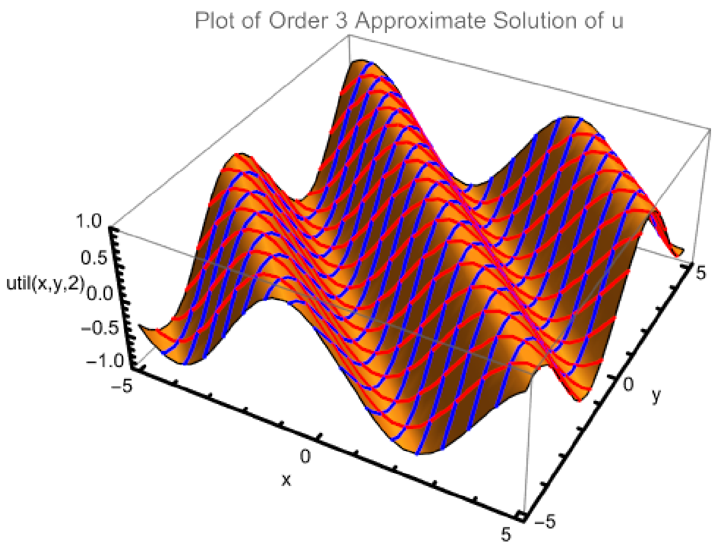

Figure 4.6: Approximate solution $u = util(x, y, 2)$ (OHAM)

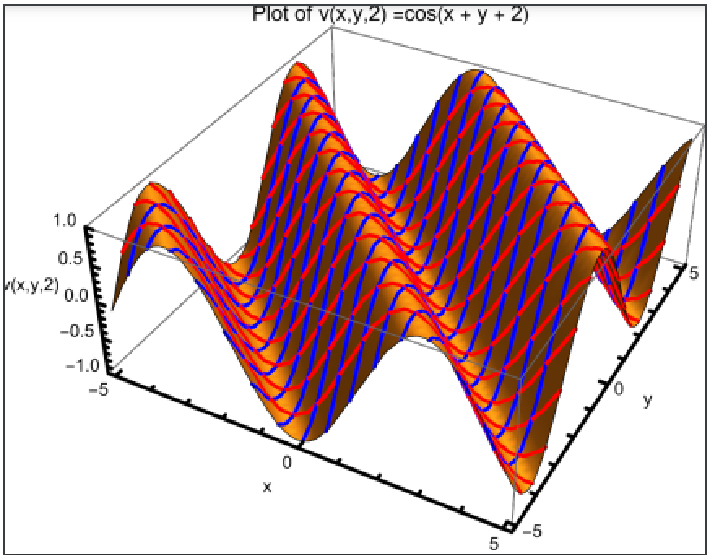

Figure 4.7: Exact solution $v(x, y, 2) = \cos(x + y + 2)$

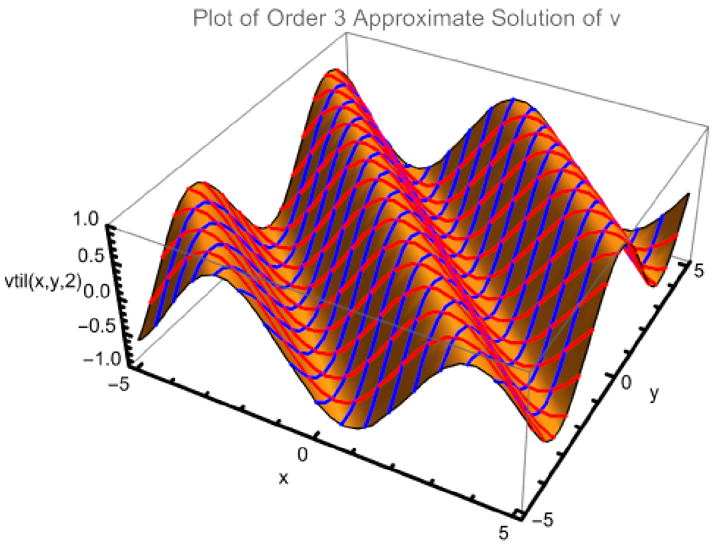

Figure 4.8: Approximate solution $v = vtil(x, y, 2)$ (OHAM)

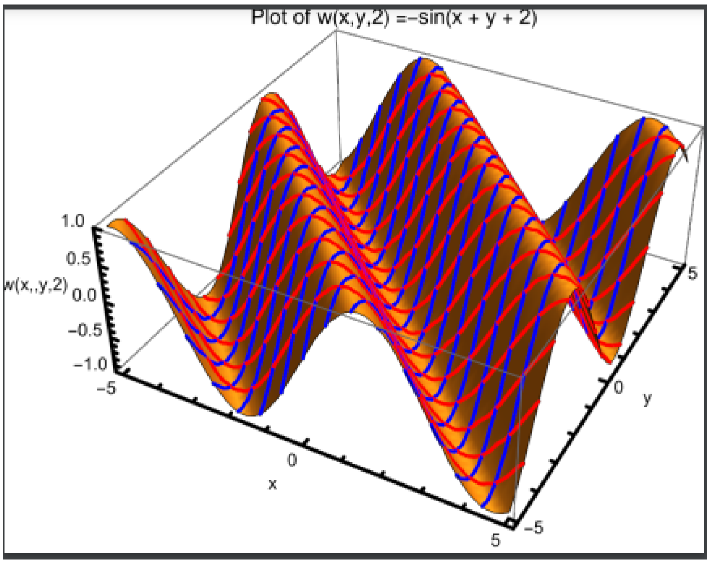

Figure 4.9: Exact solution $w(x, y, 2) = -\sin(x + y + 2)$

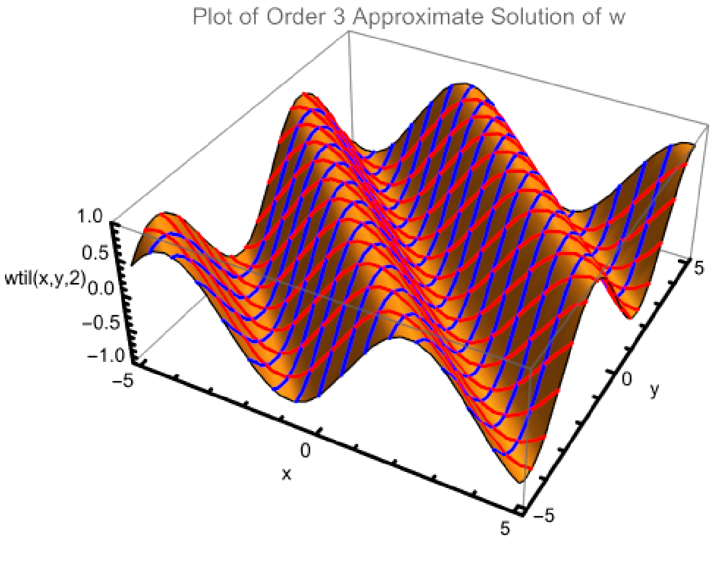

Figure 4.10: Approximate solution $w = wtil(x, y, 2)$ (OHAM)

4.4 Illustrative Example for Nonlinear Equations

Example 4.3. *We consider the two space variables nonlinear system of partial differential equations :*

$$\begin{aligned} u_t + u_x v_x + u_y v_y + u &= 0, \\ v_t + v_x w_x - v_y w_y - v &= 0, \\ w_t + w_x u_x + w_y u_y - w &= 0. \end{aligned}$$

together with initial conditions

$$\begin{aligned} u(x,y,0) &= u_0(x,y) = e^{x+y}, \\ v(x,y,0) &= v_0(x,y) = e^{x-y}, \\ w(x,y,0) &= w_0(x,y) = e^{-x+y}. \end{aligned}$$

Again, following Equation(4.3), we expand $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ in power series in λ as

$$u(x,y,t,\lambda) = u_0(x,y,t) + \sum\nolimits_{n=1}^{\infty} u_n(x,y,t)\lambda^n, \tag{4.15}$$

$$v(x,y,t,\lambda) = v_0(x,y,t) + \sum\nolimits_{n=1}^{\infty} v_n(x,y,t)\lambda^n, \tag{4.16}$$

and

$$w(x,y,t,\lambda) = w_0(x,y,t) + \sum\nolimits_{n=1}^{\infty} w_n(x,y,t)\lambda^n. \tag{4.17}$$

The operators L and N that appear in the deformation equation (4.8) are defined as

$$\begin{aligned} L(u(x,y,t,\lambda)) &= \frac{\partial}{\partial t}u(x,y,t), \\ L(v(x,y,t,\lambda)) &= \frac{\partial}{\partial t}v(x,y,t), \\ L(w(x,y,t,\lambda)) &= \frac{\partial}{\partial t}w(x,y,t), \\ N(u(x,y,t,\lambda)) &= \frac{\partial}{\partial x}u(x,y,t)\frac{\partial}{\partial x}v(x,y,t) + \frac{\partial}{\partial y}u(x,y,t)\frac{\partial}{\partial y}v(x,y,t) + u(x,y,t), \\ N(v(x,y,t,\lambda)) &= \frac{\partial}{\partial x}v(x,y,t)\frac{\partial}{\partial x}w(x,y,t) - \frac{\partial}{\partial y}v(x,y,t)\frac{\partial}{\partial y}w(x,y,t) - v(x,y,t), \\ N(w(x,y,t,\lambda)) &= \frac{\partial}{\partial x}w(x,y,t)\frac{\partial}{\partial x}u(x,y,t) + \frac{\partial}{\partial y}w(x,y,t)\frac{\partial}{\partial y}u(x,y,t) - w(x,y,t). \end{aligned}$$

The deformation equation corresponding to $u(x,y,t)$ is given by

$$(1-\lambda)L[u(x,y,t,\lambda)] = H(\lambda)\left[L\left(u(x,y,t,\lambda)\right) + N\left(u(x,y,t,\lambda)\right)\right], \tag{4.18}$$

or we write with the aid of Equations (4.15-4.17) and the definitions of the operators L and

N we write Equation (4.18) as

$$
\begin{aligned}
&(1-\lambda)\left[\frac{\partial}{\partial t}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n\right] \\
&= \left(\sum_{j=1}^{\infty}c_j\lambda^j\right)\left[\frac{\partial}{\partial t}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n\right. \\
&+ \left(\frac{\partial}{\partial x}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial x}u_n(x,y,t)\lambda^n\right)\left(\frac{\partial}{\partial x}v_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial x}v_n(x,y,t)\lambda^n\right) \\
&+ \left(\frac{\partial}{\partial y}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial y}u_n(x,y,t)\lambda^n\right)\left(\frac{\partial}{\partial y}v_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial y}v_n(x,y,t)\lambda^n\right) \\
&+ \left. u_0(x,y,t)+\sum_{n=1}^{\infty}u_n(x,y,t)\lambda^n\right] \qquad (4.19)
\end{aligned}
$$

The left and right-hand sides are rearranged in the forms

$$
\begin{aligned}
L.H.S. &= \frac{\partial}{\partial t}u_0(x,y,t)-\lambda\frac{\partial}{\partial t}u_0(x,y,t)+\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^n-\sum_{n=1}^{\infty}\frac{\partial}{\partial t}u_n(x,y,t)\lambda^{n+1} \\
&= \frac{\partial}{\partial t}u_0(x,y,t)+\left(\frac{\partial}{\partial t}u_1(x,y,t)-\frac{\partial}{\partial t}u_0(x,y,t)\right)\lambda \\
&+ \sum_{n=1}^{\infty}\left(\frac{\partial}{\partial t}u_{n+1}(x,y,t)-\frac{\partial}{\partial t}u_n(x,y,t)\right)\lambda^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
R.H.S. &= c_1\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_0(x,y,t)\right]\lambda \\
&+ \left(c_1\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial x}u_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)\right.\right.+ \\
&\qquad \left.\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_1(x,y,t)+\frac{\partial}{\partial y}u_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_1(x,y,t)\right] \\
&+ \left.c_2\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_0(x,y,t)\right]\right)\lambda^2 \\
&+ \left(c_1\left[\frac{\partial}{\partial t}u_2(x,y,t)+\frac{\partial}{\partial x}u_2(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial x}u_1(x,y,t)\frac{\partial}{\partial x}v_1(x,y,t)\right.\right.+ \\
&\qquad \frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_2(x,y,t)+\frac{\partial}{\partial y}u_2(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+\frac{\partial}{\partial y}u_1(x,y,t)\frac{\partial}{\partial y}v_1(x,y,t)+ \\
&\qquad \left.\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_2(x,y,t)+u_2(x,y,t)\right] \\
&+ c_2\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial x}u_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)\right.+ \\
&\qquad \left.\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_1(x,y,t)+\frac{\partial}{\partial y}u_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_1(x,y,t)\right] \\
&+ \left.c_3\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_0(x,y,t)\right]\right)\lambda^3 \\
&+ \ldots
\end{aligned}
$$

Equating the coefficients of the same powers of λ on both sides we get the deformation equations:
Zeroth order deformation equation:

$$\frac{\partial}{\partial t}u_0(x,y,t)=0.$$

First-order deformation equation:

$$\begin{aligned}&\frac{\partial}{\partial t}u_1(x,y,t)-\frac{\partial}{\partial t}u_0(x,y,t)\\&\quad=c_1\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_0(x,y,t)\right].\end{aligned}$$

Second-order deformation equation

$$\begin{aligned}&\frac{\partial}{\partial t}u_2(x,y,t)-\frac{\partial}{\partial t}u_1(x,y,t)\\&\quad=c_1\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial x}u_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\right.\\&\qquad\left.\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_1(x,y,t)+\frac{\partial}{\partial y}u_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_1(x,y,t)\right]+\\&\quad c_2\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_0(x,y,t)\right].\end{aligned}$$

Third-order deformation equation:

$$\begin{aligned}&\frac{\partial}{\partial t}u_3(x,y,t)-\frac{\partial}{\partial t}u_2(x,y,t)\\&\quad=c_1\left[\frac{\partial}{\partial t}u_2(x,y,t)+\frac{\partial}{\partial x}u_2(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial x}u_1(x,y,t)\frac{\partial}{\partial x}v_1(x,y,t)+\right.\\&\qquad\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_2(x,y,t)+\frac{\partial}{\partial y}u_2(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+\frac{\partial}{\partial y}u_1(x,y,t)\frac{\partial}{\partial y}v_1(x,y,t)+\\&\qquad\left.\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_2(x,y,t)+u_2(x,y,t)\right]+\\&\quad c_2\left[\frac{\partial}{\partial t}u_1(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_1(x,y,t)+\frac{\partial}{\partial x}u_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\right.\\&\qquad\left.\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_1(x,y,t)+\frac{\partial}{\partial y}u_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_1(x,y,t)\right]+\\&\quad c_3\left[\frac{\partial}{\partial t}u_0(x,y,t)+\frac{\partial}{\partial x}u_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t)+\frac{\partial}{\partial y}u_0(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t)+u_0(x,y,t)\right],\end{aligned}$$

and so on. Similarly, with the aid of Equations (4.15)-(4.17) and the definitions of the operators L and N we derive the following deformation equations for $v(x,y,t)$ and $w(x,y,t)$:
For $v(x,y,t)$:
Zeroth order deformation equation:

$$\frac{\partial}{\partial t}v_0(x,y,t)=0.$$

First-order deformation equation:

$$\frac{\partial}{\partial t}v_1(x,y,t) - \frac{\partial}{\partial t}v_0(x,y,t) = c_1\left[\frac{\partial}{\partial t}v_0(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t)\frac{\partial}{\partial x}w_0(x,y,t) - \frac{\partial}{\partial y}v_0(x,y,t)\frac{\partial}{\partial y}w_0(x,y,t) - v_0(x,y,t)\right]$$

Second-order deformation equation

$$\begin{aligned}\frac{\partial}{\partial t}v_2(x,y,t) - \frac{\partial}{\partial t}v_1(x,y,t) &= c_1\left[\frac{\partial}{\partial t}v_1(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t)\frac{\partial}{\partial x}w_1(x,y,t) + \frac{\partial}{\partial x}v_1(x,y,t)\frac{\partial}{\partial x}w_0(x,y,t) - \right.\\ &\left.\frac{\partial}{\partial y}v_0(x,y,t)\frac{\partial}{\partial y}w_1(x,y,t) - \frac{\partial}{\partial y}v_1(x,y,t)\frac{\partial}{\partial y}w_0(x,y,t) - v_1(x,y,t)\right] + \\ &c_2\left[\frac{\partial}{\partial t}v_0(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t)\frac{\partial}{\partial x}w_0(x,y,t) - \frac{\partial}{\partial y}v_0(x,y,t)\frac{\partial}{\partial y}w_0(x,y,t) - v_0(x,y,t)\right].\end{aligned}$$

Third-order deformation equation:

$$\begin{aligned}\frac{\partial}{\partial t}v_3(x,y,t) - \frac{\partial}{\partial t}v_2(x,y,t) &= c_1\left[\frac{\partial}{\partial t}v_2(x,y,t) + \frac{\partial}{\partial x}v_2(x,y,t)\frac{\partial}{\partial x}w_0(x,y,t) + \frac{\partial}{\partial x}v_1(x,y,t)\frac{\partial}{\partial x}w_1(x,y,t) + \right.\\ &\frac{\partial}{\partial x}v_0(x,y,t)\frac{\partial}{\partial x}w_2(x,y,t) - \frac{\partial}{\partial y}v_2(x,y,t)\frac{\partial}{\partial y}w_0(x,y,t) - \frac{\partial}{\partial y}v_1(x,y,t)\frac{\partial}{\partial y}w_1(x,y,t) - \\ &\left.\frac{\partial}{\partial y}v_0(x,y,t)\frac{\partial}{\partial y}w_2(x,y,t) - v_2(x,y,t)\right] + \\ &c_2\left[\frac{\partial}{\partial t}v_1(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t)\frac{\partial}{\partial x}w_1(x,y,t) + \frac{\partial}{\partial x}v_1(x,y,t)\frac{\partial}{\partial x}w_0(x,y,t) - \right.\\ &\left.\frac{\partial}{\partial y}v_0(x,y,t)\frac{\partial}{\partial y}w_1(x,y,t) - \frac{\partial}{\partial y}v_1(x,y,t)\frac{\partial}{\partial y}w_0(x,y,t) - v_1(x,y,t)\right] + \\ &c_3\left[\frac{\partial}{\partial t}v_0(x,y,t) + \frac{\partial}{\partial x}v_0(x,y,t)\frac{\partial}{\partial x}w_0(x,y,t) - \frac{\partial}{\partial y}v_0(x,y,t)\frac{\partial}{\partial y}w_0(x,y,t) - v_0(x,y,t)\right],\end{aligned}$$

and so on.

For $w(x,y,t)$:

Zeroth order deformation equation:

$$\frac{\partial}{\partial t}w_0(x,y,t) = 0.$$

First-order deformation equation:

$$\frac{\partial}{\partial t}w_1(x,y,t) - \frac{\partial}{\partial t}w_0(x,y,t) = c_1\left[\frac{\partial}{\partial t}w_0(x,y,t) + \frac{\partial}{\partial x}w_0(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t) + \frac{\partial}{\partial y}w_0(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t) - w_0(x,y,t)\right]$$

Second-order deformation equation

$$\begin{aligned}
&\frac{\partial}{\partial t}w_2(x,y,t) - \frac{\partial}{\partial t}w_1(x,y,t) \\
&\quad = c_1\left[\frac{\partial}{\partial t}w_1(x,y,t) + \frac{\partial}{\partial x}w_0(x,y,t)\frac{\partial}{\partial x}u_1(x,y,t) + \frac{\partial}{\partial x}w_1(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t) + \right. \\
&\qquad \left.\frac{\partial}{\partial y}w_0(x,y,t)\frac{\partial}{\partial y}u_1(x,y,t) + \frac{\partial}{\partial y}w_1(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t) - w_1(x,y,t)\right] + \\
&\quad c_2\left[\frac{\partial}{\partial t}w_0(x,y,t) + \frac{\partial}{\partial x}w_0(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t) + \frac{\partial}{\partial y}w_0(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t) - w_0(x,y,t)\right].
\end{aligned}$$

Third-order deformation equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}w_3(x,y,t) - \frac{\partial}{\partial t}w_2(x,y,t) \\
&\quad = c_1\left[\frac{\partial}{\partial t}w_2(x,y,t) + \frac{\partial}{\partial x}w_2(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t) + \frac{\partial}{\partial x}w_1(x,y,t)\frac{\partial}{\partial x}u_1(x,y,t) + \right. \\
&\qquad \frac{\partial}{\partial x}w_0(x,y,t)\frac{\partial}{\partial x}u_2(x,y,t) + \frac{\partial}{\partial y}w_2(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t) + \frac{\partial}{\partial y}w_1(x,y,t)\frac{\partial}{\partial y}u_1(x,y,t) + \\
&\qquad \left.\frac{\partial}{\partial y}w_0(x,y,t)\frac{\partial}{\partial y}u_2(x,y,t) - w_2(x,y,t)\right] + \\
&\quad c_2\left[\frac{\partial}{\partial t}w_1(x,y,t) + \frac{\partial}{\partial x}w_0(x,y,t)\frac{\partial}{\partial x}u_1(x,y,t) + \frac{\partial}{\partial x}w_1(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t) + \right. \\
&\qquad \left.\frac{\partial}{\partial y}w_0(x,y,t)\frac{\partial}{\partial y}u_1(x,y,t) + \frac{\partial}{\partial y}w_1(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t) - w_1(x,y,t)\right] + \\
&\quad c_3\left[\frac{\partial}{\partial t}w_0(x,y,t) + \frac{\partial}{\partial x}w_0(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t) + \frac{\partial}{\partial y}w_0(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t) - w_0(x,y,t)\right],
\end{aligned}$$

and so on. Solving the previous deformation equations by applying the initial conditions

$$u_0(x,y,0) = u_0(x,y) = e^{x+y},\ v_0(x,y,0) = v_0(x,y) = e^{x-y},\ w_0(x,y,0) = w_0(x,y) = e^{-x+y},$$

and

$$u_n(x,y,0) = v_n(x,y,0) = w_n(x,y,0) = 0,\ \ n = 1,2,3,\ldots,$$

leads to

$$
\begin{aligned}
u_1(x,y,t) &= \ c_1te^{x+y}, \ v_1(x,y,t) = -c_1te^{x-y}, \ w_1(x,y,t) = -c_1te^{-x+y},\\
u_2(x,y,t) &= \ c_1te^{x+y} + c_1^2te^{x+y} + c_1^2\frac{t^2}{2!}e^{x+y} + c_2te^{x+y},\\
v_2(x,y,t) &= -c_1te^{x-y} - c_1^2te^{x-y} + c_1^2\frac{t^2}{2!}e^{x-y} - c_2te^{x-y},\\
w_2(x,y,t) &= \ -c_1te^{-x+y} - c_1^2te^{-x+y} + c_1^2\frac{t^2}{2!}e^{-x+y} - c_2te^{-x+y},\\
u_3(x,y,t) &= \ c_1te^{x+y} + 2c_1^2te^{x+y} + c_1^2t^2e^{x+y} + c_2te^{x+y} + c_1^3te^{x+y} + c_1^3t^2e^{x+y} +\\
&\qquad c_1^3\frac{t^3}{3!}e^{x+y} + c_1c_2te^{x+y} + c_1c_2\frac{t^2}{2!}e^{x+y} + c_3te^{x+y},\\
v_3(x,y,t) &= -c_1te^{x-y} - 2c_1^2te^{x-y} + c_1^2t^2e^{x-y} - c_2te^{x-y} - c_1^3te^{x-y} + c_1^3t^2e^{x-y} -\\
&\qquad c_1^3\frac{t^3}{3!}e^{x-y} - c_1c_2te^{x-y} + c_1c_2\frac{t^2}{2!}e^{x-y} - c_3te^{x-y},\\
w_3(x,y,t) &= -c_1te^{-x+y} - 2c_1^2te^{-x+y} + c_1^2t^2e^{-x+y} - c_2te^{-x+y} - c_1^3te^{-x+y} + c_1^3t^2e^{-x+y} -\\
&\qquad c_1^3\frac{t^3}{3!}e^{-x+y} - c_1c_2te^{-x+y} + c_1c_2\frac{t^2}{2!}e^{-x+y} - c_3te^{-x+y},
\end{aligned}
$$

and so on. Combining these solutions we get, for example, the fourth-order approximate solutions of $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$, respectively:

$$u = u_0(x,y,t) + \sum_{k=1}^{4} u_k(x,y,t)$$

$$v = v_0(x,y,t) + \sum_{k=1}^{4} v_k(x,y,t)$$

$$w = w_0(x,y,t) + \sum_{k=1}^{4} w_k(x,y,t)$$

Note that in general, the deformation equations for $u(x,y,t)$ can be restated using the operators L and N as follows:
The zeroth, first, and kth deformation equations, respectively:

$$
\begin{aligned}
&L(u_0(x,y,t)) = 0,\\
&L(u_1(x,y,t) = c_1N_0(u_0(x,y,t)),\\
&L(u_k(x,y,t) - u_{k-1}(x,y,t))\\
&\qquad = c_kN_0(u_0(x,y,t)) \ +\\
&\qquad \sum_{i=1}^{k-1} c_i \left(L(u_{k-i}(x,y,t)) \ + \right.\\
&\qquad \left. N_{k-i}(u_0(x,y,t), u_1(x,y,t), \ldots, u_{k-i}(x,y,t), v_0(x,y,t), v_1(x,y,t), \ldots, v_{k-i}(x,y,t))\right),
\end{aligned}
$$

where $N_m(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots v_m)$ is the coefficient of λ^m obtained by expanding

$N(u(x,y,t,\lambda))$ in series with respect to the embedding parameter λ. In fact

$$N_0(u_0(x,y,t)) = \left(\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\frac{\partial v_0}{\partial y} + u_0\right)\Bigg|_{(x,y,t)},$$

$$\begin{aligned} N_m(u_0,u_1,\ldots,u_m,v_0,v_1,\ldots v_m)|_{(x,y,t)} \quad = \quad & \left(\frac{\partial u_0}{\partial x}\frac{\partial v_m}{\partial x} + \frac{\partial u_1}{\partial x}\frac{\partial v_{m-1}}{\partial x} + \ldots + \frac{\partial u_m}{\partial x}\frac{\partial v_0}{\partial x} + \right. \\ & \left. \frac{\partial u_0}{\partial y}\frac{\partial v_m}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial v_{m-1}}{\partial y} + \ldots + \frac{\partial u_m}{\partial y}\frac{\partial v_0}{\partial y} + u_m\right)\Bigg|_{(x,y,t)}. \end{aligned}$$

Similar zeroth, first, and kth deformation equations for $v(x,y,t)$ and $w(x,y,t)$ can be derived. As in previous examples, to determine the convergence control constants c_1, c_2, c_3 and c_4, we utilize MATHEMATICA to minimize the L_2-error, Equation(4.5), over the spacial domain $[-5,-5] \times [-5,-5]$. As a result, we obtain $c_1 = 0.586623$, $c_2 = -3.40891$, $c_3 = 7.86808$ and $c_4 = -8.83021$. In Figures 4.11, 4.13 and 4.15 we will illustrate the exact solutions $u(1,y,t) = e^{(1+y-t)}$, $v(1,y,t) = e^{(1-y+t)}$ and $w(1,y,t) = e^{(-1+y+t)}$. Meanwhile, in Figures 4.12, 4.14 and 4.16, we depict the fourth order approximate solutions u, v and w on same domain. Upon comparison with the results obtained from the HAM method shown in Figures 3.12, 3.14 and 3.16, we observe greater improvement, particularly in the u component.

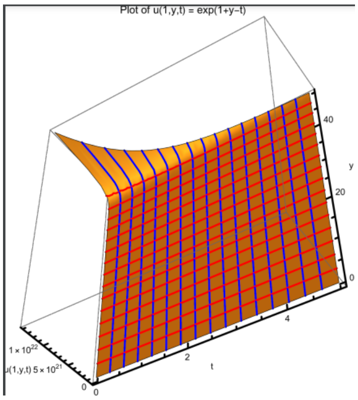

Figure 4.11: Exact solution $u(1, y, t) = e^{(1+y-t)}$

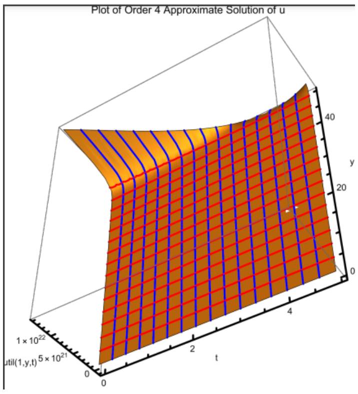

Figure 4.12: Approximate solution $u = util(1, y, t)$ (OHAM)

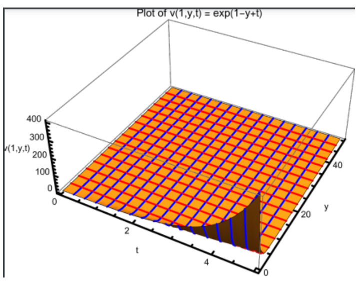

Figure 4.13: Exact solution $v(1, y, t) = e^{(1-y+t)}$

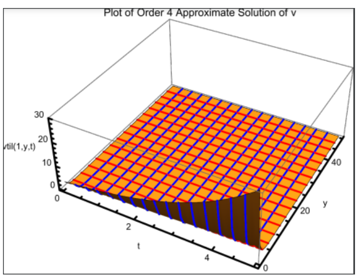

Figure 4.14: Approximate solution $v = vtil(1, y, t)$ (OHAM)

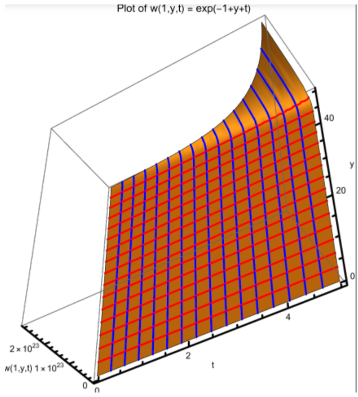

Figure 4.15: Exact solution $w(1, y, t) = e^{(-1+y+t)}$

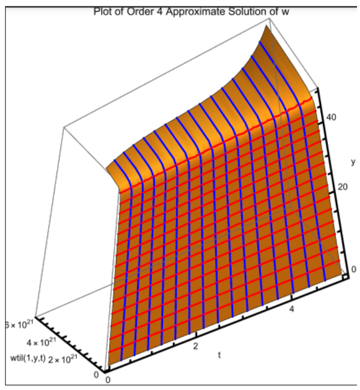

Figure 4.16: Approximate solution $w = wtil(1, y, t)$ (OHAM)

Conclusion and Outlook

The Homotopy Analysis Method (HAM) and the Optimal Homotopy Asymptotic Method (OHAM) are utilized to derive analytical approximate solutions for linear and non-linear systems of partial differential equations. A comparison between the resulting approximated solutions and the exact solutions demonstrates OHAM's reliability and capacity to offer analytical approximate solutions for such equations.

The close correspondence between the derived analytical approximated solutions and the exact solutions serves as an encouragement to explore the application of this technique in solving physical systems of partial differential equations, particularly in cases where the initial conditions lack smoothness. Furthermore, the method developed in this thesis can be analogously applied to systems in three or more spatial dimensions, just as they are applied to systems in two spatial dimensions.

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العنوان: طريقة (OHAM) لحل أنظمة المعادلات التفاضلية الجزئية من الدرجة الاولى

اعداد: ملاك توفيق عبدالفتاح مقبل

اشراف: د. يوسف زحايقه

ملخص

في هذه الأطروحة استخدمنا الأسلوب المعروف (OHAM) المطور من أسلوب (HAM) لبناء طريقة عددية شبه تحليلية تستخدم لإيجاد حلول تقريبية لأنظمة المعادلات التفاضلية الخطية وغير الخطية من الدرجة الأولى.

في إطار هذا العمل تم استخدام المفهوم الهندسي التوبولوجي (Homotopy) لبناء تلك الطريقة، حيث تم افتراض تحقق معادلة (Homotopy) محتوية على معلمة تضمينية (Embedding Parameter) تنتمي قيمها الى الفترة المغلقة بين الصفر والواحد، بحيث انه عند تغير قيمة المعلمة من الصفر الى الواحد يتغير حل تلك المعادلة (والذي هو عبارة عن متسلسلة قوة في المعلمة) بطريقة متصلة من حل يسهل ايجاده الى الحل المغلق (الحقيقي). وللحصول على الحل التقريبي يتم قطع المتسلسلة واستخدام عدد محدود من حدودها، ولتحديد الثوابت التي تظهر في الحل التقريبي (التي تعرف بثوابت التحكم في التقارب) نستخدم طريقة المربعات الصغرى.

لقد تم في هذا البحث تطبيق الطريقة المشتقة على عدد من الأمثلة التي لها حلول مغلقة معروفة وأشارت النتائج الى أن هذه الطريقة تتصف بالدقة اضافة إلى كونها طريقة بسيطة وفعالة لا تعتمد على المعلمات الصغيرة المطلوبة في أساليب الاضطراب (Perturbation Methods)، كما ان مجال التقارب الخاص بطريقة OHAM قابل للتعديل بسهولة وذلك من خلال الاعتماد على ثوابت التحكم بالتقارب التي تظهر في الحلول التقريبية، مما يعزز تنوعها.